DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

MMA, LMAC

2nd. Semester – 2018/2019
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Duration: 90 minutes **Test 1**

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 0 — Introduction to Stochastic Processes 2.5 points

Let X_n represent the quarterly growth rate of the U.S. GNP at the end of the n^{th} quarter. $\{X_n : n \in \mathbb{Z}\}$ has been frequently modelled as a second order moving average (MA(2)) process, that is, $X_n = \mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}$, where: μ, θ_1, and θ_2 are real constants; and ϵ_m, $m \in \mathbb{Z}$, are uncorrelated r.v. with zero mean and variance σ_ϵ^2.

(a) Obtain the mean and variance function of $\{X_n : n \in \mathbb{Z}\}$. (1.0)

- **Stochastic process**
  $\{X_n : n \in \mathbb{Z}\}$
  $X_n = \mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}$
  $\mu, \theta_1, \theta_2 \in \mathbb{R}$
  $$\epsilon_m : \begin{cases} E(\epsilon_m) = 0, & m \in \mathbb{Z} \\ V(\epsilon_m) = cov(\epsilon_m, \epsilon_m) = \sigma_\epsilon^2, & m \in \mathbb{Z} \\ cov(\epsilon_j, \epsilon_m) = 0, \quad j \neq m, & j, m \in \mathbb{Z} \quad \text{(uncorrelated r.v.)} \end{cases}$$
- **Mean function**
  $$\begin{aligned} E(X_n) &= E\left(\mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}\right) \\ &= \mu + E(\epsilon_n) + \theta_1 E(\epsilon_{n-1}) + \theta_2 E(\epsilon_{n-2}) \\ &\overset{E(\epsilon_m)=0,\, m\in\mathbb{Z}}{=} \mu \end{aligned}$$
- **Variance function**
  $$\begin{aligned} V(X_n) &= V\left(\mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}\right) \\ &\overset{cov(\epsilon_j,\epsilon_m)=0,\, j\neq m}{=} V(\epsilon_n) + \theta_1^2 V(\epsilon_{n-1}) + \theta_2^2 V(\epsilon_{n-2}) \\ &\overset{V(\epsilon_m)=\sigma_\epsilon^2,\, m\in\mathbb{Z}}{=} \left(1 + \theta_1^2 + \theta_2^2\right) \times \sigma_\epsilon^2. \end{aligned}$$

(b) Is $\{X_n : n \in \mathbb{Z}\}$ a second order weakly stationary process? (1.5)
Hint: Take advantage of the properties of the covariance operator.

- **Autocovariance function**
  Taking advantage of the properties of the covariance operator (it is symmetric, bilinear, etc.) and of the fact that $\{\epsilon_m : m \in \mathbb{Z}\}$ is a family of uncorrelated r.v. with variance σ_ϵ^2, we can derive the autocovariance function of $\{X_n : n \in \mathbb{Z}\}$.
  - For $k = 0$,
    $$\begin{aligned} cov(X_n, X_n) &= V(X_n) \\ &\overset{(a)}{=} \left(1 + \theta_1^2 + \theta_2^2\right) \times \sigma_\epsilon^2. \end{aligned}$$
  - For $k = 1$ (or $k = -1$),
    $$\begin{aligned} cov(X_n, X_{n+1}) &= cov\left(\mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}, \mu + \epsilon_{n+1} + \theta_1\epsilon_n + \theta_2\epsilon_{n-1}\right) \\ &\overset{cov(\epsilon_j,\epsilon_m)=0,\, j\neq m}{=} cov(\epsilon_n, \theta_1\epsilon_n) + cov(\theta_1\epsilon_{n-1}, \theta_2\epsilon_{n-1}) \\ &= \theta_1 V(\epsilon_n) + \theta_1\theta_2 V(\epsilon_{n-1}) \\ &= \theta_1(1 + \theta_2) \times \sigma_\epsilon^2. \end{aligned}$$

  - For $k = 2$ (or $k = -2$),
    $$\begin{aligned} cov(X_n, X_{n+2}) &= cov\left(\mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}, \mu + \epsilon_{n+2} + \theta_1\epsilon_{n+1} + \theta_2\epsilon_n\right) \\ &= cov(\epsilon_n, \theta_2\epsilon_n) \\ &= \theta_2 V(\epsilon_n) \\ &= \theta_2 \times \sigma_\epsilon^2. \end{aligned}$$
  - For $k > 2$ (or $k < -2$),
    $$\begin{aligned} cov(X_n, X_{n+k}) &= cov\left(\mu + \epsilon_n + \theta_1\epsilon_{n-1} + \theta_2\epsilon_{n-2}, \mu + \epsilon_{n+k} + \theta_1\epsilon_{n+k-1} + \theta_2\epsilon_{n+k-2}\right) \\ &= 0. \end{aligned}$$

  Consequently,
  $$cov(X_n, X_{n+k}) = \begin{cases} \left(1 + \theta_1^2 + \theta_2^2\right) \times \sigma_\epsilon^2, & k = 0 \\ \theta_1(1 + \theta_2) \times \sigma_\epsilon^2, & |k| = 1 \\ \theta_2 \times \sigma_\epsilon^2, & |k| = 2 \\ 0, & |k| > 2. \end{cases}$$

  [Alternatively, consider $\theta_0 = 1$ and obtain $cov(X_n, X_m) = \sum_{i=0}^2 \sum_{j=0}^2 \theta_i\theta_j cov(\epsilon_{n-i}, \epsilon_{m-j}) = \ldots$]
- **Checking whether the process is (second order weakly) stationary**
  $E(X_n)$ does not depend on n and $cov(X_n, X_{n+k})$ only depends on the time lag k, hence $\{X_n : n \in \mathbb{Z}\}$ is a second order weakly stationary process.

Group 1 — Poisson Processes 9.0 points

1. Patients arrive at the doctor's office according to a Poisson process with rate $\lambda = 6$ (patients per hour).

(a) Find the probability that exactly 3 patients arrive in the first hour, given that at least one patient arrived in the first 30 minutes. (1.5)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda)$
  $N(t)$ = number of arrivals of patients by time t (time in hours)
  $\lambda = 0.5$
- **Relevant distributions**
  $N(t) \sim \text{Poisson}(\lambda\, t)$
  $(N(s) \mid N(t) = n) \sim \text{Binomial}(n, s/t)$, $0 < s < t$ (see formulae)
- **Requested probability**
  $$\begin{aligned} P[N(1) = 3 \mid N(0.5) > 0] &\overset{Bayes'\ theo.}{=} \frac{P[N(0.5) > 0 \mid N(1) = 3] \times P[N(1) = 3]}{P[N(0.5) > 0]} \\ &= \frac{\{1 - P[N(0.5) = 0 \mid N(1) = 3]\} \times P[N(1) = 3]}{1 - P[N(0.5) = 0]} \\ &= \frac{\left[1 - \binom{3}{0} 0.5^0 (1 - 0.5)^{3-0}\right] \times \frac{e^{-6\times 1}(6\times 1)^3}{3!}}{1 - \frac{e^{-6\times 0.5}(6\times 0.5)^0}{0!}} \\ &= \frac{\left(1 - 0.5^3\right) \times \frac{e^{-6}6^3}{3!}}{1 - e^{-3}} \\ &\simeq 0.082172. \end{aligned}$$

  [Alternatively,
  $$\begin{aligned} P[N(1) = 3 \mid N(0.5) > 0] &= \frac{P[N(0.5) > 0, N(1) = 3]}{P[N(0.5) > 0]} \\ &= \frac{P[N(1) = 3] - P[N(0.5) = 0, N(1) = 3]}{1 - P[N(0.5) = 0]} \end{aligned}$$

$$\begin{aligned}
P[N(1)=3 \mid N(0.5)>0] &\overset{indep.incr.}{=} \frac{P[N(1)=3]-P[N(0.5)=0]\times P[N(1)-N(0.5)=3]}{1-P[N(0.5)=0]} \\
&\overset{station.incr.}{=} \frac{P[N(1)=3]-P[N(0.5)=0]\times P[N(1-0.5)=3]}{1-P[N(0.5)=0]} \\
&\overset{N(t)\sim Poi(6t)}{=} \frac{\frac{e^{-6\times 1}(6\times 1)^3}{3!}-\frac{e^{-6\times 0.5}(6\times 0.5)^0}{0!}\times\frac{e^{-6\times 0.5}(6\times 0.5)^3}{3!}}{1-\frac{e^{-6\times 0.5}(6\times 0.5)^0}{0!}} \\
&= \frac{\frac{e^{-6}}{3!}\times\left(6^3-3^3\right)}{1-e^{-3}} \\
&\simeq 0.082172.
\end{aligned}$$

Or $P[N(1)=3 \mid N(0.5)>0] = \frac{P[N(0.5)>0,N(1)=3]}{P[N(0.5)>0]} = \cdots = \frac{\sum_{i=1}^{3} P[N(0.5)=i,N(1)-N(0.5)=3-i]}{1-P[N(0.5)=0]} = \ldots]$

(b) Suppose that each patient is a man (resp. woman) with probability $\frac{2}{3}$ (resp. $\frac{1}{3}$) . Now suppose that 10 men arrived in the first 2 hours. How many women would you expect to have arrived in the first 2 hours? (1.0)

- **Split processes**
  The original PP with rate $\lambda=6$, $\{N(t): t\geq 0\}$, is now split in two other processes, $\{N_i(t): t\geq 0\}$ $(i=M,W)$, referring to the counts of men (M) and women (W).
- **Distributions**
  These two split processes are INDEPENDENT and also POISSON with rates
  $$\begin{aligned}
  \lambda_M &= \lambda\times p_M = 6\times\frac{2}{3}=4, \\
  \lambda_W &= \lambda\times p_W = 6\times\frac{1}{3}=2,
  \end{aligned}$$
  respectively. Consequently,
  $$N_i(t)\sim_{indep.}\text{Poisson}(\lambda_i\,t), \quad i=M,W. \tag{1}$$
- **Requested expected value**
  $$\begin{aligned}
  E[N_W(2)\mid N_M(2)=10] &\overset{(1)}{=} E[N_W(2)] \\
  &= 2\times 2 \\
  &= 4.
  \end{aligned}$$

2. Short-term power outages occur in a electrical grid according to a Poisson process with rate $\lambda=1$ (outage per month). The time (in months) it takes to report each short-term power outage to the authorities is a r.v. with a uniform distribution in the interval $(0,1)$. (2.0)

Find the probability there are at most 5 reported power outages in a year.

- **Stochastic process**
  $\{N(t): t\geq 0\}\sim PP(\lambda=2)$
  $N(t)=$ number of short-term power outages in t months
- **Non-homogenous Bernoulli splitting**
  Let R be the time it takes to report a short-term power outage to the authorities. A power outage, which occurred at time s $(0<s<t)$, is reported by time t $(t>s)$ with probability
  $$\begin{aligned}
  p(s) &= P(R\leq t-s) \\
  &= F_{U(0,1)}(t-s)=\begin{cases} t-s, & 0<t-s<1 \\ 1, & t-s\geq 1 \end{cases} = \begin{cases} 1, & 0<s\leq t-1 \\ t-s, & t-1<s<t. \end{cases}
  \end{aligned}$$
  Then the number of reported short-term power outages by month t, $N_R(t)$, results from a non-homogenous Bernoulli splitting of $\{N(t): t\geq 0\}$ and

$$N_R(t)\overset{form.}{\sim}\text{Poisson}\left(\lambda\int_0^t p(s)\,ds\right),$$

where
$$\begin{aligned}
\int_0^t p(s)\,ds &= \int_0^t F_{U(0,1)}(t-s)\,ds = \int_0^t F_{U(0,1)}(s)\,ds \\
&\overset{t=12}{=} \int_0^{11} ds+\int_{11}^{12}(12-s)\,ds = \int_0^1 s\,ds+\int_1^{12} ds \\
&= 11+\left(12s-\frac{s^2}{2}\right)\Big|_{11}^{12} = \frac{s^2}{2}\Big|_{11}^{12}+11 \\
&= 11.5.
\end{aligned}$$

- **Requested probability**
  $$\begin{aligned}
  P[N_R(12)\leq 5] &= F_{Poisson(1\times 11.5)}(5) \\
  &\overset{tables}{\simeq} 0.0277.
  \end{aligned}$$

3. A factory produces items one at a time according to a non-homogeneous Poisson process, $\{N(t): t\geq 0\}$ with intensity function $\lambda(t)=\frac{1}{t^2+1}$, $t>0$. (2.0)

Derive the associated mean value function and compute $P[N(\sqrt{2}-1)=2,\,N(1)\leq 3]$.

Hint: Note that $\frac{d\arctan(t)}{dt}=\frac{1}{t^2+1}$ and $\tan(\frac{\pi}{8})=\sqrt{2}-1$.

- **Stochastic process**
  $\{N(t): t>0\}\sim NHPP$
  $N(t)=$ number of items produced until time t
- **Intensity function**
  $\lambda(t)=\frac{1}{t^2+1}$, $t>0$
- **Requested mean value function**
  Capitalizing on the hint, we get
  $$\begin{aligned}
  m(t) &= \int_0^t \frac{1}{s^2+1}\,ds \\
  &= \arctan(s)|_0^t \\
  &= \arctan(t), \quad t>0.
  \end{aligned}$$
- **Relevant distribution**
  $N(t+s)-N(s)\sim\text{Poisson}(m(t+s)-m(s))$, $\quad t,s>0$
- **Requested probability**
  We are dealing with a counting process, thus $N(\sqrt{2}-1)\leq_{st}N(1)$ and
  $$\begin{aligned}
  P[N(\sqrt{2}-1)=2,\,N(1)\leq 3] &= \sum_{i=2}^{3} P[N(\sqrt{2}-1)=2,\,N(1)=i] \\
  &= \sum_{i=2}^{3} P[N(\sqrt{2}-1)=2,\,N(1)-N(\sqrt{2}-1)=i-2] \\
  &\overset{indep.incr.}{=} \sum_{i=2}^{3} P[N(\sqrt{2}-1)=2]\times P[N(1)-N(\sqrt{2}-1)=i-2] \\
  &\overset{N(t)\sim Poi(m(t))\ldots}{=} P_{Poi(m(\sqrt{2}-1))}(2)\times\sum_{i=2}^{3} P_{Poi(m(1)-m(\sqrt{2}-1))}(i-2)
  \end{aligned}$$

$$
\begin{aligned}
P[N(\sqrt{2}-1)=2, N(1)\le 3] &= P_{Poi(\arctan(\sqrt{2}-1))}(2) \times \sum_{i=2}^{3} P_{Poi(\arctan(1)-\arctan(\sqrt{2}-1))}(i-2) \\
&= P_{Poi(\frac{\pi}{8})}(2) \times F_{Poi(\frac{\pi}{4}-\frac{\pi}{8})}(1) \\
&= \frac{e^{-\frac{\pi}{8}}\,(\frac{\pi}{8})^2}{2!} \times \sum_{i=0}^{1} \frac{e^{-\frac{\pi}{8}}\,(\frac{\pi}{8})^i}{i!} \\
&\simeq 0.048961.
\end{aligned}
$$

4. A customer (only) makes deposits in a bank according to a Poisson process with rate λ_D per week. The sizes of successive deposits are i.i.d. r.v. with mean μ_D and variance σ_D^2.

Unknown to the customer, the customer's life partner (only) makes withdrawals from the same account according to a Poisson process with rate λ_W per week. The sizes of successive withdrawals are i.i.d. r.v. with mean μ_W and variance σ_W^2. Assume that the deposit and withdrawal processes are independent of each other and that the customer has an unlimited credit line.

(a) Obtain the mean and variance of the account balance at time t, $B(t)$. (1.0)

- **Stochastic processes**
  $\{N_D(t): t\ge 0\} \sim PP(\lambda_D)$
  $N_D(t)$ = number of deposits over $[0,t]$
  $N_D(t) \sim \text{Poisson}(\lambda_D\, t)$
  $\{N_W(t): t\ge 0\} \sim PP(\lambda_W)$ (independent of $\{N_D(t): t\ge 0\}$)
  $N_W(t)$ = number of withdrawals over $[0,t]$
  $N_W(t) \sim \text{Poisson}(\lambda_W\, t)$
- **R.v. et al.**
  D_i = amount of the i^{th} deposit
  $D_i \overset{i.i.d.}{\sim} D$
  $E(D)=\mu_D$, $V(D)=\sigma_D^2$
  W_i = amount of the i^{th} withdrawal
  $W_i \overset{i.i.d.}{\sim} W$ (independent of D_i)
  $E(W)=\mu_W$, $V(W)=\sigma_W^2$
- **Relevant stochastic process**
  $\left\{B(t)=\sum_{i=1}^{N_D(t)} D_i - \sum_{i=1}^{N_W(t)} W_i : t\ge 0\right\}$
  $B(t)$ = account balance at time t
- **Requested mean and variance**
  Since

$$
\left\{\sum_{i=1}^{N_D(t)} D_i : t\ge 0\right\} \sim \text{Compound } PP(\lambda_D, D) \quad \perp\!\!\!\perp \quad \left\{\sum_{i=1}^{N_W(t)} W_i : t\ge 0\right\} \sim \text{Compound } PP(\lambda_W, W)
$$

we can write

$$
\begin{aligned}
E[B(t)] &= E\left[\textstyle\sum_{i=1}^{N_D(t)} D_i\right] - E\left[\textstyle\sum_{i=1}^{N_W(t)} W_i\right] \\
&\overset{form.}{=} \lambda_D\, t \times E(D) - \lambda_W\, t \times E(W) \\
&= \left(\lambda_D\,\mu_D - \lambda_W\,\mu_W\right) \times t \\
V[B(t)] &= V\left[\textstyle\sum_{i=1}^{N_D(t)} D_i\right] + V\left[\textstyle\sum_{i=1}^{N_W(t)} W_i\right] \\
&\overset{form.}{=} \lambda_D\, t \times E(D^2) + \lambda_W\, t \times E(W^2) \\
&= \lambda_D\, t \times [V(D)+E^2(D)] + \lambda_W\, t \times [V(W)+E^2(W)] \\
&= \left[\lambda_D\left(\sigma_D^2+\mu_D^2\right) + \lambda_W\left(\sigma_W^2+\mu_W^2\right)\right] \times t.
\end{aligned}
$$

(b) What sort of stochastic process is $\{B(t): t\ge 0\}$? Justify. (1.5)

- **Describing** $\{B(t): t\ge 0\}$
  $\{B(t): t\ge 0\}$ is also a compound PP.
  Indeed $B(t)=\sum_{i=1}^{N(t)} Y_i$ where:
  - $\{N(t): t\ge 0\}$ results from the merging of two independent PP, $\{N_D(t): t\ge 0\} \sim PP(\lambda_D)$ and $\{N_W(t): t\ge 0\} \sim PP(\lambda_W)$, hence$\{N(t): t\ge 0\} \sim PP(\lambda=\lambda_D+\lambda_W)$;
  - $Y_i \overset{i.i.d.}{\sim} Y$, where Y is a r.v. that results from a mixture of D and $(-W)$ with weights $p_D = P(\text{deposit before withdrawal}) = \frac{\lambda_D}{\lambda_D+\lambda_W}$ and $p_W = P(\text{withdrawal before deposit}) = \frac{\lambda_W}{\lambda_D+\lambda_W}$, respectively.
    [To obtain these probabilities, we capitalize on the lack of memory and on the well-know result regarding the probability of first failure when we are comparing to independent r.v. with exponential distributions.]

  [Note
  Let $f_Y(y)$ be the p.d.f. of Y. Then

$$
\begin{aligned}
f_Y(y) &= p_D \times f_D(y) + p_W \times f_W(y) \\
E(Y) &= \int_{-\infty}^{+\infty} y \times [p_D \times f_D(y) + p_W \times f_W(y)]\,dy \\
&= p_D \times E(D) + p_W \times E(W) \\
E(Y^2) &= \int_{-\infty}^{+\infty} y^2 \times [p_D \times f_D(y) + p_W \times f_W(y)]\,dy \\
&= p_D \times E(D^2) + p_W \times E(W^2) \\
E[B(t)] &= E[\textstyle\sum_{i=1}^{N(t)} Y_i] \\
&\overset{form.}{=} \lambda\, t \times E(Y) \\
&= (\lambda_D+\lambda_W)\, t \times [p_D \times E(D) + p_W \times E(W)] \\
&= \left(\lambda_D\,\mu_D - \lambda_W\,\mu_W\right) \times t \\
V[B(t)] &= V[\textstyle\sum_{i=1}^{N(t)} Y_i] \\
&\overset{form.}{=} \lambda\, t \times E(Y^2) \\
&= (\lambda_D+\lambda_W)\, t \times [p_D \times E(D^2) + p_W \times E(W^")] \\
&= \left[\lambda_D\left(\sigma_D^2+\mu_D^2\right) + \lambda_W\left(\sigma_W^2+\mu_W^2\right)\right] \times t.
\end{aligned}
$$

  These two last results check with the ones we obtained in (a).]

Group 2 — Renewal Processes	8.5 points

1. Consider a renewal process, $\{N(t): t\ge 0\}$, consisting of all *even* arrivals of a Poisson process with rate λ.

(a) What is the long-run rate at which events occur in the renewal process $\{N(t): t\ge 0\}$? (1.0)

- **Renewal process**
  $\{N(t): t\ge 0\}$
  $N(t)$ = number of *even* arrivals until time t
- **Inter-renewal times**
  $X_i \overset{i.i.d.}{\sim} X$, $i\in\mathbb{N}$
  $X \sim \text{Gamma}(2,\lambda)$ because the time between consecutive even arrivals is a sum of two independent exponentially distributed r.v. with mean λ^{-1}.
- **Expected inter-renewal time**

$$
\begin{aligned}
\mu &= E(X) \\
&= \frac{2}{\lambda}.
\end{aligned}
$$

- **Requested long-run rate**

  According to the SLLN for renewal processes (see formulae!),

$$\lim_{t\to+\infty}\frac{N(t)}{t} \overset{w.p.1}{=} \frac{1}{\mu} = \frac{\lambda}{2}.$$

(b) Derive the renewal function $m(t)$ of $\{N(t): t\geq 0\}$. (2.5)

- **Deriving the renewal function**

  The LST of the inter-renewal d.f. of X is given by

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-sx}\,dF(x)\\ &= E\left(e^{-sX}\right)\\ &= M_X(-s)\\ &\overset{form.}{=} \left(\frac{\lambda}{\lambda+s}\right)^2\end{aligned}$$

  Moreover, the LST of the renewal function can be obtained in terms of the one of F:

$$\begin{aligned}\tilde{m}(s) &\overset{form.}{=} \frac{\tilde{F}(s)}{1-\tilde{F}(s)}\\ &= \frac{\left(\frac{\lambda}{\lambda+s}\right)^2}{1-\left(\frac{\lambda}{\lambda+s}\right)^2}\\ &= \frac{\lambda^2}{(s+0)(s+2\lambda)}.\end{aligned}$$

  Taking advantage of the LT in the formulae, we successively get:

$$\begin{aligned}\frac{dm(t)}{dt} &= LT^{-1}[\tilde{m}(s), t]\\ &= LT^{-1}\left[\frac{\lambda^2}{(s+0)(s+2\lambda)}, t\right]\\ &= \lambda^2\times\frac{e^{-0\times t}-e^{-2\lambda\times t}}{2\lambda-0}\\ &= \frac{\lambda}{2}\left(1-e^{-2\lambda t}\right)\\ m(t) &= \int_0^t \frac{\lambda}{2}\left(1-e^{-2\lambda x}\right)dx\\ &= \frac{\lambda t}{2}-\frac{\lambda}{2}\frac{1-e^{-2\lambda t}}{2\lambda}\\ &= \frac{\lambda t}{2}-\frac{1-e^{-2\lambda t}}{4}, \quad t\geq 0.\end{aligned}$$

(c) Show that the renewal function obtained in (b) verifies the elementary renewal theorem. (1.0)

- Verification of the elementary renewal theorem (ERT)

$$\begin{aligned}\lim_{t\to+\infty}\frac{m(t)}{t} &= \lim_{t\to+\infty}\frac{\frac{\lambda t}{2}-\frac{1-e^{-2\lambda t}}{4}}{t} = \lim_{t\to+\infty}\left(\frac{\lambda}{2}-\frac{1-e^{-2\lambda t}}{4t}\right)\\ &= \frac{\lambda}{2}\\ &= \frac{1}{\mu},\end{aligned}$$

  hence verifying the ERT.

2. Let $\{N(t): t\geq 0\}$ be a renewal process with inter-renewal time distribution F. Using the renewal argument derive the following renewal type equation for the k^{th} factorial moment of $N(t)$, $m_k(t) = E\{N(t)\times[N(t)-1]\times\cdots\times[N(t)-k+1]\}$, with $k\in\mathbb{N}$, $m_0(t)=1$ and $m_1(t)=m(t)$: (2.0)

$$m_k(t) = k\int_0^t m_{k-1}(t-x)\,dF(x) + \int_0^t m_k(t-x)\,dF(x).$$

Hint: For $n\in\mathbb{N}_0$ and $k\in\mathbb{N}$, $\quad (n)_k = n(n-1)\dots(n-k+1) \quad$ and $\quad (n+1)_k = k(n)_{k-1}+(n)_k$.

- **Renewal process**

  $\{N(t): t\geq 0\}$

- **Inter-renewal times**

  $X_i \overset{i.i.d.}{\sim} X_1 \sim F$

  $F(x) = P(X_1\leq x)$

- k^{th} **factorial moment of** $N(t)$

  Let $(N(t))_k = N(t)\times[N(t)-1]\times\cdots\times[N(t)-k+1]$. Then

$$\begin{aligned}m_k(t) &= E\{N(t)\times[N(t)-1]\times\cdots\times[N(t)-k+1]\}\\ &= E[(N(t))_k]\end{aligned}$$

- **Renewal-type equation**

  Applying the renewal argument, i.e., conditioning on the time of the first renewal, $X_1 = x$, we have:

  – for $0<x\leq t$, there is the renewal at x plus $N(t-x)$ renewals in the interval $(x,t]$, thus

$$E[(N(t))_k \mid X_1 = x] = E[(1+N(t-x))_k];$$

  – for $x>t$, there are no renewals in the interval $[0,t]$, therefore

$$E\{(N(t))_k \mid X_1 = x\} = 0.$$

  Consequently,

$$\begin{aligned}m_k(t) &= E[(N(t))_k]\\ &= E\{E[(N(t))_k \mid X_1]\}\\ &= \int_0^{+\infty} E[(N(t))_k \mid X_1 = x]\,dF(x)\\ &= \int_0^t E[(1+N(t-x))_k]\,dF(x) + \int_t^{+\infty} 0\,dF(x)\\ &\overset{(n+1)_k = k(n)_{k-1}+(n)_k}{=} \int_0^t E[k(N(t-x))_{k-1}+(N(t-x))_k]\,dF(x)\\ &= k\int_0^t m_{k-1}(t-x)\,dF(x) + \int_0^t m_k(t-x)\,dF(x). \quad \checkmark\end{aligned}$$

3. Consider the age replacement policy, where we replace a machine upon failure or upon reaching age T, and suppose the machine lifetimes are i.i.d. r.v. with exponential distribution with unit mean. Admit that a new machine costs C_1 monetary units and also that an additional cost of C_2 monetary units is incurred whenever a machine breaks down. (2.0)

Compute the optimal age replacement parameter T that minimizes the long run expected total cost per time unit.

- **Renewal process**

  $\{N(t): t\geq 0\}$

  $N(t)$ = number of replacements by time t

- **Inter-renewal times**

  $X_i \overset{i.i.d.}{\sim} X \overset{st}{=} \min\{T, Y\}$, $i \in \mathbb{N}_0$, where

  T = age replacement parameter

  $F_Y(y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-y}, & y > 0 \end{cases}$

- **Expected inter-renewal time**

$$\begin{aligned} E(X) &= E[\min\{T, Y\}] \\ &= \int_0^{+\infty} \min\{T, Y\}\, dF_Y(y) \\ &= \int_0^T y\, dF_Y(y) + \int_0^T T\, dF_Y(y) \\ &= \int_0^T y\, e^{-y} dy + T\, P(Y > T) \\ &= \left[-Te^{-T} + \left(1 - e^{-T}\right)\right] + T\, e^{-T} \\ &= 1 - e^{-T} \end{aligned}$$

- **Reward renewal process**

  $\{R(t) = \sum_{n=1}^{N(t)} R_n : t \geq 0\}$

  $R(t)$ = total cost incurred until time t

  $(X_n, R_n) \overset{i.i.d.}{\sim} (X, R)$, $n \in \mathbb{N}$

  $R = \begin{cases} C_1, \quad Y = T & \text{(machine replaced before it breaks down, i.e., } X \geq T) \\ C_1 + C_2, \quad Y < T & \text{(machine replaced because it broke down before T, i.e., } X < T) \end{cases}$

- **Expected cost per replacement**

  $E(R) = C_1 \times P(Y = T) + (C_1 + C_2) \times P(Y < T) = C_1 \times e^{-T} + (C_1 + C_2) \times (1 - e^{-T}) = C_1 + C_2 \times (1 - e^{-T})$

- **Long run expected total cost per time unit**

  Since $E(X), E(R) < +\infty$, we can apply the ERT for renewal reward processes and get

$$\begin{aligned} q(T) &= \lim_{t \to +\infty} \frac{E[R(t)]}{t} \\ &= \frac{E(R)}{E(X)} \\ &= \frac{C_1 + C_2 \times (1 - e^{-T})}{1 - e^{-T}} \\ &= \frac{C_1}{1 - e^{-T}} + C_2. \end{aligned}$$

- **Requested optimal age replacement parameter** T

  $q(T)$ is a decreasing function of T, hence

$$T^\star > 0 \quad : \quad q(T^\star) = \inf_{T > 0} q(T) \quad \Leftrightarrow \quad T^\star = +\infty.$$

- [**Comment** — The machines should only be replaced when they break down.]