Chapter 8

Modelling interconnected and non-linear systems

This chapter presents an overview of the modelling process.

8.1 Energy, effort and flow

Impedances

Table 8.1 presents the impedances of all the flow accumulators, effort accumulators, and energy dissipators, summing up Tables 4.1, 5.1, 6.1, and 7.1, and showing clearly the existing parallelism between systems of different types.

This is the place to notice that effort variables are measured in relation to an arbitrary value that serves as zero:

- In Table 8.1 this is explicit for thermal systems, since temperature is denoted as ΔT, as what matters is the temperature difference.
- In the case of electrical systems, what matters is always the electrical tension at the extremities of the component.
- In the case of pipe flow, resistance and inductance depend on the pressure difference at the extremities. Reservoirs with a free surface also depend on a pressure difference, between the pressure of the liquid at the bottom and the atmospheric pressure.
- In the case of mechanical systems, the energy dissipated by a damper depends on the relative velocities of its extremities, and the energy accumulated by a spring depends on the relative position of its extremities.

Notice that there may be values of these variables that we can think of as absolute zeros, such as temperature -273.15 °C $= 0$ K, pressure 0 Pa of complete vacuum, or position and velocity measured in an inertial frame of reference. Still, it is often far more practical to use other values, such as atmospheric pressure, room temperature, or resting position, as zero.

Dealing with initial conditions

Example 8.1. A 300 kg dirigible balloon flies at constant altitude $z = 200$ m, because its impulsion cancels its weight. It can move vertically thanks to two electrical propulsors, each of which provides a force given by $F_p(t) = \gamma U(t)$,

Type of system	Mechanical, translation	Mechanical, rotation	Electrical	Fluidic	Thermal
effort e flow f	velocity $\dot{x}$ force F	angular velocity $\dot{\omega}$ torque τ	voltage U current I	pressure P volume flow rate Q	temperature T heat flow rate q
effort accumulator impedance	spring $\frac{sX(s)}{F(s)} = \frac{s}{K}$	angular spring $\frac{s\Omega(s)}{T(s)} = \frac{s}{\kappa}$	inductor $\frac{U(s)}{I(s)} = Ls$	fluidic inductance $\frac{P(s)}{Q(s)} = Ls$	— —
flow accumulator impedance	mass $\frac{sX(s)}{F(s)} = \frac{1}{Ms}$	moment of inertia $\frac{s\Omega(s)}{T(s)} = \frac{1}{Js}$	capacitor $\frac{U(s)}{I(s)} = \frac{1}{Cs}$	reservoir $\frac{P(s)}{Q(s)} = \frac{1}{Cs}$	heat accumulator $\frac{\Delta T(s)}{Q(s)} = \frac{1}{mC_p s}$
dissipator impedance	damper $\frac{sX(s)}{F(s)} = \frac{1}{b}$	rotary damper $\frac{s\Omega(s)}{T(s)} = \frac{1}{b}$	resistor $\frac{U(s)}{I(s)} = R$	fluidic resistance $\frac{P(s)}{Q(s)} = R$	thermal resistance $\frac{\Delta T(s)}{Q(s)} = R$

Table 8.1: Effort, flow, accumulators and dissipators in different types systems

where $U(t)$ is the tension applied (control input) and the gain is $\gamma = 15$ N/V (the force is upwards when $U > 0$). When the balloon moves, there is a viscous drag force with coefficient $c = 30$ N s/m. How does the altitude change with time when a 20 V tension is applied during 10 s?

A balance of forces shows that

$$\underbrace{300\ddot{z}(t)}_{\text{mass}\times\text{acceleration}} = \underbrace{2 \times 15U(t)}_{\text{propulsors}} \underbrace{-30\dot{z}(t)}_{\text{drag force}} \tag{8.1}$$

We know that initial conditions are $z(0) = 200$ and $\dot{z}(0) = 0$, so we could be tempted to apply the Laplace transform as

$$300\left(Z(s)s^2 - 200s\right) = 30U(s) - 30\left(Z(s)s - 200\right) \tag{8.2}$$

Rearranging terms,

$$(300s^2 + 30s)Z(s) = 30U(s) + 6 \times 10^4 s + 6 \times 10^3 \tag{8.3}$$

$$\Leftrightarrow Z(s) = \frac{1}{(10s+1)s}U(s) + \frac{2 \times 10^3 s}{(10s+1)s} + \frac{2 \times 10^2}{(10s+1)s} = \frac{1}{(10s+1)s}U(s) + \frac{2 \times 10^2}{s}$$

Notice that it is impossible to find a transfer function $\frac{Z(s)}{U(s)}$ relating the (Laplace transforms of) the input and the output. To obtain a transfer function, make $z^*(t) = z(t) - (0)$, and then

No transfer function if initial conditions are not zero

$$300Z^*(s) = 30U(s) - 30Z^*(s)s \Leftrightarrow \frac{Z^*(s)}{U(s)} = \frac{1}{(10s+1)s} \tag{8.4}$$

The result will of course be the same, but this allows us to use many results established for transfer functions, such as those in Chapters 9 and 10. It also allows us to use MATLAB to find the answer as follows:

```
>> G = tf(1, [10 1 0]);
>> Ts = 0.001; t = 0 : Ts : 50;
>> U = zeros(size(t)); U(1:10/Ts) = 20*ones(1, 10/Ts);
>> z = lsim(G, U, t); z = z + 200;
>> figure, plot(t,Z)
>> xlabel('t [s]'), ylabel('z [m]')
```

Notice how we had to add 200 to the result (or else we would have to bear in mind that the plot would show oscillations around 200 m). See Figure 8.1. □

Remark 8.1. Remember that we already did something similar in Example 7.1. □

8.2 System interconnection

Transfer functions are of great aid when modelling several interconnected systems, of the same or of different types.

Example 8.2. Consider the system in Figure 8.2. The force exerted by the inductance in the handle that undergoes displacement x_2 is given by $F_2(t) = \alpha i(t)$, where $i(t)$ is the current in the inductance. Find $\frac{X_1(s)}{V_i(s)}$.

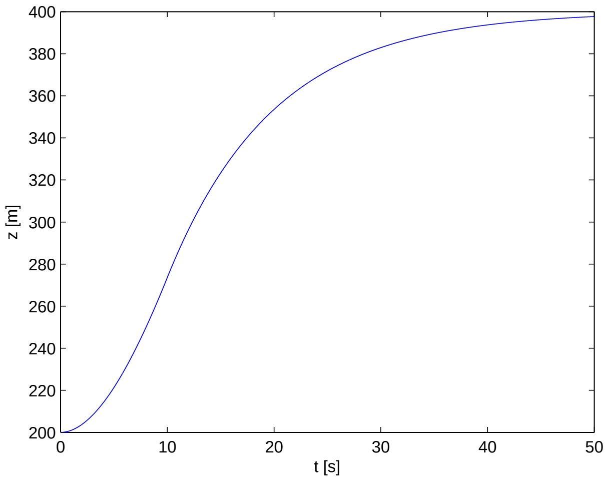

Figure 8.1: Results of Example 8.1.

We can of course write all the equations, and obtain the desired result with successive replacements. Transfer functions allow us to model each system separately, making such replacements easier.

As to the electrical system, remembering (5.45),

$$R + Ls = \frac{\frac{n_2}{n_1} V_i(s)}{I(s)} \Leftrightarrow \frac{I(s)}{V_i(s)} = \frac{\frac{n_2}{n_1}}{R + Ls} \tag{8.5}$$

As to the lever, and letting F_1 be the force exerted on mass m_1,

$$F_1(t)a = F_2(t)b \Leftrightarrow \frac{F_1(s)}{F_2(s)} = \frac{b}{a} \tag{8.6}$$

As to the mass,

$$F_1(t) - Kx_1(t) = m_1 \ddot{x}_1(t) \Leftrightarrow \frac{X_1(s)}{F_1(s)} = \frac{1}{m_1 s^2 + K} \tag{8.7}$$

Finally,

$$\frac{X_1(s)}{V_i(s)} = \frac{X_1(s)}{F_1(s)} \frac{F_1(s)}{F_2(s)} \frac{F_2(s)}{I(s)} \frac{I(s)}{V_i(s)} = \frac{\frac{b}{a} \alpha \frac{n_2}{n_1}}{(m_1 s^2 + K)(R + Ls)} \tag{8.8}$$

This way, we are also able to study each transfer function separately, analysing its influence in the final result. □

Bond graphs

Bond graphs are another tool that can be used to assist the modelling of interconnected systems. They consist in a graphical representation of what happens with energy in a system, based upon the concepts of effort and flux. These are written above and below arrows (of which, by convention, only half the tip is drawn). Figure 8.3 shows two examples of bond graphs in which several elements have the same flux and different efforts; the corresponding junction

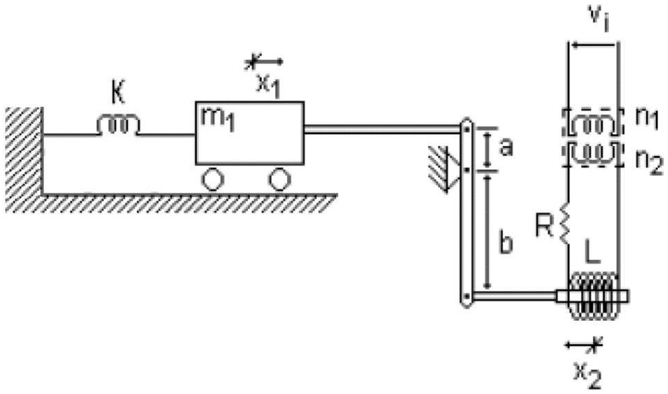

Figure 8.2: System of Example 8.2.

of the several efforts in one system is by convention denoted by number 1. Figure 8.4 shows two examples of bond graphs in which several elements have the same effort and different flows; the corresponding junction of the several flows in one system is by convention denoted by number 0. Also notice how sources of energy are denoted by SE. We will not study bond graphs more complicated than these, nor further explore the ability of this graphical tool to assist in the modelling.

8.3 Dealing with non-linearities

Non-linearities can be classified as hard or soft, as they are respectively more or less severe. Though no uniform definition is universally accepted, we will say that a **soft non-linearity** is one that is differentiable, while a **hard non-linearity** is not. Figure 8.5 presents two examples. *Soft non-linearity* *Hard non-linearity*

Non-linearities are very common. They may be part of the design of a system, even of a control system. In other courses you will learn how to deal with hard non-linearities in control. What is important here is to notice that soft non-linearities can be approximated by a first order approximation around the operating point. Estimating how large the approximation error may be is important; we will do that in Chapters 11–13.

Example 8.3. In Figure 8.6, mass $m = 10$ kg rests on a non-linear spring and is pulled by force F applied simultaneously on a linear spring with $k = 10^3$ N/m and on a linear damper with $b = 500$ N s/m. The non-linear force of the spring is given by $F_k = 5000 - \frac{500}{\Delta y + 0.1}$ (SI), where the Δy is the variation of length around the uncompressed value. We want a linear model for this system around nominal conditions of rest when $F = 0$.

Figure 8.7 shows the non-linear force. When $F = 0$, the non-linear spring is compressed by the weight of m, which is -9.8×10 N (notice the minus sign, since the weight is downwards and the positive sign of y corresponds to an upwards direction), corresponding to

$$-98 = 5000 - \frac{500}{\Delta y + 0.1} \Leftrightarrow \Delta y = -1.9 \times 10^{-3} \text{ m} \quad (8.9)$$

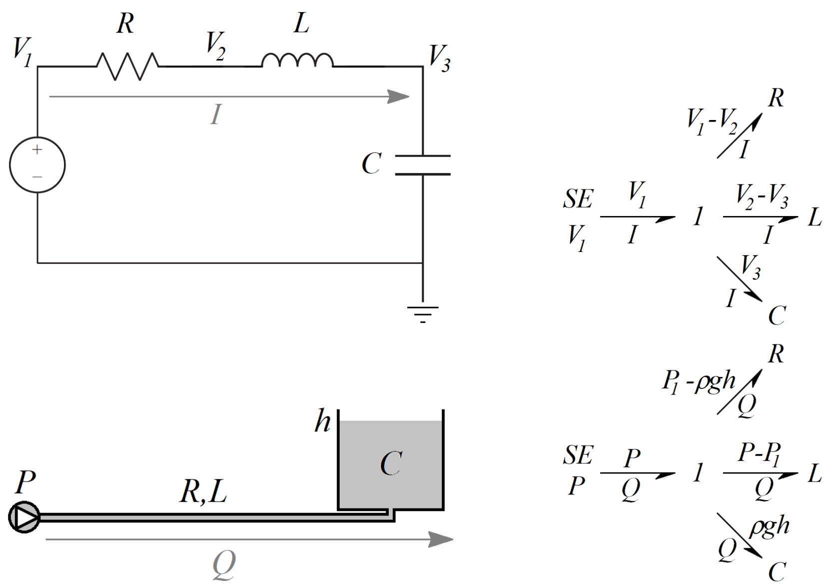

Figure 8.3: Two bond graphs of systems where elements have the same flux and there is an effort junction. Top: electrical circuit. Notice that $V_1 = (V_1 - V_2) + (V_2 - V_3) + V_3$. Bottom: fluidic system. Since the pipe has both resistance and inductance, the pressure change from the pump delivering a constant pressure P to the bottom of the reservoir where the hydraulic head is h and the pressure is ρgh is split into two, as if the fluid would first go through an inductance without resistance and then through a resistance without inductance, so that $P = (P - P_1) + (P_1 - \rho gh) + \rho gh$.

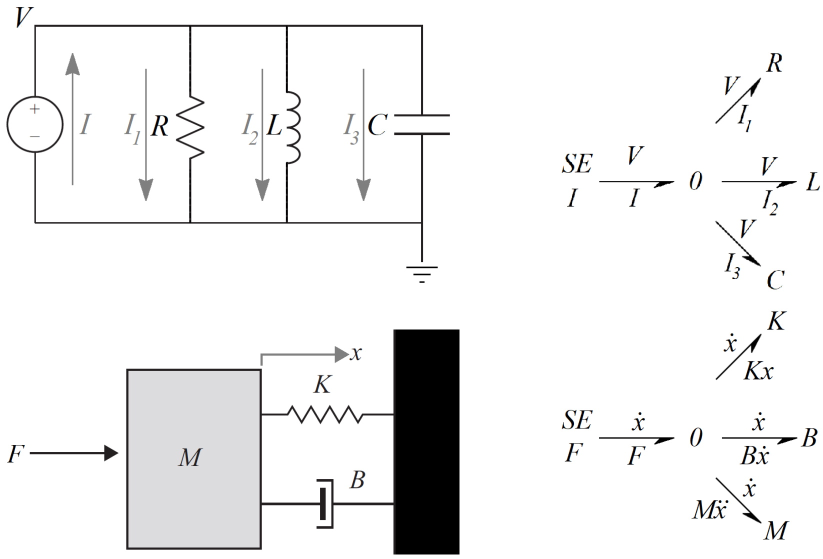

Figure 8.4: Two bond graphs of systems where elements have the same effort and there is a flux junction. Top: electrical circuit. Notice that $I = I_1 + I_2 + I_3$. Bottom: mechanical system. Notice that $F - Kx - B\dot{x} = M\ddot{x} \Leftrightarrow F = Kx + B\dot{x} + M\ddot{x}$.

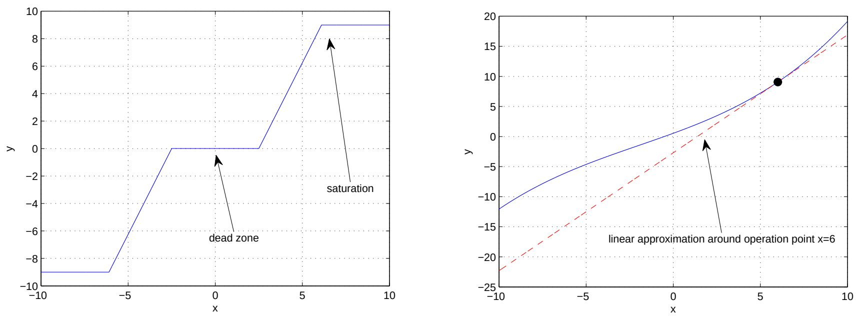

Figure 8.5: Left: hard non-linearities (dead zone and saturation; in practice limits need not be symmetric for positive and negative values, though this is assumption is frequent). Right: soft non-linearity and one of its linear approximations.

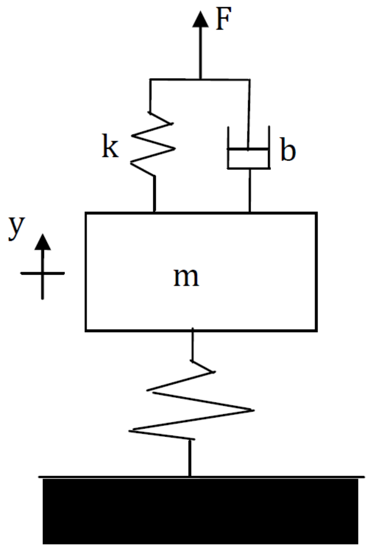

Figure 8.6: System of Example 8.3.

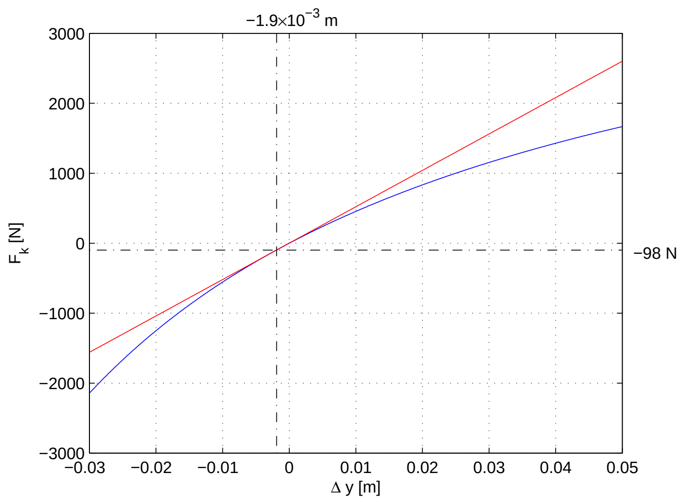

Figure 8.7: Non-linear force of Example 8.3.

The linearised law is

$$F_k \approx \left.\frac{\mathrm{d}F_k}{\mathrm{d}(\Delta y)}\right|_{\Delta y=-1.9\times 10^{-3}\ \mathrm{m}} y = \left.\frac{500}{(\Delta y+0.1)^2}\right|_{\Delta y=-1.9\times 10^{-3}\ \mathrm{m}} y = 5.2\times 10^4 y \text{ (SI)} \tag{8.10}$$

where $y = \Delta y + 1.9 \times 10^{-3}$ m, or, if you prefer, the variation of length around $\Delta y = -1.9 \times 10^{-3}$ m. Furthermore, the linear components are assumed to have no mass, and hence transmit force F to mass m. Thus

$$m\ddot{y} = F - 5.2 \times 10^4 \text{ (SI)} \tag{8.11}$$

It should be stressed that linear model (8.11) is only an approximation. □

Glossary

> Desejoso ainda o Fucarãdono, como mais douto q̃ os outros, de leuar a sua auante cõ pregũtas q̃ embaraçassẽ o padre, lhe veyo arguindo de nouo q̃ porq̃ razão punha nomes torpes ao Criador de todas as cousas, & aos Sãtos q̃ no ceo assistião em louuor seu, infamãdoo de mẽtiroso, pois elle, como todos criaõ, era Deos de toda a verdade ? & para q̃ se entenda dõde naceo a este dizer isto, se ha de saber q̃ na lingoa do Iapaõ se chama a mẽtira diusa, & porq̃ o padre quãdo pregaua dezia q̃ aq̃ella ley q̃ elle vinha denũciar era a verdadeira ley de Deos, o qual nome elles pela grossaria da sua lingoa não podião pronũciar taõ claro como nos & por dizerẽ Deos dezião diùs, daquy veyo que estes seruos do diabo tomaraõ motiuo de dizerẽ aos seus que o padre era demonio em carne q̃ vinha infamar a Deos põdo-lhe nome de mentiroso: (...) E porque tambem se saiba a razaõ porque lhe este bonzo disse que punha nomes torpes aos santos, foy, porque tinha o padre por custume quando acabaua de dizer missa rezar com todos hũa Ladaynha para rogar a N. Senhor pela augmẽtação da fé Catholica, & nesta ladainha dezia sempre, como nella se custuma, *Sancte Petre ora pro nobis*, *Sancte Paule ora pro nobis*, & assi dos mais Santos. E porq̃ tambem este vocablo santi na lingoa Iapoa he torpe & infame, daquy veyo arguyr este ao padre q̃ punha maos nomes aos Sãtos, (...) & daly por diãte mãdou o padre q̃ se naõ dissesse mais *sancte*, senaõ *beate Petre*, *beate Paule*, & assi aos outros Santos, porque já dantes tinhaõ os bonzos todos perante el Rey feito peçonha disto.
>
> Fernão Mendes Pinto (1509? — †1583), *Peregrinaçam*, CCXIII

bond graph grafo de ligação
linearisation, linearization (US) linearização

Exercises

1. Draw the bond graph of the system in Figure 8.8.
2. Draw the bond graph of the balloon from Example 8.1.

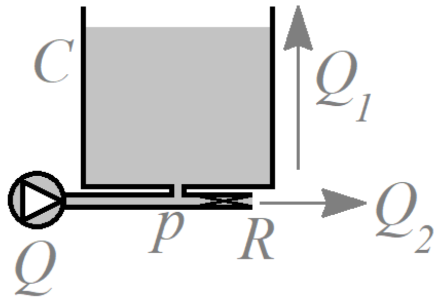

Figure 8.8: System of Exercise 1.

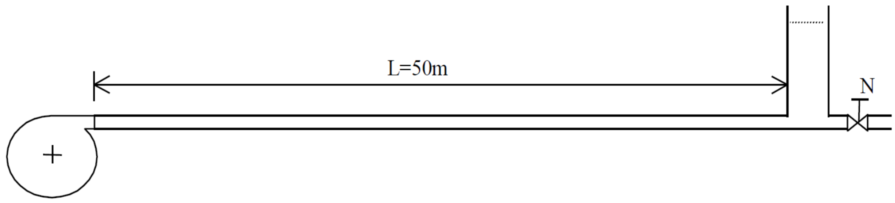

Figure 8.9: System of Exercise 3.

3. The system in Figure 8.9 is fed by a water pump with a characteristic curve given by $P(t) = 10^5 - 2 \times 10^6 Q(t)$, where P and Q are the pressure (Pa) and the volumetric flow (m^3/s) provided.

   The pipe has a 0.01 m^2 diameter and a length of 50 m. Its flow resistance is neglectable; its inertance is not.

   The tank has a free surface and 1 m^2 cross-section.

   The valve is non-linear and verifies relation

$$Q_v(t) = 0.3 \times 10^{-4} N(t) \sqrt{P_v(t)} \tag{8.12}$$

   where Q_v is the flow through the valve (m^3/s), N is the opening of the valve (dimensionless), and P_v is the pressure (Pa) at the entrance of the valve, which is also the pressure at the bottom of the tank.

   In nominal conditions, $\overline{P_v} = 8 \times 10^4$ Pa and $\overline{Q_v} = 0.01$ m^3/s.

   (a) Show that the pipe's inertance is $L = 5 \times 10^6$ kg m^{-4}.

   (b) Show that in nominal conditions the height of water in the tank is $\bar{h} = 8.16$ m.

   (c) Show that the non-linear relation of the valve (8.12) can be linearised as

$$Q_v(t) = \overline{Q_v(t)} + 0.085 \left(N(t) - \overline{N(t)} \right) + 6.25 \times 10^{-7} \left(P_v(t) - \overline{P_v(t)} \right) \tag{8.13}$$

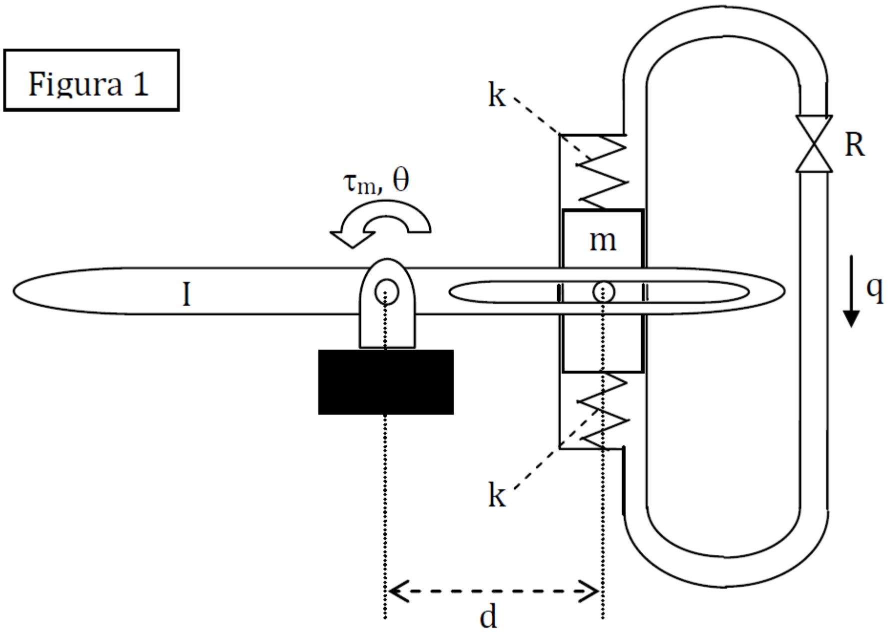

Figure 8.10: System of Exercise 4.

(d) Show that the system can be modelled by (8.13) together with

$$\begin{cases} P_v(t) - \overline{P_v} = \rho g \left(h - \bar{h}\right) \\ \left(Q(t) - \bar{Q}\right) - \left(Q_v(t) - \overline{Q_v}\right) = A\frac{\mathrm{d}\left(h-\bar{h}\right)}{\mathrm{d}t} \\ \left(P_b(t) - \overline{P_b}\right) - \left(P_v(t) - \overline{P_v}\right) = L\frac{\mathrm{d}\left(Q(t)-\bar{Q}\right)}{\mathrm{d}t} \end{cases} \tag{8.14}$$

(e) Find transfer function $\frac{\Delta P_v(s)}{\Delta N(s)}$, relating variations around nominal conditions.

4. In Figure 8.10, the lever with inertia I oscillates around the horizontal position (i.e. $\theta(t) = 0$) and is moved by torque τ_m. Mass m moves vertically, at distance d from the fulcrum of the lever, inside a cylinder with two springs of constant k, filled with incompressible oil. The pressure difference $\Delta p(t)$ between the two chambers of the cylinder moves the oil through fluidic resistance R. Thanks to oil lubrication, friction inside the cylinder is neglectable.
   (a) Write linearised equations for the dynamics of the system.
   (b) Find transfer function $\frac{\Theta(s)}{T_m(s)}$.
5. In Figure 8.11, the lever with inertia I is actuated by force $F(t)$ and supported on the other side by a spring and a damper. On the lever there is a car with mass m, moving to sidewards due to gravity, without friction. When $F = 0$ and the car is on the fulcrum (i.e. its position is $x = 0$), the lever remains in the horizontal position. There is no friction at the fulcrum.

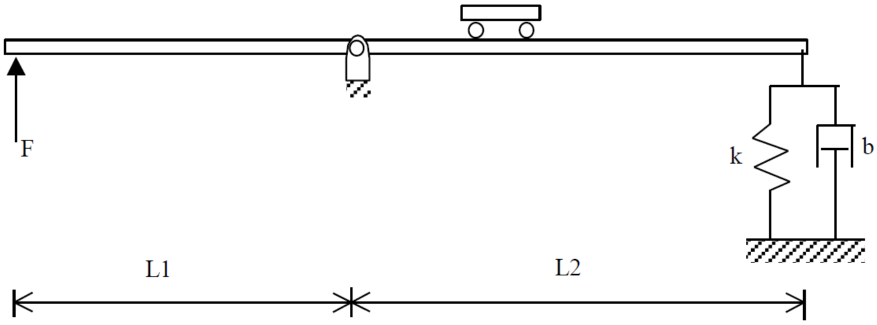

Figure 8.11: System of Exercise 5.

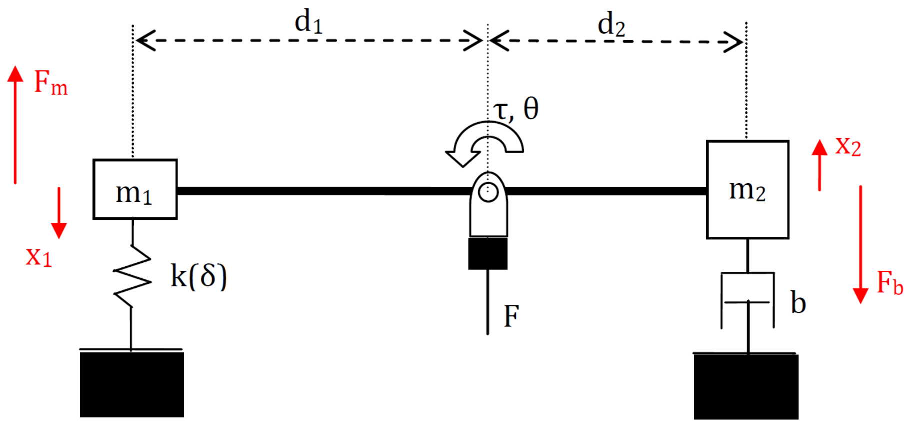

Figure 8.12: System of Exercise 6.

(a) Write linearised equations for the dynamics of the system.

(b) Find transfer function $\frac{X(s)}{F(s)}$.

6. The lever in Figure 8.12, with neglectable mass, is moved by a torque τ applied on the fulcrum, in the absence of which the lever is horizontal (i.e. $\theta = 0$). F is the force exerted on the fulcrum.

   It is known that $m_1 = 1.5$ kg, $m_2 = 2.0$ kg, $d_1 = 0.6$ kg, $d_2 = 0.4$ m, and $b = 20$ N s/m. The spring obeys the non-linear law in Figure 8.12, where δ is the length variation in mm (with $\delta > 0$ corresponding to compression), and F_m is the resulting force in N.

   (a) Show that, in nominal conditions, $\overline{F_m} = 1.63$ N.

   (b) Show from the plot in Figure 8.13 that the force of the spring can be linearised as $F_m = 1.63 + 3.26 \times 10^3 x_1$.

   (c) Write linearised equations for the dynamics of the system.

   (d) Find transfer function $\frac{F(s)}{T(s)}$.

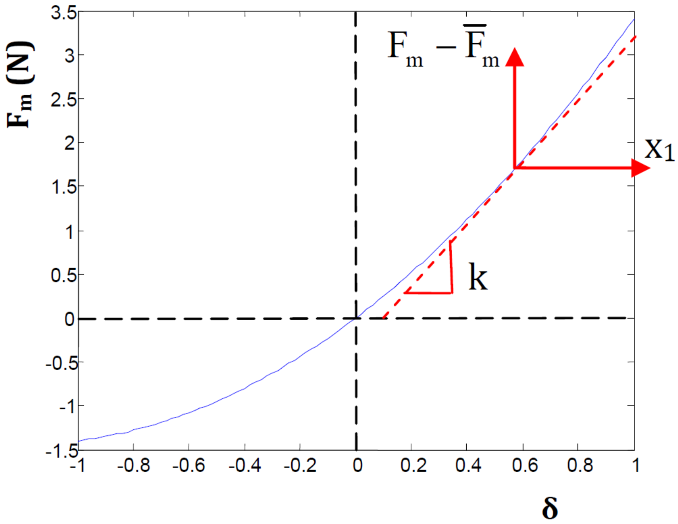

Figure 8.13: Non-linear law of the spring of Exercise 6.

7. In Figure 8.14, the fresh water ($\rho = 1000$ kg/m^3) tank in the left is big enough to keep a constant liquid height $h_0 = 5$ m, while the tank in the right has a 10 m^2 cross-section and a variable liquid height $h_1(t)$.

   Flow $q_c(t)$ bleeds this tank and does not depend on pressure; flow $q_1(t)$ passes through a non-linear valve that verifies

$$q_1(t) = 0.15 x_v(t) \sqrt{\Delta p(t)} \text{ (SI)} \tag{8.15}$$

   where $\Delta p(t)$ is the pressure difference on both sides of the valve and $x_v(t)$ is mechanically actuated by $h_1(t)$ through a rigid lever with $a = 0.4$ m and $b = 4$ m.

   In nominal conditions, $q_c(t) = q_1(t) = 0.2$ m^3/s and $h_1(t) = 3$ m.

   (a) Show that the model of the flow through the valve (8.15) can be linearised around nominal conditions as

$$q_1(t) = 21 x_v(t) + 5.09 \times 10^{-6} \Delta p \text{ (SI)} \tag{8.16}$$

   (b) Write linearised equations for the dynamics of the system.

   (c) Find transfer function $\frac{\Delta H_1(s)}{Q_c(s)}$.

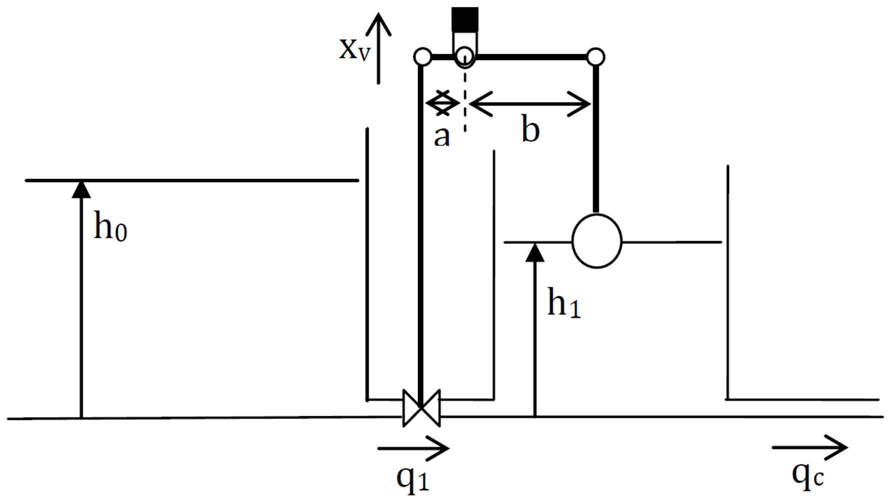

Figure 8.14: System of Exercise 7.