Probability Theory

LMAC, MMA

1st. Semester – 2021/2022
2021/11/09 — 09:10
MAP30 #2

Duration: **30** minutes

- Write your number and name below.
- Add your answers on this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and THREE QUESTIONS. The total of points is 4.0.

Number: **Name**:

1. Let X be a positive r.v. **(1.0)**

Show that $Y = \frac{1}{X}$ is a Borel measurable function, therefore also a r.v.

- **R.v.**

  Let $(\Omega, \mathscr{F})$ and $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be two measurable spaces. Then, $X : \Omega \to \mathbb{R}$ and

  $$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}, \quad \forall B \in \mathscr{B}(\mathbb{R}).$$

  In particular, $X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathscr{F}$, $\forall x \in \mathbb{R}$.

- **Proof**

  Firstly, a function $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff

  $$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\} \in \mathscr{B}(\mathbb{R}), \quad \forall B \in \mathscr{B}(\mathbb{R}).$$

  Secondly, if

  $$g^{-1}((-\infty, y]) = \{x \in \mathbb{R} : g(x) \leq y\} \in \mathscr{B}(\mathbb{R}), \quad \forall y \in \mathbb{R},$$

  then $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

  Thirdly, since $X > 0$ we have:

  - for $y \leq 0$,
    $$g^{-1}((-\infty, y]) = \{x \in \mathbb{R} : g(x) \leq y\} = \emptyset \in \mathscr{B}(\mathbb{R});$$
  - for $y > 0$,
    $$g^{-1}((-\infty, y]) = \left\{x \in \mathbb{R} : g(x) = 1/x \leq y\right\} = \left\{x \in \mathbb{R} : x \geq 1/y\right\} = \left[1/y, +\infty\right) \in \mathscr{B}(\mathbb{R}).$$

  As a result, $Y = \frac{1}{X}$ is a Borel measurable function and therefore a r.v. ✓

2. Admit X is an absolutely continuous r.v. with p.d.f. **(1.5)**

$$f_X(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2}, & 0 < x \leq 1 \\ \frac{1}{2x^2}, & x > 1. \end{cases}$$

Derive the p.d.f. of $Y = g(X) = \frac{1}{X}$ and show that X and Y are identically distributed.

- **R.v., p.d.f., and range**

  X; $f_X(x)$ see text above; $\mathbb{R}_X = \mathbb{R}^+$

- **Transformation and its range**

  $Y = g(X) = \frac{1}{X}$

  $\mathbb{R}_Y = g(\mathbb{R}_X) = \mathbb{R}^+$

- **Pointwise inverse of g and its derivative**

$$\begin{aligned} y &= g(x) = \frac{1}{x} \\ g^{-1}(y) &= \frac{1}{y} \\ \frac{d\,g^{-1}(y)}{dy} &= \frac{d\,(1/y)}{dy} = -\frac{1}{y^2} \end{aligned}$$

- **P.d.f. of Y**

Since $g(x) = \frac{1}{x}$ is a continuously differentiable, strictly decreasing function of $x \in \mathbb{R}_X$

$$\begin{aligned} f_Y(y) &= f_X\left[g^{-1}(y)\right] \times \left|\frac{d\,g^{-1}(y)}{dy}\right| \\ &= f_X(1/y) \times \left|-1/y^2\right| \\ &= \begin{cases} 0, & y \leq 0 \\ \frac{1}{2} \times \left|-1/y^2\right| = \frac{1}{2y^2}, & 0 < x = \frac{1}{y} \leq 1 \Leftrightarrow y \geq 1 \\ \frac{1}{2\,(1/y)^2} \times \left|-1/y^2\right| = \frac{1}{2}, & x = \frac{1}{y} > 1 \Leftrightarrow 0 < y < 1 \end{cases} \\ &\equiv f_X(y). \end{aligned}$$

X and $Y = g(X) = \frac{1}{X}$ are indeed identically distributed. ✓

3. Let (X, Y) be a random vector with joint p.d.f. $f_{X,Y}(x, y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, $(x, y) \in \mathbb{R}^2$. **(1.5)**
Derive the p.d.f. of $U = X - Y$ to show that $U \sim \text{normal}(0, 2)$.

- **Random vector and range**

(X, Y)

$\mathbb{R}_{X,Y} = (\mathbb{R})^2$

Transformation of (X, Y) and its range

$U = g(X, Y) = X - Y$

$\mathbb{R}_U = g(\mathbb{R}_{X,Y}) = \mathbb{R}$

- **P.d.f. of U**

$$\begin{aligned} f_U(u) &= f_{X-Y}(u) \\ &= \int_{-\infty}^{+\infty} f_{X,Y}(x, x-u)\,dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-(x-u)^2/2}\,dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \exp\left(-\frac{x^2 + x^2 - 2xu + u^2}{2}\right) dx \\ &= \frac{1}{\sqrt{4\pi}} e^{-\frac{u^2}{4}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi \times (1/\sqrt{2})^2}} \exp\left[-\frac{(x-u/2)^2}{2 \times (1/\sqrt{2})^2}\right] dx \\ &= \frac{1}{\sqrt{2\pi \times (\sqrt{2})^2}} e^{-\frac{u^2}{2\times(\sqrt{2})^2}} \int_{-\infty}^{+\infty} f_{normal(u/2,(1/\sqrt{2})^2)}(x)\,dx \\ &= \frac{1}{\sqrt{2\pi \times (\sqrt{2})^2}} e^{-\frac{u^2}{2\times(\sqrt{2})^2}}, \quad u \in \mathbb{R}. \end{aligned}$$

Consequently, $U = X - Y \sim \text{normal}(0, (\sqrt{2})^2)$.