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Advanced Plasma Physics

MEFT 2021/22

Problem Class 3

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

(To practice at home) Bernstein-Kruskal-Green modes. Consider a uniform plasma, magnetized by an externally imposed magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$. As you must remember from the previous plasma course, within the fluid description of plasma, there is an electron mode (electrostatic) propagating at the transverse direction of the magnetic field with dispersion relation

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 + 3k^2 v_e^2,$$

where $\omega_{ce} = eB_0/m_e$ is the electron cyclotron frequency. The quantity $\omega_{\rm UH} = \sqrt{\omega_{pe}^2 + \omega_{ce}^2}$ receives the name of *upper hybrid frequency*. The goal of the present exercise is to look for a kinetic formulation for the transverse electrostatic modes that can take place in this system. A nice treatment of this problem can be found in the book "Principles of Plasma Physics", by Krall & Trievelpiece.

a) Start by linearizing the Vlasov equation for the electrons (neglect the motion of the ions). Show that, in order to an equilibrium f_{e0} to exist, the following condition must hold

$$\frac{\partial f_{e0}}{\partial \theta} = 0,$$

i.e. that the equilibrium is a function of v_z and $v_\perp \equiv \sqrt{v_x^2 + v_y^2}$ only.

b) At lowest (i.e. unperturbed) order, the velocity of the particles must satisfy the equation

$$\dot{\mathbf{v}} = -\frac{e}{m_e}\mathbf{v}\times\mathbf{B}_0.$$

With that in mind, show that the linearized Vlasov equation leads to

$$\frac{D}{Dt}f_{e1} = \frac{e}{m_e}\left(\mathbf{E}_1 + \mathbf{v}'\times\mathbf{B}_1\right)\cdot\frac{\partial f_{e0}}{\partial \mathbf{v}'},$$

where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla} - \dfrac{e}{m_e}\left(\mathbf{v}\times\mathbf{B}_0\right)\cdot\boldsymbol{\nabla}_{\mathbf{v}}$ is the total phase-space derivative at lowest order, and $\mathbf{B}_1$ is the perturbation to the magnetic field lines due to the wave (which slightly changes the helicoidal trajectories of the particles). Explain why the formal solution of the first order equation is therefore given by

$$f_{e1}(\mathbf{r},\mathbf{v},t) = \frac{e}{m_e}\int_{-\infty}^{t}\left[\mathbf{E}_1(\mathbf{r}',t') + \mathbf{v}'\times\mathbf{B}_1(\mathbf{r}',t')\right]\cdot\frac{\partial f_{e0}}{\partial \mathbf{v}'}dt',$$

where the primes stands for quantities perturbed in phase by the wave.

c) At lowest order, the velocity may be decomposed into its components as $\mathbf{v} = (v_\perp\cos\theta, v_\perp\sin\theta, v_z)$. In the presence of perturbations, it is easy to understand that a phase proportional to the cyclotron frequency must be added, such that the most generic ansatz for the velocities is

$$\mathbf{v}' = (v_\perp\cos\chi, v_\perp\sin\chi, v_z),$$

where $\chi = \theta + \omega_{ce}(t-t')$. Use the chain rule to eliminate $\mathbf{v}'$ in the equations and show that, for the case of electrostatic waves ($\mathbf{B}_1 = 0$), we have

$$f_{e1}(\mathbf{r},\mathbf{v},t) = -\frac{e}{m_e}\int_{-\infty}^{t}\left(E_x\frac{\partial f_{e0}}{\partial v_\perp}\cos\chi + E_y\frac{\partial f_{e0}}{\partial v_\perp}\sin\chi\right)dt'.$$

d) For simplicity, let us assume propagation along xx, $E_{1x}(x',t') = \tilde{E}_1 e^{ik(x-x')-i\omega(t-t')}$. First, integrate the velocity transformation to obtain the relation

$$x - x' = \frac{v_\perp}{\omega_{ce}}(\sin\theta - \sin\chi) = -\frac{v_\perp}{\omega_{ce}}\sin[\omega_{ce}(t-t')],$$

where we have taken $\theta = 0$. Second, make use of the Bessel identity,

$$e^{ia\sin x} = \sum_{n=-\infty}^{+\infty} J_n(a)e^{inx},$$

to show that

$$\begin{aligned}\tilde{f}_{e1} &= -\frac{e}{m_e}\int_{-\infty}^{t}\left[\tilde{E}_1\frac{\partial f_{e0}}{\partial v_\perp}\cos[\omega_{ce}(t-t')]\sum_{n=-\infty}^{+\infty}J_n\left(\frac{kv_\perp}{\omega_{ce}}\right)e^{-i(n\omega_{ce}-\omega)(t-t')}\right]dt'.\\ &= -\frac{ie}{m_e}\left[\tilde{E}_1\frac{\partial f_{e0}}{\partial v_\perp}\sum_{n=-\infty}^{+\infty}J_n\left(\frac{kv_\perp}{\omega_{ce}}\right)\frac{n(\omega-\omega_{ce})}{n^2(\omega-\omega_{ce})^2-\omega_{ce}^2}\right].\end{aligned}$$

e) Make use of the Poisson equation to show that the dielectric function reads

$$\epsilon(k,\omega) = 1 - \frac{\omega_{pe}^2}{k}\sum_{n=-\infty}^{+\infty}\int J_n\left(\frac{kv_\perp}{\omega_{ce}}\right)\frac{n(\omega-\omega_{ce})}{n^2(\omega-\omega_{ce})^2-\omega_{ce}^2}\frac{\partial g_{e0}}{\partial v_\perp}dv_\perp.$$

The dispersion relation for the case of a Maxwellian plasma must be found numerically. However, it is possible to obtain analytical solutions for the cold plasma case, $g_{0e}(v_\perp) = \delta(v_\perp)$.

Problem 1. The Kortweg-de Vries equation. Let us consider the propagation of nonlinear ion-acoustic waves in uniform, unmagnetized plasmas. For that, we should rely on a fluid description of the problem (consider one-dimensional electrostatic waves, for simplicity)

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial (n_\alpha u_\alpha)}{\partial x} = 0, \quad \frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} = \frac{q_\alpha}{m_e} E - \frac{1}{m_\alpha n_\alpha} \frac{\partial P_\alpha}{\partial x}.$$

a) At the scale of the ion motion, the electrons are not at rest. On the contrary, they move so fast that they follow the ions adiabatically, therefore being in thermal equilibrium. Show that the linearlized Poisson equation yields

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\lambda_{De}^2} \right) \phi_1 = -\frac{e}{\epsilon_0} n_1.$$

b) Make use of the equations of motion for the ions to show that, in the limit $T_i \ll T_e$, we obtain

$$\omega = \frac{c_s k}{\sqrt{1 + k^2 \lambda_{De}^2}}. \tag{1}$$

c) We now come back to the original equations, but keeping the nonlinearity appearing in the momentum conservation equation (the so-called convective term). Show that

$$\mathcal{F}\left[\left(\frac{\partial}{\partial t} + u_i \frac{\partial u_i}{\partial x} \right) u_i \right] = -i \frac{\omega_{pi}^2}{\omega} \frac{k^2 \lambda_{De}^2}{1 + k^2 \lambda_{De}^2} \tilde{u}_{i1}, \tag{2}$$

where $\mathcal{F}[A(x,t)] \equiv \tilde{A}(k, \omega)$ is the Fourier transform of a certain quantity $A(x,t)$.

d) We are interested in the region of the ion spectrum where the dispersion starts loosing its acoustic character, $k\lambda_{De} \simeq 1$. For that, we replace ω in the denominator of Eq. (2). Then, we expand the denominator in the second factor of the RHS to first order. Upon replacing $k \to -i \frac{\partial}{\partial x}$ (momentum operator in quantum mechanics, right?), show that Eq. (2) reduces to the *Kortweg-de Vries equation*,

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + c_s \frac{\partial u_i}{\partial x} + \frac{1}{2} c_s \lambda_{De}^2 \frac{\partial^3 u_i}{\partial x^3} = 0.$$

e) Make use of the Mathematica script available at our webpage to observe what happens in the following cases: i) neglecting the nonlinear term, ii) neglecting the dispersive term. Discuss with your colleagues the physics of both numerical solutions.

f) Simulate the case of two solitons colliding against each other. Observe the features of such collisions. Do the wavepackets break at anytime? What happens to the original form of the solitons after the collision? Maybe you are ready to explain to your colleagues why these nonlinear waves receive the name of *solitons*.

Problem 2. Trievelpiece-Gould waves. Consider a plasma produced at the interior of a cylindrical container of radius a. Let us assume, for definiteness, that such a container is metallic. In the following calculations, we make use of the cylindrical coordinates (r, θ, z), and consider waves propagating along the column axis, i.e. the $z-$ direction.

a) Start by showing that the Poisson equation can be written as

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2}\right)\Phi = \frac{e}{\epsilon_0}\left(n_e - n_i\right),$$

where $\Phi = \Phi(r, \theta, z) = \phi(r, \theta)\varphi(z)$, and $n_i(x, y, z)$ and $n_e(x, y, z)$ are the 3D ionic and electronic densities, respectively.

b) Consider the homogeneous Poisson (or Laplace) equation, resulting from the plasma approximation. Making use of the separation of variables above, show that the transverse component of the potential satisfies the Helmholtz equation

$$\nabla_\perp^2\phi + q^2\phi = 0, \tag{3}$$

where q is some arbitrary constant.

c) For symmetry reasons, we may expect $\phi(r, \theta)$ to display radial symmetry. As such, it can be decomposed as

$$\phi(r, \theta) = \sum_\ell R_\ell(r)e^{i\ell\theta},$$

where ℓ is an integer (why?). Show that the $R_\ell(r)$ satisfy the Bessel equation,

$$x^2 R_\ell'' + xR_\ell' + \left(x^2 - \ell^2\right)R_\ell = 0,$$

where $x = qr$.

d) Make use of the appropriate boundary conditions to show that the formal profile of the transverse potential is given by

$$\phi(r, \theta) = \sum_{n,\ell} \mathcal{A}_\ell J_\ell(k_{n,\ell}r)e^{i\ell\theta},$$

where $k_{n,\ell} = \alpha_{n,\ell}/a$ and $\alpha_{n,\ell}$ is the nth zero of the ℓth Bessel function of the first kind, $J_\ell(x)$.

e) We now restrict the discussion to the first harmonic, i.e. $\ell = 0$, corresponding to the lowest excitation along the transverse direction (i.e. the potential vanishes only at the border of the container). In what follows, we show that the longitudinal electron waves inherit the structure of the transverse potential. First, convince yourself that the resulting potential along the $z-$ direction reads

$$\left(\frac{\partial^2}{\partial z^2} - k_n^2\right)\varphi = \frac{e}{\epsilon_0}\left(n_e - n_i\right),$$

where $k_n \equiv \alpha_{n,0}/a$. Then, work out the fluid equations to obtain the dispersion relation of the Trivelpiece-Gould waves

$$\omega^2 = \omega_{pe}^2\frac{k^2}{k^2 + k_n^2} + \gamma_e v_e^2 k^2, \tag{4}$$

where $v_e = \sqrt{k_B T_e/m_e}$. Plot the dispersion relation for the first and second harmonics ($n = 0$ and $n = 1$) and explain what is happening physically. What is apparently strange with these waves? Does it remind you of something?

More on this issue... The Trivelpiece-Gould waves have been obtained in the quantum case in Physics of Plasmas **15**, 072109 (2008).