On the H-Manipulability of Fuzzy Social Choice Functions

Fouad Ben Abdelaziz [*], José Figueira [†], Olfa Meddeb [‡]

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[*]Institut Supérieur de Gestion, Université de Tunis, 41, Rue de la liberté, Cité Bouchoucha, Le Bardo 2000, Tunisie. Phone: (+216) 71 561 854, Fax: (+216) 71 568 767, E-mail: fabdelaz@sharjah.ac.ae

[†]CEG-IST, Center for Management Studies, Instituto Superior Técnico, Departamento de Engenharia e Gestão, Office: 2N818, IST - Taguspark, Av. Prof. Cavaco Silva, 2780 - 990, Porto Salvo, Portugal, Phone: (+351) 214 233 507, Fax: (+351) 214 233 568, E-mail: figueira@ist.utl.pt

[‡]Institut Supérieur de Gestion, Université de Tunis, 41, Rue de la liberté, Cité Bouchoucha, Le Bardo 2000, Tunisie. Phone: (+216) 71 561 854, Fax: (+216) 71 568 767, E-mail: Olfa.Meddeb@isg.rnu.tn

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Abstract

In voting theory the well-known Gibbard-Satterthwaite theorem is about the manipulability of aggregators which consists of the aggregation of individual preferences expressed as a complete ordering over the set of alternatives. This paper deals with the generalization of such a theorem in a context where each individual expresses a fuzzy preference (weak) ordering. This extension is called H-manipulability. The concept of fuzzy game form as the generalization of Gibbard's concept game form is also introduced here in a fuzzy framework. The proof of the H-manipulability theorem is relied on the connection of fuzzy social choice functions with fuzzy aggregation rules. In particular, the dictatorship of fuzzy aggregation rules corresponds to the H-dictatorship fuzzy social choice.

Keywords: Fuzzy preference orderings, Fuzzy game form, Gibbard-Satterthwaite theorem, H-manipulability, Dutta's impossibility theorem.

Contents

Introduction

Group decision making In collective group decision making real-world situations, a collective decision regarding the choice of the "best" alternative from a finite set is usually obtained by applying an aggregation function or operator. Our attention is restricted to a function associating a single alternative to a collection of individual preference ordering over the set of alternatives. Modelling preferences is often a very hard task in the field of group decision making. Traditionally, the individual preferences can be modelled through binary relations. They can be crisp or fuzzy according to the circumstances (e.g. Orlovsky, 1978; Basu, 1984).

On the manipulability in voting theory In voting theory, the strategic manipulation of non-dictatorial aggregation functions is always possible wherever the model contains at least three alternatives and the objective consists of selecting a single one. Each individual should give an ordering (preference pre-order or weak order) over the set of alternatives modelled through a crisp relation (Gibbard, 1973; Satterthwaite, 1975) (Henceforth G-S). Gibbard (1973) deduced this manipulability result from the Arrow's impossibility theorem (Arrow, 1963).

A key question Is it possible to find a collective decision representing sincere individual preferences when those are modelled through fuzzy relations over the set of alternatives, *i.e.* is the collective decision obtained by a non-dictatorial aggregation function?

About fuzzy preference relations We are interested in defining the manipulation of aggregation functions starting with a collection of fuzzy individual preference relations. In fact, when comparing alternatives, individuals are often affected by the presence of uncertainty, ambiguity or imprecision due to imperfect knowledge of data. Consequently, an individual might not be able to clearly state a preference relation for any pair of alternatives. In the literature, it has been argued that fuzzy relations incorporate inherent subjectivity and imprecision of human thinking (Goguen, 1967; Barrett and Pattanaik, 1985; Ovchinnikov and Owernoy, 1988). Given two alternatives, it is assumed that an individual has a degree of preference over such a given pair of alternatives.

Fuzzy Social Choice Functions ($FSCF$) In addition, the uncertainty affects the type of aggregation functions. The $FSCF$ are used for mapping a collection of fuzzy preference relations into a chosen collective alternative. It can be viewed as a two-step procedure (Barrett *et al.*, 1986; Barnerjee, 1994; Garcia-Lapresta and Llamazares, 2000). There exist two types of $FSCF$ decompositions:

1. *Aggregation and defuzzification*

   (a) Apply a fuzzy aggregation rule (FAR) that leads to a comprehensive (collective) fuzzy relation.

   (b) Generate from the fuzzy relation the best alternative by applying a choice function that leads to a collective choice.

2. *Defuzzification and aggregation*

   (a) Apply a choice function for each individual fuzzy relation to obtain an individual choice set.

   (b) Aggregate the individual choices using an aggregation function.

Purpose of the paper In this paper, we restrict our attention to the set of fuzzy relations satisfying reflexivity, connectedness, and *max-min* transitivity. They are called fuzzy orderings (Dutta *et al.*, 1986). The fuzzy counterpart of G-S manipulability is introduced and called H-manipulability. The purpose of this paper is to prove the H-manipulability theorem relying on the fuzzy counterpart impossibility result for FARs (Dutta, 1987).

Outline of the paper The organization of this paper is as follows. Section 2 is devoted to elementary concepts, related to $FSCF$ and FAR; the fuzzy counterpart of Arrow's impossibility theorem (Dutta, 1987) is introduced. Section 3 deals with the generalization of the game form concept in the fuzzy framework. Section 4 introduces the H-manipulability theorem and resumes the different steps of its proof. Finally, conclusions and avenues for future research are provided.

1 Mathematical background

The focus of the paper is in the context where a group of several individuals has to choose an alternative from a finite set of alternatives. Consider that the preferences of each individual are modelled by using a fuzzy binary relation.

1.1 Elementary concepts

This section is consecrated to the basic data of the model, the standard definitions on fuzzy set theory, and some elementary properties.

1.1.1 Basic data

Let

- $X = \{x_1, x_2, \ldots, x_j, \ldots, x_m\}$ denote a finite set of alternatives, with $|X| \geq 3$;
- $\chi = \{S \mid S \subseteq X \text{ and } S \neq \emptyset\}$ denote the set of all non-empty subsets of X;
- $N = \{1, 2, \ldots, i, \ldots, n\}$ denote a group of n individuals, with $n \geq 2$.

1.1.2 Fuzzy relations

Definition 1. (**Fuzzy binary relation**)
A fuzzy binary relation, R, over X is a function $R : X \times X \longrightarrow [0, 1]$.

A fuzzy relation can be considered as a fuzzy set in $X \times X$ with a membership function R, introduced to model vagueness or imprecision. Generally, the *imprecision* or *vagueness* is detected when there are some difficulties to express clearly our knowledge (is the turquoise color green or blue?). In our settings, the vagueness affects the preferences of an individual. Thus, $R(x, y)$ represents the degree to which the crisp weak preference "the alternative x is at least as preferred as alternative y" (Zadeh, 1965). Consequently, for each pair of alternatives x and y belonging to X, we have a number $R(x, y) \in [0, 1]$ interpreted as the degree of preferences of x over y. The fuzzy relation R is considered as a weak fuzzy preference relation.

Definition 2. (**fuzzy ordering**)
A fuzzy relation, R, is said to be a fuzzy ordering, if it fulfills the following properties,

1. *Connectedness:* $R(x, y) + R(y, x) \geq 1, \forall\, x, y \in X$;
2. *Reflexivity:* $R(x, x) = 1, \forall\, x \in X$;

3. *Max-min transitivity*: $R(x, z) \geq \min\{R(x, y), R(y, z)\}, \forall\, x, y, z \in X$.

Let H denote the set of all fuzzy orderings.

Definition 3. (**Union and intersection of fuzzy subsets**)
Let $A : X \rightarrow [0, 1]$ and $B : X \rightarrow [0, 1]$ denote two fuzzy subsets,

1. *The union between A and B is the fuzzy set $A \cup B$ such that*

$$\forall\, x \in X, A \cup B(x) = \max\{A(x), B(x)\};$$

2. *The intersection between A and B is the fuzzy set $A \cap B$ such that*

$$\forall\, x \in X, A \cap B(x) = \min\{A(x), B(x)\}.$$

Remark 1. If $\min\{A(x), B(x)\} = 0$ for all $x \in X$, then $A \cap B = \emptyset$.

Given a fuzzy preference relation, R, there is no a clear way for deriving the indifference relation (I) and the strict preference relation (P) from R. Dutta (1987) stated a manner of derivation of P and I in Proposition 1 as follows.

Proposition 1. (**Fuzzy indifference and fuzzy strict preference**)
Let R denote a connected fuzzy preference relation, R, satisfying the following conditions,

1. *$R = P \cup I$;*
2. *I is symmetric, i.e., $\forall\, x, y \in X, I(x, y) = I(y, x)$;*
3. *P is anti-symmetric, i.e., $\forall\, x, y \in X, P(x, y) > 0 \Rightarrow P(y, x) = 0$.*

Then, $\forall\, x, y \in X$,

$$P(x, y) = \begin{cases} R(x, y), & \text{if } R(x, y) > R(y, x) \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$I(x, y) = \min\{R(x, y), R(y, x)\}.$$

Throughout this paper, we will follow the above derivation for P and I. In what follows, the following notation is also needed.

Let,

- R_N, denote the n-tuple individual fuzzy relations $(R_1, R_2, \ldots, R_i, \ldots, R_n) \in T^n$;
- $R_N \mid R_i'$, represent the fuzzy profile $(R_1, R_2, \ldots, R_{i-1}, R_i', R_{i+1}, \ldots, R_n)$;
- $R_N \mid R_1', R_2'$, represent the fuzzy profile $(R_1', R_2', R_3, \ldots, R_i, \ldots, R_n)$;
- $X_H(R_N \mid R_i')$, denote the set $\{f(R_N \mid R_i'), \forall\, R_i' \in H\}$.

1.2 Fuzzy aggregation rules

When a group of n individuals has to express their collective opinion on each pair of alternatives, an FAR can be used starting with fuzzy individual preference relations.

Definition 4. (**Fuzzy aggregation rule**)
A fuzzy aggregation rule (FAR) is a function $h : T^n \longrightarrow T$,

$$R_N \mapsto R^s = h(R_N).$$

where, T is a non-empty set of fuzzy preference relations.

In what follows T corresponds to H. The properties used by Dutta (1987) define the fuzzy counterpart of impossibility Arrow's theorem.

1.2.1 Impossibility conditions

Let $h : T^n \longrightarrow T$ denote an FAR. The function h satisfies,

1. *The independence of irrelevant alternative (IIA)*
   If $\forall\, R_N, R_N' \in T^n$, and $\forall\, x, y \in X$ with $x \neq y$,
   - $R_i(x, y) = R_i'(x, y)$ and,
   - $R_i(y, x) = R_i'(y, x)$, $\forall i \in N$, then
   - $R^s(x, y) = R'^s(x, y)$ and $R^s(y, x) = R'^s(y, x)$.

2. *The Pareto criterion (PC)*

   If $\forall\ R_N \in T^n$, aand $\forall\ x, y \in X$ with $x \neq y$, then $P^s(x, y) \geq \min_{i \in N} P_i(x, y)$.

3. *The positive responsiveness (PR)*

   If $\forall\ R_N, R'_N \in T^n$, and $\forall\ x, y \in X$ with $x \neq y$,

   - $R_i(x, y) = R'_i(x, y), \forall\ i \neq j$ and,
   - $R^s(x, y) = R^s(y, x)$ and,
   - $(P_j(x, y) = 0$ and $P'_j(x, y) > 0)$ or $(P_j(y, x) > 0$ and $P'_j(y, x) = 0)$, then
   - $P'^s(x, y) > 0$

4. *Dictatorship (D)*

   If there exists an individual $k \in N$ such that

   $$\forall\ R_N \in T^n, \text{ and } \forall\ x, y \in X \text{ with } x \neq y, P_k(x, y) > 0, \text{ then } P^s(x, y) > 0.$$

Condition (IIA) states that when two distinct fuzzy profiles are considered, if for a given pair of alternatives the degrees of preferences for each individual are identical for both fuzzy profiles, then the social aggregate preference degrees are the same for the considered pair of alternatives.

Condition (PC) states that the strict social preference degree of an alternative over another must be at least as high as each strict individual preference degree of the given pair of alternatives.

Condition (PR) states that when two distinct fuzzy profiles are considered, if the individual preference degrees of a given pair of alternatives (x, y) are identical for each individual in both fuzzy profiles, except for the individual j and the preference of j changes in favor of the alternative x, then, if the social aggregate preference degrees of (x, y) and (y, x) of the first profile are identical, x must be socially preferred to y with a strict preference degree for the second profile.

Condition (D) states that when there exists an individual k such that for each fuzzy profile, if he/she has a strict preference degree over a given pair of alternatives (x, y), then x must be socially preferred to y with a strict preference degree.

1.2.2 Impossibility theorem

Moreover, Dutta (1987) showed the following fuzzy counterpart of the Arrow's impossibility theorem.

Theorem 1. (**Impossibility theorem**)
Let $n \geq 3$ and $h : T^n \longrightarrow T$ denote an FAR. If h satisfies IIA, PC, and PR, then h is dictatorial.

1.3 Fuzzy social choice functions

When a group of n individuals has to make a social choice from a finite set of alternatives, a fuzzy social choice function can be used starting with fuzzy individual preference relations.

Definition 4. (**Fuzzy social choice function**)
A fuzzy social choice function ($FSCF$) is a function $f : T^n \longrightarrow X$, where T is a non-empty set of fuzzy binary relations over X.

Now, consider some properties of $FSCF$s needed to state the fuzzy counterpart of G-S-theorem. They are defined from the concept of dominance degree, proposed by Basu (1984) as follows,

$$G(R, S)(x) = \min_{y \in S} R(x, y), \forall\, x \in S.$$

We consider that the manipulator uses,

$$B_H(S, R) = \{x \in S \mid G(S, R)(x) \geq G(S, R)(y), \forall\, y \in S\}$$

as a choice function. Such an individual is said to be *H-manipulator* since he/she is H-rational (Dutta *et al.*, 1986). Therefore, the definition of the manipulability of an $FSCF$ is defined as follows.

Definition 5. (**H-manipulability and H-dictatorship**)
Let $f : H^n \longrightarrow X$ denote an $FSCF$. The function f is

1. *H-manipulable by individual $R_N \in H^n$ if there is $R'_i \in \mathcal{H}$ such that*

$$G(R_i, X_H(R_N \mid R'_i))(f(R_N \mid R'_i)) \geq G(R_i, X_H(R_N \mid R'_i))(f(R_N))$$

2. *H-strategy-proof if there is no $R_N \in H^n$ at which f is H-manipulable.*
3. *H-dictatorial if there exists a dictator $k \in N$, i.e. for every $R \in H^n$ and all*

$$x \neq f(R_N) \in X, G(R_k, X)(f(R_N)) \geq G(R_k, X)(x)$$

Some necessary conditions, however, are needed for a choice function $C(S)$ to be H-rationalizable in terms a fuzzy ordering, R, starting with a subset S of the set of alternatives X. In other words, the basic relation R_C of C, defined as follows: for all $x, y \in S, xR_Cy$ if and only if $x \in C(\{x, y\})$, must satisfy the following proposition (Dutta, 1987).

Proposition 2. (Necessary condition for H-rationalizability)
Let C denote an exact choice function which is H-rationalizable in terms a fuzzy ordering. Then,

1. *C satisfies property β_+, i.e.,*
$$\forall\, x, y \in X, \forall\, A, B \in \chi, [x, y \in C(A),\ \textit{and}\ x \in C(B)] \Rightarrow [y \in C(B)];$$
2. *R_C is quasi-transitive, i.e. P_C is transitive;*
3. *$\forall\, x, y, z \in X$, if (xP_Cy, yI_Cz, xP_Cz), then $C(\{x, y, z\}) = \{x\}$.*

2 Fuzzy game form

In this section, we adopt the concept of game form by Gibbard (1973) and extend it to the fuzzy framework.

Definition 6. (**Fuzzy game form**)
A game form is a function $g : (S_1, S_2, \dots, S_i, \dots S_n) \longrightarrow Z$, where S_i represents a set of strategies assigned to each player $i \in N$. The values of g are called outcomes, and denoted by Z. The function g is said to be a fuzzy game form, when the set of strategies for each player corresponds to the set of fuzzy binary relations.

Remark 2.
an $FSCF$, $f : H^n \longrightarrow X$ is a fuzzy game form such that, the set of strategy for each player is a fuzzy binary ordering over X and the set of outcomes, Z, is included in X.

It should be noted, however, that nothing in the structure of a fuzzy game form tells us which strategy "honestly" represents any given preference ordering. Manipulability is then a property of a game $g(s_1, s_2, \dots, s_i, \dots, s_n)$ plus n functions $\sigma_1, \sigma_2, \dots, \sigma_i, \dots \sigma_n$, where for each individual k and a fuzzy preference ordering R_k, $\sigma_k(R_k)$ is the strategy for k which honestly represents R_k. In other words, where a decision making system is characterized by a fuzzy social choice function, f, it is characterized also by functions, $g, \sigma_1, \sigma_2, \dots, \sigma_i, \dots \sigma_n$, such that, the collective decision with the fuzzy profile, R_N, corresponds to $g(\sigma_1(R_1), \sigma_2(R_2), \dots, \sigma_i(R_i), \dots \sigma_n(R_n))$.

Now, in order to establish the H-manipulability result in terms of fuzzy game forms, we define the strategy dominance as follows.

Definition 7. (**Dominant strategy**)
A strategy s^* is dominant for a given player, k, and a certain fuzzy preference ordering, R, over Z, if there exists no $S_N = (s_1, s_2, \ldots, s_i, \ldots, s_n) \in T^n$ such that

$$G(R, Z)(g(S_N)) \geq G(R, Z)(g(S_N \mid s^*)).$$

In other words, a strategy s^* is R-dominant for player k if, no matter what strategies are fixed for every one else, strategy s^* for k produces an outcome with a degree of dominance at least as high as the degrees of the remaining outcomes. Formally, a strategy s^* is R-dominant for player k if, for every n-tuple strategy $S_N \in T^n$,

$$G(R, Z)(g(S_N \mid s^*)) \geq G(R, Z)(g(S_N)).$$

Therefore, the definition of H-straightforward fuzzy game forms is presented as follows.

Definition 8. (**H-straightforward fuzzy game form**)
A fuzzy game form is H-straightforward if for every player, k, and a given fuzzy preference ordering, R, over Z, there is a strategy which is R-dominant for k.

Moreover, the concept of H-dictatorship for a game form g is defined in the same manner for an $FSCF$, *i.e.*, a player k is H-dictator for g, if for every $S_N \in T^n$,

$$\forall x \neq g(S_N) \in Z, G(R_k, X)(g(S_N)) \geq G(R_k, X)(x)$$

3 H-manipulability theorem

The objective of this section is to prove the H-manipulability theorem. The following steps will be followed.

1. Stating by proving that if a fuzzy game form with at least three possible outcomes is H-straightforward, then it is H-dictatorial (Theorem 2).
2. Then, proving that if a fuzzy social choice function, f, with at least three possible outcomes is non H-dictatorial, then it is H-manipulable (Theorem 3).

Theorem 2. (H**-dictatorship of fuzzy game form**)
If a fuzzy game form with at least three possible outcomes is H-straightforward, then it is H-dictatorial.

Proof

Let g denote a H-straightforward game form with at least three outcomes. We must prove that g is H-dictatorial.

Since g is H-straightforward, for every i and R, there is a strategy s which is R-dominant for i.

For each fuzzy profile R_N, set $\sigma(R_N) = \langle \sigma_1(R_1), \sigma_2(R_2), \ldots, \sigma_i(R_i), \ldots \sigma_n(R_n) \rangle$. The functions σ and $\sigma_1, \sigma_2 \ldots, \sigma_i, \ldots, \sigma_n$ will be fixed throughout the proof; the function f will be the composition of g and σ, so that for all R_N,

$$f(R_N) = g(\sigma_1(R_1), \sigma_2(R_2), \ldots, \sigma_i(R_i), \ldots, \sigma_n(R_n)).$$

The proof of this theorem will follow the following steps

1. Define a fuzzy aggregation rule h such that for each fuzzy profile R_N, a fuzzy ordering $R^s = h(R_N)$;
2. Prove that this function, h, satisfies all impossibility conditions except non-dictatorship;
3. Apply the impossibility theorem by Dutta (1987) to deduce the dictatorship of h;
4. Finally, prove that the dictatorship of h corresponds to H-dictatorship of g.

Let us now detail steps of the proof.

1. *Construction of a fuzzy aggregation rule*

   Let h denote a fuzzy aggregation rule, $h : H^n \longrightarrow H$ with $R^s = h(R_N)$, such that
   $$B_H(\{x, y\}, R^s) = f(R_N * \{x, y\})$$
   where,
   $$B_H(S, R) = \{x \in S \mid G(R, S)(x) \geq G(R, S)(y), \forall y \in S\}.$$

In other words, if n individuals with R_N fuzzy preferences have to choose between x and y using f, their choice will correspond to the set $B_H(\{x, y\}, R^s)$. Thus, the outcome of f is H-rationalizable in terms of the fuzzy ordering R^s.

Therefore, if $f(R_N * \{x, y\}) = x$, we have $R^s(x, y) > R^s(y, x)$. Also, if $f(R_N * \{x, y\}) = y$, we have $R^s(y, x) > R^s(x, y)$. The choice function, defined by the outcome of f must verify, however, Proposition 2 to be H-rationalizable in terms a fuzzy ordering R.

More precisely, we fix the function $h : H^n \longrightarrow H$ such that,

$$\forall\, x \in X, R^s(x, x) = 1$$
$$\text{and } \forall\, R_N, R'_N \in H^n, \text{ and } \forall\, x, y \in X \text{ with } x \neq y,$$
$$R^s(x, y) = \begin{cases} 1, & \text{if } R_i(x, y) > R_i(y, x), \forall i \in N \\ \alpha \in]1/2, 1], & \text{if } f(R_N * \{x, y\}) = x \\ \beta \in [0, 1/2[, & \text{if } f(R_N * \{x, y\}) = y \end{cases}$$

where, $\alpha + \beta \geq 1$.

In addition, we suppose that $R^s(x, y) = R^s(y, x)$ only if $R_i(x, y) = R_i(y, x)$, for all $i \in N$. We can now easily show that R^s satisfies reflexivity, connectedness and *max-min* transitivity. We shall show that h is an FAR which satisfies the Dutta's conditions except dictatorship.

2. *Verification of impossibility conditions by Dutta, except dictatorship*

   (a) *IIA*

   If $\forall\, R_N, R'_N \in T^n$, and $\forall\, x, y \in X$ with $x \neq y$,
   - $R_i(x, y) = R'_i(x, y)$ and,
   - $R_i(y, x) = R'_i(y, x)$, $\forall i \in N$, then
   - $f(R_N * \{x, y\}) = f(R'_N * \{x, y\})$.

   Thus, $R^s(x, y) = R'^s(x, y)$ and $R^s(y, x) = R'^s(y, x)$. Therefore, we can conclude that IIA holds for the function h.

(b) *PC*

If $\forall\ R_N, R'_N \in T^n$, and $\forall\ x, y \in X$ with $x \neq y$, we have

$$P^s(x,y) = \begin{cases} R^s(x,y), & \text{if } R^s(x,y) > R^s(y,x) \\ \\ 0, & \text{otherwise.} \end{cases}$$

However, $R^s(x,y) > R^s(y,x)$ is verified only when $f(R_N * \{x,y\}) = x$. Thus,

$$R^s(x,y) = \begin{cases} 1, & \text{if } R_i(x,y) > R_i(y,x), \forall i \in N \\ \\ \alpha \in]1/2, 1], & \text{otherwise} \end{cases}$$

Therefore, if $R_i(x,y) > R_i(y,x), \forall i \in N$, then

$$P_i(x,y) > 0, \forall i \in N \text{ and } P^s(x,y) = 1 \geq \min_{i \in N} P_i(x,y).$$

Otherwise, $P^s(x,y) = \alpha$ or 0 and $\min_{i \in N} P_i(x,y) = 0$, since there exists an $i \in N$ such that $R_i(y,x) \geq R_i(x,y)$. Therefore, we can conclude that *PC* holds for the function h.

(c) *PR*

We suppose that $\forall\ R_N, R'_N \in T^n$, and $\forall\ x, y \in X$ with $x \neq y$,

- $R_i(x,y) = R'_i(x,y)$ for all $i \neq j$; and,
- $R_s(x,y) = R_s(y,x)$, then

we have necessarily that $\forall i \in N$, $R_i(x,y) = R_i(y,x)$. Thus, we have, $P_j(x,y) = 0$.

Consequently, we must now show if $P'_j(x,y) > 0$, then $P'^s(x,y) > 0$.

Indeed, if $P'_j(x,y) > 0$, then $R'_j(x,y) > R'_j(y,x)$. Thus,

$$G(\{x,y\}, R'_j)(x) \geq G(\{x,y\}, R'_j)(y).$$

In addition, for all R_N, $f(R_N) = g(\sigma_1(R_1), , \sigma_2(R_2), \ldots, \sigma_i(R_i), \ldots \sigma_n(R_n))$ and, g is supposed to be H-straightforward game form.

Thus, $g(\sigma_1(R'_1), \sigma_2(R_2), \ldots, \sigma_j(R'_j), \ldots \sigma_n(R'_n))$ is equivalent to x. Then, $R'^s(x,y) = \alpha > R'^s(y,x) = \beta$ and $P'^s(x,y) = R'^s(x,y) > 0$. Therefore, we can conclude that (PR) holds for the function h.

3. *Dictatorship of the function h*

Since the function h fulfills all the impossibility conditions, *i.e. IIA, PC,* and *RP*, it must be dictatorial from Theorem 1.

4. *Equivalence dictatorship with H-dictatorship*

We prove now that a dictator for h is a H-dictator for g. We suppose that h is dictatorial, then there exists an individual $k \in N$ such that $\forall\ R_N \in T^n$, and $\forall\ x, y \in X$ with $x \neq y$, $P_k(x, y) > 0$, then $P^s(y, x) = 0$.

Thus, $G(\{x, y\}, R^s) = x$ and $G(\{x, y\}, R_k)(x) > G(\{x, y\}, R_k)(y)$. Consequently, k is H-dictatorial for g when $X = \{x, y\}$.

We suppose now that

- $X = \{x, y, z\}$;
- $P_k(x, y) > 0$;
- $P_k(x, z) > 0$ and;
- $P_k(y, z) > 0$.

Since k is dictatorial for h, then we have

$$P^s(x, y) > 0, P^s(x, z) > 0, \text{ and } P^s(y, z) > 0.$$

We shall show that $f(R_N) = x$. It is sufficient to prove that

(*i*) $G(X, R^s)(x) \geq G(X, R^s)(y)$ and,

(*ii*) $G(X, R^s)(x) \geq G(X, R^s)(z)$.

To prove the first inequality, three cases are considered. We will prove our result based on the fact that, if for all $x, y, z \in X$(Dasgupta and Deb, 1996),

$$\begin{cases} R^s(x, y) > R^s(y, x), \\ \\ R^s(y, z) > R^s(z, y), \end{cases} \Rightarrow R^s(x, z) \geq \min\{R^s(x, y), R^s(y, z)\}$$

when R^s satisfies the *max-min* transitivity.

(a) $R^s(x, y) > R^s(y, z) > R^s(y, x)$

Since we have

$$R^s(x, z) \geq \min\{R^s(x, y), R^s(y, z)\} = R^s(y, z),$$

$$G(X, R^s)(x) = \min\{R^s(x, y), R^s(y, x)\} \geq R^s(y, z) > R^s(y, x) = G(X, R^s)(y)$$

Then,

$$G(X, R^s)(x) \geq G(X, R^s)(y).$$

(b) $R^s(y, z) > R^s(x, y) > R^s(y, x)$.
Since we have

$$R^s(x, z) \geq \min\{R^s(x, y), R^s(y, z)\} = R^s(x, y),$$

$$G(X, R^s)(x) = R^s(x, y) \geq R^s(y, z) > R^s(y, x) = G(X, R^s)(y).$$

Then,

$$G(X, R^s)(x) \geq G(X, R^s)(y).$$

(c) $R^s(x, y) > R^s(y, x) > R^s(y, z)$

Since we have

$$R^s(x, z) \geq \min\{R^s(x, y), R^s(y, z)\} = R^s(y, z),$$

$$G(X, R^s)(x) = R^s(x, y) \geq R^s(y, z) = G(X, R^s)(y).$$

Then,

$$G(X, R^s)(x) \geq G(X, R^s)(y).$$

The same reasoning is used to prove that $G(X, R^s)(x) \geq G(X, R^s)(z)$ and we obtain that if $P^s(x, y) > 0, P^s(x, z) > 0$ and $P^s(y, z) > 0$, then $f(R_N) = x$. Also, if the same conditions about the fuzzy relation of individual k, we can conclude that$f(R_N) = x$. Consequently, k is H-dictatorial for f. □

The above result is used to prove the H-manipulability theorem.

Theorem 3. (**H-manipulability theorem**)
If a fuzzy social choice function, f, with at least three possible outcomes is non H-dictatorial, then it is H-manipulable.

Proof

Suppose that f is non H-dictatorial and has at least three possible outcomes. Then, f is a game form, f is not H-straightforward (Theorem 2) and thus for player k and R, no strategy is R-dominant for k. The relation R here is a fuzzy ordering over the set of outcomes, Z, and a strategy is a fuzzy ordering R^* over the set X, $Z \subseteq X$.

Let R^* extend R to X, such that for all x in Z,

$$G(X, R^*)(x) = G(Z, R)(x). \tag{1}$$

Then, in particular, R^* is not R-dominant for k. Hence, for some n-tuple strategy R_N of fuzzy orderings of X, it is not the case that,

$$G(Z, R)(f(R_N \mid R^*) \geq G(Z, R)(f(R_N)).$$

But, since $(f(R_N \mid R^*)$ and $f(R_N)$ are in Z, from (1),

$$G(X, R^*)(f(R_N)) \geq G(X, R^*)(f(R_N \mid R^*).$$

and f is H-manipulable. Assuming Theorem 2, we have proved Theorem 3. □

Conclusions

We established the fuzzy counterpart of G-S manipulability theorem. The proof is strongly relied on the Dutta's impossibility theorem established for fuzzy aggregation rules. A possible avenue for future research is to extend the fuzzy counterpart of G-S manipulability theorem for other domains of fuzzy social choice functions.

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References

[1] Arrow K. J. (1963) *Social Choice and Individual Values.* New York: Wiley.

[2] Barnerjee A. (1995) Fuzzy choice functions, revealed prefrence and rationality. *Fuzzy Sets and Systems*, 70: 31-43.

[3] Barrett C. R. and P. K. Pattanaik (1985) On vague preferences. In: J. Kacprzyk and M. Federizzi, Eds., *Multiperson Decision Making Using Fuzzy Sets and Possibility Theory*, Kluwer Academic Publishers, Dordrecht, 155-162.

[4] Barrett C. R. Pattanaik P. K. and M. Salles (1986) On the structure of fuzzy social welfare functions. *Fuzzy Sets and Systems*, 19: 1-10.

[5] Basu K. (1984) Fuzzy revealed preference theory. *Journal of Economic Theory*, 32: 212-227.

[6] Dutta B. (1987) Fuzzy preferences and social choice. *Mathematical Social Sciences*, 13: 215-229.

[7] Dutta B. Panda SC. and P. K. Pattanaik (1986) Exact choices and fuzzy preferences. *Mathematical Social Sciences*, 11: 53-68.

[8] Garcia-Lapresta J. L. and Llamazares B. (2000) Aggregation of fuzzy preferences: Somes rules of the mean. *Social Choice and Welfare*, 17: 673-670.

[9] Geslin S. (2003) *Aggrégration de Préférences Floues et Domaines Économiques.* Thèse de doctorat. Faculté de Sciences Économiques et de Gestion, Université de Caen, France.

[10] Gibbard A. (1973) Manipulation of social choice schemes: A general Result. *Econometrica*, Vol. 41, No. 4: 587-601.

[11] Goguen, J. A. (1967) L-fuzzy sets. *Journal of Mathematical Analysis*, 18: 145-174.

[12] Orlovsky S.A. (1978) Decision-making with a fuzzy prefrence relation. *Fuzzy Sets and Systems*, 1: 155-167.

[13] Ovchinnikov S. and V. M. Ozernoy (1988) Identifying noninferior decision alternatives based on fuzzy binary relations. in: J. Kacprzyk, M. Roubens (Eds.), Non *conventional Prefrences Relations in Decision Making*, Springer, Berlin, pp. 82-95.

[14] Satterthwaite M.A. (1975) Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, Vol. 10: 187-217.

[15] Zadeh, L. A. (1971) Quantitative fuzzy semantics. *Information Sciences*. Vol. 3: 159-176.