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Advanced Plasma Physics

MEFT 2021/22

Problem Class 2

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

(To practice at home) Landau damping. As we have seen in class, the kinetic dispersion relation requires the evaluation of the integral over the complex plane, since $\omega = \omega_r + i\omega_i$,

$$\epsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{k}\int_L \frac{g'_{0,e}(v)}{\omega - kv}dv.$$

a) Assuming $\omega_i \ll \omega_r$, make use of Plemelj's formula,

$$\lim_{a\to 0}\frac{1}{x \pm ia} = \text{P.V.}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

to show that the kinetic dielectric function may be written as

$$\epsilon(k,\omega) \simeq 1 + \frac{\omega_{pe}^2}{k}\text{P.V.}\int_{-\infty}^{+\infty}\frac{g'_{0,e}(v)}{\omega - kv}dv - i\pi\frac{\omega_{pe}^2}{k^2}g'_{0,e}(v)\bigg|_{v=\omega/k}.$$

Identify the first and the second terms with the real and imaginary components of the dielectric function, $\epsilon_r(k,\omega)$ and $\epsilon_i(k,\omega)$.

b) Since $\omega_i \ll \omega_r$, we may expand ϵ_r and ϵ_i around ω_r at first order in ω_i. Obtain the formula to compute the imaginary part of the frequency explicitly,

$$\omega_i \simeq \pi\frac{\omega_{pe}^2}{k^2}\frac{g'_{0,e}(\omega/k)}{\partial\epsilon_r/\partial_{\omega_r}}$$

c) Concretize for the case of the Langmuir waves in a Maxwell-Boltzmann plasma, and show that

$$\omega_i \simeq -\sqrt{\frac{\pi}{8}}e^{-3/2}\frac{\omega_{pe}}{(k\lambda_D)^2}e^{-1/(2k^2\lambda_D^2)}.$$

Define $y \equiv \omega_i/\omega_{pe}$ and $x \equiv k\lambda_D$ to plot the previous result. Observe its features and interpret them physically.

Problem 1. Beam-plasma instability. Consider a cold, homogeneous plasma composed by ions and electrons, where the ions are at rest and the electrons are streaming with velocity $\mathbf{v}_0 = v_0 \mathbf{e_x}$. Consider electrostatic perturbations only.

a) Discuss the form of the equilibrium functions $g_{0,e}(v)$ and $g_{0,i}(v)$ and show that the dielectric function for this problem reads

$$\epsilon(k, \omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - \omega_0)^2}, \tag{1}$$

with $\omega_0 = kv_0$ being the streaming frequency.

b) The computation of the dispersion relation involves a fourth-order polynomial, for which we may expect four real roots. Plot $\epsilon(k, \omega)$ and observe that it only contains two real roots for $\omega_0 < \omega_c$, where ω_c is a certain critical value. Discuss with your colleagues how this relates to the onset of a dynamical instability in the plasma and determine the value of ω_c.

c) Show that the instability terminates at the cut-off wavevector k_c given by

$$k_c \simeq \frac{\omega_{pe}}{v_0} \left(1 + \frac{1}{2} \left(\frac{m_e}{m_i} \right)^{1/3} \right).$$

What happens for modes $k > k_c$?

d) Its is expected that the instability driven in the ion motion happens at a much slower scale than that of the streaming mode, i.e. $\omega \ll \omega_0$ (why?). So, we may look for the most unstable mode, $k_{\max}$, which maximizes the imaginary part of the frequency ($\omega_{i,\max} \equiv \max(\omega_i(k)) = \omega_i(k_{\max})$). Expand Eq. (1) and show that

$$\omega_{i,\max} \simeq \frac{\sqrt{3}}{2^{4/3}} \left(\frac{m_e}{m_i} \right)^{1/3} \omega_{pe}.$$

e) With the help of Mathematica, solve the kinetic dispersion relation numerically and obtain the $\omega_r(k)$ and $\omega_i(k)$ for a certain value of ω_{pi}/ω_{pe} (or, equivalently, for a certain mass ration m_i/m_e). Identify the features that you estimated analytically in the previous points. Discuss the results with your colleagues.

Problem 2. The Krook collision integral. Assume that your plasma is sufficiently dense such that collisions start to become important. A way to take them into account is by adding a collision integral within the *relaxation-time (Krook) approximation* to the RHS of the Vlasov equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \right) f_e + \frac{\mathbf{F}}{m_e} \cdot \boldsymbol{\nabla}_{\mathbf{v}} f_e = -\nu \left(f_e - f_{0,e} \right),$$

where $f_{0,e}(\boldsymbol{x}, \boldsymbol{v}, t)$ is the equilibrium distribution function, as usual. We neglect the effect of the ions, which are considered to be at rest.

a) Show that, at first order in $f_e - f_{0,e}$, we may write

$$f_e \approx f_{0,e} - \frac{1}{\nu} \left(\mathbf{v} \cdot \boldsymbol{\nabla} f_0 + \frac{\mathbf{F}}{m_e} \cdot \boldsymbol{\nabla}_{\mathbf{v}} f_0 \right).$$

b) Consider that a constant electric field $\mathbf{E}$ is applied to a *homogeneous*, *unmagnetized* plasma. Use the previous result to derive Ohm's law,

$$\mathbf{J}_e = \sigma_e \mathbf{E} \tag{2}$$

where $\sigma_e = e^2 n_0/\nu m_e$ is the electron conductivity and n_0 is the plasma density. If the calculations were repeated in the presence of a transverse magnetic field ($\mathbf{B} \perp \mathbf{E}$), what kind of effect would Eq. (2) be describing (argue without calculations)?

c) Consider now the case of particle transport in such a collisional plasma. For that, neglect the electric field and assume that a temperature gradient $\boldsymbol{\nabla} T$ is present at the terminal of the plasma. You may expect that the system is no longer homogenous (think about the microscopic meaning of "temperature gradient"). Show that the particle current $\mathbf{J} = \mathbf{J}_e/e$ is given by Fick's Law,

$$\mathbf{J} = -\kappa \boldsymbol{\nabla} T, \quad \kappa = \frac{2n_0}{3\nu m_e} C_V,$$

where κ is the *heat conductivity* and $C_V = \partial \langle E \rangle / \partial T$ is the specific heat.