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Advanced Plasma Physics

MEFT 2021/22

Problem Class 1

[Solutions]

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

Problem 1. Vlasov equation. As derived in the theory class, the Klimontovich equation for the α-species of the plasma reads

$$\frac{\partial N_\alpha}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} N_\alpha + \frac{\mathbf{F}_\alpha^{(m)}}{m_\alpha} \cdot \boldsymbol{\nabla}_\mathbf{v} N_\alpha = 0, \tag{1}$$

where $\mathbf{F}_\alpha^{(m)}$ is the microscopic force (due to the microscopic fields $\mathbf{E}^{(m)}$ and $\mathbf{B}^{(m)}$). Define the *smooth distribution function* $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, in terms of which we may rewrite the microscopic distribution function as

$$N_\alpha(\mathbf{r}, \mathbf{v}, t) = f_\alpha(\mathbf{r}, \mathbf{v}, t) + \delta N_\alpha(\mathbf{r}, \mathbf{v}, t).$$

a) Discuss in class with your colleagues the physical meaning of both f_α and δN_α.

As discussed in class, $N_\alpha(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N_0} \delta(\mathbf{r} - \mathbf{r}_i(t))\delta(\mathbf{v} - \mathbf{v}_i(t))$, represents a "fuzzy" distribution function in the $6N-$ dimensional phase-space for the $\alpha-$species of the plasma (also known as the *Klomontovich function*). By defining the "smooth" distribution function as $f_\alpha(\mathbf{r}, \mathbf{v}, t) = \langle N_\alpha(\mathbf{r}, \mathbf{v}, t)\rangle$, $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ appears as the averaged distribution function. As such, since we may decompose the Klimontovich function as

$$N_\alpha(\mathbf{r}, \mathbf{v}, t) = f_\alpha(\mathbf{r}, \mathbf{v}, t) + \delta N_\alpha(\mathbf{r}, \mathbf{v}, t),$$

we may interpret the last term as the fluctuations on top of the smooth function. Notice that the latter decomposition is exact, with f_α describing the physics at the time scales $t \gtrsim 1/\omega_{p,\alpha}$, i.e. it embodies the collective response of the plasma, while δN_α contains informations about short-scale collisions taking place at the timescales $t \ll 1/\omega_{p,\alpha}$ (collisions).

b) Average out the Klimontovich equation and show that the equation for the smooth function f_α now reads

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_\alpha + \frac{\mathbf{F}_\alpha}{m_\alpha}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_\alpha = \mathcal{C}[\delta N_\alpha],$$

specifying the form (and the physical meaning) of $\mathcal{C}[\delta N_\alpha]$. Is the latter equation more or less accurate than Eq. (1)?

Plugging the decomposition of the previous point in Eq. (1), we obtain

$$\frac{(\partial f_\alpha + \delta N_\alpha)}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\left(f_\alpha + \delta N_\alpha\right) + \frac{\mathbf{F}_\alpha + \delta\mathbf{F}_\alpha^{(m)}}{m_\alpha}\cdot\boldsymbol{\nabla}_{\mathbf{v}}\left(f_\alpha + \delta N_\alpha\right) = 0,$$

where we have also defined $\mathbf{F}_\alpha = \langle \mathbf{F}_\alpha^{(m)}\rangle$. We now take the average of the whole equation. Noticing that the average value of the fast oscillating contributions vanishes, $\langle \delta N_\alpha\rangle = \langle \delta\mathbf{F}_\alpha^{(m)}\rangle = 0$, we have

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_\alpha + \frac{\mathbf{F}_\alpha}{m_\alpha}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_\alpha = -\Big\langle \frac{\delta\mathbf{F}_\alpha^{(m)}}{m_\alpha}\cdot\boldsymbol{\nabla}_{\mathbf{v}}\delta N_\alpha\Big\rangle.$$

The last term can be readily identified with the collision integral $\mathcal{C}[\delta N_\alpha]$.

c) Consider the case of a fully ionized, dilute plasma, for which the free mean path is sufficiently large, i.e. under the condition $n_\alpha \ell_\alpha^3 \gg 1$ (understand the physical meaning of this approximation). Show that the plasma is appropriately described by the Vlasov equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_\alpha + \frac{\mathbf{F}_\alpha}{m_\alpha}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_\alpha \simeq 0.$$

Discuss how you would relate the mean-field force $\mathbf{F}_\alpha$ to the EM-fields and the distribution function f_α, when dealing with both electrostatic and electromagnetic phenomena.

In the dilute conditions described above, a fully ionized plasma should not undergo too many short-range collisions, which happens for two main reasons: i) the neutrals are absent, and ii) the short-range $e-e$, $e-i$ and $i-i$ Coulomb collisions are not important at the scales $L \gtrsim \lambda_D$ (i.e. scales larger than the Debye length, above which quasi-neutrality can be assumed). This last condition is achieved provided that $\lambda_D \lesssim \ell_\alpha$ (i.e. shielding occurs at sufficiently short distances, and the rare collisions take place outside the Debye sphere). In this *collisionless limit*, we may neglect $\mathcal{C}[\delta N_\alpha]$. The force acting on each species should be the Lorentz form,

$$\mathbf{F}_\alpha(\mathbf{r},t) = \frac{q_\alpha}{m_\alpha}\left(\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right),$$

with the fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ governed by Maxwell's equations. In the latter, the source terms (charge density, ρ, and charge current, $\mathbf{J}$) are related to the phase-space distribution functions $f_\alpha(\mathbf{r},\mathbf{v},t)$ as

$$\rho(\mathbf{r},t) = e\int\left\{f_i(\mathbf{r},\mathbf{v},t) - f_e(\mathbf{r},\mathbf{v},t)\right\}d\mathbf{v}, \quad \mathbf{J}(\mathbf{r},t) = e\int\left\{\mathbf{v}f_i(\mathbf{r},\mathbf{v},t) - \mathbf{v}f_e(\mathbf{r},\mathbf{v},t)\right\}d\mathbf{v}.$$

Problem 2. Electrostatic waves. Let us consider small fluctuations around a certain initial distribution (that we here assume to be the thermal equilibrium) as $f_\alpha = f_{0,\alpha} + f_{1,\alpha}$, where $f_{1,\alpha} \ll f_{0,\alpha}$ is a small perturbation.

c) Show that the dielectric function reads

$$\epsilon(k,\omega) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k} \int_{-\infty}^{+\infty} \frac{g'_{0,\alpha}(v)}{\omega - kv} dv, \tag{2}$$

where $v = \boldsymbol{v}\cdot\hat{\boldsymbol{k}}$ and $f_{0,\alpha}(\boldsymbol{v}) = n_0 g_{0,\alpha}(\boldsymbol{v})$ for homogeneous plasmas (quasi-neutrality is assumed here, so $n_{0,e} = n_{0,i} \equiv n_0$.

At linear order, i.e. by neglecting terms of the order $\mathcal{O}\left(f_{1,\alpha}^2\right)$ the Vlasov equation for an unmagnetized plasma reads

$$\frac{\partial f_{1,\alpha}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_{1,\alpha} + \frac{q_\alpha}{m_\alpha}\mathbf{E}_1\cdot\boldsymbol{\nabla}_\mathbf{v} f_{0,\alpha} \simeq 0.$$

Here, we have assumed that the plasma is not electrified, $\mathbf{E}_0 = 0$, which is compatible with the assumption that the plasma is quasi-neutral. Any electric field appearing must be of first order, being associated with the fluctuations. Since the resulting equation is linear, it is convenient to introduce the double Fourier transform of a generic quantity $A(\mathbf{r},\mathbf{t})$ over the spacial and temporal variables, as

$$\tilde{A}(\mathbf{k},\omega) = \int A(\mathbf{r},t) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}.$$

After Fourier-transforming the Vlasov equation, we get, for homogeneous plasmas, $f_{0,\alpha}(\mathbf{r},\mathbf{v}) = n_{0,\alpha} g_{0,\alpha}(\mathbf{v})$,

$$-i\left(\omega - \mathbf{k}\cdot\mathbf{v}\right)\tilde{f}_{1,\alpha} + \frac{q_\alpha n_{0,\alpha}}{m_\alpha}\tilde{\mathbf{E}}_1\cdot\boldsymbol{\nabla}_\mathbf{v} g_{0,\alpha} = 0 \quad \Rightarrow \quad \tilde{f}_{1,\alpha} = -\frac{i q_\alpha n_{0,\alpha}}{m_\alpha}\frac{\tilde{\mathbf{E}}_1\cdot\boldsymbol{\nabla}_\mathbf{v} g_{0,\alpha}}{\omega - \mathbf{k}\cdot\mathbf{v}}.$$

At first order, the Poisson equation relates the electric field to the phase-space distributions as

$$i\mathbf{k}\cdot\mathbf{E}_1 = \frac{e}{\epsilon_0}\left(n_i - n_e\right) \simeq \frac{e}{\epsilon_0}\int\left(\tilde{f}_{1,i} - \tilde{f}_{1,e}\right) d\mathbf{v},$$

where we have assumed the quasi-neutrality condition, $n_{e,0} \equiv \int d\mathbf{v} f_{e,0} = \int d\mathbf{v} f_{i,0} \equiv n_{i,0}$. Putting things together, we get

$$i\mathbf{k}\cdot\mathbf{E}_1 = -i\left(\frac{e^2 n_0}{\epsilon_0 m_i}\int\frac{\tilde{\mathbf{E}}_1\cdot\boldsymbol{\nabla}_\mathbf{v} g_{0,i}}{\omega - \mathbf{k}\cdot\mathbf{v}} d\mathbf{v} + \frac{e^2 n_0}{\epsilon_0 m_e}\int\frac{\tilde{\mathbf{E}}_1\cdot\boldsymbol{\nabla}_\mathbf{v} g_{0,e}}{\omega - \mathbf{k}\cdot\mathbf{v}} d\mathbf{v}\right).$$

$$i\mathbf{k}\cdot\mathbf{E}_1 = -i\mathbf{k}\cdot\left(\frac{\omega_{p,i}^2}{k}\int\frac{\tilde{\mathbf{E}}_1 g'_{0,i}(v)}{\omega - kv} dv + \frac{\omega_{p,e}^2}{k}\int\frac{\tilde{\mathbf{E}}_1 g'_{0,e}}{\omega - kv} dv\right),$$

where $v = \mathbf{v}\cdot\mathbf{k}/k$ and $k = |\mathbf{k}|$. Finally, by identifying the latter equation as the Poisson equation for the displacement vector in the plasma, $i\mathbf{k}\cdot\tilde{\mathbf{D}}_1 = 0$, with $\tilde{\mathbf{D}}_1 = \epsilon_0\epsilon(k,\omega)\tilde{\mathbf{E}}_1$, we recover the stated result.

b) Let us focus on the case of electronic waves only. As such, we take the limit in which ions are inertia-less, $m_i \to \infty$. Assuming that electrons follow the Maxwell-Boltzmann distribution,

$$g_{0,e} = \frac{1}{\sqrt{2\pi}v_e} e^{-v^2/(2v_e^2)},$$

where $v_e = \sqrt{k_B T_e/m_e}$ is the electron thermal speed. Moreover, it is expected for electron plasma waves to feature very large phase speeds in the long-wavelength limit $k \to 0$ (why?), i.e. they satisfy the condition $\omega/k \gg v_e$. Obtain the dispersion relation for the Langmuir waves,

$$\omega = \sqrt{\omega_{pe}^2 + 3v_e^2 k^2}.$$

Discuss this result in the light of what you have learned from the hydrodynamic formulation of plasmas, with Prof. Jorge Vieira.

Because of the smallness of the electron-to-ion mass ratio, $m_e/m_i \ll 1$, we can assume the ions to remain immobile at the scale of the electron oscillations. This is the so-called the *inertialess limit* of the ions, $m_i \to \infty$, which is equivalent to set $\omega_{pi} \to 0$ in the expression for the dielectric permittivity,

$$\epsilon(k,\omega) \simeq 1 + \frac{\omega_{p\alpha}^2}{k} \int_{-\infty}^{+\infty} \frac{g_{0,e}'(v)}{\omega - kv} dv = 1 - \frac{\omega_{p\alpha}^2}{k^2} \int_{-\infty}^{+\infty} \frac{g_{0,e}'(v)}{v - v_\varphi} dv = 1 - \frac{\omega_{p\alpha}^2}{k^2} \int_{-\infty}^{+\infty} \frac{g_{0,e}(v)}{(v - v_\varphi)^2} dv,$$

where $v_\varphi = \omega/k$ is the phase velocity of the wave (the last step is achieved upon integration by parts). Since the electron waves feature large phase speeds in the long-wavelength limit $k \to 0$,

$$v_\varphi = \frac{\omega}{k} \simeq \frac{\omega_{pe}}{k} \gg v_e,$$

with $v_e = \sqrt{k_B T_e/m_e}$ denoting the electron thermal speed, we may expand the denominator in the dielectric function in powers of v/v_φ, reads

$$\epsilon(k,\omega) \simeq 1 - \frac{\omega_{p\alpha}^2}{k^2 v_\varphi^2} \int_{-\infty}^{+\infty} g_{0,e}(v) \left(1 + 2\frac{v}{v_\varphi} + 3\frac{v^2}{v_\varphi^2} + \dots\right) dv.$$

The integrals are straight forwardly computed, with the first term yiealding 1 (normalization condition) and the second term being zero for the obvious reasons (the distribution is an even function). The third term contributes as $3v_e^2/v_\varphi^2 = 3v_e^2 k^2/\omega^2$, such that

$$\epsilon(k,\omega) \simeq 1 - \frac{\omega_{pe}^2}{\omega^2} \left(1 + 3\frac{v_e^2 k^2}{\omega^2}\right).$$

The dispersion relation can then be found by setting $\epsilon(k,\omega) = 0$. For convenience, we approximate the term ω^4 in the denominator as $\omega^4 \simeq \omega^2 \omega_{pe}^2$, thus circumventing the handling of a polynomial equation of fourth order (which is not that complicated in our case, but we do it for historical reasons). We should finally get the dispersion relation in the statement,

$$\omega_p = \sqrt{\omega_{pe}^2 + 3v_e^2 k^2} \simeq \omega_{pe} + \frac{3}{2} v_e^2 k^2.$$

Without this approximation, we would get

$$\omega = \frac{1}{\sqrt{2}} \sqrt{\omega_{pe}^2 + \sqrt{\omega_{pe}^2 \left(\omega_{pe}^2 + 12 v_e^2 k^2\right)}} \simeq \omega_{pe} + \frac{3}{4} v_e^2 k^2.$$

Not bad, right?

c) Consider now oscillation taking place in the ion sector. For that task, we may anticipate that some of the previous considerations for the electrons remain valid. However, we can no longer assume the electrons to be inertialess (why?). On the contrary, we assume that electrons follow the motion of the ions adiabatically, therefore remaining in thermal equilibrium at all times. Make the proper adjustments to Eq. (2) to show that the dispersion relation of ion-acoustic waves is given by

$$\omega \simeq \frac{c_s k}{\sqrt{1 + k^2 \lambda_D^2}}.$$

Obtain explicit expressions for c_s and λ_D in terms of the basic parameters of the system and discuss their physical meaning.

Since we are now interested in the motion of the ions, we take into account their mass: they are not inertialess any longer. As such, we repeat the previous arguments for the ion contribution and write the dielectric perttimivity as

$$\epsilon(k, \omega) \simeq 1 - \frac{\omega_{pi}^2}{\omega^2} \left(1 + 3\frac{v_i^2 k^2}{\omega^2}\right) - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} \frac{g_{0,e}'(v)}{v - v_\varphi} dv.$$

We notice, however, that the phase speed of the ion waves is much smaller than the thermal speed of the electrons, $v_\varphi \ll v_e$, which allows us to approximate the last integral as

$$\int_{-\infty}^{+\infty} \frac{g_{0,e}'(v)}{v - v_\varphi} dv \simeq \int_{-\infty}^{+\infty} \frac{g_{0,e}'(v)}{v} dv = \frac{1}{\sqrt{2\pi} v_e^2} \int_{-\infty}^{+\infty} \frac{-\not{v} e^{-v^2/2v_e^2}}{\not{v}} dv = \frac{1}{v_e^2}.$$

The dielectric permittivity now reads

$$\epsilon(k, \omega) \simeq 1 - \frac{\omega_{pi}^2}{\omega^2} \left(1 + 3\frac{v_i^2 k^2}{\omega^2}\right) + \frac{1}{k^2 \lambda_D^2},$$

where $\lambda_D = v_e/\omega_{pe} = \sqrt{\epsilon_0 k_B T_e / e^2 n_0}$. Using again the trick $\omega^4 \simeq \omega^2 \omega_{pi}^2$, we get set $\epsilon(k, \omega) = 0$ to obtain the dispersion relation

$$\omega = \frac{\omega_{pi} \lambda_D k}{\sqrt{1 + k^2 \lambda_D^2}} \left(1 + 3\frac{T_i}{T_e} k^2 \lambda_D^2\right)^{1/2}.$$

The dispersion relation in the statement can be found in the limit of cold ions, $T_i \ll T_e$, and upon definition of the *ion-acoustic sound speed*, $c_s \equiv \omega_{pi} \lambda_D = \sqrt{k_B T_e / m_i}$. The latter contains a very peculiar information: the typical speed of the ions is dictated by the ion inertia (m_i) but contains information about the electron temperature (T_e). The reason for that is easy to guess: ions are much slower than electrons; as such, the latter are able remain in thermal equilibrium as the former move (in other words, the electrons follow the ion motion *adiabatically*). Ions therefore feature an acoustic mode because of the Debye shielding. It is now understandable why the contrary does not happen: the ions do not provide such a shielding for the electrons.

d) Plot the dispersion relation ω vs k and digress over its features in both limits $k\lambda_D \ll 1$ and $k\lambda_D \gg 1$. Vividly discuss your conclusions with your colleagues.

In the long-wavelength limit, $k\lambda_D \ll 1$, the dispersion relation of the ion waves is acoustic

$$\omega \simeq c_s k.$$

This is so since the wavelength is much larger than the Debye length, $\lambda \gg \lambda_D$, which means that the wave does not have enough resolution to "see" what happens inside the Debye sphere. In this limit, Debye shielding is effective and the resulting ion-ion interaction is of the form $\sim e^{-r/\lambda_D}/r$, that of a short-range potential of scale λ_D (the potential dies out very quickly outside the Debye sphere). Physically, this means that excitation of ion waves is *local*, and requires only an infinitesimal amount of energy to be produced, $\omega \to 0$ as $k \to 0$. This is a general feature of short-range potentials, and it is ultimately related with the fact that acoustic modes are a consequence of the Goldstone theorem. Remember what the latter states: for each continuous symmetry that is broken in the system, there is a *massless* field kicking in the theory. Here, the massless field would be ion-acoustic wave, which one could express in terms of a classical field, while the symmetry (much harder to identify here) is that of a global phase.

In the short-wavelength limit, however, the dispersion relation reads

$$\omega \simeq \frac{c_s k}{\lambda_D k} = \omega_{pi}.$$

As we can observe, the spectrum is gapped, $\omega \to \omega_{pi}$, similarly to what happens for the electron plasma (Langmuir) waves. The reason for that stems in the fact that short-wavelength fluctuations resolve the Debye sphere, does preventing ion shielding to take place effectively: the ions are no longer shielded and, therefore, feature a potential of the form $1/r$, the so-called "bare" Coulomb potential. As you can imagine, the cost of energy associated to the excitation of such a mode is finite, since the ions participating in that motion interact all over the system (there is no typical scale for the interaction).