DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Reliability and Quality Control

MMA / LMAC

TEST 1
2nd. Semester — **2017/18**
2018/05/05 — **8AM**

- Duration: **1h30m**
- **Please justify your answers.**
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. The 2−out−of−4 mooring system for a floating offshore facility includes multiple lines and foundation anchors. Admit that the reliabilities of the mooring sub-systems in 1000m, 1500m, 2000m and 3000m water depths are equal to p_1, p_2, p_3, p_4 (resp.).

(a) Identify the minimal path sets and the minimal cut sets of the mooring system. Provide two expressions for the structure function (DO NOT simplify them!). Comment on these two expressions. **(3.0)**

- **Minimal path sets**

$$\begin{aligned} \mathscr{P}_1 &= \{1,2\} \\ \mathscr{P}_2 &= \{1,3\} \\ \mathscr{P}_3 &= \{1,4\} \\ \mathscr{P}_4 &= \{2,3\} \\ \mathscr{P}_5 &= \{2,4\} \\ \mathscr{P}_6 &= \{3,4\} \\ p^* &= 6 \text{ minimal path sets} \end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned} \mathscr{K}_1 &= \{1,2,3\} \\ \mathscr{K}_2 &= \{1,2,4\} \\ \mathscr{K}_3 &= \{1,3,4\} \\ \mathscr{K}_4 &= \{2,3,4\} \\ q &= 4 \text{ minimal path sets} \end{aligned}$$

- **Structure function** (in terms of the minimal path sets)

$$\begin{aligned} \phi_{minpathsets}(\underline{X}) &\stackrel{Th.1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathscr{P}_j} X_i\right) \\ &= 1-(1-X_1X_2)\times(1-X_1X_3)\times(1-X_1X_4)\times(1-X_2X_3) \\ &\quad \times(1-X_2X_4)\times(1-X_3X_4). \qquad (1) \end{aligned}$$

- **Structure function** (in terms of the minimal path sets)

$$\begin{aligned} \phi_{mincutsets}(\underline{X}) &\stackrel{Th.1.30}{=} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathscr{K}_j}(1-X_i)\right] \\ &= [1-(1-X_1)(1-X_2)(1-X_3)]\times[1-(1-X_1)(1-X_2)(1-X_4)] \\ &\quad \times[1-(1-X_1)(1-X_3)(1-X_4)] \\ &\quad \times[1-(1-X_2)(1-X_3)(1-X_4)]. \qquad (2) \end{aligned}$$

- **Comments**
  - Even though the expressions in (1) and (2) differ, $\phi_{minpathsets}(\underline{x}) \equiv \phi_{mincutsets}(\underline{X})$, for all $(x_1,\dots,x_4) \in \{0,1\}^4$[; after all we are dealing with binary state variables, $X_i \stackrel{index.}{\sim}$ Bernoulli(p_i), $i=1,2,3,4$.]
  - Capitalizing on the minimal cut sets would be a slightly more cumbersome way of obtaining the structure function.

(b) When the four mooring sub-systems operate independently, the reliability of the system is equal to $p_1p_2+p_1p_3+p_1p_4+p_2p_3+p_2p_4+p_3p_4-2p_1p_2p_3-2p_1p_2p_4-2p_1p_3p_4-2p_2p_3p_4+3p_1p_2p_3p_4$. Compute the importance of the reliability of each mooring sub-system. **(1.5)**

- **Reliability**

$$\begin{aligned} r(\underline{p}) &= E[\phi(\underline{X})] \\ &= p_1p_2+p_1p_3+p_1p_4+p_2p_3+p_2p_4+p_3p_4 \\ &\quad -2p_1p_2p_3-2p_1p_2p_4-2p_1p_3p_4-2p_2p_3p_4+3p_1p_2p_3p_4 \end{aligned}$$

- **Importance of the reliability of the each mooring sub-systems**

$$\begin{aligned} I_r(i) &\stackrel{(1.29)}{=} \frac{\partial r(\underline{p})}{\partial p_i} \\ &= \begin{cases} p_2+p_3+p4-2p_2p_3-2p_2p_4-2p_3p_4+3p_2p_3p_4, & i=1 \\ p_1+p_3+p4-2p_1p_3-2p_1p_4-2p_3p_4+3p_1p_3p_4, & i=2 \\ p_1+p_2+p4-2p_1p_2-2p_1p_4-2p_2p_4+3p_1p_2p_4, & i=3 \\ p_1+p_2+p3-2p_1p_2-2p_1p_3-2p_2p_3+3p_1p_2p_3, & i=4 \end{cases} \end{aligned}$$

(c) Obtain the approximate perturbation in the reliability of the mooring system when p_i increases to $p_i+\epsilon$, for $i=1,2,3,4$ and $\epsilon \in (0, \min_{i=1,2,3,4}\{1-p_i\})$. **(1.0)**

- **Perturbation in the reliability of the system**
  According to Note 1.61, the perturbation in the reliability of the mooring system — due to changes of $\Delta p_i = \epsilon$ in the reliability of the sub-systems $i=1,2,3,4$ — is given by:

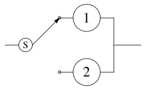
$$\begin{aligned} \Delta r(\underline{p}) &\simeq \sum_{i=1}^{n} I_r(p_i)\,\Delta p_i \\ &= \sum_{i=1}^{n} I_r(p_i)\times\epsilon \\ &= [3(p_1+p_2+p_3+p_4)-4(p_1p_2+p_1p_3+p_1p_4+p_2p_3+p_2p_4+p_3p_4) \\ &\quad 3(p_1p_2p_3+p_1p_2p_4+p_1p_3p_4+p_2p_3p_4)]\times\epsilon. \end{aligned}$$

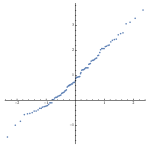
(d) Determine the min-max upper bound for the reliability of the original mooring system now with positively associated mooring sub-systems with reliabilities $p_i = 1-10^{-i}$, $i=1,2,3,4$. **(1.0)**

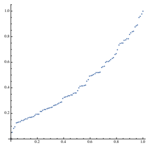
- **Min-max upper bound**
  Since we are dealing with a coherent system with sub-systems operating in a positively associated fashion, we can apply Theorem 1.70 and obtain

$$\begin{aligned} r(\underline{p}) &\stackrel{Th.,1.70}{\le} \min_{j=1,\dots,q}\left[1-\prod_{i\in\mathscr{K}_j}(1-p_i)\right] \\ &\stackrel{1-p_i=10^{-i/3}}{=} \min_{j=1,\dots,q}\left[1-\prod_{i\in\mathscr{K}_j}10^{-i}\right] \\ &= \min_{j=1,\dots,q}\left[1-10^{-\sum_{i\in\mathscr{K}_j} i}\right] \\ &= 1-10^{-\min_{j=1,\dots,q}\sum_{i\in\mathscr{K}_j} i} \\ &= 1-10^{-(1+2+3)} \\ &\stackrel{p=0.9}{=} 0.999999. \end{aligned}$$

2. Consider a cold standby system composed of two identical pumps each with constant failure rate λ failures/hour. If the probability that the switch (S) will fail to activate the standby pump is equal to p then the reliability function of the pump system shown below (picture on the left) at time t is equal to $[1+(1-p)\lambda t]\times e^{-\lambda t}$, $t \ge 0$.







(a) Compute the probability that this system does not fail in the first 1 000 h, when $\lambda = 10^{-3}$ and $p = 0.015$. Obtain the expected time to failure of this pump system. **(1.5)**

- **R.v.**
  T = time to failure of the pump system

- **Requested reliability**

$$\begin{aligned} R(t) &= [1+(1-p)\lambda\, t] \times e^{-\lambda\, t} \\ &\overset{t=1000,\,etc.}{=} [1+(1-0.015)\times 10^{-3}\times 1000]\times e^{-10^{-3}\,1000}\,dt \\ &= 1.985\times e^{-1} \\ &\simeq 0.730241. \end{aligned}$$

- **Expected value of T**

$$\begin{aligned} E(T) &\overset{(2.10)}{=} \int_0^{+\infty} R(t)\,dt \\ &= \int_0^{+\infty} [1+(1-p)\,\lambda\, t]\times e^{-\lambda\, t}\,dt \\ &= \frac{1}{\lambda}\int_0^{+\infty} \lambda\, e^{-\lambda\, t}\,dt + (1-p)\int_0^{+\infty} t\,\lambda\, e^{-\lambda\, t}\,dt \\ &= \frac{1}{\lambda}\int_0^{+\infty} f_{Exp(\lambda)}(t)\,dt + (1-p)\times E[Exp(\lambda)] \\ &= \frac{1}{\lambda} + (1-p)\frac{1}{\lambda} \\ &= \frac{2-p}{\lambda} \\ &= \frac{2-0.015}{10^{-3}} \\ &= 1985 \text{ hours.} \end{aligned}$$

(b) Prove that the time to failure of the pump system is IHR when $p = 0$. **(2.0)**

- **Reliability function of T when $p = 0$**
  $R(t) = (1+\lambda\, t)\times e^{-\lambda\, t},\ t \geq 0$

- **P.d.f. of T when $p = 0$**

$$\begin{aligned} f(t) &= -\frac{dR(t)}{dt} \\ &= -\frac{d[(1+\lambda\, t)\times e^{-\lambda\, t}]}{dt} \\ &= -\left(-\lambda\, e^{-\lambda\, t} + \lambda\, e^{-\lambda\, t} - \lambda^2\, t\, e^{-\lambda\, t}\right) \\ &= \lambda^2\, t\, e^{-\lambda\, t},\ t \geq 0 \end{aligned}$$

- **Hazard rate function of T when $p = 0$**

$$\begin{aligned} \lambda(t) &= \frac{f(t)}{R(t)} \\ &= \frac{\lambda^2\, t\, e^{-\lambda\, t}}{(1+\lambda\, t)\times e^{-\lambda\, t}} \\ &= \frac{\lambda^2\, t}{1+\lambda\, t},\ t \geq 0. \end{aligned}$$

- **Devising the stochastic ageing character of T**
  Since $\lambda(t) \equiv \frac{\lambda^2}{\frac{1}{t}+\lambda}$ we immediately conclude that $\lambda(t)\uparrow_t$, therefore $T \in IHR$.
  [Alternatively: 1) $\frac{d\,\lambda(t)}{dt} = \frac{\lambda^2(1+\lambda\, t) - \lambda^2\, t\,\lambda}{(1+\lambda\, t)^2} = \frac{\lambda^2}{(1+\lambda\, t)^2} > 0 \Rightarrow T \in IHR$.
  2) In fact, $R(t) \equiv R_{Gamma(\alpha=2>1,\lambda)}(t) \Rightarrow T \in IHR$.]

(c) How many identical pumps would you have to set in parallel so that the reliability exceeds 0.730241 for a period of 1 000 h? **(2.5)**
Is the time to failure of a system with 3 pumps set in parallel an IHR r.v.?

- **Individual operation times; common hazard and reliability functions**
  $T_i^\star$ = time to failure of pump i, $i = 1, \ldots, n$
  $T_i^\star$ are i.i.d. r.v. with common hazard function $\lambda(t) = \lambda = 10^{-3}$, $t \geq 0$, therefore:

$$\begin{aligned} T_i^\star &\overset{i.i.d.}{\sim} \text{Exponential}(\lambda); \\ R_i(t) &= P(T_i^\star > t) \\ &= R^\star(t) \\ &= e^{-\lambda\, t},\ t \geq 0. \end{aligned}$$

- **New failure time**
  $T^\star = T^\star_{(n)}$ = failure time of the system with n pumps set in parallel

- **Reliability function of $T^\star$**

$$\begin{aligned} R_{T^\star}(t) &= R_{=T^\star_{(n)}}(t) \\ &\overset{(2.5)}{=} 1-[1-R^\star(t)]^n \\ &= 1-[1-e^{-\lambda\, t}]^n,\ t \geq 0 \end{aligned}$$

- **Requested number of pumps**

$$\begin{aligned} n \quad : \quad & R_{T^\star}(1000) > 0.730241 \quad [\simeq R(1000)] \\ & 1-(1-e^{-\lambda\, t})^n > 0.730241 \\ & 1-0.730241 > [1-e^{-\lambda\, t}]^n \\ & n > \frac{\ln(1-0.730241)}{\ln(1-e^{-1})} \simeq 2.85654, \end{aligned}$$

  thus we need at least $n^\star = 3$ pumps.

- **Devising the stochastic ageing character of $T^\star$**
  According to (3.8),

$$\begin{aligned} \lambda^\star(t) &= \frac{n\,\lambda(t)}{\sum_{j=0}^{n-1}[1-R(t)]^{-j}} \\ &= \frac{n\,\lambda}{\sum_{j=0}^{n-1}(1-e^{-\lambda\, t})^{-j}}. \end{aligned}$$

  Moreover, we successively have

$$\begin{aligned} & 1-e^{-\lambda\, t} && \uparrow_t \\ & \sum_{j=0}^{n-1}(1-e^{-\lambda\, t})^{-j} && \downarrow_t \\ \lambda^\star(t) = & \frac{n\,\lambda}{\sum_{j=0}^{n-1}(1-e^{-\lambda\, t})^{-j}} && \uparrow_t, \end{aligned}$$

  that is, $T^\star \in IHR$ for any $n > 1$.

3. Compressors are very important for the operation of process plants, and every effort is taken to restart a failed compressor as soon as possible.

(a) After having collected a sample of 99 repair times (in hours) of a certain type of compressor, an engineer obtained the ordered sample $(0.23, 0.37, \ldots, 36.94)$, the lognormal plotting paper and the TTT plot shown above (center and right pictures, resp.). **(2.5)**

Derive the abscissae and the ordinates of the lognormal probability plot and exemplify the obtention of its two first points.

What conclusions can the engineer draw from the lognormal plotting paper and from the TTT plot.

- **R.v.**
  X_i = repair time i, $i = 1, \ldots, n$ $\quad (n = 99)$
  $X_i \overset{i.i.d.}{\sim} X$, $i = 1, \ldots, n$
- **Ordered random sample and its realisation**
  $(X_{(1)}, \ldots, X_{(n)}) \qquad (x_{(1)}, \ldots, x_{(n)})$
- **Postulated model**
  $\{\text{Lognormal}(\mu, \sigma^2), \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$
- **Deriving the lognormal probability plot**
  For any absolutely continuous model

  $$F_X(X_{(i)}) \sim \text{Beta}(i, n-i+1).$$

  Thus, by considering as an estimate of

  $$p_i = F_X(x_{(i)}) \overset{(4.19)}{=} \Phi\left[\frac{\ln(x_{(i)}) - \mu}{\sigma}\right]$$

  the following expected value

  $$\hat{p}_i = E[F_X(X_{(i)})] = E[\text{Beta}(i, n-i+1)] = \frac{i}{n+1},$$

  we are supposed to confront (indirectly and) graphically $\hat{p}_i$ and $F_X(x_{(i)})$, that is,

  $$\frac{i}{n+1} \rightarrow \Phi\left[\frac{\ln(x_{(i)}) - \mu}{\sigma}\right]$$
  $$\Phi^{-1}\left(\frac{i}{n+1}\right) \rightarrow \frac{\ln(x_{(i)}) - \mu}{\sigma}$$
  $$\mu + \sigma \times \Phi^{-1}\left(\frac{i}{n+1}\right) \rightarrow \ln(x_{(i)}).$$

  Consequently, the points of the lognormal probability plot have abcissae and ordinates:

  $$\left(\phi^{-1}\left(\frac{i}{n+1}\right), \quad \ln(x_{(i)})\right), \; i = 1, \ldots, n.$$

- **Two first points of the lognormal probability plot**

i	$\frac{i}{n+1}$	$\phi^{-1}\left(\frac{i}{n+1}\right)$	$\ln(x_{(i)})$
1	$\frac{1}{100} = 0.01$	$\phi^{-1}(0.01) = -\Phi^{-1}(1-0.01) = -2.3263$	$\ln(0.23) = -1.469676$
2	$\frac{2}{100} = 0.02$	$\phi^{-1}(0.02) = -\Phi^{-1}(1-0.02) = -2.0537$	$\ln(0.37) = -0.994252$

- **Comment on the lognormal probability plot**
  The points sort of fall along a straight line in the lognormal probability plot, hence providing some evidence that the repair times come from the lognormal model.
- **Comment on the TTT plot**
  According to Note 5.5 of the lecture notes (LN), if a TTT plot suggests a:

  – 45^o line, then the data can be modeled by a distribution with CHR, the exponential distribution;
  – concave (resp. convex) curve above the 45^o line, then the data can be fitted by an IHR (resp. DHR) distribution.

  In this particular case, we deal with a TTT plot which suggests a concave segment (above the 45^o line) followed by a convex one (below the 45^o line). Therefore we are led to believe that a distribution with non monotonic hazard rate function suits the data, namely any lognormal distribution (see page 98 of the LN), thus concurring with the lognormal probability plot.

(b) A sample of 100 operation times of that same type of compressor was collected by the engineer, who used the R software to perform the Atkinson test for exponentiality. Comment on the $p-$value the engineer obtained, 0.6091. **(1.0)**

- **Comment on the $p-$value of the Atkinson test for exponentiality**
  Recall that the $p-$value is the largest significance level leading to the non rejection of the null hypothesis. Thus, for these particular data set and null hypothesis $H_0 : T \sim \text{Exponential}(\lambda)$, $\lambda > 0$:
  - we should not reject H_0 for any significance levels $\alpha_0 \le p-\text{value} = 0.6091$, namely the usual significance levels $(1\%, 5\%, 10\%)$;
  - we should reject H_0 for any significance levels $\alpha_0 > p-\text{value} = 0.6091$.

  The exponential model seems to be very reasonable in light of the data set.

(c) A sample of operation times, $(t_1, \ldots, t_{100})$, led to $\sum_{i=1}^{100} t_i = 22791.1$ hours. **(4.0)**

Compute: (i) the UMVU estimate of $E(T)$, the expected value of the operation time; (ii) the UMVU estimate of $P(T > 10\,\text{days})$; (iii) a 90% confidence interval (CI) for this same probability.

List the underlying assumptions that allowed you to obtain these point estimates and the CI.

- **Distribution assumption**
  In light of (b), it is fairly reasonable to admit that the operation times T_i are such that $T_i \overset{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n^\star$ $(n^\star = 100)$.
- **Life test**
  Complete data is available!
  $\sum_{i=1}^{100} t_i = 22791.1$
- **UMVU estimates of $E(T) = \lambda^{-1}$ and $R_T(240) = e^{-240\lambda}$**
  Since it is fairly reasonable to admit that the T is an exponentially distributed r.v. and $t = 240 < \sum_{i=1}^{100} = 22791.1$, we capitalize on the comment after Table 5.10 and on (5.21) to get the requested UMVU estimates:

$$\begin{aligned}
UMVUE[E(T)] &= \hat{\lambda}^{-1} \\
&= \bar{t} \quad \text{(sample mean)} \\
&= \frac{22791.1}{n^\star} \\
&= 227.911; \\
UMVUE[R_T(t)] &= \left(1 - \frac{\hat{\lambda}\, t}{n^\star}\right)^{n^\star - 1} \\
&= \left(1 - \frac{t}{\sum_{i=1}^{n^\star} t_i}\right)^{n^\star - 1} \\
&\overset{t=240, etc.}{=} \left(1 - \frac{240}{22791.1}\right)^{100-1} \\
&\simeq 0.350624.
\end{aligned}$$

- 90% **confidence interval (CI) for** $R_T(240) = e^{-240\lambda}$
  - **Pivotal quantity**
    $Z = 2\lambda \sum_{i=1}^{n} T_i \sim \xi^2_{(2n)}$ (check lecture note just before Exercise 5.26)
  - **Percentage points**
    They are represented by a and b, balanced and such that $P(a \leq Z \leq b) = 1 - \alpha$, where $\alpha = 0.05$:

$$\begin{aligned} a &= F^{-1}_{\chi^2_{(2n)}}(\alpha/2) = F^{-1}_{\chi^2_{(200)}}(0.05) \overset{table}{=} 168.3 \\ b &= F^{-1}_{\chi^2_{(2n)}}(1-\alpha/2) = F^{-1}_{\chi^2_{(200)}}(0.95) \overset{table}{=} 234.0 \end{aligned}$$

  - **Inverting the inequality** $a \leq Z \leq b$
    Since $R_T(240) = e^{-240\lambda}$ is a decreasing function of $\lambda > 0$, we can add that

$$\begin{aligned} P(a \leq Z \leq b) &= 1-\alpha \\ P\left(a \leq 2\lambda \sum_{i=1}^{n} T_i \leq b\right) &= 1-\alpha \\ &\vdots \\ P\left(\frac{a}{2\sum_{i=1}^{n} T_i} \leq \lambda \leq \frac{b}{2\sum_{i=1}^{n} T_i}\right) &= 1-\alpha \\ P\left(e^{-\frac{240\times b}{2\sum_{i=1}^{n} T_i}} \leq e^{-240\lambda} \leq e^{-\frac{240\times a}{2\sum_{i=1}^{n} T_i}}\right) &= 1-\alpha. \end{aligned}$$

  - **Requested CI**

$$\begin{aligned} CI_{95\%}(R_T(240)) &= \left[e^{-\frac{240\times b}{2\sum_{i=1}^{n} t_i}}; e^{-\frac{240\times a}{2\sum_{i=1}^{n} t_i}}\right] \\ &= \left[e^{-\frac{240\times 234.0}{2\times 22791.1}}; e^{-\frac{240\times 168.3}{2\times 22791.1}}\right] \\ &\simeq [0.2917; 0.4123]. \end{aligned}$$