Many-Sorted Equivalence of Shiny and Strongly Polite Theories

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Abstract

Herein we close the question of the equivalence of shiny and strongly polite theories by establishing that, for theories with a decidable quantifier-free satisfiability problem, the set of many-sorted shiny theories coincides with the set of many-sorted strongly polite theories. Capitalizing on this equivalence, we obtain a Nelson-Oppen combination theorem for many-sorted shiny theories.

Keywords: Nelson-Oppen method, combination of satisfiability procedures, shiny theories, polite theories, strongly polite theories, first-order logic, many-sorted logic

1 Introduction

The Nelson-Oppen method is a well-known method for modularly combining satisfiability procedures of given theories. The method was proposed by Nelson and Oppen in 1979, [12], and provides a way of deciding the satisfiability of quantifier-free formulas in the union of two (one-sorted) theories, as long as both of them have their own procedure for deciding the satisfiability problem of quantifier-free formulas. After a correction, see [13], the two main conditions of the Nelson-Oppen[1] method are that:

- the theories are *stably infinite*,
- their signatures are disjoint.

Concerned about the fact that many theories of interest, such as those admitting only finite models, are not stably infinite, Tinelli and Zarba, in [17], showed that the Nelson-Oppen combination procedure still applies when the stable infiniteness condition is replaced by the requirement that all but one of the theories are *shiny*. However, a shiny theory must be equipped with a function that computes minimal cardinalities of models of formulas, which is inherently hard to compute.

In order to overcome the problem of computing this function and in proving that a theory is shiny, as well as to generalize these combination methods to the many-sorted case, Ranise, Ringeissen and Zarba proposed an alternative requirement, *politeness*, in [14], and analyzed its relationship with shininess. A polite theory has to be equipped with a $\mathsf{witness}$ function,

[1] A correctness proof of the method was presented by Tinelli and Harandi in [15].

which was thought to be easier to compute than the mincard function of shiny theories. They showed that given a polite theory and an arbitrary one, the Nelson-Oppen combination procedure is still valid when the theories have disjoint signatures and both have their own procedure for deciding the satisfiability problem of quantifier-free formulas. Some time later, in [10], Jovanović and Barrett reported that the politeness notion provided in [14] allowed, after all, witness functions that are not sufficiently strong to prove the combination theorem. To overcome this issue they provided a stronger notion of politeness, in the sequel called *strong politeness*, equipped with a stronger witness function, s-witness, that allowed to prove the combination theorem. However, in [10], the relationship between strong politeness and shininess was not studied. This motivated the work in [3], where the authors investigated the relationship between shiny and strongly polite theories in the one-sorted case. They showed that a shiny theory with a decidable quantifier-free satisfiability problem is strongly polite, and, for the other direction, they provided two different sets of conditions under which a polite theory is shiny.

In [8], Fontaine introduced *gentle* theories that are a natural generalization of shiny theories. In [8] and [1], the authors further showed that the union of gentle theories is gentle, and classified several theories in decidable fragments of first-order logic in terms of gentleness and shininess. Gentle theories can be combined with a very broad class of theories, although there is no Nelson-Oppen theorem for the combination of gentle theories with an arbitrary theory. Furthermore, gentle theories have also played a role on the area of non-disjoint combination of theories [19].

Herein we settle the loose end from [3] and show that the class of many-sorted shiny theories coincides, with respect to any set of sorts, with the class of strongly polite theories, when the theory is equipped with a quantifier-free satisfiability solver. We begin by adapting the notion of shininess to the many-sorted case. This adaptation is by no means immediate. For example, the stably finite notion, when adapted to the many-sorted case, has to include a condition on the cardinality of the finite domains and the mincard function had to be replaced by a more general function, minmods, that returns tuples of the cardinalities of the *minimal* models of the theory that satisfy a given formula. Nonetheless these generalized notions coincide with the usual ones that they extend when seen in the one-sorted case. Then, we extend the results in [3] in a two-fold manner: on one hand we prove the equivalence between shiny and strongly polite theories unconditionally, and on the other hand we obtain this result in the many-sorted context. These results do not have the restriction imposed in [14] when relating polite theories with shiny theories, that the set of sorts has to be the full set of sorts in the signature. Capitalizing on this equivalence and on the Nelson-Oppen combination theorem for many-sorted strongly polite theories in [10], we establish a Nelson-Oppen combination theorem for many-sorted shiny theories.

1.1 Organization of the Paper

The paper is organized as follows: in Section 2 we introduce the main notions and definitions used throughout the paper. In Section 3 we show that when equipped with a quantifier-free satisfiability solver, the classes of many-sorted shiny and strongly polite theories coincide for any given finite set of sorts. In Section 4 we capitalize on the proved equivalence between strongly polite and shiny, and on the combination theorem for strongly polite theories and establish the Nelson-Oppen combination theorem for many-sorted shiny theories. In Section 5 we conclude the paper and provide some directions for further research.

2 Preliminaries

The results in this paper concern theories of many-sorted first-order logic with equality. For each sort, we assume a disjoint countably infinite set of variables. We follow the many-sorted presentation of first-order logic with equality as is done in [7].

2.1 Syntax

A *signature* is a tuple $\Sigma = \langle \Sigma^S, \Sigma^F, \Sigma^P, \alpha, \tau \rangle$ where Σ^S is the non-empty finite set of sorts, Σ^F is the set of function symbols, Σ^P is the set of predicate symbols, α is a map that for each function and predicate symbol returns its arity and τ is a map that for each function and predicate symbol returns its type. When applied to variables or sets of variables, τ returns the sorts of the variables. For each sort $\sigma \in \Sigma^S$, we use $\cong_\sigma$ to denote the equality logic symbol over pairs of terms of sort σ and assume the standard many-sorted definitions of Σ-*atom* and Σ-*term*. A Σ-*formula* is inductively defined as usual over Σ-atoms using the connectives $\wedge, \vee, \neg, \rightarrow$ or the quantifiers $\forall$ and $\exists$. We denote by $\mathsf{QF}(\Sigma)$ the set of Σ-formulas with no occurrences of quantifiers and, given a Σ-formula φ, by $\mathsf{vars}(\varphi)$ the set of free variables of φ, i.e., the set of variables not under the scope of a quantifier. Furthermore, we denote by $\mathsf{vars}_\sigma(\varphi)$ the set of free variables of sort σ occurring in φ. Given a set of terms T and a sort σ, we denote by T_σ the set of terms in T of sort σ, and say that a Σ-formula is a Σ-*sentence* if it has no free variables. In the sequel, when there is no ambiguity, we may omit the reference to the signature when referring to atoms, terms, formulas and sentences.

Given a finite set of variables Y over a set of sorts S and $E \subseteq Y^2$, we write

$$E \sqsubseteq Y^2$$

to denote that E is a family of sort-wise equivalence relations over Y, i.e.,

$$E = \bigcup_{\sigma \in S} E_\sigma,$$

and for each sort $\sigma \in S$, E_σ is an equivalence relation on Y_σ^2.

Definition 2.1 (Arrangement formula)**.** *Given a finite set of variables Y over a set of sorts S and $E \sqsubseteq Y^2$, the* arrangement formula *induced by E over Y, denoted by*

$$\delta_E^Y$$

is the formula

$$\bigwedge_{\sigma \in S} \delta_{E_\sigma}^{Y_\sigma}$$

where $\delta_{E_\sigma}^{Y_\sigma}$ is

$$\bigwedge_{(x,y) \in E_\sigma} (x \cong_\sigma y) \wedge \bigwedge_{(x,y) \in Y_\sigma^2 \setminus E_\sigma} \neg(x \cong_\sigma y) \ .$$

In the sequel, when there is no ambiguity, we may simply denote δ_E^Y by δ_E.

2.2 Semantics

Given a signature Σ, a Σ-*interpretation* $\mathcal{A}$ over a set of variables X is a map that interprets:

- each sort $\sigma \in \Sigma^S$ as a non-empty set A_σ;
- each variable $x \in X$ with sort σ as an element $x^\mathcal{A} \in A_\sigma$;
- each function symbol $f \in \Sigma^F$ of arity n and type $\sigma_1 \times \ldots \times \sigma_n \to \sigma$ as a map $f^\mathcal{A} : A_{\sigma_1} \times \ldots \times A_{\sigma_n} \to A_\sigma$, and
- each predicate symbol $p \in \Sigma^P$ of arity n and type $\sigma_1 \times \ldots \times \sigma_n$ as a subset $p^\mathcal{A}$ of $A_{\sigma_1} \times \ldots \times A_{\sigma_n}$.

We denote the domain of a Σ-interpretation $\mathcal{A}$ by A, i.e., the collection of the domains A_σ for each sort σ. In the sequel, when there is no ambiguity, we may omit the reference to the signature when referring to interpretations.

Given an interpretation $\mathcal{A}$ and a term t, we denote by $t^\mathcal{A}$ the interpretation of t under $\mathcal{A}$. Similarly, we denote by $\varphi^\mathcal{A}$ the truth value of the formula φ under the interpretation $\mathcal{A}$. Furthermore, given a set T of terms, we denote by $[\![T]\!]^\mathcal{A}$ the set $\{t^\mathcal{A} : t \in T\}$. Finally, we write $\mathcal{A} \models \varphi$ when the formula φ is true under the interpretation $\mathcal{A}$, i.e., $\mathcal{A}$ satisfies φ.

A formula φ is *satisfiable* if it is true under some interpretation. It is *unsatisfiable* otherwise.

Given sets of variables Y and X, we say that two interpretations $\mathcal{A}$ and $\mathcal{B}$ over X are *equivalent modulo* Y whenever $A = B$, $f^\mathcal{A} = f^\mathcal{B}$ for each function symbol f, $p^\mathcal{A} = p^\mathcal{B}$ for each predicate symbol p, and $x^\mathcal{A} = x^\mathcal{B}$ for each variable x in $X \setminus Y$.

We also say that an *interpretation* $\mathcal{A}$ *is finite (resp. infinite)* with respect to a set S of sorts when, for each sort $\sigma \in S$, the set A_σ is finite (resp. infinite).

2.3 Theories

Given a signature Σ, a Σ-*theory* is a set of Σ-sentences, and given a Σ-theory $\mathcal{T}$, a $\mathcal{T}$-*model* is a Σ-interpretation that satisfies all the sentences of $\mathcal{T}$. A formula φ is $\mathcal{T}$-*satisfiable* when there is a $\mathcal{T}$-model that satisfies it, and two formulas are $\mathcal{T}$-*equivalent* if they are interpreted to the same truth value in every $\mathcal{T}$-model. Given a Σ_1-theory $\mathcal{T}_1$ and a Σ_2-theory $\mathcal{T}_2$, their union, $\mathcal{T}_1 \cup \mathcal{T}_2$, is a $\Sigma_1 \cup \Sigma_2$-theory defined by the union of the sentences of $\mathcal{T}_1$ with the sentences of $\mathcal{T}_2$. In the sequel, when there is no ambiguity, we may omit the reference to the signature when referring to theories.

Definition 2.2 (Smoothness, [14])**.** *A Σ-theory $\mathcal{T}$ is* smooth *with respect to a set of sorts $S \subseteq \Sigma^S$ if for every quantifier-free formula φ, $\mathcal{T}$-model $\mathcal{A}$ satisfying φ and cardinals $\kappa_\sigma \geq |A_\sigma|$ for each $\sigma \in S$, there exists a $\mathcal{T}$-model $\mathcal{B}$ satisfying φ such that $|B_\sigma| = \kappa_\sigma$ for all $\sigma \in S$.*

Definition 2.3 (Strong finite witnessability, [10])**.** *A Σ-theory $\mathcal{T}$ is* strongly finitely witnessable *with respect to a set of sorts $S \subseteq \Sigma^S$ if there exists a computable function* $\mathsf{s\text{-}witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ *such that for every quantifier-free formula φ the following conditions hold:*

- *φ and $\exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$ are $\mathcal{T}$-equivalent, where $\vec{w}$ are the variables in the formula $\mathsf{s\text{-}witness}(\varphi)$ which do not occur in φ;*

- *for every finite set Y of variables*[2] *with sorts in S and $E \sqsubseteq Y^2$, if* $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ *is $\mathcal{T}$-satisfiable then there exists a $\mathcal{T}$-model $\mathcal{A}$ of* $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ *such that* $A_\sigma = [\![\mathsf{vars}_\sigma\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right)]\!]^{\mathcal{A}}$*, for all $\sigma \in S$.*

A function satisfying the above properties is called a *strong witness function for $\mathcal{T}$* with respect to S.

Definition 2.4 (Strong politeness, [10])**.** *A Σ-theory is* strongly polite *with respect to a set of sorts $S \subseteq \Sigma^S$ whenever it is smooth and strongly finitely witnessable with respect to S.*

Shiny theories were introduced by Tinelli and Zarba in [17] and extended to the many-sorted case by Ranise, Ringeissen and Zarba in [14]. In the one-sorted case, these theories are characterized by having a computable function which given a satisfiable formula returns the cardinality of the smallest model of the theory that satisfies the formula. However, when we are dealing with many-sorted theories, unless orderings are imposed on sorts, there is more than one minimal model. By minimal models, we refer to the models that cannot be spanned by smoothness from other models. In a sense, they form a basis to the set of models of smooth theories.

Here, instead of defining the cardinality of the smallest model in terms of some measure as in [14], where models are compared by their maximum cardinality, we consider the cardinalities of minimal models.

Given a tuple $\bar{k}$ indexed by a set of sorts S, we denote by $\bar{k}[\sigma]$ its σ-component for each $\sigma \in S$.

Definition 2.5 ($\mathsf{minmods}$ function)**.** *Given a Σ-theory $\mathcal{T}$, a set of sorts $S \subseteq \Sigma^S$ and a $\mathcal{T}$-satisfiable quantifier free formula φ, $\mathsf{minmods}_{\mathcal{T},S}(\varphi)$ is the set of S-tuples defined as follows:*

$\bar{k} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$ *iff*

- *there is a model $\mathcal{A}$ of $\mathcal{T} \cup \{\varphi\}$ with $|A_\sigma| = \bar{k}[\sigma]$ for each $\sigma \in S$;*
- *for each model $\mathcal{B}$ of $\mathcal{T} \cup \{\varphi\}$ with $\langle |B_\sigma| \rangle_{\sigma \in S} \neq \bar{k}$, there is a sort $\sigma \in S$ such that $\bar{k}[\sigma] < |B_\sigma|$.*

We also provide a notion of many-sorted stable finiteness different from the one in [14].

Definition 2.6 (Stable finiteness)**.** *A Σ-theory $\mathcal{T}$ is* stably finite *with respect to a set of sorts $S \subseteq \Sigma^S$ if for every quantifier-free formula φ satisfied by a $\mathcal{T}$-model $\mathcal{A}$ there exists a finite $\mathcal{T}$-model $\mathcal{A}'$ with respect to S satisfying φ such that $|A'_\sigma| \leq |A_\sigma|$ for each $\sigma \in S$.*

Notice how this formulation of many-sorted stable finiteness coincides with the usual stable finiteness property on one-sorted theories. Furthermore, with this formulation we guarantee that, for a stably finite theory, the set $\mathsf{minmods}_{\mathcal{T},S}(\varphi)$ only contains tuples with finite cardinalities. Observe that this is not the case with the many-sorted extension of the stable finiteness notion in [14], which states that if a formula has a model then it has a finite model.

[2] As in [10], we do not restrict Y to be the set of variables in $\mathsf{s\text{-}witness}$ since this generality is needed to show Lemma A.2 and Theorem 3.7 of [10].

As an example of a theory stably finite according to [14] (but not stably finite according to the notion proposed in Definition 2.6), with minimal models with infinite cardinalities, consider a two-sorted theory that accepts all models $\mathcal{A}$ with cardinalities such that

$$\text{either } |A_1| \geq 2 \text{ and } |A_2| = \infty \text{ or } |A_1| \geq 3 \text{ and } |A_2| \geq 3 \ .$$

Then, in this theory and with the notion of stable finiteness of [14], the formula $x \cong_1 x$, has a minimal model with an infinite component, $\langle 2, \infty \rangle$.

We now show that the $\mathsf{minmods}$ set for a given formula is finite, by checking that there can only be a finite number of finite tuples that satisfy the second property of the $\mathsf{minmods}$ notion.

Proposition 2.7. *Let $\mathcal{T}$ be a many-sorted stably finite Σ-theory with respect to a set of sorts $S \subseteq \Sigma^S$. For any $\mathcal{T}$-satisfiable quantifier-free formula φ, the set $\mathsf{minmods}_{\mathcal{T},S}(\varphi)$ is finite and only contains tuples with finite cardinalities.*

Proof: Observe that by the second property of the $\mathsf{minmods}$ function, for any two different elements $\bar{k}_1, \bar{k}_2 \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$, there exists a sort σ such that $\bar{k}_1[\sigma] < \bar{k}_2[\sigma]$. This also holds for the other direction, i.e., there exists a sort σ' such that $\bar{k}_2[\sigma'] < \bar{k}_1[\sigma']$. Thus, the elements in $\mathsf{minmods}$ are incomparable in the product order, and as such, we can bound the size of the $\mathsf{minmods}$ set by the size of the largest set of incomparable elements in the product order in $\mathbb{N}^{|S|}$ (we only need to take into account finite cardinalities because the theory is stably finite). It happens that there are no infinite sets of incomparable elements in the product order in $\mathbb{N}^{|S|}$, and so the $\mathsf{minmods}$ set is finite.

The fact that there are no infinite sets of incomparable elements in the product order in $\mathbb{N}^{|S|}$ can be restated in terms of partial order theory: there are no infinite antichains (sets of incomparable elements) in the partial ordered set $\mathbb{N}^{|S|}$. Take $|S| = k$ and view an element $\langle a_1, \ldots, a_k \rangle \in \mathbb{N}^k$ as the monomial $x_1^{a_1} \cdots x_k^{a_k}$ in the polynomial ring $\mathbb{Z}_2[x_1, \ldots, x_k]$. Suppose there is an infinite antichain, and consider the ideal generated by its elements. By Hilbert's Basis theorem [5, 6], this ideal is finitely generated. Since all elements generated by each basis monomial are comparable to it, the only possible incomparable elements are the basis elements, which are in finite number. Thus there are no infinite antichains. QED

In the sequel, we use the following useful lemmas:

Lemma 2.8. *For each $\mathcal{T}$-satisfiable quantifier-free formula φ, $\mathsf{minmods}_{\mathcal{T},S}(\varphi) \neq \emptyset$.*

Proof: Let $\mathcal{A}$ be a $\mathcal{T}$-model of φ. Then, either $\langle |A_\sigma| \rangle_{\sigma \in S}$ is in $\mathsf{minmods}_{\mathcal{T},S}(\varphi)$ or it is not and so there is another $\mathcal{T}$-model of φ smaller than $\mathcal{A}$ which is in $\mathsf{minmods}_{\mathcal{T},S}(\varphi)$. In either case, the set is not empty. QED

Lemma 2.9. *Given a many-sorted stably finite Σ-theory $\mathcal{T}$ with respect to a set of sorts $S \subseteq \Sigma^S$ and a $\mathcal{T}$-satisfiable quantifier free formula φ, for any $\mathcal{T}$-model $\mathcal{A}$ of φ, there is a tuple $\bar{k} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$ such that*

$$\bar{k}[\sigma] \leq |A_\sigma|, \text{ for all } \sigma \in S \ .$$

Proof:

Consider the following set M of tuples of S-cardinalities of finite $\mathcal{T}$-models of φ:

$$\{\langle|B_\sigma|\rangle_{\sigma\in S} : \mathcal{B} \text{ is a finite } \mathcal{T}\text{-model of } \varphi \text{ and } |B_\sigma| \leq |A_\sigma| \text{ for all } \sigma \in S\} \subseteq \mathbb{N}^{|S|} \ .$$

This set is non-empty due to the stable finiteness of $\mathcal{T}$. Consider the product order over $\mathbb{N}^{|S|}$ defined as usual, i.e., $\bar{k} \leq \bar{k}'$ iff $\bar{k}[\sigma] \leq \bar{k}'[\sigma]$ for all $\sigma \in S$. Since $\mathbb{N}^{|S|}$ is lower-bounded, and M is a non-empty subset of it, we conclude that M must have a minimal element $\bar{m}$, that is, an element which has no smaller element than it. We claim that $\bar{m} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$: if it were not, by definition of $\mathsf{minmods}$ and since $\bar{m}$ corresponds to a tuple of cardinalities of a $\mathcal{T}$-model of φ, it meant that there existed a model $\mathcal{C}$ such that for all $\sigma \in S$, $|C_\sigma| \leq \bar{m}[\sigma]$ and $\langle|C_\sigma|\rangle_{\sigma\in S} \neq \bar{m}$. But this contradicts the fact that $\bar{m}$ is a minimal element of M. QED

Equipped with the notions of smoothness, stable finiteness, as well as the $\mathsf{minmods}$ function (which plays the role of the $\mathsf{mincard}$ function in the one-sorted case) we are able to define shininess in the many-sorted case. We emphasize that each of these notions coincide with their original notions in the one-sorted case as introduced in [17].

Definition 2.10 (Shininess)**.** *A Σ-theory $\mathcal{T}$ is* shiny *with respect to a set of sorts $S \subseteq \Sigma^S$ whenever it is smooth and stably finite with respect to S, and the function $\mathsf{minmods}_{\mathcal{T},S}$ is computable.*

3 Shiny and Strongly Polite Theories

In this section we analyze the relationship between many-sorted shiny and strongly polite theories. We start by showing that a shiny theory with respect to a set of sorts is strongly polite with respect to the same set, assuming that the theory has a decidable quantifier-free satisfiability problem.

Proposition 3.1. *A shiny theory with respect to a set of sorts S with a decidable quantifier-free satisfiability problem is strongly polite with respect to S.*

Proof: Let $\mathcal{T}$ be a shiny Σ-theory with respect to a set $S \subseteq \Sigma^S$ of sorts and Sat an algorithm that solves its quantifier-free satisfiability problem. Since a shiny theory is by definition smooth, in order to conclude that $\mathcal{T}$ is strongly polite with respect to S, we are left to prove that $\mathcal{T}$ is strongly finitely witnessable with respect to S. In the sequel, given a $\mathcal{T}$-satisfiable quantifier-free formula φ and a family of equivalence relations $E \sqsubseteq \mathsf{vars}(\varphi)^2$, if $\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$ is $\mathcal{T}$-satisfiable, we denote by $\mathsf{MM}(\varphi, E)$ the set $\mathsf{minmods}_{\mathcal{T},S}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)$.

In other words, $\mathsf{MM}(\varphi, E)$ has the tuples of cardinalities of the minimal $\mathcal{T}$-models that satisfy $\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$. Given a tuple $\bar{k} \in \mathsf{MM}(\varphi, E)$, we denote the cardinality of the σ domain by $\bar{k}[\sigma]$. Observe that all the $\bar{k}[\sigma]$ are finite due to the stable finiteness of $\mathcal{T}$ with respect to S.

The proof is structured in the following manner: we begin by proposing a strong witness function $\mathsf{s\text{-}witness}$ and in Lemma 3.2 we show that it is computable. In Lemma 3.3 we show that this function satisfies the first condition of strong finite witnessability and finally in Lemma 3.4 that it satisfies the second condition.

Let $\mathsf{s\text{-}witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ be such that

$$\mathsf{s\text{-}witness}(\varphi) = \varphi \wedge \Omega,$$

where Ω is

$$\bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]} \right)$$

and $\gamma_{\bar{k}[\sigma]}$ is

$$\begin{cases} \displaystyle\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{\bar{k}[\sigma]} \neg(w_{i,\sigma} \cong_\sigma w_{j,\sigma}), & \text{if } \bar{k}[\sigma] > 1 \\ (w_{1,\sigma} \cong_\sigma w_{1,\sigma}), & \text{if } \bar{k}[\sigma] = 1 \end{cases}$$

and $w_{1,\sigma}, \ldots, w_{\bar{k}[\sigma],\sigma}$ are distinct σ-variables not occurring in φ and in $\gamma_{\bar{k}'[\sigma]}$ for all $\bar{k}' \neq \bar{k}$.

Lemma 3.2. *The function* $\mathsf{s\text{-}witness}$ *is a computable function.*

Proof: It is immediate to conclude that $\mathsf{s\text{-}witness}$ is computable since:

- there is a finite number of sets E with $E \sqsubseteq \mathsf{vars}(\varphi)^2$ since $\mathsf{vars}(\varphi)$ is finite;
- the formula $\delta_E^{\mathsf{vars}(\varphi)}$ can be computed in a finite number of steps since E and $\mathsf{vars}(\varphi)$ are finite;
- the set $\mathsf{MM}\,(\varphi, E)$ is computable since the $\mathsf{minmods}$ function is computable and finite by Proposition 2.7;
- the formula $\gamma_{\bar{k}[\sigma]}$ is computable since $\bar{k}[\sigma]$ is a natural number due to the stable finiteness of $\mathcal{T}$ with respect to S.

QED

We now show that $\mathsf{s\text{-}witness}$ satisfies the first condition of strong finite witnessability.

Lemma 3.3. *Let φ be a quantifier-free formula. Then, φ and $\exists \vec{w}$ $\mathsf{s\text{-}witness}(\varphi)$ are $\mathcal{T}$-equivalent.*

Proof: Let $\mathcal{A}$ be a $\mathcal{T}$-model. Assume that $\mathcal{A} \models \exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$. Then $\mathcal{A} \models \varphi \wedge \exists \vec{w}\ \Omega$, and so $\mathcal{A} \models \varphi$.

For the other direction, assume $\mathcal{A} \models \varphi$. We need to show that

$$\mathcal{A} \models \exists \vec{w} \bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]} \right) .$$

Let $\mathcal{A}'$ be an interpretation equivalent modulo $\vec{w}$ to $\mathcal{A}$ (and so with the same domain and the same interpretation of functions, predicates and of all variables except possibly those in $\vec{w}$) such that for each sort $\sigma \in S$:

- if A_σ is infinite then $w_{1,\sigma}^{\mathcal{A}'} \neq w_{2,\sigma}^{\mathcal{A}'}$ for every $w_{1,\sigma}, w_{2,\sigma} \in \vec{w}_\sigma$;
- if A_σ is finite then for each $E \sqsubseteq \mathsf{vars}(\varphi)^2$ with $\mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right) = 1$:
  - if $\bar{k}[\sigma] \leq |A_\sigma|$ then $w_i^{\mathcal{A}'} \neq w_j^{\mathcal{A}'}$ for every $w_i, w_j \in \mathsf{vars}(\gamma_{\bar{k}[\sigma]})$;
  - otherwise, $w_i^{\mathcal{A}'} = w_j^{\mathcal{A}'}$ for every $w_i, w_j \in \mathsf{vars}(\gamma_{\bar{k}[\sigma]})$.

Then

$$\mathcal{A}' \models \bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}\right)$$

since for each $E \sqsubseteq \mathsf{vars}(\varphi)^2$ with $\mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right) = 1$ either

- $\mathcal{A}' \not\models \delta_E^{\mathsf{vars}(\varphi)}$ and so $\mathcal{A}' \models \delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$; or
- $\mathcal{A}' \models \delta_E^{\mathsf{vars}(\varphi)}$. Observe that $\mathcal{A}' \models \varphi$ because $\mathcal{A}$ and $\mathcal{A}'$ may only differ in the interpretation of the variables in $\vec{w}$ which do not occur in φ. So $\mathcal{A}' \models \varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$. Since $\mathcal{A}'$ is a model for $\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}$, by Lemma 2.9, there is a $\bar{k} \in \mathsf{MM}(\varphi, E)$ such that the cardinality of A_σ has to be greater or equal than $\bar{k}[\sigma]$ for each sort σ in S. Hence $\mathcal{A}' \models \gamma_{\bar{k}[\sigma]}$ for each $\sigma \in S$ and so $\mathcal{A}' \models \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$. Thus, $\mathcal{A}' \models \delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$.

QED

We now show that $\mathsf{s\text{-}witness}$ satisfies the second condition of strong finite witnessability.

Lemma 3.4. *Let Y be a finite set of variables with sorts in S, φ a quantifier-free formula and $\psi = \mathsf{s\text{-}witness}(\varphi)$. Given a family $E \sqsubseteq Y^2$, if $\psi \wedge \delta_E^Y$ is $\mathcal{T}$-satisfiable, then there exists a $\mathcal{T}$-model $\mathcal{A}$ that satisfies $\psi \wedge \delta_E^Y$ such that $A_\sigma = [\![\mathsf{vars}_\sigma(\psi \wedge \delta_E^Y)]\!]^{\mathcal{A}}$, for each sort σ in S.*

Proof:

Let Y be a finite set of variables with sorts in S and $E \sqsubseteq Y^2$ such that $\psi \wedge \delta_E^Y$ is $\mathcal{T}$-satisfiable. For each $\sigma \in S$, let p_σ be the number of equivalence classes induced by δ_E^Y, and $Y_1^\sigma, \ldots, Y_{p_\sigma}^\sigma$ those classes. These form a partition of the set of variables Y_σ. Furthermore, let $\mathcal{A}$ be a $\mathcal{T}$-model of $\psi \wedge \delta_E^Y$ and let $\delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ be the arrangement formula induced by

$$E_\mathcal{A} = \bigcup_{\sigma \in S} \left\{(x, y) : x, y \in \mathsf{vars}_\sigma(\varphi) \text{ and } x^\mathcal{A} = y^\mathcal{A}\right\} \quad .$$

Then, obviously, $\delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ is satisfied by $\mathcal{A}$. Hence, $\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ is $\mathcal{T}$-satisfiable and thus has a $\mathcal{T}$-model with cardinality tuple $\bar{k}$ in $\mathsf{MM}(\varphi, E_\mathcal{A})$ since $\mathcal{T}$ is stably finite with respect to S. Let $K_\sigma = \max\left\{\bar{k}[\sigma], p_\sigma\right\}$. By the smoothness of $\mathcal{T}$ and since $\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ is $\mathcal{T}$-satisfiable, let $\mathcal{B}$ be a $\mathcal{T}$-model such that

$$\mathcal{B} \models \varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)} \quad \text{and} \quad |B_\sigma| = K_\sigma \text{ for each sort } \sigma \in S,$$

and, for each sort σ in S, let $d_1^\sigma, \ldots, d_{p_\sigma}^\sigma$ be distinct elements of B_σ such that for $i = 1, \ldots, p_\sigma$

$$d_i^\sigma = y^{\mathcal{B}} \text{ for all } y \in Y_i^\sigma \cap \mathsf{vars}(\varphi),$$

and assuming that the variables of $\gamma_{\bar{k}[\sigma]}$ are $w_1, \ldots, w_{\bar{k}[\sigma]}$ let $e_1^\sigma, \ldots, e_{\bar{k}[\sigma]}^\sigma$ be distinct elements of B_σ such that

$$e_j^\sigma = d_i^\sigma \text{ if } w_j \in Y_i^\sigma$$

for $j = 1, \ldots, \bar{k}[\sigma]$ and $i = 1, \ldots, p_\sigma$. Observe that distinct variables in $w_1, \ldots, w_{\bar{k}[\sigma]}$ are in distinct sets in $Y_1^\sigma, \ldots, Y_{p_\sigma}^\sigma$ if they are in any Y_i^σ, since $\mathcal{A} \models \delta_E^Y$ and $\mathcal{A} \models \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$. Let $\mathcal{B}'$ be the $\mathcal{T}$-model equivalent modulo $(\vec{w} \cup (Y \setminus \mathsf{vars}(\varphi)))$ to $\mathcal{B}$ such that

$$x^{\mathcal{B}'} = \begin{cases} d_i^\sigma & \text{if } x \in Y_i^\sigma \text{ for some } i \in \{1, \ldots, p_\sigma\} \text{ and } \sigma \in S \\ e_j^\sigma & \text{if } x \notin Y \text{ and } x \text{ is } w_j \text{ with } w_j \in \mathsf{vars}\left(\gamma_{\bar{k}[\sigma]}\right) \text{ and } \sigma \in S \\ x^{\mathcal{B}} & \text{if } x \notin Y \text{ and } x \notin \mathsf{vars}\left(\gamma_{\bar{k}[\sigma]}\right) \text{ for all } \sigma \in S \end{cases}$$

for each $x \in \vec{w} \cup (Y \setminus \mathsf{vars}(\varphi))$. Let us now prove that $\mathcal{B}' \models \varphi \wedge \Omega \wedge \delta_E^Y$:
(a) $\mathcal{B}' \models \varphi$. This follows immediately taking into account that $\mathcal{B} \models \varphi$ and that $\mathcal{B}$ and $\mathcal{B}'$ may only differ in variables in $\vec{w} \cup (Y \setminus \mathsf{vars}(\varphi))$, hence not occurring in φ;
(b) $\mathcal{B}' \models \Omega$. Observe that $\mathcal{B}' \models \varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$ since $\mathcal{B}$ and $\mathcal{B}'$ may only differ in variables in $\vec{w} \cup (Y \setminus \mathsf{vars}(\varphi))$, hence not occurring in $\varphi \wedge \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$. Moreover $\mathcal{B}' \models \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$ by definition, and so $\mathcal{B}' \models \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi, E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$. Since $\mathcal{B}' \models \delta_{E_\mathcal{A}}^{\mathsf{vars}(\varphi)}$, we have that $\mathcal{B}' \not\models \delta_E^{\mathsf{vars}(\varphi)}$ for all $E \neq E_\mathcal{A}$ with $E \sqsubseteq \mathsf{vars}(\varphi)^2$. Hence $\mathcal{B}' \models \delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi, E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}$ for all $E \sqsubseteq \mathsf{vars}(\varphi)^2$ with $\mathsf{Sat}(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}) = 1$ and so $\mathcal{B}' \models \Omega$;
(c) $\mathcal{B}' \models \delta_E^Y$. We only need to verify that $\mathcal{B}'$ satisfies the equalities and disequalities induced by E. This holds since by construction, for each $\sigma \in S$, $\mathcal{B}'$ assigns the same value to variables in Y_i^σ, and assigns different values to variables in different sets Y_i^σ.

Finally it remains to show that $B'_\sigma = [\![\mathsf{vars}_\sigma(\varphi \wedge \Omega \wedge \delta_E^Y)]\!]^{\mathcal{B}'}$ for each sort $\sigma \in S$:
($\subseteq$): Let $d \in B'_\sigma$. Then d is either a d_i^σ for some $i = 1, \ldots, p_\sigma$ or a e_j^σ for some $j = 1, \ldots, \bar{k}[\sigma]$. In the case that $d = d_i^\sigma$ then we have that $d = x^{\mathcal{B}'}$ for all $x \in Y_i^\sigma$. On the other hand, if $d = e_j^\sigma$ then $d = w_j^{\mathcal{B}'}$ for the w_j variable in $\mathsf{vars}_\sigma(\gamma_{\bar{k}[\sigma]})$;
($\supseteq$): Obviously, $[\![\mathsf{vars}_\sigma(\varphi \wedge \Omega \wedge \delta_E^Y)]\!]^{\mathcal{B}'} \subseteq B'_\sigma$ by definition. QED

Combining Lemmas 3.2, 3.3 and 3.4 we conclude that a shiny theory with respect to a set S is strongly finitely witnessable with respect to S, hence strongly polite with respect to S since it is smooth by definition. QED

We find relevant to remark that the given proof is constructive in the way that we provide a way to build a strong witness function for a shiny theory, provided that it has a decidable quantifier-free satisfiability problem.

We now turn our attention to showing that a strongly polite theory with respect to a set of sorts is shiny with respect to that set, assuming that the theory has a decidable quantifier-free satisfiability problem. The result holds for any set of sorts S since the computation of the $\mathsf{minmods}_{\mathcal{T},S}$ function will not rely on enumerating interpretations. This circumvents the restriction that $S = \Sigma^S$ imposed in [14]. We begin by proving lemmas relating the $\mathsf{minmods}$ function with equivalence classes.

Lemma 3.5. *Let Y be a finite set of variables and $E \sqsubseteq Y^2$ a sort-wise family of equivalence relations over Y. Given the arrangement formula δ_E^Y and a sort $\sigma \in \tau(Y)$, the number of equivalence classes in the quotient set of Y_σ by E_σ is computable in at most $\mathcal{O}(|Y_\sigma|^2)$ steps.*

Proof:
Consider Algorithm 1 to compute the number of equivalence classes in the quotient set of Y_σ by E_σ, denoted in the sequel by $\left|{}^{Y_\sigma}/_{E_\sigma}\right|$, given the arrangement formula δ_E^Y and sort σ.

Algorithm 1 – Counting equivalence classes given an arrangement formula and a sort

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Input: an arrangement formula δ_E^Y, and a sort σ
Output: cardinality of the quotient set $\left|{}^{Y_\sigma}/_{E_\sigma}\right|$

1: Construct a graph G_σ with nodes Y_σ and edge set $\left\{(x,y) : (x \cong_\sigma y) \in \delta_{E_\sigma}^{Y_\sigma}\right\}$
2: Compute connected components of G_σ
3: Return number of connected components of the graph

In step 1, the adjacency list for G_σ is built from $\delta_{E_\sigma}^{Y_\sigma}$. This can be made in $\mathcal{O}(|Y_\sigma|^2)$. To compute the number of connected components of the graph, we refer to the algorithm described by Hopcroft and Tarjan in [9] which has time complexity of the order of

$$\max\{|Ed|, |V|\} = \max\left\{\left|\left\{(x,y) : (x \cong_\sigma y) \in \delta_{E_\sigma}^{Y_\sigma}\right\}\right|, |Y_\sigma|\right\} \leq |Y_\sigma|^2 \quad .$$

So the time complexity of Algorithm 1 is $\mathcal{O}(|Y_\sigma|^2)$. QED

We now show that a strongly polite theory with respect to a set of sorts is stably finite with respect to that set.

Lemma 3.6. *A strongly polite theory with respect to a set of sorts S is stably finite with respect to that set.*

Proof: Let φ be a quantifier-free formula satisfied by a model $\mathcal{A}$ and let FS be the subset of sorts of S for which A is finite. Then $\mathcal{A}$ satisfies the formula $\exists\vec{w}\ \mathsf{s\text{-}witness}(\varphi)$ by the first condition of strong finite witnessability. Let $\mathcal{A}'$ be a model equivalent modulo $\vec{w}$ to $\mathcal{A}$ satisfying $\mathsf{s\text{-}witness}(\varphi)$ and consider the family of equivalence relations of variables of $\mathsf{s\text{-}witness}(\varphi)$ induced by $\mathcal{A}'$, i.e.,

$$E_{\mathcal{A}'} = \bigcup_{\sigma \in \mathsf{FS}} \{(x,y) : x^{\mathcal{A}'} = y^{\mathcal{A}'} \text{ and } x, y \in \mathsf{vars}_\sigma(\mathsf{s\text{-}witness}(\varphi))\} \quad .$$

Take $Y = \bigcup_{\sigma \in \mathsf{FS}} \mathsf{vars}_\sigma(\mathsf{s\text{-}witness}(\varphi))$. Then $\mathcal{A}'$ satisfies $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E_{\mathcal{A}'}}^Y$. Observe that $\left|{}^{Y_\sigma}/_{E_{\mathcal{A}'\sigma}}\right| \leq |A'_\sigma| = |A_\sigma|$ for all $\sigma \in \mathsf{FS}$. By the second property of strong finite witnessability, we obtain that there is a $\mathcal{T}$-model $\mathcal{B}$ such that

$$B_\sigma = \left[\!\left[\mathsf{vars}_\sigma\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E_{\mathcal{A}'}}^Y\right)\right]\!\right]^{\mathcal{B}}, \text{ for all } \sigma \in S \quad .$$

From this, we obtain that all $|B_\sigma|$ are finite for $\sigma \in S$, and that $|B_\sigma| = \left|{}^{Y_\sigma}\!/_{E_{\mathcal{A}'\sigma}}\right|$ for all $\sigma \in \mathsf{FS}$. So $|B_\sigma| \leq |A_\sigma|$ for all $\sigma \in \mathsf{FS}$ as desired. QED

All the following lemmas assume that $\mathcal{T}$ is a strongly polite theory with respect to a set of sorts S and that φ is a quantifier-free satisfiable formula.

Lemma 3.7. *Let $Y = \mathsf{vars}(\mathsf{s\text{-}witness}(\varphi))$ and $E \sqsubseteq Y^2$. If $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is $\mathcal{T}$-satisfiable, then $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ has a model with $\left|{}^{Y_\sigma}\!/_{E_\sigma}\right|$ as the cardinality of the σ-domain, for each $\sigma \in S$.*

Proof: By the second property of strong finite witnessability, since $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is $\mathcal{T}$-satisfiable, there exists a $\mathcal{T}$-model $\mathcal{A}$ such that

$$A_\sigma = [\![\mathsf{vars}_\sigma\left(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y\right)]\!]^{\mathcal{A}}, \text{ for all } \sigma \in S \ .$$

Since $Y = \mathsf{vars}(\mathsf{s\text{-}witness}(\varphi))$, the cardinality of A_σ is exactly $\left|{}^{Y_\sigma}\!/_{E_\sigma}\right|$ for each $\sigma \in S$. QED

Lemma 3.8. *Let $Y = \mathsf{vars}(\mathsf{s\text{-}witness}(\varphi))$. For all $\bar{k} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$ there is an $E \sqsubseteq Y^2$ such that*

$$\mathsf{Sat}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y) = 1$$

and

$$\left|{}^{Y_\sigma}\!/_{E_\sigma}\right| = \bar{k}[\sigma] \text{ for each } \sigma \in S \ .$$

Proof: Choose $\bar{k}$ and let $\mathcal{A}$ be a $\mathcal{T}$-model of φ such that $|A_\sigma| = \bar{k}[\sigma]$ for each $\sigma \in S$. We have that $\mathcal{A}$ satisfies $\exists \vec{w}\ \mathsf{s\text{-}witness}(\varphi)$ by the first property of strong finite witnessability. Take $\mathcal{A}'$ as a model that satisfies $\mathsf{s\text{-}witness}(\varphi)$ and that is equivalent modulo $\vec{w}$ to $\mathcal{A}$. Observe that by definition of equivalent modulo $\vec{w}$, $|A_\sigma| = |A'{}_\sigma|$.

Now, consider the equivalence relation induced by $\mathcal{A}'$, $E_{\mathcal{A}'}$,

$$E_{\mathcal{A}'} = \bigcup_{\sigma \in S} \{(x, y) : x^{\mathcal{A}'} = y^{\mathcal{A}'} \text{ and } x, y \in \mathsf{vars}_\sigma(\mathsf{s\text{-}witness}(\varphi))\} \ .$$

Clearly, $\mathcal{A}'$ satisfies $\delta_{E_{\mathcal{A}'}}^Y$ and so it satisfies $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E_{\mathcal{A}'}}^Y$. Observe that $\left|{}^{Y_\sigma}\!/_{E_{\mathcal{A}'\sigma}}\right| \leq |A'_\sigma| = |A_\sigma| = \bar{k}[\sigma]$. Hence, by Lemma 3.7, we obtain that there is a $\mathcal{T}$-model $\mathcal{B}$ of $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E_{\mathcal{A}'}}^Y$ such that $|B_\sigma| = \left|{}^{Y_\sigma}\!/_{E_{\mathcal{A}'\sigma}}\right| \leq |A'_\sigma| = \bar{k}[\sigma]$ for each $\sigma \in S$. So, since $\mathcal{B}$ satisfies φ and $\bar{k} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$ we can conclude that $|B_\sigma| = \bar{k}[\sigma]$ for each $\sigma \in S$. That is, $\left|{}^{Y_\sigma}\!/_{E_{\mathcal{A}'\sigma}}\right| = \bar{k}[\sigma]$ for each $\sigma \in S$. QED

Proposition 3.9. *A strongly polite theory with respect to a set of sorts S with a decidable quantifier-free satisfiability problem is shiny with respect to S. Moreover, Algorithm 2 computes the $\mathsf{minmods}_{\mathcal{T},S}$ function.*

Proof:

Let $\mathcal{T}$ be a strongly polite theory with respect to a set of sorts S. By definition of strong politeness, $\mathcal{T}$ is smooth with respect to S. Moreover, by Lemma 3.6, $\mathcal{T}$ is stably finite with respect to S and so, in order to show that $\mathcal{T}$ is shiny with respect to S, it remains to prove that $\mathsf{minmods}_{\mathcal{T},S}$ is computable. Consider Algorithm 2 that returns, given an input formula φ, a set denoted in the sequel by $\mathsf{MM\_alg}(\varphi)$. The algorithm starts by building a set of

Algorithm 2 – $\mathsf{MM\_alg}$ algorithm – computes the $\mathsf{minmods}_{\mathcal{T},S}$ function for a strongly polite theory $\mathcal{T}$ with respect to a set of sorts S

Input: φ, a $\mathcal{T}$-satisfiable quantifier-free formula
Output: $\mathsf{minmods}_{\mathcal{T},S}(\varphi)$

1: $Y = \mathsf{vars}_S(\mathsf{s\text{-}witness}(\varphi))$
2: $\mathsf{Card} = \left\{ \langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S} : E \sqsubseteq Y^2 \text{ and } \mathsf{Sat}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y) = 1 \right\}$
3: $\mathsf{MM} = \emptyset$
4: **for** $\bar{k} \in \mathsf{Card}$ do
5: **if** $\neg \exists \bar{m} \in \mathsf{MM} : \bar{k}[\sigma] \geq \bar{m}[\sigma]$ for all $\sigma \in S$
6: **then** $\mathsf{MM} = \{\bar{k}\} \cup \mathsf{MM} \setminus \{\bar{k}' \in \mathsf{MM} : \bar{k}[\sigma] \leq \bar{k}'[\sigma] \text{ for all } \sigma \in S\}$
7: return MM

relevant tuples of S-cardinalities of models, and proceeds by finding the minimal elements of this set with respect to the product order defined as usual, i.e., $\bar{k} \leq \bar{k}'$ iff $\bar{k}[\sigma] \leq \bar{k}'[\sigma]$ for all $\sigma \in S$. The method used for finding the minimal elements in a poset is described in [18].

(1) $\mathsf{MM\_alg}$ terminates:
It is enough to see that Y is a finite set and so also the set Card.

(2) $\mathsf{MM\_alg}(\varphi) \subseteq \mathsf{minmods}_{\mathcal{T},S}(\varphi)$:
By Lemma 3.7, we know that Card only contains tuples of S-cardinalities of models of φ. Let $E \sqsubseteq Y^2$ be such that $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is satisfiable and $\langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S} \in \mathsf{MM\_alg}(\varphi)$. Suppose by contradiction that

$$\langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S} \notin \mathsf{minmods}_{\mathcal{T},S}(\varphi) \ .$$

Then, by Lemma 2.9, there is a $\bar{k} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$ such that $\bar{k} \neq \langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S}$ and $\bar{k}[\sigma] \leq \left| {}^{Y_\sigma}/_{E_\sigma} \right|$ for each $\sigma \in S$. By Lemma 3.8, let $E' \sqsubseteq Y^2$ be such that $\mathsf{Sat}(\mathsf{s\text{-}witness}(\varphi) \wedge \delta_{E'}^Y) = 1$ and $\left| {}^{Y_\sigma}/_{E'_\sigma} \right| = \bar{k}[\sigma]$ for each $\sigma \in S$. Then $\left| {}^{Y_\sigma}/_{E'_\sigma} \right| \leq \left| {}^{Y_\sigma}/_{E_\sigma} \right|$ for each $\sigma \in S$. Observe that $\langle \left| {}^{Y_\sigma}/_{E'_\sigma} \right| \rangle_{\sigma \in S} \in \mathsf{Card}$ at step 2. Then, either $\langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S}$ would be removed from MM at step 6 when $\langle \left| {}^{Y_\sigma}/_{E'_\sigma} \right| \rangle_{\sigma \in S}$ is added or it would have never been added to the MM set if $\langle \left| {}^{Y_\sigma}/_{E'_\sigma} \right| \rangle_{\sigma \in S}$ was already there, since it fails the condition at step 5. Both these cases contradict with $\langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S} \in \mathsf{MM\_alg}(\varphi)$ and thus, $\langle \left| {}^{Y_\sigma}/_{E_\sigma} \right| \rangle_{\sigma \in S} \in \mathsf{minmods}_{\mathsf{T},S}(\varphi)$.

(3) $\mathsf{minmods}_{\mathcal{T},S}(\varphi) \subseteq \mathsf{MM\_alg}(\varphi)$:
Let $\bar{k} \in \mathsf{minmods}_{\mathcal{T},S}(\varphi)$ and, by Lemma 3.8, let $E \sqsubseteq Y^2$ be such that $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_E^Y$ is satisfiable and $\left| {}^{Y_\sigma}/_{E_\sigma} \right| = \bar{k}[\sigma]$ for each $\sigma \in S$. Suppose by contradiction that $\bar{k} \notin \mathsf{MM\_alg}(\varphi)$. Then, by definition of $\mathsf{MM\_alg}$, there is an $\langle \left| {}^{Y_\sigma}/_{R_\sigma} \right| \rangle_{\sigma \in S} \in \mathsf{MM}$ such that $\left| {}^{Y_\sigma}/_{R_\sigma} \right| \leq \bar{k}[\sigma] = \left| {}^{Y_\sigma}/_{E_\sigma} \right|$ for each σ. By Lemma 3.7, let $\mathcal{A}$ be the model of $\mathsf{s\text{-}witness}(\varphi) \wedge \delta_R^Y$ such that $|A_\sigma| = \left| {}^{Y_\sigma}/_{R_\sigma} \right|$ for each $\sigma \in S$. Then $\mathcal{A}$ is a model of φ and so, by definition of $\mathsf{minmods}$, the tuple $\bar{k} \notin \mathsf{minmods}_{\mathcal{T},S}(\varphi)$, which is a contradiction. So $\bar{k} \in \mathsf{MM\_alg}(\varphi)$.

So the proposed algorithm $\mathsf{MM\_alg}$ terminates and computes the $\mathsf{minmods}_{\mathcal{T},S}$ function. QED

Combining Propositions 3.1 and 3.9 we obtain the equivalence between shininess and strong politeness in the many-sorted case when the theories are equipped with a quantifier-free satisfiability solver. It should be made clear that, in practice, the requirement that the theories have satisfiability solvers is not a significant restriction since shiny and strongly polite theories were proposed in view of a more general Nelson-Oppen result, which is about combination of satisfiability solvers.

Theorem 3.10. *Let $\mathcal{T}$ be a first-order Σ-theory with a decidable quantifier-free satisfiability problem and $S \subseteq \Sigma^S$ a set of sorts. Then, the following statements are equivalent:*

1. *$\mathcal{T}$ is shiny with respect to S;*
2. *$\mathcal{T}$ is strongly polite with respect to S.*

4 Combination method for many-sorted shiny theories

In the previous section we showed the equivalence between many-sorted shiny and strongly polite theories. Because of this and the fact that there exists a Nelson-Oppen combination method for many-sorted strongly polite theories, see [10], we get a Nelson-Oppen combination theorem for many-sorted shiny theories as a consequence.

Furthermore, we obtain two combination methods – one based on the constructive translation to strongly polite theories and the other based on directly making use of the $\mathsf{minmods}_{\mathcal{T},S}$ function of shiny theories. The latter method, in the one-sorted case, will coincide with the combination method for shiny theories introduced in [17].

Theorem 4.1 (Combination theorem for many-sorted shiny theories)**.** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be theories with decidable quantifier-free satisfiability problems with no function or predicate symbols in common, and denote their set of sorts in common by S. If $\mathcal{T}_2$ is shiny with respect to S, then $\mathcal{T}_1 \cup \mathcal{T}_2$ has a decidable quantifier-free satisfiability problem.*

This theorem is a simple consequence of the following proposition, Proposition 4.2, where the satisfiability procedure for $\mathcal{T}_1 \cup \mathcal{T}_2$ is explicitly constructed using the satisfiability procedures for each of the theories.

We recall from the proof of Proposition 3.1 that a shiny theory $\mathcal{T}$ with respect to a set of sorts S with a decidable quantifier-free satisfiability problem has a strong witness function $\mathsf{s\text{-}witness} : \mathsf{QF}(\Sigma) \to \mathsf{QF}(\Sigma)$ defined as follows

$$\mathsf{s\text{-}witness}(\varphi) = \varphi \wedge \Omega,$$

where Ω is

$$\bigwedge_{\substack{E \sqsubseteq \mathsf{vars}(\varphi)^2 \\ \mathsf{Sat}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)=1}} \left(\delta_E^{\mathsf{vars}(\varphi)} \to \bigvee_{\bar{k} \in \mathsf{MM}(\varphi,E)} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]} \right) ,$$

where $\mathsf{MM}(\varphi, E)$ is the set $\mathsf{minmods}_{\mathcal{T},S}\left(\varphi \wedge \delta_E^{\mathsf{vars}(\varphi)}\right)$ and $\gamma_{\bar{k}[\sigma]}$ is

$$\begin{cases} \displaystyle\bigwedge_{\substack{i,j=1 \\ i \neq j}}^{\bar{k}[\sigma]} \neg(w_{i,\sigma} \cong_\sigma w_{j,\sigma}), & \text{if } \bar{k}[\sigma] > 1 \\ (w_{1,\sigma} \cong_\sigma w_{1,\sigma}), & \text{if } \bar{k}[\sigma] = 1 \end{cases}$$

with $w_{1,\sigma}, \ldots, w_{\bar{k}[\sigma],\sigma}$ being distinct fresh σ-variables.

Proposition 4.2. *Let $\mathcal{T}_i$ be Σ_i-theories with a decidable quantifier-free satisfiability problem, for $i = 1, 2$, such that $\Sigma_1^P \cap \Sigma_2^P = \emptyset$ and $\Sigma_1^F \cap \Sigma_2^F = \emptyset$. Assume that $\mathcal{T}_2$ is shiny with respect to $S = \Sigma_1^S \cap \Sigma_2^S$. Then for every conjunction Γ_1 of Σ_1-literals and Γ_2 of Σ_2-literals the following are equivalent:*

1. *$\Gamma_1 \wedge \Gamma_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiable;*
2. *there exists $E \sqsubseteq Y^2$, where Y is $\mathsf{vars}_S(\mathsf{s\text{-}witness}_{\mathcal{T}_2,S}(\Gamma_2))$, such that*

$$\begin{aligned}\mathcal{T}_1 &\models \Gamma_1 \wedge \delta_E^Y \text{ and}\\ \mathcal{T}_2 &\models \mathsf{s\text{-}witness}_{\mathcal{T}_2,S}(\Gamma_2) \wedge \delta_E^Y \ .\end{aligned}$$

3. *there exists $E \sqsubseteq Y^2$, where Y is $\mathsf{vars}_S(\Gamma_1) \cap \mathsf{vars}_S(\Gamma_2)$, such that*

$$\begin{aligned}\mathcal{T}_1 &\models \Gamma_1 \wedge \delta_E^Y \wedge \Big(\bigvee_{\bar{k} \in \mathsf{MM}} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]}\Big) \quad \text{and}\\ \mathcal{T}_2 &\models \Gamma_2 \wedge \delta_E^Y \ ,\end{aligned}$$

   where MM is $\mathsf{minmods}_{\mathcal{T}_2,S}(\Gamma_2 \wedge \delta_E^Y)$.

Proof: The equivalence between 1. and 2. follows from the combination proposition, Proposition 2, in [10], capitalizing on the equivalence of many-sorted shiny and strongly polite theories established in Theorem 3.10.

We now show 1.→3. and 3.→1. separately. Suppose $\Gamma_1 \wedge \Gamma_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiable. Then there is a $\mathcal{T}_1 \cup \mathcal{T}_2$-model $\mathcal{A}$ that satisfies $\Gamma_1 \wedge \Gamma_2$. Furthermore, let $Y = \mathsf{vars}_S(\Gamma_1) \cap \mathsf{vars}_S(\Gamma_2)$ and δ_E^Y be the arrangement formula induced by

$$E_{\mathcal{A}} = \bigcup_{\sigma \in S} \{(x, y) : x^{\mathcal{A}} = y^{\mathcal{A}} \text{ and } x, y \in Y_\sigma\} \ .$$

Obviously $\mathcal{A}$ is a $\mathcal{T}_1$-model of $\Gamma_1 \wedge \delta_E^Y$ and a $\mathcal{T}_2$-model of $\Gamma_2 \wedge \delta_E^Y$. Finally, since $\mathcal{A}$ is a model of $\Gamma_2 \wedge \delta_E^Y$, by Lemma 2.9, the cardinalities of the domains A_σ for sorts $\sigma \in S$ must be larger or equal than $\bar{k}[\sigma]$ for some $\bar{k} \in \mathsf{minmods}_{\mathcal{T}_2,S}(\Gamma_2 \wedge \delta_E^Y)$. Hence, $\mathcal{A} \models (\bigvee_{\bar{k} \in \mathsf{MM}} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]})$.

For the other direction, suppose we have a $\mathcal{T}_1$-model $\mathcal{A}_1$ satisfying $\Gamma_1 \wedge \delta_E^Y \wedge (\bigvee_{\bar{k} \in \mathsf{MM}} \bigwedge_{\sigma \in S} \gamma_{\bar{k}[\sigma]})$ and a $\mathcal{T}_2$-model $\mathcal{A}_2$ satisfying $\Gamma_2 \wedge \delta_E^Y$. Take the tuple $\bar{K}$ of cardinalities of domains of $\mathcal{A}_1$ with sorts in S. Clearly, there is a $\bar{k} \in \mathsf{minmods}_{\mathcal{T}_2,S}(\Gamma_2 \wedge \delta_E^Y)$ such that $\bar{k}[\sigma] \leq \bar{K}[\sigma]$ for all $\sigma \in S$. By smoothness of $\mathcal{T}_2$, we know there is a $\mathcal{T}_2$-model $\mathcal{B}'$ of $\Gamma_2 \wedge \delta_E^Y$ such that $|B'_\sigma| = \bar{K}[\sigma]$ for all $\sigma \in S$. Since both $\mathcal{A}_1$ and $\mathcal{B}'$ satisfy δ_E^Y and have the same cardinalities of domains for each $\sigma \in S$, we can construct a $\mathcal{T}_1 \cup \mathcal{T}_2$-model of $\Gamma_1 \wedge \Gamma_2$ via Theorem 2.5 in the extended version, see [11], of [10]. QED

This result provides a way to effectively construct the satisfiability procedure for $\mathcal{T}_1 \cup \mathcal{T}_2$, and so Theorem 4.1, which states the existence of such procedure, follows immediately.

5 Conclusion and Further Research

In this paper we proved a Nelson-Oppen theorem for the combination of a many-sorted shiny theory with an arbitrary theory, extending to the many-sorted case the work in [17]. For this, we investigated the relationship between the notions of shininess and strong politeness in the many-sorted case. We showed that, in the many-sorted case, a shiny theory with respect to a set of sorts (and equipped with a decidable quantifier-free satisfiability problem) is strongly polite with respect to the same set. On the other hand we were also able to prove that a strongly polite theory with respect to a set of sorts (with a decidable quantifier-free satisfiability problem) is shiny with respect to the same set. These results show that the classes of shiny and strongly polite theories with a decidable quantifier-free satisfiability problem are, in fact, the same.

We intend to investigate in the future more general conditions that a theory should satisfy in order to be combined with an arbitrary theory by a Nelson-Oppen method. More concretely, we leave as future work the investigation of a class of theories strictly containing the shiny/strongly polite theories for which there exists an indiscriminate Nelson-Oppen method, in the sense that they can be combined with an arbitrary theory with a decidable quantifier-free satisfiability problem.

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