The Fuzzy Manipulability and The Rationality of Choice

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Abstract

In this paper, our attention is focused on some choice functions with fuzzy preferences. They verified some plausible properties presented by Barrett *et al.* (1990) and Sengupta (1999). The main objective of this paper is to make the connection between the manipulability of fuzzy social choice functions and the rationality of choice. On one hand, we establish the link between Tang's result and PO-rationality. On the other hand, H and $D[\alpha]$ rationality by Dutta *et al.* (1986) allows us to prove both generalizations of Gibbard-Sattertherwaite manipulation theorem.

Keywords: Fuzzy preference relations, Fuzzy choice functions, Manipulability of fuzzy social choice functions, H and $D[\alpha]$ rationalizability.

Contents

Introduction

Group decision making In group decision making real-world situations, a collective decision regarding the choice of the "best" alternative (candidate) from a finite set is usually obtained by applying an aggregation function or operator. Our attention is restricted to a function associating a single alternative to a collection of individual preference orderings over the set of alternatives. Modelling preferences is often a very hard task in the field of decision making. Traditionally, the individual preferences can be modelled through binary relations. They can be crisp or fuzzy according to the circumstances (e.g. Orlovsky, 1978; Basu, 1984).

On the manipulability in voting theory In voting theory, the strategic manipulation of non-dictatorial aggregation functions is always possible wherever the model contains at least three alternatives and the objective consists of selecting a single one. Each individual should give an ordering (preference pre-order or weak order) over the set of alternatives modelled through a crisp relation (Gibbard, 1973; Satterthwaite, 1975) (Henceforth G-S).

A key question Is it possible to find a collective decision representing sincere individual preferences when those are modelled through fuzzy relations over the set of alternatives, *i.e.* is the collective decision obtained by a non-dictatorial aggregation function?

About fuzzy preference relations We are interested in defining the manipulation of aggregation functions starting with a collection of fuzzy individual preference relations. In fact, when comparing alternatives, individuals are often affected by the presence of ambiguity or imprecision due to imperfect knowledge of data. Consequently, an individual might not be able to clearly state a preference relation for any pair of alternatives. In the literature, it has been argued that fuzzy relations incorporate inherent subjectivity and imprecision of human thinking (Goguen, 1967; Barrett and Pattanaik, 1985; Ovchinnikov and Owernoy, 1988). Given two alternatives, it is assumed that an individual has a degree of preference over such a given pair of alternatives.

Fuzzy Social Choice Functions ($FSCF$) In addition, the uncertainty affects the type of aggregation functions. The $FSCF$ are used for mapping a collection of fuzzy preference relations into a chosen collective alternative. It can be viewed as a two-step procedure (Barrett *et al.*, 1986; Barnerjee, 1994; Garcia-Lapresta and Llamazares, 2000). There exist two types of $FSCF$ decompositions:

1. *Aggregation and defuzzification*

   (a) Apply a fuzzy aggregation rule (FAR) that leads to a comprehensive (collective) fuzzy relation.

   (b) Generate from the fuzzy relation the best alternative by applying a choice function that leads to a collective choice.

2. *Defuzzification and aggregation*

   (a) Apply a choice function for each individual fuzzy relation to obtain an individual choice set.

   (b) Aggregate the individual choices using an aggregation function.

A rational choice of an individual When dealing with exact preferences, the manipulability of the aggregation functions is assumed to be made by rational manipulator(s). In the standard model, the manipulator is assumed to have his/her own preference order. If he/she is alone confronted to the set of alternatives, there is no ambiguity to generate a subset containing the best alternatives, called *the choice set*. The mechanisms to define choice sets over a set of alternatives, are called *choice functions*. In the classical choice theory a large number of choice functions have been proposed. We only cite here the ones that can be extended to fuzzy choice theory: *greatest set, non-dominated set, strictly dominating set, and maximal set.* All these sets coincide for a fixed preference order and a fixed alternative set. In the literature, this choice set is called *the rational choice* (Sen, 1971; Suzumura, 1976).

About the manipulation of aggregation functions In group decision making context, however, for each individual his/her preference relation and the selected alternative depends on all individual preferences. An individual can be considered as a manipulator only when he/she knows other individual preference relations and the used aggregation procedure. With such information, the manipulator misrepresent his/her preference relation to secure an outcome he/she prefers to the honest one. Indeed, the first sept for such a manipulator is to look at the set of possible outcomes with the remaining individual preference relations fixed. The second step is to determine the best choice from the possible outcome set for his/her preference relation. Finally, he/she declares a binary relation assuring his/her best choice as a collective decision.

Fuzzy rational choice The interesting question is how to determine the best choice from a fixed set based on a fuzzy preference relation. In the literature of this field (e.g. Dutta *et al.*, 1986; Dutta, 1987; Kulshreshtha and Sheckar, 2000) fuzzy preference-based choice functions are linked with rationality concepts.

Purpose of the paper Our purpose in this paper is to establish the relationship of the manipulability of $FSCF$ with some rationality concepts in a fuzzy preference structure framework. The Tang's result will be interpreted and new extensions of the G-S theorem will be stated in the fuzzy framework.

Outline of the paper The organization of this paper is as follows. Section 2 is devoted to elementary definitions related to $FSCF$. The main choice functions introduced in the literature are presented, their characterizations by plausible properties of Barrett *et al.* (1990) and Sengupta (1999) as well as. In addition, the related rationalizability notions are introduced. Section 2 deals with the existing fuzzy manipulation result of Tang (1994). In sections 4 and 5, new results on the fuzzy manipulability will be introduced. They can be viewed as the generalization of Gibbard-Sattertherwaite manipulation theorem. Their connections with $D[\alpha]$- and H- rationalities are also established. Finally, a summary is presented to conclude.

1 Mathematical background

The focus of the paper is on the context where a group of several individuals has to choose an alternative from a finite set of alternatives. Consider that the preferences of each individual are modelled by using a fuzzy binary relation.

1.1 Elementary concepts

This section is consecrated to the basic data of the model, the standard definitions on fuzzy set theory, and some elementary properties.

1.1.1 Basic data

Let,

- $X = \{x_1, x_2, \ldots, x_j, \ldots, x_m\}$, denote a finite set of alternatives, with $|X| \geq 3$;
- $\chi = \{S \mid S \subseteq X \text{ and } S \neq \emptyset\}$, denote the set of all non-empty subsets of X;
- $N = \{1, 2, \ldots, i, \ldots, n\}$, denote a group of n individuals, with $n \geq 2$.

1.1.2 Fuzzy relations

Definition 1. (**Fuzzy binary relation**)
A fuzzy binary relation, (FBR)*, over* X *is a function* $R : X \times X \longrightarrow [0, 1]$.

A fuzzy relation can be considered as a fuzzy set in $X \times X$ with a membership function R, introduced to model vagueness, or imprecision. Generally, the *imprecision* or *vagueness* is detected when there are some difficulties to express clearly our knowledge (is the turquoise color green or blue?). In our settings, the vagueness affects the preferences of an individual. Thus, $R(x, y)$ represents the degree to which the crisp weak preference "the alternative x is at least as preferred as alternative y" (Zadeh, 1965). Consequently, for each pair of alternatives x and y belonging to X, we have a number $R(x, y) \in [0, 1]$ interpreted as the degree of preferences of x over y. In the fuzzy set theory, most properties of crisp relations have been considered in multiple versions.

Here, we introduce the definitions that are by far the most widely used in the literature (e.g. Dutta *et al.*, 1986; Dutta, 1987; Banerjee, 1993).

The properties of fuzzy relations are the following:

An FBR is

1. *Crisp:*
$$R(x, y) \in \{1, 0\}, \forall\, x, y \in X.$$
This property implies that R corresponds to an exact preference relation.

2. *Reflexive:*
$$R(x, x) = 1, \forall\, x \in X.$$
This assumption is considered in order to have weak preference relation. For any ordered pair of alternatives (x, y), $R(x, y)$ is the degree to which "x is at least preferred to y".

3. *Anti-reflexive:*
$$R(x, x) = 0, \forall\, x \in X.$$
This assumption is considered in order to have strict preference relation. For any ordered pair of alternatives (x, y), $R(x, y)$ is the degree to which "x is strictly preferred to y". Thus, x cannot be strictly preferred to y with any positive degree of confidence.

4. *Weak anti-symmetric*:

$$\text{if } R(x,y) = 1, \text{ then } R(y,x) = 0, \forall\, x, y \in X.$$

   This property is appropriate to a strict preference relation. It amounts that if x is strictly preferred to y with a maximum degree of confidence, then y can not be strictly preferred to x with positive degree of confidence.

5. *Connected:*

$$R(x,y) + R(y,x) \geq 1, \forall\, x, y \in X.$$

   An anti-symmetric FBR cannot satisfy the connectedness.

6. *Max-min transitive* $(T_{max-min})$:

$$\forall\, x, y, z \in X, R(x,z) \geq \min\{R(x,y), R(y,z)\}.$$

7. *Weak max-min transitive*, *i.e.* for all $x, y \in X$,

$$[(R(x,y) \geq R(y,x)) \wedge (R(y,z) \geq R(z,y))], \Rightarrow R(x,z) \geq \min[R(x,y), R(y,z)].$$

   Sengupta (1999) stated that the *max-min* transitivity implies the weak *max-min* transitivity.

8. *Strong transitive*: for all $x, y \in X$,

$$R(x,y) > R(y,x) \text{ and } R(y,z) > R(z,y), \text{ then } R(x,z) > R(z,x).$$

   Kolodziejcyk (1986) stated that the *max-min* transitivity implies the strong transitivity.

Definition 2. (**Fuzzy ordering**)
An FBR, R, is said to be fuzzy ordering if it fulfills reflexivity, connectedness, and max-min transitivity.

1.2 Fuzzy aggregation rules

When a group of n individuals has to express their collective opinion on each pair of alternatives, an FAR can be used starting with fuzzy individual preference relations.

Definition 4. (**Fuzzy aggregation rule**)
A fuzzy aggregation rule (FAR) is a function $h : T^n \longrightarrow T$,

$$R_N = (R_1, R_2, \ldots, R_i, \ldots, R_n) \mapsto R^s = h(R_N).$$

where, T is a non-empty set of fuzzy preference relations.

The preference degree $R^s(x, y)$ reflects an overall opinion on each pair of alternatives. The function h can be borrowed from the multiple criteria decision making literature. The individuals can be viewed as the criteria and values to be aggregated are the fuzzy individual relation values.

1.3 Choice functions with fuzzy preferences

Consider an individual who preferences are presented by an FBR, R. He/she has to make exact choices faced with a set of alternatives, X. Thus, the rule to indicate exact choice sets, called *choice function*, has to be defined. We discuss below some choice functions introduced in Dutta *et al.* (1986).

1. *The pairwise optimal choice function B_{PO}*

   Given a subset of X containing two elements x and y, if $R(x, y) \geq R(y, x)$, then the individual would specify x as an alternative chosen from $\{x, y\}$.
   In this case, it is said that the alternative x is *pairwise optimal vis-à-vis* the alternative y.

   Under this interpretation, the exact choice set can be identified with the set of all alternatives x in S such that the alternative x is always chosen from every two elements subset of S containing x.

   The set $B_{PO}(S, R)$ can be defined as follows

   $$B_{PO}(S, R) = \{x \in S \mid \ R(x, y) \geq R(y, x), \ \forall y \in S\}. \tag{1}$$

2. *The α-dominance choice function $B_{D[\alpha]}$*

   The individual has, firstly, to specify a confidence threshold α in $[0, 1]$. Given a subset of X containing two elements x and y, if $R(x, y) \geq \alpha$, then the individual would specify x as an alternative chosen from $\{x, y\}$.

In this case, it is said that the alternative x *α-dominates* y.

Given a reflexive FBR, R, the set of all x in S which α-dominate every y in S can be considered as the exact choice set generated by R for a subset S.

The set $B_{D[\alpha]}(S, R)$ can be defined as follows

$$B_{D[\alpha]}(S, R) = \{x \in S \mid \ R(x, y) \geq \alpha, \ \forall y \in S\}. \tag{2}$$

3. *The G-dominance choice function* B_H

Given a reflexive FBR, R, the R-greatest set in S is defined to be the fuzzy set $G(S, R) : S \to [0, 1]$ such that

$$G(R, S)(x) = \min_{y \in S} R(x, y), \forall\, x \in S. \tag{3}$$

For all x in S, $G(S, R)(x)$ can be viewed as a degree of dominance of x and the exact choice set for S corresponds to the set of all x in S which score the highest with the function $G(R, S)$.

Formally, the choice set $B_H(S, R)$ can be identified with

$$B_H(S, R) = \{x \in S \mid G(S, R)(x) \geq G(S, R)(y), \forall\, y \in S\}. \tag{4}$$

A first characterization of the above choice sets is given in Dutta *et al.* (1986) as follows.

Proposition 1.

1. *For every S in χ and every fuzzy ordering R,*

$$B_H(S, R) \neq \emptyset, \ \textit{and } B_{PO}(S, R) \neq \emptyset.$$

2. *For every S in χ and every exact ordering R,*

$$B_H(S, R) = B_{PO}(S, R) = B_{D[\alpha]}(S, R).$$

1.4 Characterization of choice functions with fuzzy preferences

1. *Rationality properties by Bareet et al. (1990)*

   Bareet et al. (1990) presented the following plausible rationality properties to be satisfied by choice functions. Let $\mathfrak{R}$ be the set of all fuzzy preference relations satisfying reflexivity and connectedness and $\mathcal{H}$ be the set of all fuzzy orderings.

   The choice function $C : \chi \times \mathfrak{R}' \rightarrow \chi$ such that $\mathfrak{R}'$ is non-empty subset of $\mathfrak{R}$. satisfies the following conditions.

   (a) *Conditions for choosing an alternative*

   i) *Reward for pairwise weak dominance* ($RPWD$) if and only if $\forall\, S \subseteq X, \forall\, R \in \mathfrak{R}',$ and $\forall\, x \in S$,

   $$[R(x,y) \geq R(y,x), \forall\, y \in S \setminus \{x\}] \Rightarrow [x \in C(S,R)].$$

   ii) *Reward for pairwise strict dominance* ($RPSD$) if and only if $\forall\, S \subseteq X, \forall\, R \in \mathfrak{R}',$ and $\forall\, x \in S$,

   $$[R(x,y) > R(y,x), \forall\, y \in S \setminus \{x\}] \Rightarrow [x \in C(S,R)].$$

   (b) *Conditions for rejecting an alternative*

   i) *Strict rejection* ($SREJ$) if and only if $\forall\, S \subseteq X, \forall\, R \in \mathfrak{R}',$ and $\forall\, x \in S$

   $$\begin{cases} R(y,x) \geq R(x,y), & \forall\, y \in S \setminus \{x\} \\ R(y,x) > R(x,y), & \text{for some } y \in S \setminus \{x\} \end{cases}$$

   $$\Rightarrow [x \notin C(S,R)].$$

   ii) *Weak rejection* ($WREJ$) if and only if $\forall\, S \subseteq X, \forall\, R \in \mathfrak{R}',$ and $\forall\, x \in S$,

   $$[R(y,x) > R(x,y), \forall\, y \in S \setminus \{x\}] \Rightarrow [x \notin C(S,R)].$$

	RPWD	RPSD	WREJ	SREJ	UF	LF
			$\mathfrak{R}' = \mathfrak{R}$			
B_{PO}	$\star$	$\star$				$\star$
B_H					$\star$	$\star$
			$\mathfrak{R}' = \mathcal{H}$			
B_{PO}	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
B_H	$\star$	$\star$			$\star$	$\star$

Table 1: $(\star)$ indicates that the choice function satisfies the property under consideration

(c) *Faithfulness*

i) *Upper faithfulness* (UF) if and only if $\forall\ S \subseteq X, \forall\ R \in \mathfrak{R}'$,

$$G(S,R) = \{x \in S, R(x,y) = 1,\ \forall\ y \in S\} \neq \emptyset \Rightarrow [C(S,R) \subset G(S,R)].$$

ii) *Lower faithfulness* (LF) if and only if $\forall\ S \subseteq X, \forall\ R \in \mathfrak{R}'$,

$$G(S,R) = \{x \in S, R(x,y) = 1,\ \forall\ y \in S\} \neq \emptyset \Rightarrow [G(S,R) \subset C(S,R)]).$$

The evaluation of the choice functions B_{PO} and B_H (1, 4) on these rationality conditions in Barrett *et al.* (1990) can summarized in the table 1.

2. *Axiomatic characterization by Sengupta (1999)*

Sengupta (1999) presented some plausible axioms to characterize choice functions for every FBR in $\mathfrak{R}$.

(a) *Independence* (I) :

$\forall\ S \subseteq X, \forall\ R, R' \in \mathfrak{R}$,

$$[R(x,y) = R'(x,y), \forall\ x,y \in S] \Rightarrow C(S,R) = C(S,R').$$

This property secures that for two identical individual fuzzy relations on a fixed set, the associated choice sets are also the same.

(b) *Neutrality* (N) :

For all $R, R' \in \mathfrak{R}$, and all $x, y, z, w \in X$ (not necessarily distinct)

$$[R(x,y) = R'(z,w), \text{ and } R(y,x) = R'(w,z)] \Rightarrow$$

$$\begin{cases} [x \in C(\{x,y\},R) \Leftrightarrow z \in C(\{z,w\},R')], \\ [y \in C(\{x,y\},R) \Leftrightarrow w \in C(\{z,w\},R')] \end{cases}$$

(c) *Monotonicity* (M) :

$$\forall\, R, R' \in \mathfrak{R}, \forall\, x, y, \in X, [R(x,y) \leq R'(x,y), \text{ and } R(y,x) \geq R'(y,x)] \Rightarrow$$

$$\begin{cases} \{x\} = C(\{x,y\},R) \Rightarrow \{x\} = C(\{x,y\},R') \\ x \in C(\{x,y\},R) \Rightarrow x \in C(\{z,w\},R'). \end{cases}$$

It is important to mention that neutrality and monotonicity axioms are imposed only for choice functions satisfying independence.

According to Sengupta (1999), neutrality says that as long as the preference between two pairs of alternatives (x, y) and (z, w) are symmetric across two different preference relations, the choice in one situation should be sufficient to determine the choice in the other situation. This amounts to the fact that when determining choices the preference is important and the identity of the alternatives is not.

Condition M requires that if $R(x, y)$ does not go down and $R(y, x)$ does not go up, then x does not get treated less favorably (or y more favorably) by the choice rule when compared to the situation before the change in R. The property M seems to be an extremely plausible one of any choice function.

(d) *Continuity* (CON)

$\forall\, R^1, R^2, \ldots, R^j, \ldots, \in \mathfrak{R}, \forall\, x, y \in X,$

$$\begin{cases} x \in C(\{x,y\},R^i), \forall\, R^i \\ \text{the sequence } \langle R^i(x,y), R^i(y,x)\rangle \text{ converges to some } \langle \overline{R}(x,y), \overline{R}(y,x)\rangle \end{cases}$$

then

$$x \in \{x, y\}, \overline{R}$$

(e) *Chernoff Condition* (CC)
$\forall x, y \in X$, and $S \in \chi$ such that $\{x, y\} \subset S$,

$$[x \notin C(\{x, y\}, R)] \Rightarrow [x \notin C(S, R)]$$

(f) *Beta Condition* (β)
$\forall x, y \in X$, and $S \in \chi$,

$$[\{x, y\} = C(\{x, y\}, R)] \Rightarrow [x \in C(S, R) \Leftrightarrow y \in C(S, R)]$$

Using the above properties, Sengupta (1999) proved the following proposition.

Proposition 2. (**Sengupta's characterization of** $B_{D[\alpha]}$)
For all fuzzy ordering, R, the choice function, $C(S, R)$, satisfies I, CC, β, N, M, and CON if and only if there exists $\alpha \leq 1/2$ such that

$$C(S, R) = B_{D[\alpha]}(S, R) = \{x \in S \mid R(x, y) \geq \alpha, \ \forall y \in S\}.$$

The choice function $B_{D[\alpha]}$ is already defined by Dutta *et al.* (1986) (2). This result represents a plausible characterization of Dutta *et al.*'s choice function.

1.5 Concepts of rationalizability according to Dutta *et al.* (1986)

Assuming that the individual preferences are fuzzy, Dutta *et al.* (1986) considered that a choice set can be rationalizable in terms of a fuzzy preference, if there exists a fuzzy relation such that the observed set coincide with one of their exact choice sets. Therefore, they proposed several manners of rationalization. We present here the most interesting rationalizability concepts.

1. *PO-rationalizabity*

   $C(X)$ is PO-rationalizable in terms of fuzzy relation, R, if and only if for all $S \in \chi$, his/her choice set $C(S)$ corresponds to $B_{PO}(S, R)$ (1).

2. *$D(\alpha)$-rationalizabity*

   $C(X)$ is $D(\alpha)$-rationalizable in terms of fuzzy relation, R, if and only if for all $S \in \chi$, his/her choice set $C(S)$ corresponds to $B_{D(\alpha)}(S, R)$ (2).

3. *H-rationalizabity*

   $C(X)$ is H-rationalizable in terms of fuzzy relation, R, if and only if for all $S \in \chi$, his/her choice set $C(S)$ corresponds to $B_H(S, R)$(4).

2 Tang's approach

To our best knowledge, the only work concerned with the manipulability of fuzzy social choice functions is the one by Tang (1994).

2.1 Tang's domain: definitions and notation

Tang (1994) assumed that all the individual preferences are represented by fuzzy binary relations satisfying the following properties:

1. *Anti-reflexivity*: for all $x \in X, R_i(x, x) = 0$
2. *Weak anti-symmetric*: if $R_i(x, y) = 1$, then $R_i(y, x) = 0, \forall\, x, y \in X$.
3. *Strong transitivity*: for all $x, y \in X$,

$$R_i(x, y) > R_i(y, x) \text{ and } R_i(y, z) > R_i(z, y), \text{ then } R_i(x, z) > R_i(z, x).$$

Moreover, the following notation is considered.

Let,

- D, denote the set of all fuzzy relations satisfying anti-reflexivity, weak anti-symmetric and strong transitivity, is denoted by D.
- D_+, denote the set all $d \in D$, such that for all distinct $x, y \in X$,

$$R(x, y) > R(y, x) \text{ or } R(x, y) < R(y, x).$$

- D_E, (respectively D_{E+}) denote the set of all crisp $R \in D$ (respectively D_+); all relations in D_E are strict orders and D_{E+} denotes the set all crisp $R \in D_+$.
- R_N, denote the fuzzy profile $(R_1, R_2, \ldots, R_i, \ldots, R_n) \in D^n$.
- $R_N \mid R'_i$, denote the fuzzy profile $(R_1, \ldots, R_{i-1}, R'_i, R_{i+1}, \ldots, R_n)$.
- $R_N \mid R'_1, R'_2$, denote the fuzzy profile $(R'_1, R'_2, R_3, \ldots, R_i, \ldots, R_n)$.
- $J_0 = \{(t_1, t_2) \in [0,1]^2 \mid$ for some $R \in D$ and some distinct $x, y \in X, R(x,y) = t_1$ and $R(y,x) = t_2\}$.
- $X(R_N \mid R'_i)$, denote the set $\{\nu(R_N \mid R'_i), \forall\ R'_i \in D\}$ where, ν is an $FSCF$.

Moreover the manipulability, strategy-proofness, and dictatorship are defined by Tang in the fuzzy framework as well as the non-narrow domain.

Definition 3.
An $(FSCF)$ $\nu : \mathcal{D}^n \subseteq D^n \to X$ is,

1. *Manipulable* by individual $i \in N$ at $R_N \in \mathcal{D}^n$ if there exists $R'_i \in \mathcal{D}$ such that

$$R_i(\nu(R_N \mid R'_i), \nu(R_N)) \geq R_i(\nu(R_N)), \nu(R_N \mid R'_i))$$

2. *Strategy-proof* if there exists no $R_N \in \mathcal{D}^n$ at which ν is manipulable
3. *Dictatorial* if there exists a dictator k for ν, *i.e.*, for every $R_N \in \mathcal{D}^n$ and all

$$x \neq \nu(R_N) \in X, R_k(\nu(R_N), x) \geq R_k(x, \nu(R_N))$$

4. The function ν has *a non-narrow domain* if for all $(t_1, t_2) \in J_0$ and all distinct $a, b, c \in X$, there exists $R \in D$ such that

$$R(a,b) = R(a,c) = 1, R(b,c) = t_1, R(c,b) = t_2 \tag{5}$$

   and also there exists $d' \in D$ such that

$$R'(b,a) = R'(c,a) = 1, R'(b,c) = t_1, d'(c,b) = t_2. \tag{6}$$

   In this case, $\mathcal{D}$ can coincide with D, D_+, D_E, D_{E+}.

2.2 Tang's theorem

Theorem 1. (Tang's theorem)
Every strategy-proof $(FSCF)$ *with a non-narrow domain,* ν *is dictatorial.*

It should be remarked that Tang's work is typically technical, no indication is specified about the outcome of fuzzy social choice function. The way of translating fuzzy preferences into an exact social choice is not discussed. Just a technical definition of manipulability of such functions as well as the proof of the above theorem are provided. We append his proof at the end of the report.

3 On the connection of Tang's manipulability with PO-rationality

Starting with Tang's approach, we can deduce that an $FSCF$ is not manipulable by the individual i at $R_N \in D$, if

$$R_i(\nu(R_N \mid R'_i), \nu(R_N)) \leq R_i(\nu(R_N), \nu(R_N \mid R'_i)), \forall\ R'_i \in D \tag{7}$$

Thus, an $FSCF$ is not manipulable by i at d_N, if

$$\forall\ x \in X(R_N \mid R'_i), R_i(x, \nu(R_N)) \leq R_i(\nu(R_N), x).$$

In other words, $\nu(R_N)$ is in $B_{PO}(X(R_N \mid R'_i), R_i)$. We recall that

$$B_{PO}(X(R_N \mid R'_i), R_i) = \{y \in X(R_N \mid R'_i) \mid R_i(y, x) \geq R_i(x, y), \forall x \in X(R_N \mid R'_i)\}.$$

Moreover, we can see that this amounts to PO-rationality of $\nu(R_N)$ for the individual i when the individual i have to choose from the set $X(R_N \mid R'_i)$.
In other words, the manipulation is made when an individual i tries to obtain his/her PO-rational choice generated by his/her own preference relation. It is chosen from the set of possible outcomes obtained when he/she varies his/her fuzzy relation in the $FSCF$'s domain. Thus, the connection of the definition of the manipulability with one type of rationality of choice is welly established. Therefore, we can deduce that an $FSCF$ is strategy-proof, if each $\nu(R_N)$ is PO-rational for all the individuals in X.

Now, we present an illustrative example to understand the Tang's manipulability concept.

Example 1.
Let $X = \{x_1, x_2, x_3\}$ denote a finite set of alternatives. We are interested in the context where a group of 3 individuals, $N = \{1, 2, 3\}$, has to choose an alternative from X. We consider that the preferences of each individual i over X are modelled by using a fuzzy binary relation R_i as follows.

R_1	x_1	x_2	x_3		R_2	x_1	x_2	x_3		R_3	x_1	x_2	x_3
x_1	0	0.6	0.7		x_1	0	0.5	0.6		x_1	0	0.6	0.5
x_2	0.5	0	0.8		x_2	0.7	0	0.9		x_2	0.2	0	0.4
x_3	0.4	0.3	0		x_3	0.2	0.3	0		x_3	0.8	0.7	0

By using the arithmetic mean aggregation operator, followed by the choice function B_{PO}, we obtain x_1 as the collective choice. Indeed, the collective fuzzy preference relation is

R^s	x_1	x_2	x_3
x_1	0	0.56	0.6
x_2	0.46	0	0.7
x_3	0.46	0.43	0

- For the individual 1, the alternative x_1 corresponds to a PO-rational choice. This is equivalent to the non necessity to manipulate the used $FSCF$.

- When the individual 2 changes his/her fuzzy relation in D, he/she can obtain each alternative from X as a collective choice. The relations of individuals 1 and 3 are supposed to be fixed as above. His/her PO-rational choice from X corresponds to x_2. Thus, the alternative x_1 does not correspond to a PO-rational choice from X. It is a sufficient and necessary condition to have the opportunity to manipulate the used $FSCF$.
  The individual 2 can declare the following fuzzy relation d'_2 to obtain x_2 as a collective choice.

R'_2	x_1	x_2	x_3
x_1	0	**0.2**	0.6
x_2	**0.9**	0	0.9
x_3	0.2	0.3	0

Conclusion
Through this example, the connection of PO-rationality with the Tang's manipulability can be observed. We call the manipulability defined by Tang, the PO-*manipulability.* In fact, the rationality of exact choice with fuzzy preferences exists in many versions. Thus, the manipulability of an $FSCF$ can also be deduced in the same ways as we will see in the following section.

4 $D[\alpha]$-manipulability

4.1 Motivation and definitions

We observed that Tang's definition of the $FSCF$ manipulability can be considered when the manipulator is PO-rational. However, the manipulator can be $D[\alpha]$-rational. In other words, he/she has $B_{D[\alpha]}$ as a choice function for all S in χ. Thus, the way of $FSCF$ manipulation by such an individual is different to the first one. Tang's manipulability must be redefined and we will specify the following fuzzy relation domain to restate the manipulability theorem.

Let,

- $\mathcal{H}_W$, denote the set of all fuzzy relations fulfilling connectedness, reflexivity, and weak max-min transitivity.
- $\mathcal{H}_W^E$, denote all crisp relations in $\mathcal{H}_W$.
- $X_\alpha(R_N \mid R_i')$, denote the set $\{\nu(R_N \mid R_i'), \forall\, R_i' \in \mathcal{H}_W\}$ where, ν is an $FSCF$.

Definition 5. **($D[\alpha]$-manipulability and $D[\alpha]$-dictatorship)**
For $\alpha \leq 1/2$, An $(FSCF)$ $\nu : \mathcal{D}^n \subseteq \mathcal{H}_W^n \rightarrow X$ is

1. *$D[\alpha]$-manipulable by individual $R_N \in \mathcal{D}^n$ if there is $R_i' \in \mathcal{D}$ such that*
$$R_i(\nu(R_N \mid R_i'), \nu(R_N)) \geq \alpha$$
2. *$D[\alpha]$-strategy-proof if there is no $R_N \in \mathcal{D}^n$ at which ν is $D[\alpha]$-manipulable.*
3. *$D[\alpha]$-dictatorial if there exists a dictator $k \in N$, i.e. for every $R_N \in \mathcal{D}^n$*
$$x \neq \nu(R_N) \in X, R_k(\nu(R_N), x) \geq \alpha$$

4.2 $D[\alpha]$-manipulability theorem

We establish the following theorem using the $D[\alpha]$ rationality concept.

Theorem 2. ($D[\alpha]$-manipulability)
If an FSCF is $D[\alpha]$-strategy-proof with $\mathcal{H}_W$ domain, then it is $D[\alpha]$-dictatorial.

Proof
Consider a strategy-proof ($FSCF$) $\nu : \mathcal{D}^n \subseteq \mathcal{H}_W^n \to X$. Note that $\mathcal{H}_W^E \subseteq \mathcal{H}_W$, Gibbard-Sattertherwaite manipulation theorem can be applicable. Hence, there exists an individual, we assume individual 1, as the dictator for ν restricted on the domain $(\mathcal{H}_W^E)^n$.

For any $R_N \in \mathcal{D}^n$, let B_1 denote the α-greatest elements in X for the individual 1, *i.e.*,

$$B_1 = \{y \mid R_1(y,x) \geq \alpha, \forall\, x \in X\}.$$

It should be remarked that B_1 is non-empty since B_1 is equivalent to $B_{D[\alpha]}(X, R_1)$ and Sengupta (1999) proved that for all $R \in \mathcal{H}_W$,

$$S_\alpha(S,R) = \{x \in S \mid \text{ for no } y \in S, R(y,x) > R(y,x) \text{ and } R(x,y) < \alpha\} \neq \emptyset$$

and

$$\forall\, \alpha \leq 1/2, S_\alpha(S,R) = B_{D[\alpha]}(S,R).$$

We can deduce that

$$\forall\, \alpha \leq 1/2, \forall\, R \in \mathcal{H}_W, B_{D[\alpha]}(S,R) \neq \emptyset, \forall\, S.$$

Let $R'_N \in (\mathcal{H}_W^E)^n$ such that for all $y \in B_1$ and all $z \in X - B_1$,

$$R'_1(y,z) = 1 \text{ and } R'_i(z,y) = 1 (i \neq 1).$$

Since $R'_N \in \mathcal{H}_W^E$, 1 is dictator. Thus,

$$\forall\, x \neq \nu(R'_N) \in X, R'_1(\nu(R'_N), x) = 1 \Rightarrow \nu(R'_N) \in B_1.$$

Let w_0 be $\nu(R_N)$ and $w_k = \nu(R_N \mid R'_1, R'_2, \ldots, R_i, \ldots, R'_k)$ with $1 \leq k \leq n$.

Let j denote the least k in $\{0, 1, \ldots, i, \ldots, n\}$ such that $w_k \in B_1$. We will prove that j must be equal to 0 since ν is supposed to be non-manipulable.

If $j = 1$, *i.e.* $w_1 = \nu(R_N \mid R'_1) \in B_1$ and $w_0 = \nu(R_N), w_0 \notin B_1$. Thus, for 1,

$$R_1(w_1, w_0) \geq \alpha \text{ and } R_1(w_0, w_1) < \alpha.$$

Thus, ν is manipulable by 1 at R_N.

If $j > 1$, *i.e.*,

$$w_j = \nu(R_N \mid R'_1, R'_2, \ldots, R'_j) \in B_1, \text{ and } w_{j-1} = \nu(R_N \mid R'_1, \ldots, R'_i, \ldots, R'_{j-1}), w_{j-1} \notin B_1.$$

Thus, for j, $R'_j(w_{j-1}, w_j) = 1 \geq \alpha$. Consequently, the function ν is manipulable by j at $\nu(R_N \mid R'_1, \ldots, R'_i, \ldots, R'_j)$.

This leads to conclude that j must be equal to 0. Thus, $w_0 = \nu(R_N) \in B_1$, *i.e.* $R_1(w_0, x) \geq \alpha, \forall\, x \in X$. It follows that 1 is also the dictator for ν.

4.3 On the connection of $D[\alpha]$-manipulability with $D[\alpha]$-rationality

With $D[\alpha]$-manipulability, we can deduce that an $FSCF$ is not manipulable by the individual i at $R_N \in \mathcal{H}_W^n$, if

$$R_i(\nu(R_N \mid R'_i), \nu(R_N)) < \alpha, \forall\, R'_i \in \mathcal{H}_W \tag{8}$$

Thus, an $FSCF$ is not manipulable by i at d_N, if

$$\forall\, x \in X_\alpha(R_N, R'_i), R_i(x, \nu(R_N)) < \alpha.$$

Since $R_i(x, \nu(R_N)) + R_i(\nu(R_N), x) \geq 1$, we have $R_i(x, \nu(R_N)) \geq 1 - R_i(\nu(R_N), x)$.

Thus,

$$1 - R_i(\nu(R_N), x) \leq R_i(x, \nu(R_N)) < \alpha \Rightarrow \alpha > 1 - R_i(\nu(R_N), x).$$

We obtain

$$\forall\, x \in X_\alpha(R_N, R'_i), R_i(\nu(R_N), x) > 1 - \alpha$$

Relying on choice functions given in Section 1.3, we can say that $\nu(R_N)$ is in $B_{D[1-\alpha]}(X_\alpha(R_N, R'_i), R_i)$ (5).
Moreover, it can be seen that this amounts to $D[1-\alpha]$-rationality of $\nu(R_N)$ for the individual i when the individual i have to choose from the set $X(R_N \mid R'_i)$.

Example 2.
Let $X = \{x_1, x_2, x_3\}$ denote a finite set of alternatives. We are interested in the context where a group of 3 individuals, $N = \{1, 2, 3\}$, has to choose an alternative from X. We consider that the preferences of each individual i over X are modeled by using a fuzzy binary relation R_i as follows.

R_1	x_1	x_2	x_3
x_1	1	0.7	0.8
x_2	0.4	1	0.6
x_3	0.4	0.4	1

R_2	x_1	x_2	x_3
x_1	1	0.5	0.8
x_2	0.7	1	0.75
x_3	0.6	0.5	1

R_3	x_1	x_2	x_3
x_1	1	0.6	0.7
x_2	0.5	1	0.8
x_3	0.4	0.4	1

By using the arithmetic mean aggregation operator, followed by the choice function B_{PO}, we obtain x_1 as the collective choice. Indeed, the collective fuzzy preference relation is

R^s	x_1	x_2	x_3
x_1	1	0.6	0.766
x_2	0.53	1	0.71
x_3	0.46	0.43	1

Let α be equal to 0.45.

- For the individual 1, the alternative x_1 corresponds to a $D[1-\alpha]$-rational choice and $R_1(x_2, x_1)$ and $R_1(x_3, x_1)$ are inferior to α. This is equivalent to the non necessity to manipulate the used $FSCF$.

- When the individual 2 changes his/her fuzzy relation in D, he/she can obtain each alternative from X as a collective choice. The relations of individuals 1 and 3 are supposed fixed as above. His/her $D[1-\alpha]$-rational choice from X corresponds to x_2. Thus, the alternative x_1 doesn't correspond to a $D[1-\alpha]$-rational choice from X. Moreover, $R_2(x_2, x_1)$ and $R_2(x_3, x_1)$ are superior to α. All this is a sufficient and necessary conditions to have the opportunity to manipulate the used $FSCF$.

  The individual 2 can declare the following fuzzy relation R_2' to obtain x_2 as a collective choice.

R_2'	x_1	x_2	x_3
x_1	1	0.5	0.8
x_2	**0.95**	1	0.75
x_3	0.6	0.5	1

- When the individual 3 changes his/her fuzzy relation in D, he/she can obtain each alternative from X as a collective choice. The relations of individuals 1 and 2 are supposed fixed as above. His/her $D[1-\alpha]$-rational choice set corresponds to $\{x_1, x_2\}$. Moreover, $R_3(x_2, x_1)$ is superior to α. Thus, the individual 3 has the opportunity to manipulate the used $FSCF$, through x_1 is in $B_{D[1-\alpha]}$. The individual 3 can declare the following fuzzy relation R'_3 to obtain x_2 as a collective choice.

R'_3	x_1	x_2	x_3
x_1	1	0.6	0.7
x_2	**0.9**	1	0.8
x_3	0.4	0.4	1

This example shows that the $D[1-\alpha]$-rationality of the sincere collective choice represents a necessary and non sufficient condition of non $D[\alpha]$-manipulability.

5 H-manipulability

5.1 Motivation and definitions

We observed that the PO and $D(\alpha)$-manipulability are considered following the rationality type of the manipulator. However, the manipulator can also be H-rational. Thus, the way of $FSCF$ manipulation by such an individual may be different. A new manipulability definition must be considered and we will specify the following fuzzy relation domain to restate the manipulability theorem.

Let,

- $\mathcal{H}^E$, denote all crisp relations in $\mathcal{H}$.
- $X_H(R_N \mid R'_i)$, denote the set $\{\nu(R_N \mid R'_i), \forall\ R'_i \in \mathcal{H}\}$.

Definition 6. (H-manipulability and H-dictatorship)
An $(FSCF)$ $\nu : \mathcal{D}^n \subseteq \mathcal{H}^n \to X$ is

1. *H-manipulable by individual $R_N \in \mathcal{D}^n$ if there is $R'_i \in \mathcal{D}$ such that*

$$G(R_i, X_H(R_N \mid R'_i))(\nu(R_N \mid R'_i)) \geq G(R_i, X_H(R_N \mid R'_i))(\nu(R_N))$$

2. *H-strategy-proof if there is no $R_N \in \mathcal{D}^n$ at which ν is H-manipulable.*

3. H-dictatorial if there exists a dictator $k \in N$, i.e. for every $R \in D^n$ and all

$$x \neq \nu(R_N) \in X, G(R_k, X)(\nu(R_N)) \geq G(R_k, X)(x)$$

Basu (1984) defined $G(S, R)$ is such that

$$G(R, S)(x) = \min_{y \in S} R(x, y), \forall\, x \in S.$$

5.2 H-manipulability theorem

Theorem 3. **(H-manipulability)**
Every H-strategy-proof $(FSCF)$ with $\mathcal{H}$ domain, ν is H-dictatorial.

Proof
Consider a strategy-proof $(FSCF)$ $\nu : \mathcal{D}^n \subseteq H^n \to X$. Noting that $\mathcal{H}^E \subseteq \mathcal{H}$, Gibbard-Sattertherwaite manipulation theorem can be applicable. Hence there exists an individual, we assume individual 1, who is the dictator for ν restricted on the domain $(\mathcal{H}^E)^n$.

For any $R_N \in \mathcal{D}^n$, let $G(R_1)$ denote the greatest elements in X for the individual 1, *i.e.*,

$$G(R_1) = \{y \mid G(R_1, X)(y) \geq G(R_1, X)(x) \forall x \in X\}.$$

It should be remarked that $G(R_1)$ is non-empty, since $G(R_1)$ is equivalent to $B_H(X, R_1)$ and Dutta *et al.*(1986) proved that for all $R \in H, B_H(S, R) \neq \emptyset, \forall\, S$.

Let $R'_N \in (\mathcal{H}^E)^n$ such that for all $y \in G(R_1)$ and all $z \in X - G(R_1)$,

$$R'_1(y, z) = 1 \text{ and } R'_i(z, x) = 1, \forall\, x \in X, (i \neq 1).$$

Since $R'_N \in \mathcal{H}^E_W$, 1 is dictator. Thus, $\forall\, x \neq \nu(R'_N) \in X$,

$$R'_1(\nu(R'_N), x) = 1 \Rightarrow \nu(R'_N) \in G(R_1).$$

Let w_0 be $\nu(R_N)$ and $w_k = \nu(R_N \mid R'_1, R'_2, \ldots, R'_i, \ldots, R'_k)$ with $1 \leq k \leq n$. Let j denote the least k in $\{0, 1, \ldots, i, \ldots, n\}$ such that $w_k \in G(d_1)$.

If $j = 1$, *i.e.* $w_1 = \nu(R_N \mid R'_1) \in G(R_1)$ and $w_0 = \nu(R_N), w_0 \notin G(R_1)$. Thus, for 1, $G(R_1, X)(w_1) \geq G(R_1, X)(w_0)$. So, ν is manipulable by 1 at R_N.

If $j > 1$, *i.e.* $w_j = \nu(R_N \mid R'_1, \ldots, R'_j) \in G(R_1)$ and $w_{j-1} = \nu(R_N \mid R'_1, \ldots, R'_{j-1})$, $w_{j-1} \notin G(R_1)$.
Thus, for j, $R'_j(w_{j-1}, x) = 1, \forall\, x \in X \Rightarrow G(R'_j, X)(w_{j-1}) = \min_{x \in X} R'_j(w_{j-1}, x)$ is equal to 1.
Then, $G(R'_j, X)(w_{j-1}) \geq G(R'_j, X)(w_j)$. Therefore, the function ν is manipulable by j at $\nu(R_N \mid R'_1, \ldots, R'_j)$.

This leads to conclude that j must be equal to 0. Thus, $w_0 = \nu(R_N) \in G(R_1)$, *i.e.* $G(R_1, X)(w_0) \geq G(R_1, X)(x), \forall\, x \in X$. It follows that 1 is also the dictator for ν.

5.3 On the connection of H-manipulability with H-rationality

With H-manipulability, we can deduce that an $FSCF$ is not manipulable by the individual i at $R_N \in \mathcal{H}$, if

$$G(R_i, X_H(R_N \mid R'_i))(\nu(R_N \mid R'_i)) < G(R_i, X_H(R_N \mid R'_i))(\nu(R_N)), \forall\, R'_i \in \mathcal{H} \quad (9)$$

Thus, an $FSCF$ is not manipulable by i at R_N, if

$$\forall\, x \in X_H(R_N, R'_i), G(R_i, X_H(R_N \mid R'_i))(x) < G(R_i, X_H(R_N \mid R'_i))(\nu(R_N)).$$

Relying on choice functions given in section(2.3), we can say that $\nu(R_N)$ is in $B_H(X_H(R_N \mid R'_i), R_i)$.

We recall that
$B_H(X_H(R_N \mid R'_i), R_i) = \{x \in X_H(R_N \mid R'_i) \mid$
$G(R_i, X_H(R_N \mid R'_i))(x) = \max_{y \in X_H(R_N \mid R'_i)} G(R_i, X_H(R_N \mid R'_i))(y)\}$.

Moreover, it can be seen that this amounts to H-rationality of $\nu(R_N)$ for the individual i when the individual i have to choose from the set $X_H(R_N \mid R'_i)$.

Example 3.
We reconsider the same example of $D[\alpha]$-manipulability, since fuzzy individual relations also satisfy *max-min* transitivity.

- For the individual 1, the alternative x_1 corresponds to a H-rational choice. This is equivalent to the non necessity to manipulate the used $FSCF$.

- When the individual 2 changes his/her fuzzy relation in D, he/she can obtain each alternative from X as a collective choice. The relations of individuals

1 and 3 are supposed fixed as above. His/her H-rational choice from X corresponds to x_2. Thus, the alternative x_1 doesn't correspond to a H-rational choice from X. All this is a sufficient and necessary conditions to have the opportunity to manipulate the used $FSCF$.
The individual 2 can declare the following fuzzy relation d'_2 to obtain x_2 as a collective choice.

R'_2	x_1	x_2	x_3
x_1	1	0.5	0.8
x_2	**0.9**	1	**0.95**
x_3	0.6	0.5	1

This example shows that the H-rationality of the sincere collective choice represents a necessary and sufficient condition of non H-manipulability.

Conclusions

In this paper, we have presented the main choice functions with fuzzy preferences. Also, the characterization of these choice function is introduced and the definition rationalizability concept defined by Dutta *et al.* (1986) is considered. The purpose of this paper is to make the connection between the manipulability of fuzzy social choice functions and the rationality of choice. This is well established for Tang's result and allows us to give two new results. The key point is to look at fuzzy choice function domains. Therefore, for adequate domain the generalization of Gibbard-Sattertherwaite manipulation theorem can be established.

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Proof of Tang's Theorem

Consider an strategy-proof $(FSCF)$ $\nu : \mathcal{D}^n \to X$ with a non-narrow domain $\mathcal{D}^n$. Noting that $D_E \subseteq \mathcal{D}$, Gibbard-Sattertherwaite manipulation theorem can be applicable. Hence there exists an individual, noted by 1, who is the dictator for ν restricted on the domain D_E^n.

For any $R_N \in R^n$, let B denote the R_1-maximal elements in X, viz. $R_1(x,y) \geq R_1(x,y)$ for any y in B and any $x \in X$.

Let $R'_N \in D_E^n$ such that for all $y \in B$ and all $z \in X - B$, both $R'_1(y,z) = 1$ and $R'_i(z,y) = 1 (i \neq 1)$.

Since $R'_N \in D_E^n, \nu(R'_N) \in B$ (1 is dictator when $R'_N \in D_E^n$).

Let w_0 be $\nu(R_N)$ and $w_k = \nu(R_N \mid R'_1, R'_2, \ldots, R'_k)$ with $1 \leq k \leq n$. Let j denote the least k in $\{0, 1, \ldots, n\}$ such that $w_k \in B$.

If $j = 1$, *i.e.* $w_1 = \nu(R_N \mid R'_1) \in B$ and $w_0 = \nu(R_N), w_0 \notin B$. Thus, for 1, $R_1(w_1, w_0) \geq R_1(w_0, w_1)$. So, ν is manipulable by 1 at R_N.

If $j > 1$, *i.e.* $w_j = \nu(R_N \mid R'_1, \ldots, R'_j) \in B$ and $w_{j-1} = \nu({}_R N \mid R'_1, \ldots, R'_{j-1}), w_{j-1} \notin B$. Thus, for j, $R'_j(w_{j-1}, w_j) \geq R'_j(w_j, w_{j-1})$. Consequently, the function ν is manipulable by j at $\nu(R_N \mid R'_1, ..., R'_j)$.

This leads to conclude that j must be equal to 0. Thus, $w_0 = \nu(R_N) \in B$, i.e. $R_1(w_0, x) \geq R_1(x, w_0), \forall\, x \in X$. It follows that 1 is also the dictator for ν with the non-narrow domain $\mathcal{D}^n$.

To conclude, we remark that the strong transitivity assumption assures that B is always non-empty, as we verified. This is a necessary condition to make the above proof. Also, the assumption of having non-narrow domain, is just necessary to make attention that we can have a fuzzy relation in D with some values equal to zero. Such relation is used in the Tang's proof as we saw. .