TÉCNICO LISBOA

12th January 2022
Lecturers: F. Nabais, H. Terças

Advanced Plasma Physics

MEFT 2021/22

Problem Class 4

> Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

Problem 1. Surface plasmon-polaritons. Consider electromagnetic (EM) waves that propagate at the *interface* of two materials. Consider that a cold plasma, of dielectric constant $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$ occupies the lower half-plane $z < 0$, while a dielectric material (e.g. a polymer) characterized by a relative dielectric constant $\epsilon = \epsilon_2$ fills the upper half-plane, $z > 0$. We are looking for "polaritons", a new surface mode resulting from the hybridization between the plasma oscillation (plasmons) and the transverse waves in the dielectric (photons).

a) Start from the relevant Maxwell's equations in the absence of sources to show that the electric field at a generic point z of the space obeys the Helmholtz equation

$$\left(\nabla^2 + k_0^2\epsilon(z)\right)\mathbf{E} = 0, \quad \text{with} \quad \epsilon(z) = \begin{cases} \epsilon_2, & z > 0 \\ 1 - \dfrac{\omega_p^2}{\omega^2}, & z < 0 \end{cases}$$

and $k_0 = \omega/c$ standing for the vacuum wavevector of a EM wave of frequency ω.

b) By choosing the direction of propagation along x, $\mathbf{k} = k\hat{\mathbf{x}}$, we may split the electric field as $\mathbf{E}(x, y, z) = e^{ikz}\mathbf{E}(y, z)$. Combine both Faraday's and Ampère's law to obtain

$$\begin{cases} \dfrac{\partial E_y}{\partial z} = -i\omega B_x, \\ \dfrac{\partial E_x}{\partial z} - ikE_z = i\omega B_y, \\ ikE_y = i\omega B_z \end{cases} \qquad \begin{cases} \dfrac{\partial B_y}{\partial z} = i\dfrac{\omega}{c^2}\omega E_x, \\ \dfrac{\partial B_x}{\partial z} - ikB_z = -i\dfrac{\omega}{c^2}E_y, \\ ikB_y = -i\dfrac{\omega}{c^2}E_z. \end{cases}$$

c) Consider a TM-polarized mode, in which $\mathbf{B} = (0, B_y, 0)$ and $\mathbf{E} = (E_x, 0, E_z)$. Show that the wave equation is given by

$$\frac{\partial^2 B_y}{\partial z^2} + \left(\epsilon(z)k_0^2 - k^2\right)B_y = 0.$$

Look for evanescent waves $B_y(z) \sim e^{-\kappa_i}|z|$, where $\kappa_i = 1/\lambda_i$ and λ_i is the skin depth of the $i-$th medium.

d) Apply the appropriate continuity conditions for the magnetic fields **H** and displacement vector **D** at the interface to show that dispersion relation is given by

$$\frac{c^2k^2}{\omega^2} = \frac{\left(1 - \frac{\omega_p^2}{\omega^2}\right)\epsilon_2}{1 - \frac{\omega_p^2}{\omega^2} + \epsilon_2}.$$

e) Consider the dielectric medium to be the vacuum, $\epsilon_2 = 1$. Find the two modes $\omega(k)$ supported by the interface and show that there is a forbidden gap given by the condition

$$\frac{\omega_p}{\sqrt{2}} \leq \omega \leq \omega_p$$

Problem 2. Mie resonances. Another interesting class of plasma modes in finite-sized systems concerns the oscillations taking place in a plasma sphere. They can be both of electrostatic and electromagnetic nature. The latter was first investigated by Gustav Mie in the context of scattering of EM waves by metallic nanoparticles.

a) Consider a spherical plasma of radius R, immersed in a dielectric medium of dielectric constant ϵ_2. Show that the electrostatic oscillations obey the equation

$$\boldsymbol{\nabla} \cdot (\epsilon(\omega)\boldsymbol{\nabla}\phi_1) = 0, \quad \text{with} \quad \epsilon(\omega) = \begin{cases} 1 - \dfrac{\omega_p^2}{\omega^2}, & r \leq R \\ \epsilon_2, & r > R \end{cases}.$$

b) Solve Laplace's equation via the variable separation method $\phi_1(r, \theta, \varphi) = R(r)Y_{m,\ell}(\theta, \varphi)$, where $Y_{m,\ell}(\theta, \varphi)$ are the spherical harmonics typical of a problem with spherical symmetry,

$$Y_{m,\ell}(\theta, \varphi) = e^{im\varphi}P_{\ell,m}(\cos\theta),$$

ans $P_{\ell,m}(x)$ the associated Legendre polynomials. Show that the radial equation obeys

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \ell(\ell+1)R = 0.$$

c) Use the polynomial method to find a general solution to the radial equation in the form

$$R(r) = \sum_n a_n r^{n+\alpha}.$$

Find the two possible solutions for the exponent α.

d) Based on the solutions you found for α, argue that the solution for the problem in hand can be parametrized as

$$R(r) = \begin{cases} \displaystyle\sum_\ell a_\ell \, r^\ell, & r \leq R \\ \displaystyle\sum_\ell b_\ell \, r^{-(\ell+1)}, & r > R \end{cases}.$$

e) Apply the continuity of ϕ_1 and $\mathbf{D}$ at the interface $r = R$ and show that oscillations exist provided that

$$\omega = \omega_p\sqrt{\frac{\ell}{\ell + \epsilon_2(\ell+1)}}.$$

Observe that, for vacuum ($\epsilon_2 = 1$), the modes occur in the range

$$\frac{\omega_p}{\sqrt{3}} \leq \omega \leq \frac{\omega_p}{\sqrt{2}}.$$

Comment on this result.