Comparing two Territory Partitions in Districting Problems: Measures and Practical Issues

Fernando Tavares-Pereira *‡§, José Figueira †‡§,
Vincent Mousseau,§ and Bernard Roy§

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*Departamento de Matemática, Universidade da Beira Interior, Rua Marquês D'Ávila e Bolama, 6201-001 Covilhã, Portugal, Phone/Fax: (+351) 275 319 732, E-mail: fpereira@noe.ubi.pt

†CEG-IST, Departamento de Engenharia e Gestão, Office: 2N818, Instituto Superior Técnico, TagusPark, Av. Prof. Cavaco Silva, 2780 - 990 Porto Salvo, Portugal. Phone: +351 21 423 32 99 Fax: +351 21 423 35 68 E-mail: figueira@ist.utl.pt

‡INESC - Coimbra, Rua Antero de Quental, 199, 3000-033 Coimbra, Portugal, Phone: (+351) 239 851 040, Fax: (+351) 239 824 692

§LAMSADE, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75 775 Paris Cedex 16, France, Phone: (+33) 1 44 05 44 01, Fax: (+33) 1 44 05 40 91, E-mail: {fpereira,figueira,mousseau,roy}@lamsade.dauphine.fr

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Abstract

Planning in Public Sector decision-making situations with a socio-economic dimension is an activity of the uttermost importance, mainly for urban planners. The ramifications of such decisions have strong effect on the life of populations. This paper deals with the comparison between district maps of a territory in the context of districting problems having a strong social-economic component. The theoretical problem context is about comparison between two partitions in a connected, non-oriented, and planar graph according to a certain attribute. To our best knowledge this problem is not so common in the literature. The question how to compare two partitions led us to the introduction of three new concepts: *compatibility*, *inclusion*, and *distance*. This is the main novelty of the paper. The proposed measures are strongly dependent on the real-world applications we were face with. Numerical experiments were done on the Paris region territory.

Keywords: *Decision-Making, Districting Problems, Comparison Indices, Combinatorics.*

1 Introduction

Planning in Public Sector decision-making situations with a socio-economic dimension is an activity of the uttermost importance, mainly for urban planners. The ramifications of such decisions have strong effect on the life of populations. The vast broad of applications with a socio-economic nature, the wide panoply of techniques, methods, and methodologies usually implemented, and the very nature of the problem context features have strong implications in a way a decision study should be designed and conducted. This paper deals with districting problems which have a strong social-economic component.

On the districting problems and some practical concerns. Over the last three decades, many researchers, academics and practitioners from distinct fields (not necessary urban planners) have developed models, built algorithms and implemented solutions concerning the so-called *districting problem*. It can be viewed as a grouping process of *elementary units* or *atoms* of a *territory* into larger pieces of land or *zones*, giving rise to a *partition*, also called *district map*.

A substantial and important growth in the application of the operations research and decision aiding models arose in a broad scope of areas.

There are many practical questions related to districting problems: to define the electoral districts of a country [2, 9, 11, 12]; to establish the different working zones for a travel salesperson team [6, 10, 16, 17]; to define areas in metropolitan internet networks to install hubs [15]; to define the areas for manufactured and consumer goods [8]. But, the same kind of questions occur also in police districting [4]; school districting [7]; districting of salt spreading operations [14]; defining electrical power zones [1], defining public transportation network pricing system [13], and many other domains. These are barely some frequent real-world decision making questions and concerns in territory partition problems that appear mainly in Public Sector decision making situations.

On the comparison of two partitions. Quantitative analysts had mainly consecrated their attention to the problem of forming political districts. Among the most useful criteria one deserves the particular attention of the scientific community working on this very topic: comparing and evaluating the "differences" between an alternative proposal and the current partition (in general, the one that was implemented a long time ago and for historical reasons gained some popularity).

Concerning the *political districting problem*, these needs come from the fact that the question of implementing a new partition is more or less frequent in order to regain or create a voting power balance among the set of political districts that forms the political district map. In such a situation, the natural objective is to build up a new partition that "minimizes" changes with respect to the previous one. There are two main reasons to "minimize" the differences between a new partition and the current one. On the one hand, the main concern consists of keeping "good" performances for the remaining criteria: compactness, socio-economic homogeneity, integrity, etc. On the other hand, in systems using single-member electoral districts each candidate prefer to keep his/her electoral district as it was previously defined since he/she has now a good comprehensive knowledge about problems and needs of his/her district [2].

The need for comparing two partitions occurs also from different fields as it is the case of the definition of school zones. The very nature of the criterion can be strongly different from the previous problem. The need for comparing two partitions may occur, however. Let us consider the problem of the definition of the pricing system of the network public transportation in a big city. It is advisable that the users (students) of the school system should not be compelled to change and frequent schools in a different transportation zone from their own residence zone. Thus a comparison between the school map and the partition transportation pricing system map is important.

Measuring the difference between two partitions. When analyzing the different districting problems in the literature we can identify different needs leading to the comparison of two partitions based on a different nature.

In our study we propose to classify the needs of comparing two partitions of a territory into three classes:

1. *Compatibility*. It is related to the cases where the comparison between two partitions is made through the verification if each zone of the first partition results from a group of zones of the second one, or if each zone along with some other zones of the same partition define a single zone of the second partition.

2. *Inclusion*. This class comprises all the cases or measures where the objective is to evaluate the differences between two partitions based on the notion of fineness, *i.e.*, when a partition is composed only of zones that results from a splitting out operation or division of the zones of the second partition.

3. *Distance*. This measure aims to reflect (modelling) every difference between two partitions.

Scope and purpose of the paper. The theoretical problem context is about the comparison between two partitions in a connected, non-oriented, and planar graph according to a certain attribute. To our best knowledge this problem is not so common in the literature. Only a few studies were identified and they deal with a quite different cases. More precisely, only two works were found. Both are attempts to measure the degree of similarity between partitioning political districting problems [2, 3].

The question how to compare two partitions led us to the introduction of three new concepts: *compatibility*, *inclusion*, and *distance*. This is the main novelty of the paper. The proposed measures are strongly dependent on the real-world applications we were faced with.

Outline of the paper Section 2 presents the main concepts and notation. Section 3, 4, and 5 are devoted to the three measures, compatibility, inclusion, and distance, respectively. Section 6 concerns the computational experiments and results. Section 7 is devoted to the use of the comparison indices in managerial problems. Finally, Section 8 presents the main conclusions and avenues for future research.

2 Concepts and notation

Consider the following notation:

- $A = \{a_1, a_2, \ldots, a_i, \ldots, a_n\}$ denote a *territory*, where each a_i represents an indivisible *elementary units*;
- $y = \{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_i, \ldots, \hat{a}_L\}$ denote a set of contiguous elementary units, called *zone*;
- $Y = \{y_1, y_2, \ldots, y_u, \ldots, y_K\}$ denote a *partition* (or *district map*) of territory A; for each elementary unit a_i there is one and only one zone $y_u \in Y$ such that a_i belongs to y_u;
- $\mathcal{Y} = \{Y', Y'', \ldots, Y^{(m)}, \ldots, Y^{(M)}\}$ denote the set of all the *feasible partitions* of territory A.

For the sake of simplicity, an elementary unit of territory, a_i, is also represented by its index i. Figure 1 (a) represents a territory, composed of 16 elementary units, divided into 4 zones, $Y = \{y_1, y_2, y_3, y_4\}$.

Given a territory $A = \{1, 2, \ldots, i, \ldots, n\}$, a *contiguity graph* is associated to A as an *undirected*, *connected*, and *planar* graph $G = (V, E)$, where $V =$

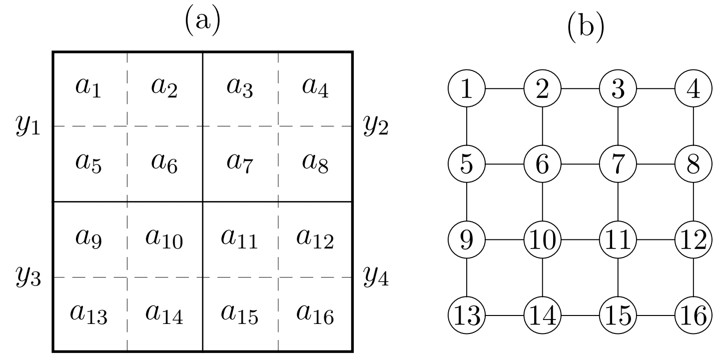


Figure 1: A territory and the associated contiguity graph .

$\{1, 2, \ldots, i, \ldots, n\}$ denotes the set of *vertices* representing elementary territorial units and $E = \{e_1, e_2, \ldots, e_k, \ldots, e_m\} \subset V \times V$ denotes the set of *edges*, where $e_k = \{i, j\}$, represents a border between two adjacent elementary units i and j. Figure 1 (b) shows the contiguity graph G corresponding to territory, of Figure 1 (a). In the remaining of this paper we will consider indifferently the territory A and the set of vertices V.

Definition 2.1 (Attribute) *Consider a contiguity graph $G = (V, E)$. An attribute P in V is a real-valued function defined in V, such that, for each $i \in V$, $P(i) \equiv p_i \in \mathbb{R}^+$. The value p_i is thus a non-negative real.*

For any subset $\bar{y} \subseteq V$ of elementary units, $P_{\bar{y}} = \sum_{i \in \bar{y}} p_i$ is the overall value of the attribute P in $\bar{y}$, and $\mathcal{P} = \sum_{i \in V} p_i$ represents the overall value of attribute P, for the graph G (assume that $P_{\bar{y}} = 0$ when $\bar{y} = \emptyset$).

Let $Y = \{y_1, y_2, \ldots, y_u, \ldots, y_K\}$ and $Y' = \{y'_1, y'_2, \ldots, y'_v, \ldots, y'_{K'}\}$, denote two partitions, where $|Y| = K$ and $|Y'| = K'$.

Definition 2.2 (Inclusion between two zones) *Consider two zones $y \in Y$ and $y' \in Y'$. The zone y is included into y', according to P (denoted $y \subseteq_P y'$), if $P_{y \cap y'} > 0$ and for all $i \in y$ such that $i \notin y'$ ($i \in y \setminus y'$), $p_i = 0$,* i.e., $P_{y \setminus y'} = 0$.

Definition 2.3 (Equality between two zones) *A zone $y \in Y$ is considered equal to $y' \in Y'$, according to attribute P (denoted $y =_P y'$), if $y \subseteq_P y'$ and $y' \subseteq_P y$.*

Definition 2.4 (Reference zone) *Consider two partitions Y, $Y' \in \mathcal{Y}$. The function $R_{Y'}$ is called* reference zone function,

$$\begin{array}{rccl} R_{Y'}: & Y & \longrightarrow & Y' \\ & y & \longmapsto & R_{Y'}(y) \end{array}$$

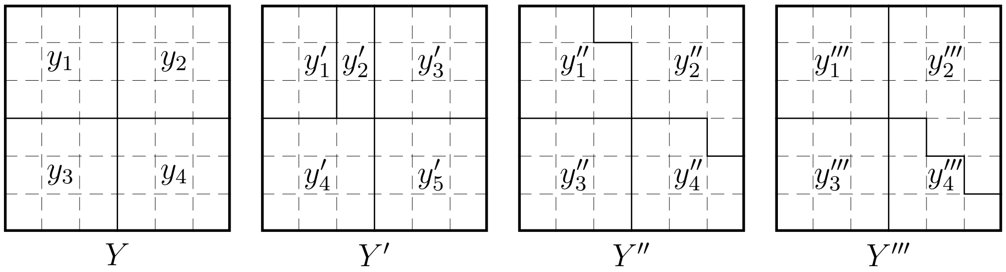


Figure 2: Four different partitions, Y, Y', Y'', and Y'.

1	1	0	1	1	1
1	1	0	1	1	1
1	1	0	1	1	1
1	1	1	1	1	10
1	1	1	1	1	1
1	1	1	1	1	1

P^1

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

P^2

1	1	0	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	0	0
1	1	1	1	1	0
1	1	1	1	1	1

P^3

Figure 3: Three different attributes, P^1, P^2, and P^3

where $R_{Y'}(y) \in Y'$ maximizes $P_{y \cap y'_v}$, $v = 1, \ldots, K'$. And, $R_{Y'}(y)$ is called the reference zone *of y in Y'.*

In other words, $R_{Y'}(y)$ is the zone belonging to Y' that contains the largest quantity of attribute P that is common to y.

Remark 2.1 *It should be noticed that when $P_{y \cap y'_v}$ is maximal for several $y'_v \in Y'$, $R_{Y'}(y)$, is defined arbitrarily, as a zone whose index is minimal among the zones for which $P_{y \cap y'_v}$ is maximal.*

Figure 2 along with Figure 3 illustrates the previous definitions. Zone y'_1 is included in y''_1, whatever the attribute. Considering the attribute P^1, y''_1 is also included in y'_1, *i.e.*, $y''_1 \subseteq_{P^1} y'_1$. In this case, the equality between y'_1 and y''_1, according to P^1, is verified. When considering P^2, the reference zone of y''_2 in Y' is $R_{Y'}(y''_2) = y'_3$. But, this does not occur when taking into account P^1. Note that, in such a case, $R_{Y'}(y''_2) = y'_5$, because $P^1_{y''_2 \cap y'_5} = 10 = \max\limits_{1 \leq u \leq 5} \{P_{y''_2 \cap y'_u}\}$.

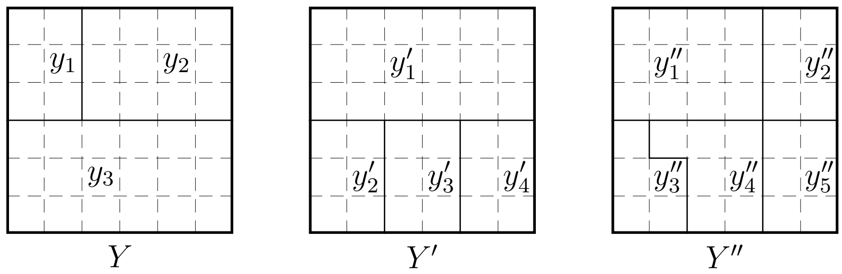


Figure 4: Compatibility example

3 Compatibility index

In this section we consider an index that evaluates the degree of compatibility between two partitions. Two partitions Y and Y' are said to be totally compatible if for any pair of zones $y \in Y$ and $y' \in Y'$ that overlap either y is included in y' or y' is included in y (Y and Y' in Figure 4 represent two totally compatible partitions).

3.1 Definition and structural properties

Definition 3.1 (Total compatibility between two partitions) *Two partitions Y and Y' are* totally compatible, according to the attribute P (*denoted* $Y \equiv_P Y'$), *if for any pair of zones $\{y, y'\} \in Y \times Y'$ such that $P_{y \cap y'} > 0$, one of the following inclusions occurs, $y \subseteq_P y'$ or $y' \subseteq_P y$.*

In Figure 4, Y and Y' are totally compatible, whatever the attribute P. However, the total compatibility between Y and Y'' will depend on the attribute P.

Definition 3.2 (Overlapping pairs) *A pair of zones $\{y, y'\} \in Y \times Y'$ for which $P_{y \cap y'} > 0$ and such that $y \nsubseteq_P y'$ or $y' \nsubseteq_P y$, is called an* overlapping pair.

Remark 3.1 *Two factors should have to contribute for the degradation of the degree of proximity to the total compatibility:*

- *The number of overlapping pairs: The distance to the total compatibility should be greater when there is an increase of the number of the overlapping pairs;*
- *The way the pairs of zones overlap: In Figure 5(a) y' is "almost" included in y. Hence this pair of zones should not contribute too much to the compatibility index. The same holds for Figure 5(c), as the intersection between y and y' is*

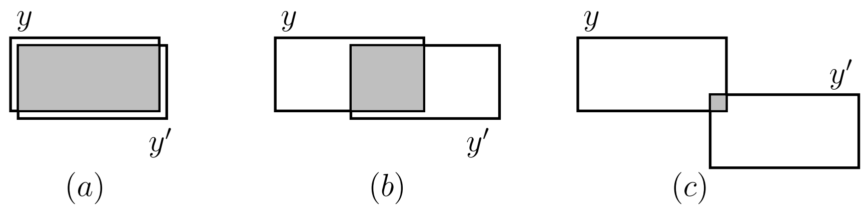


Figure 5: Overlapping pairs of zones

"almost" empty. On the contrary, Figure 5(b) depicts two overlapping zones such that $y \cap y'$, $y \setminus y'$ and $y' \setminus y$ "almost" contain the same quantity of attribute P.

Remark 3.2 *Moreover, we aim at defining a compatibility index $C_P(Y, Y')$ that satisfies the following properties:*

1. Total compatibility*: $Y \equiv_P Y'$ iff $C_P(Y, Y') = 0$;*
2. Idempotence*: $\forall\, Y \in \mathcal{Y}$, $C_P(Y, Y) = 0$;*
3. Symmetry*: $C_P(Y, Y') = C_P(Y', Y)$.*

The meaning of the properties are as follows:

1. The first property means that the compatibility index, $C_P(Y, Y')$, has a maximum value when Y and Y' are totally compatible and only in this case.
2. The second means that any partition Y will be compatible with itself.
3. The last one means that this index must be symmetric.

3.2 Implementation

The proposed implementation for the compatibility index is defined, taking into account, the minimum value between the three elements: $P_{y\setminus y'}$, $P_{y\cap y'}$ and $P_{y'\setminus y}$.

$$C_P(Y, Y') = 1 - \frac{1}{\mathcal{P}} \sum_{y \in Y} \sum_{y' \in Y'} \min\{P_{y\setminus y'}, P_{y\cap y'}, P_{y'\setminus y}\} \tag{1}$$

Considering the examples of Figure 2, the compatibility index between Y and Y'' taking into account two overlapping pairs of zones, $\{y_1, y_2''\}$, and $\{y_4, y_2''\}$, when $P = P^2$. The value of the index will be $C_{P^2}(Y, Y') = 1 - \frac{1}{36}(\min\{8, 1, 9\} + \min\{8, 1, 9\}) = 1 - \frac{2}{36} = \frac{34}{36}$. Concerning the partitions Y' and Y''', there is only one overlapping pair, $\{y_5', y_2'''\}$, then $C_{P^2}(Y', Y''') = 1 - \frac{1}{36}\min\{6, 3, 9\} = \frac{33}{36}$. When $P = P^3$, Y' and Y''' are totally compatible, so $P^3_{y_5' \cap y_2'''} = 0$.

3.3 Analysis of the index

Consider the territory A as a set, where each elementary unit is an element. It is obvious that a partition of the territory A is also a partition of the set A, in terms of *Set Theory*. It is well-known that the set $\Pi = \{y \cap y' \neq \emptyset : \; y \in Y, \; y' \in Y'\}$ constitutes a partition of the set A, where Y and Y' represents two partitions of A. Then,

$$\sum_{\{y,y'\} \in Y \times Y'} P_{y \cap y'} = \mathcal{P}$$

The following proposition shows that the index $C_P(Y, Y')$ is bounded from above and below.

Proposition 3.1 *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, Y, $Y' \in \mathcal{Y}$. The index $C_P(Y, Y') \in [0, 1]$ and the minimum and maximum values for C_P are 0 and 1, respectively.*

Proof. Since, for any $\{y, y'\} \in Y \times Y'$, $\min\{P_{y \setminus y'}, P_{y \cap y'}, P_{y' \setminus y}\} \leq P_{y \cap y'}$, then $\frac{1}{\mathcal{P}} \sum_{y \in Y} \sum_{y' \in Y'} \min\{P_{y \setminus y'}, \; P_{y \cap y'}, P_{y' \setminus y}\} \leq 1$. Thus, $C_P(Y, Y') \geq 0$. Since the attribute P is non-negative, the summation in $C_P(Y, Y')$ is non-negative. Therefore, $C_P(Y, Y') \leq 1$.

Let us now to prove that 0 and 1 are also the minimum and maximum value of $C_P(Y, Y')$. Suppose that Y and Y' are totally compatible. Then, when $P_{y \cap y'} > 0$, one of the inclusions, $y \subseteq_P y'$, $y' \subseteq_P y$, is verified, *i.e.*, when $P_{y \cap y'} > 0$, either $P_{y \setminus y'} = 0$ or $P_{y' \setminus y} = 0$. Thus, $\sum_{y \in Y} \sum_{y' \in Y'} \min\{P_{y \setminus y'}, \; P_{y \cap y'}, P_{y' \setminus y}\} = 0$, and, consequently, $C_P(Y, Y') = 1$. Now, suppose that Y, Y' are two partitions such that, for any pair $\{y, y'\} \in Y \times Y'$, $P_{y \cap y'} = \min\{P_{y \setminus y'}, \; P_{y \cap y'}, P_{y' \setminus y}\}$, see Figure 6 and consider attribute P^2 in Figure 3; it represents the worse case. Then $C_P(Y, Y') = 1 - \frac{1}{\mathcal{P}} \sum_{y \in Y} \sum_{y' \in Y'} P_{y \cap y'} = 1 - \frac{1}{\mathcal{P}} \mathcal{P} = 0$. $\square$

The following proposition states the properties of Section 3.1.

Proposition 3.2 *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, $Y, Y' \in \mathcal{Y}$. The index C_P verifies the following properties:*

1. Total compatibility: $Y \equiv_P Y'$ *iff* $C_P(Y, Y') = 1$;
2. Idempotence: $C_P(Y, Y) = 1$;
3. Symmetry: *If* $C_P(Y, Y') = 1$ *then* $C_P(Y', Y) = 1$.

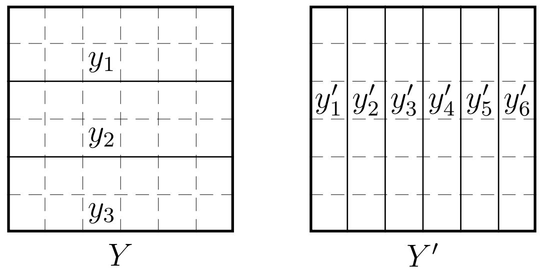


Figure 6: Totally "incompatible" partitions , Y and Y'

Proof.

1. The implication "if $Y \equiv_p Y'$ then $C_P(Y, Y') = 1$", was proved in Proposition 3.1.

   If $C_P(Y, Y') = 1$ then $\sum_{y \in Y} \sum_{y' \in Y'} \min\{P_{y \setminus y'},\ P_{y \cap y'}, P_{y' \setminus y}\} = 0$. Since all of its elements are non-negative, then, for any pair $\{y, y'\} \in Y \times Y'$, $\min\{P_{y \setminus y'},$ $P_{y \cap y'}, P_{y' \setminus y}\} = 0$. Therefore, if $P_{y \cap y'} > 0$ then $P_{y \setminus y'} = 0$ or $P_{y' \setminus y} = 0$, *i.e.*, $y \subseteq_P y'$ or $y' \subseteq_P y$. Thus $Y \equiv_p Y'$.

2. Since for any two zones $y, z \in Y$, when $P_{y \cap z} > 0$ then $y \subseteq_p z$, $z \subseteq_p y$; therefore, $C_P(Y, Y) = 1$.

3. Since the sum, intersection, and the min operator are commutative, then $C_P(Y', Y) = C_P(Y, Y')$. □

4 Inclusion index

In this section we aim at defining an inclusion index, $I_P(Y, Y')$, that measures to which extend the zones of partition Y that are included into the zones of Y'. In other words, we aim at evaluating the degree of "inclusion" of any zone $y \in Y$ into Y'. This degree of inclusion of Y in Y' is grounded, for each zone $y \in Y$, on the extend to which y is included in its reference zone in Y'. The concept of inclusion between two partitions is close to the compatibility but it differs by the fact that it is asymmetric (the compatibility being symmetric).

4.1 Definition and structural properties

Definition 4.1 (Total Inclusion between two partitions) *The partition Y is totally included in Y', according to P (denoted, $Y \subseteq_P Y'$), if $\forall\, y \in Y,\ \exists\, y' \in Y'$ such that $y \subseteq_P y'$.*

In other words, $Y \subseteq_P Y'$ if each zone of Y is totally included in a zone of Y'.

In Figure 2, the partition Y' is totally included in Y, whatever the attribute P. The reverse inclusion, $Y \subseteq_P Y'$ however, is not verified when we consider attribute P^2 (see Figure 3). But, it holds when considering P^1. It should also be remarked that the two partitions Y'' and Y''' are totally included in Y, according to P^3.

Remark 4.1 *We aim at defining an inclusion index that verifies the following properties:*

1. Total inclusion*: $Y \subseteq_P Y'$ iff $I_P(Y, Y') = 1$;*
2. Idempotence*: $\forall\, Y \in \mathcal{Y},\ I_P(Y, Y) = 1$;*
3. Anti-symmetry*: If $I_P(Y, Y') = 1$ and $Y \neq Y'$ then $I_P(Y', Y) < 1$;*
4. Transitivity*: If $I_P(Y, Y') = 1$ and $I_P(Y', Y'') = 1$ then $I_P(Y, Y'') = 1$.*

The interpretation of each property is as follows:

1. The first property means that, $I_P(Y, Y')$, has a maximal value when Y is totaly included in Y' and only in this case.
2. The second property means that any partition Y will be contained in itself.
3. Property 3 means that, when total inclusion between two different partitions, Y and Y', is verified, then the total reverse inclusion is false.
4. The last property means that when total inclusion between two partitions, Y and Y', and also between the second, Y', and the third one, Y'', are verified then the first partition, Y, is completely included in the third one, Y''.

4.2 Implementation

The proposed index to evaluate the inclusion of Y into Y', $I_P(Y, Y')$, measures the inclusion of Y in Y' and it is modelled as follows:

$$I_P(Y, Y') = \frac{1}{\mathcal{P}} \sum_{y \in Y} P_{y \cap R_{Y'}(y)}$$

Thus $I_P(Y, Y')$, is the proportion of the sum, for each zone y, of the quantity P_y that belongs to its reference zone. It is obvious that its upper bound value is equal to 1.

Remark 4.2 *It should be remarked that the value of $I_P(Y, Y')$ is independent of the choice of the reference zone, when for one $y \in Y$, there is more than one zone in Y' with the same maximum value of attribute common to y.*

Consider again Figures 2 and 3. The inclusion index, $I_{P^2}(Y'', Y)$, is $\frac{8+9+9+8}{36} = \frac{34}{36}$, according to P^2. Note that only, y_2'', is not totaly included in some zone of Y; y_2 is its reference zone in Y. When P^1 is considered, the reference zone of y_2'' will change. The overall quantity $P^1_{y_2''}$ is 19. Its largest part, 10 units, belongs to y_4. Obviously, as y_1'', y_3'', and y_4'' are included in y_1, y_3, and y_4, respectively, $I_{P^1}(Y'', Y) = \frac{6+10+9+8}{42} = \frac{33}{42}$. The same value is obtained for $I_{P^1}(Y'', Y')$. It should be remarked that, $y_1'' \subseteq_{P_1} y_1'$.

4.3 Analysis of the index

The following proposition states that $I_P(Y, Y')$ is bounded from above and below.

Proposition 4.1 *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, $Y, Y' \in \mathcal{Y}$. The index $I_P(Y, Y') \in [\frac{1}{n}, 1]$ and the minimum and maximum values for $I_P(Y, Y')$ are $\frac{1}{n}$ and 1, respectively.*

Proof. Since $P_{y \cap R_{Y'}(y)} \leq P_y$, for all $y \in Y$, $\sum_{y \in Y} P_{y \cap R_{Y'}(y)} \leq \sum_{y \in Y} P_y$. Then $I_P(Y, Y') = \frac{1}{\mathcal{P}} \sum_{y \in Y} P_{y \cap R_{Y'}(y)} \leq \frac{1}{\mathcal{P}} \sum_{y \in Y} P_y = 1$. It is also the maximum value of I_P, because when $Y \subseteq_P Y'$, $y \subseteq_P R_{Y'}(y)$, for all $y \in Y$, and, therefore, $P_{y \cap R_{Y'}(y)} = P_y$. Thus, $I_P(Y, Y') = 1$.

For each $y \in Y$, $P_{y \cap R_{Y'}(y)}$ is minimal when the overall amount of its attribute is equally distributed by all the zones of Y'. Thus, for any $y \in Y$, $P_{y \cap y'} = \frac{P_y}{K'}$, for all $y' \in Y'$. Consequently, $P_{y \cap R'_Y(y)} = \frac{P_y}{K'}$. In this case, $I_P(Y, Y') = \frac{1}{\mathcal{P}} \sum_{y \in Y} \frac{P_y}{K'} = \frac{1}{K'} \frac{1}{\mathcal{P}} \mathcal{P} = \frac{1}{K'}$. Therefore, the greater the number of zones in Y', the more degradation of $I_P(Y, Y')$ occurs. The condition, "*the overall amount of attribute from each $y \in Y$ is equally distributed by all the zones of Y'*", imposes an upper bound to the number of zones, K', in Y'. Its maximum possible value for K', is $\frac{n}{K}$, if n is a multiple of K and $p_i = \frac{\mathcal{P}}{n}$, *i.e.*, any elementary unit has the same amount of attribute. Consequently, the maximum value that K' can take is reached when K is minimum, *i.e.*, when Y has only one zone ($K = 1$). Finally, assuming that each elementary unit has the same amount of attribute, $K = 1$ and $K' = n$, the index $I_P(Y, Y')$ reaches its minimal value, $\frac{1}{n}$. $\square$

The following proposition states for the properties of Section 4.1.

Proposition 4.2 *Consider a territory A, composed of n elementary units, an attribute P defined on A, and two partitions, $Y, Y' \in \mathcal{Y}$. The index $I_P(Y, Y')$ verifies the following properties:*

1. Total inclusion: $Y \subseteq_P Y'$ *iff* $I_P(Y, Y') = 1$;
2. Idempotence: $I_P(Y, Y) = 1$;
3. Anti-symmetry: *if* $I_P(Y, Y') = 1$ *and* $Y \neq Y'$ *then* $I_P(Y', Y) < 1$;
4. Transitivity: *if* $I_P(Y, Y') = 1$ *and* $I_P(Y', Y'') = 1$ *then* $I_P(Y, Y'') = 1$.

Proof.

1. The implication "if $Y \subseteq_P Y'$ then $I_P(Y, Y') = 1$" was proved in the Proposition 4.1. Let us now consider the reverse implication. If there is a $y_{u_0} \in Y$ such that $y_{u_0} \nsubseteq_P y'$ for any $y' \in Y'$, *i.e.*, $Y \nsubseteq_P Y'$, then $P_{y_{u_0} \cap R_{Y'}(y_{u_0})} < P_{y_{u_0}}$. Therefore, since $P_{y \cap R_{Y'}(y)} \leq P_y$, $\sum_{y \in Y} P_{y \cap R_{Y'}(y)} < \sum_{y \in Y} P_y = \mathcal{P}$. That is, $I_P(Y, Y') < 1$. Thus, if $I_P(Y, Y') = 1$ then $Y \subseteq_P Y'$.

2. For all $Y \in \mathcal{Y}$, $Y \subseteq_P Y$, then $I_P(Y, Y) = 1$.

3. If $I_P(Y, Y') = 1$ and $Y \neq Y'$ then there is at least one $y'_{v_0} \in Y'$ for which, there are $y_{u_1}, \ldots, y_{u_{K_{v_0}}} \in Y$ such that $y'_{v_0} =_P \bigcup_{u=1}^{K_{v_0}} y_{u_i}$, and $P_{y'_{v_0} \cap y_{u_i}} > 0$, for more than one y_{u_i}. Consequently, $P_{y'_{v_0} \cap R_Y(y'_{v_0})} < P_{y'_{v_0}}$. Then $I_P(Y', Y) < 1$

4. If $I_P(Y, Y') = 1$ and $I_P(Y', Y'') = 1$, then $Y \subseteq_P Y'$ and $Y' \subseteq_P Y''$. Therefore, for each $y \in Y$ there is $y' \in Y'$ such that $y \subseteq_P y'$ and for which there is $y'' \in Y''$ that verifies, $y' \subseteq_P y''$. We will check now that $y \subseteq_P y''$. Since $y \subseteq y'$ then $P_{y \cap y'} > 0$, *i.e.*, $\exists\, i \in y \cap y'$ such that $p_i > 0$. Since $y' \subseteq y''$ then, also, $i \in y''$. So $P_{y \cap y''} > 0$. Let us suppose that there is an elementary unit $i \in y$

such that $i \notin y''$. By *reductio ad absurdum*, suppose that $p_i \neq 0$. Therefore, $i \in y'$ because $y \subseteq_P y'$. So, by $y' \subseteq_P y''$, $p_i = 0$. Contradiction! This means that $p_i = 0$. Then $y \subseteq_P y''$ and, consequently, $Y \subseteq_P Y''$, *i.e.*, $I_P(Y, Y'') = 1.\square$

5 Distance index

In this section we aim at defining a distance index, $D_P(Y, Y')$, that evaluates "how different" two partitions can be. The attribute P is considered here as a strictly positive function, *i.e.*, $p_i > 0$, for all $i \in V$.

5.1 Definition and structural properties

Definition 5.1 (Equality between two partitions) *Two partitions $Y, Y' \in \mathcal{Y}$ are equal, according to attribute P (denoted $Y \approx_P Y'$), if both inclusions $Y \subseteq_P Y'$ and $Y' \subseteq_P Y$ hold.*

Remark 5.1 *Note that the equality between partition $\approx_P$ refers to the attribute P. For example, in Figure 2 the partitions Y and Y' are equal according to P^3 (see also Figure 3).*

Remark 5.2 *We aim at defining a distance $D_P(Y, Y')$ in $\mathcal{Y}$ that fulfills the metric properties. Consider the following three partitions $Y, Y', Y'' \in \mathcal{Y}$,*

1. $D_P(Y, Y') = 0$ *iff* $Y \approx_P Y'$;
2. *Symmetry:* $D_P(Y, Y') = D_P(Y', Y)$;
3. *Triangular inequality:* $D_P(Y, Y'') \leq D_P(Y, Y') + D_P(Y', Y'')$.

5.2 Implementation

The proposed implementation for the distance index is defined taking into account all the edges $\{i, j\} \in E$, corresponding to the border zones in one and only one of the partitions.

Consider the following notation:

- $I_Y = \{\{i, j\} \in E : \exists\, y \in Y,\, i, j \in y\}$;
- $B_Y = \{\{i, j\} \in E : \forall\, y \in Y,\, i, j \notin y\}$;

where, for any $Y \in \mathcal{Y}$, I_Y represents the set of edges that are included into some zone, and B_Y represents the set of all edges corresponding to border zones. It should be noticed that $I_Y \cup B_Y = E$ and $I_Y \cap B_Y = \emptyset$.

Consider now, the set $IB_{YY'} \subseteq E$, defined as follows:

$$IB_{YY'} = B_Y \cap I_{Y'} \cup B_{Y'} \cap I_Y$$

where $Y, Y' \in \mathcal{Y}$. In other words, the set $IB_{YY'}$, represents the set of edges for which their adjacent vertices belong to the same zone in one of the partitions and pertain to different zones in the second one.

We will define a distance D_P between Y and Y', according to an attribute P, as follows:

$$D_P(Y, Y') = \frac{1}{\Delta} \sum_{e \in IB_{YY'}} \delta_e \qquad (2)$$

where, for each edge $e = \{i, j\} \in E$, $\delta_e = \min\{p_i, p_j\}$ and $\Delta = \sum_{e \in E} \delta_e$.

The distance, $D_P(Y, Y')$, between Y and Y', represented in Figure 2, and according P^1, is equal to zero. Note that the set $IB_{YY'}$ has three edges, the edges corresponding to the border between y'_1 and y'_2, however, the values δ_e are equal to zero, so $P^1_{y'_2} = 0$. Considering now Y and Y'', and also P^1, the set $IB_{YY''}$ has six edges. Three of them has the value δ_e equal to zero and for the others three the value is equal to one. Therefore $D_{P^1}(Y, Y'') = \frac{1}{51}(0 + 0 + 0 + 1 + 1 + 1) = \frac{3}{51}$.

5.3 Analysis of the index

Proposition 5.1 *Consider a territory A, composed of n elementary units, an attribute P defined on A, and Y, $Y' \in \mathcal{Y}$. The index $D_P(Y, Y') \in [0, 1]$ and the minimum and maximum values for D_P are 0 and 1, respectively.*

Proof. Since that $p_i \geq 0 \ \forall i \in V$, then $\delta_e \geq 0$. Consequently, $D_P(Y, Y') \geq 0$. Since $IB_{YY'} \subseteq E$, then $\sum_{e \in IB_{YY'}} \delta_e \leq \Delta$. Therefore, $D_P(Y, Y') \leq 1$.

Let us to prove that 0 and 1 are also the minimum and maximum value of $D_P(Y, Y')$, respectively. Obviously, $IB_{YY} = \emptyset$, then $D_P(Y, Y) = 0$. Suppose now that Y has only one zone and Y' has n zones (see Figure 7). Obviously, $IB_{YY'} = E$. Therefore $D_P(Y, Y') = 1$. □

Let us now to prove the properties presented in Section 5.1.

Proposition 5.2 *The operator D_P, defined in (2) is a "true" distance,* i.e., *D_P verifies the properties:*

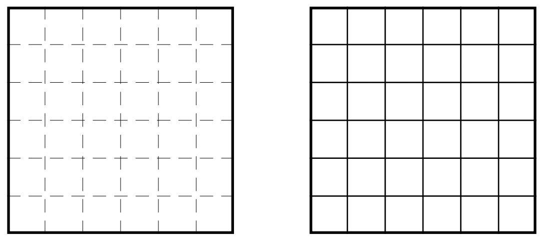


Figure 7: Trivial partitions

1. *$D_P(Y, Y') = 0$ iff $Y \approx_P Y'$;*

2. *Symmetry: $D_P(Y, Y') = D_P(Y', Y)$;*

3. *Triangular inequality: $D_P(Y, Y'') \leq D_P(Y, Y') + D_P(Y', Y'')$.*

Proof.

1. ($\Rightarrow$) Suppose that $Y \not\approx_P Y'$, *i.e.*, $Y \not\subseteq_P Y'$ or $Y' \not\subseteq_P Y$. Without loss of generality, let us suppose that $Y \not\subseteq_P Y'$, *i.e.*, $\exists\, y_0 \in Y : \forall\, y' \in Y'$, $y_0 \not\subseteq_P y'$. Thus, there is at least one $y'_0 \in Y'$, such that $P_{y \cap y'_0} > 0$ and $\sum_{i \in y_0 \setminus y'_0} p_i > 0$. Therefore, because each zone is contiguous, there is at least an edge $e = \{i, j\}$, such that, $i \in y_0 \setminus y'_0$ and $j \in y_0 \cap y'_0$, *i.e.*, $e \in IB_{YY'}$. Since the attribute is strictly positive then $D_P(Y, Y') > 0$.
($\Leftarrow$) If $D_P(Y, Y') > 0$ then there is $e = \{i, j\} \in IB_{YY'}$, such that, $\min\{p_i, p_j\} > 0$. Suppose that $e \in B_Y \cap I_{Y'}$. Then $\exists\, y'_0 \in Y' : i, j \in y'_0$ and $\forall\, y \in Y$, $i, j \notin Y$. Thus $y'_0 \not\subseteq_P y$ for all $y \in Y$, *i.e.*, $Y' \not\subseteq_P Y$. Consequently, $Y \not\approx_P Y'$.

2. Since $IB_{YY'} = IB_{Y'Y}$, then $D_P(Y, Y') = D_P(Y', Y)$.

3. It is obvious that if $IB_{YY''} \subseteq IB_{YY'} \cup IB_{Y'Y''}$ then $D_P(Y, Y'') \leq D_P(Y, Y') + D_P(Y', Y'')$. Therefore, let us to prove that $IB_{YY''} \subseteq IB_{YY'} \cup IB_{Y'Y''}$. Consider $e \in IB_{YY''}$. By the definition of $IB_{YY''}$, either

$$e \in B_Y \cap I_{Y''} \tag{3}$$

or

$$e \in B_{Y''} \cap I_Y \tag{4}$$

Concerning the partition Y', either $e \in I_{Y'}$ or $e \in B_{Y'}$. In the first case, if (3) is true then $e \in IB_{YY'}$. Otherwise, by (4), $e \in IB_{Y'Y''}$. Similarly, in the second

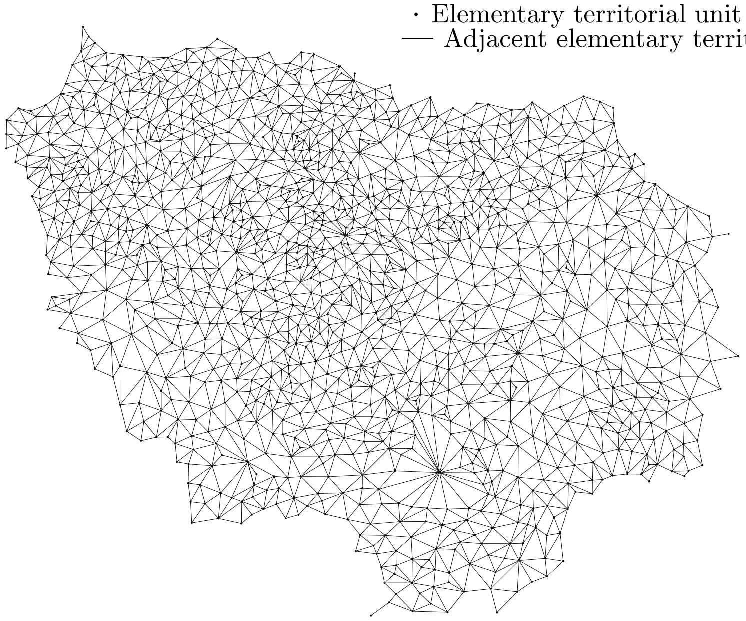


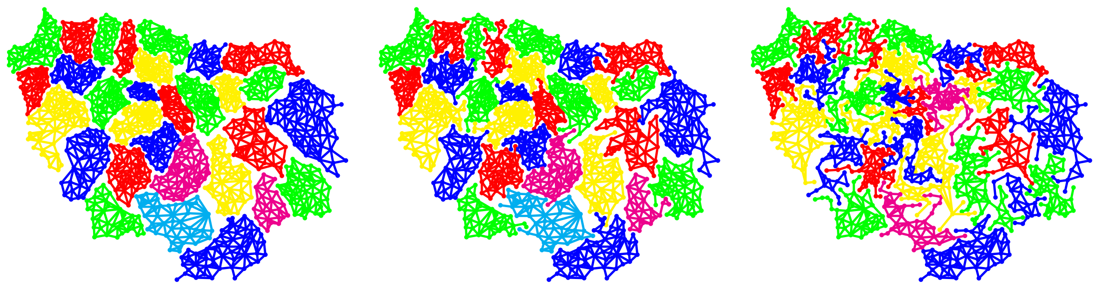
Figure 8: The contiguity graph of Paris region

case ($e \in B_{Y'}$), if (3) is true then $e \in IB_{Y'Y''}$. Otherwise, by (4), $e \in IB_{YY'}$. Therefore, in all the possibilities, $e \in IB_{YY'} \cup IB_{Y'Y''}$. $\square$

In this section it was necessary to consider P as being a strictly positive function. Without this constraint Property 1 of Proposition 5.2 is not valid. This restriction does not represent a considerable loss of applicability in real-world problems. Therefore, in a large number of cases the attribute of a territory is represented by a strictly positive number. If the implementation of distance suggested in 2 seems inadequate, it can be modified, without changing the properties proven in this section. This alteration passes through the redefinition of the sets I_Y and B_Y as follows.

- $I'_Y = \{\{i,j\} \in V \times V : \exists\, y \in Y, \, i,j \in y\}$;
- $B'_Y = \{\{i,j\} \in V \times V : \forall\, y \in Y, \, i,j \notin y\}$.

That is, I'_Y e B'_Y, are now subsets of pairs of vertices. The set $IB_{YY'}$ keeps the same definition. This fact increases the effort of calculation of the distance index. The number of elementary operations is bounded by $\mathcal{O}(n^2)$. With the previous definition the calculation of $D_P(Y,Y')$ is done in $\mathcal{O}(n)$ elementary operations.



(a) Initial partition (b) After 100 EPs (c) After 500 EPs

Figure 9: Successive elementary perturbations (EP's) of a partition

6 Numerical experiments and the behavior of the indices

In this section, we present some numerical experiments that aim at investigating the behavior of the three indices. We consider data concerning the territory of the Paris region, composed of 1300 elementary territorial units (see Figure 8, the contiguity graph $G = (V, E)$ where $|V| = 1300$ and $|E| = 3719$). The attribute of territorial units considered in the experiment is the "working population" ($P =$ working population).

In order to analyze the behavior of the indices, we consider two partitions: a *root* partition Y^R and a *current* one, Y. In the experiments, Y^R remains the same, while Y is progressively modified through successive elementary perturbations. An *elementary perturbation* (EP) consists of moving an elementary unit between two neighboring zones. Figure 9 shows how successive elementary perturbations modify an initial partition formed by 30 zones (a), after 100 EPs (b), and after 500 EPs (c).

6.1 Compatibility index

We consider an initial pair of partitions (Y^R, Y) which fulfill the property of total compatibility ($Y^R \equiv_P Y$ hence $C_P(Y^R, Y) = 1$). When applying α successive elementary perturbations to Y lead to Y^α. The compatibility index between Y^R and Y^α was computed for 100 randomly generated instances of Y^α.

The results are provided in Figure 10. It depicts how the value of $C_P(Y^R, Y^\alpha)$ evolve when $\alpha = 100, 200, \ldots, 1000$. As expected, the index $C_P(Y^R, Y^\alpha)$ decreases as α increases.

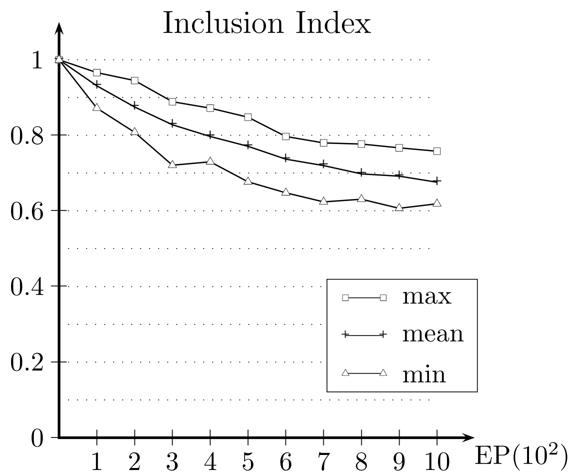


Figure 10: Behavior of the inclusion and compatibility indices

6.2 Inclusion index

We consider an initial pair of partitions (Y^R, Y) which fulfills the property of total inclusion ($Y^R \subseteq_P Y$, hence $I_P(Y^R, Y) = 1$). When applying α successive elementary perturbations to Y lead to a partition denoted Y^α. The inclusion index between Y^R and Y^α was computed for 100 randomly generated instances of Y^α.

The results are provided in Figure 10. It depicts how the value of $I_P(Y^R, Y^\alpha)$ evolve when $\alpha = 100, 200, \ldots, 1000$. As expected, the index $I_P(Y^R, Y^\alpha)$ decreases as α increases. Moreover, the observed minimal value for $I_P(Y^R, Y^\alpha)$ was 0.619, which is far from the minimal possible value for $I_P(Y, Y')$. But, it should be remarked that the minimum corresponds to very specific cases (for example Y being composed of 1 zone and the zones of Y' being the elementary units).

6.3 Distance index

We consider the pair of partitions (Y^R,Y^R) which obviously verifies $D_P(Y^R, Y^R) = 0$. When applying α successive elementary perturbations to Y^R leads to Y^{R^α}. The distance index between Y^R and Y^{R^α} was computed for 100 randomly generated instances of Y^{R^α}.

The results are provided in Figure 11. It depicts how the value of $D_P(Y^R, Y^{R^\alpha})$ evolve when $\alpha = 100, 200, \ldots, 1000$. As expected, the index $D_P(Y^R, Y^{R^\alpha})$ increases as α increases.

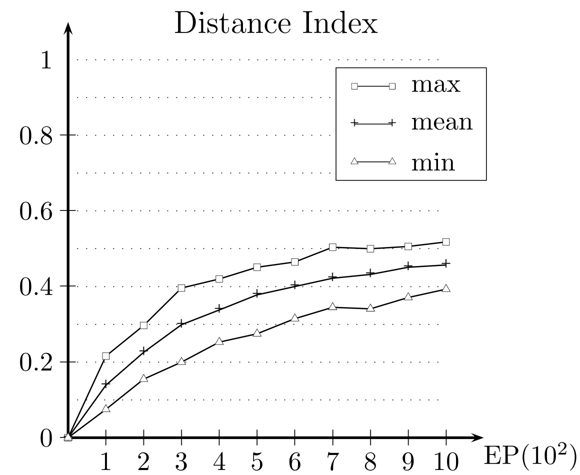


Figure 11: Behavior of the distance index

7 On the use of the comparison indices in managerial problems

We have presented in the previous three sections the indices that capture three different ways by which territory partitions can be compared: compatibility, inclusion, and distance. In this section, we illustrate the potential interest of these indices in management problems involving territory partitions.

7.1 Compatibility index

Territory partitions are frequently used in the field of salesperson management in which a commercial zone is assigned to each salesperson (see for example [6, 17, 8]). Consider a company that commercializes products grouped into several ranges, salespersons being specialized in a specific range of product. Hence, there exists a sales territory partition for each range of product. Obviously, clients can buy products from different ranges. As the demand is not geographically homogenous among product ranges, the sales territory partitions do not usually match.

Consider two ranges of products A and B and the corresponding partitions Y^A and Y^B. In order to optimize customer relations, Y^A and Y^B should be defined in a way such that a vendor share clients with a limited number of vendors in the other team. More precisely, the commercial zones should be defined so that a vendor of the range A should share clients either with one vendor of the range B, or with several vendors of the range B who share clients only with him/her.

This requires that any zone of Y^A is either totally included into a zone of Y^B, or corresponds to the union of zones of Y^B. Such a property perfectly matches the concept of total compatibility between two partitions (see Section 3), when the attribute considered is the number of clients in each territorial unit. Hence the compatibility index $C_P(Y^A, Y^B)$ appears to be useful in this context to evaluate to which extend partitions Y^A and Y^B optimizes customer relation management between ranges A and B.

7.2 Inclusion index

Let us illustrate here the interest of the inclusion index in the context of the pricing of public transportation. The pricing of the public transportation in the Paris region (France) is grounded on the definition of pricing zones. A reform of this pricing system has been undertaken by the STIF: the Paris region transportation authority (see [13]). The current partition in zones consists of concentric rings (the Paris city being the center) is considered as unsatisfactory as it no longer corresponds to travel patterns or needs. The STIF wishes to define a new partition in which the zones are geographical autonomous entities with respect to transportation. Such a partition is to be used to ground the pricing system.

In the definition of this new pricing partition, the STIF considers an existing partition: the "school map". In the French educational system, the "school map" (see [5]) defines to which high school a student should be assigned: hence each student should be sent to the school associated to his/her zone of residence. Such a school map allows to plan in which school to open/close teaching positions according to the demographic evolution of the corresponding zone. The size of the zones in this school map refers to the dimension of the associated high school and the density of population.

An important quality for a new transport pricing partition concerns its ability to take into account the "school map". Ideally, each student should be able to go to the school within the same pricing zone. This requires that the pricing partition should be included in the school partition denoted Y^{sch}. Hence, in order to evaluate and compare alternative pricing partitions $Y^1, Y^2, ...$, it is relevant to consider the inclusion index $I_P(Y^i, Y^{sch})$, $i = 1, 2, ...$ In this way, we can seek for a partition that "minimizes" the student journeys between different pricing zones.

7.3 Distance index

A large proportion of research dealing with territorial partitions has been devoted to political districting (e.g. [2, 9, 11, 12]). In modern democracies, the representatives in the parliaments represent voters attached to an electoral district, hence defining an electoral partition of the territory. Basic democratic principles impose that each district should contain approximately the same number of voters.

Moreover, the demographic evolution of populations imposes to revise regularly political districts. Obviously, such a revision is a highly sensitive issue. Namely, electors would not understand that such a revision would lead to drastically different district, and political manipulation would be suspected. Moreover, candidates are involved in the political life of their respective district and a complete change in the districts would run counter local political debate.

Therefore, when revising the electoral partition, the new partition should be as close as possible to the previous one. A criterion to be minimized can be the distance (as defined in Section 5) with respect to the number of voters.

8 Conclusion and avenues for future research

Over the last years there was a tremendous growth on the use of models and software for partitioning a territory into zones. The following examples belong perhaps to the most representative areas covered by this type of problems: electoral systems, salesperson teams planning and management, and school systems. This increasing of models and attention devoted to such a kind of problems brought by itself the need of comparing two different partitions of the same territory along with its translation into numerical figures. In political districting problems is very frequent to find as a measure of comparing two partitions a criterion that consists of minimizing the "differences" between the existing partition and a new proposal.

This research represents an initial attempt to characterize the comparison indices in districting problems. Specifically, we identified three main classes: *compatibility*, *inclusion*, and *distance*. We strongly believe that this classification can incorporate any kind of measure. The inclusion class covers all the measures devoted to evaluate the degree of fineness of a given partition with respect to a second one. The compatibility class incorporates all the measures that checks if each zone of the first partition results from a group of zones of the second one, or if each zone along with the remaining zones of the same partition define a single zone of the second partition. Finally, the distance class comprises all the measures applied to evaluate any difference between two partitions.

For each class we provided a set of elementary properties related to the very nature of the respective class. We also suggested a possible implementation for each class. Each one is analyzed in view of the expected properties.

The concepts and measures introduced in this paper do not consider a territory by itself, but an attribute associated to each elementary unit according to the problem we are dealing with. In this way it was developed the abstraction of each measures and the universe of applicability

We also implemented the measures on a real-world network and after analyzing the results we could demonstrate the good performance of those implementations. The experiments dealt with the progressive degradation of the similarity between two partitions and the consequent evaluation of such an effect on the value of each measure considered.

An interesting avenue for future research consists of increasing the flexibility of the indices in view of a better adaptation to different realities. For such propose, it is possible to consider a set of parameters to approximate each measure towards the major particularities of the different problems. We also believe that it is possible to generalize the three indices for more than one attribute.

Acknowledgements

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