A primal-dual algorithm for bi-criteria network flow problems

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Abstract

In this paper we develop a primal-dual simplex algorithm for the biobjective linear minimum cost network flow problem. This algorithm improves the general primal-dual simplex algorithm for multiobjective linear programs by [3]. We illustrate the algorithm with an example and provide numerical results.

Keywords: *Bi-criteria network flow problem, Primal-dual simplex algorithm.*

1 Introduction

Multiobjective linear programs have been studied for more than 40 years. Extensions of the simplex algorithm to deal with multiple objectives have been proposed by various authors. Algorithms to solve multiobjective linear programs in objective space are motivated by the fact that the dimension of the objective space is usually much smaller than the one of the decision space, and therefore the number of nondominated extreme points in objective space is much smaller than the number of efficient basic feasible solutions in decision space. After the discovery of interior point algorithms to solve linear programs in polynomial time, efforts to apply such methods to deal with multiple objectives are evident. For more information see the references in [5].

In single objective linear programming primal-dual simplex algorithms have proven to be very efficient for several classes of single objective linear programming problems, in particular those related to network optimization problems. Such efficient algorithms include the Hungarian method for assignment problems, the augmenting path method for minimum cost flow problems, Dijsktra's algorithm for shortest path problems, etc. (see, e.g., [1]).

A primal-dual simplex algorithm has only recently been published in [4]. In this paper we apply that algorithm to the special case of bicriteria linear network flow problems, and propose an improvement of the original algorithm.

2 Definitions and Notation

In this section we introduce necessary definitions and notation from graph theory, network flows, convex and polyhedral sets, and multi-criteria optimization that we use in this paper. For an in depth introduction to these subjects see [1] and [3]).

Let $\mathcal{G} = (\mathcal{S}, \mathcal{A})$ denote a *directed* and *connected graph*, where $\mathcal{S}$ is a finite set of *nodes* or *vertices* with cardinality $|\mathcal{S}| = m$, and $\mathcal{A}$ is a collection of ordered pairs of elements of $\mathcal{S}$ called *arcs*, with cardinality $|\mathcal{A}| = n$.

A graph $\mathcal{G}' = (\mathcal{S}', \mathcal{A}')$ is called a *subgraph* of $\mathcal{G} = (\mathcal{S}, \mathcal{A})$ if $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathcal{A}' \subseteq \mathcal{A}$. It is a *spanning subgraph* of $\mathcal{G}$ if $\mathcal{S}' = \mathcal{S}$. A *path* $\mathcal{P}$ is a sequence of vertices and arcs, $i_1 - a_1 - i_2 - a_2 - \ldots - i_{s-1} - a_{s-1} - i_s$, without repetition of vertices and where for $1 \leq k \leq s-1$ either $a_k = (i_k, i_{k+1}) \in \mathcal{A}$, or $a_k = (i_{k+1}, i_k) \in \mathcal{A}$. A *directed path* is a path without backwards arcs. A *cycle* $\mathcal{C}$ is a closed path where the only repeated vertex is the start and the end point, which coincide. A *directed cycle* is a closed directed path. If a graph $\mathcal{G}$ contains paths linking any two different vertices of $\mathcal{G}$, the graph is called *connected*. A *tree* $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is a connected subgraph without cycles where $\mathcal{V} \subseteq \mathcal{S}$ and $\mathcal{E} \subseteq \mathcal{A}$. A tree $\mathcal{T}$ is called a *spanning tree* if it spans the set of vertices $\mathcal{S}$ of $\mathcal{G}$, i.e. $\mathcal{V} = \mathcal{S}$. A spanning tree is denoted by $\mathcal{T} = (\mathcal{S}, \mathcal{E})$. Consider (k,l) a given arc belonging to the set $\mathcal{A}$ but not to $\mathcal{E}$. Then, there is a unique cycle $\mathcal{C}$ if the arc (k,l) is added to $\mathcal{E}$. The direction of $\mathcal{C}$ is defined to be the same as (k,l). The arcs of a cycle $\mathcal{C}$ can be partitioned into two subsets by distinguishing the arcs having the same direction as $\mathcal{C}$ from the arcs in the opposite direction. The collection of all possible cycles of this type is called *fundamental cycle basis* of $\mathcal{G}$.

A directed graph with numerical values assigned to its vertices and/or arcs is called *network*. Let $\mathcal{N} = (\mathcal{G}, c, l, u, b)$ be a network with a *"cost"* c_{ij}, a *lower bound* l_{ij} and an *upper bound* or *capacity* u_{ij} associated with every arc $(i,j) \in \mathcal{A}$. The numerical values l_{ij} and u_{ij}, respectively, denote the minimum and the maximum amount that must flow on the arc (i,j). Finally, let x_{ij} denote the *amount of flow* on the arc (i,j). A numerical value b_i is also associated with each vertex $i \in \mathcal{S}$ denoting its *supply* (if $b_i > 0$) or its *demand* (if $b_i < 0$). A vertex with $b_i = 0$ is called a *transshipment vertex*.

The minimum cost network flow problem is a linear programming problem on a network $\mathcal{N}$ formulated as follows

$$\begin{array}{lll} \text{minimize} & f(x) = \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij} & \\ \text{subject to:} & \sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k \qquad \forall k \in \mathcal{S} & \\ & l_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j) \in \mathcal{A} & \end{array} \tag{1}$$

or in matrix notation

$$\begin{array}{ll} \text{minimize} & f(x) = c^T x \\ \text{subject to:} & Ax = b, \\ & l \le x \le u \end{array} \tag{2}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, $c \in \mathbb{N}_0^n$, and $l, u \in \mathbb{N}_0^n$ are the cost and capacity vectors, respectively, $b \in \mathbb{N}_0^m$ is the right-hand-side vector and A the node-arc incidence matrix. Each column (i,j) of A contains exactly two nonzero coefficients: $+1$ in row i, and -1 in row j. The special structure of node-arc incidence matrices is explouted in the network simplex algorithm, that solves the minimum cost flow problem faster and with less resource consumption than standard linear programming algorithms. Moreover, node-arc incidence matrices are totally unimodular and as long as b, l, and u are integer vectors, all extreme points of the feaible set have integer coordinates (see [1] and [2]).

The dual linear programming problem associated with (1) is

$$\begin{array}{ll} \text{maximize} & \sum_{k \in \mathcal{S}} b_k \pi_k + \sum_{(i,j) \in \mathcal{A}} (l_{ij}\gamma_{ij} - u_{ij}\mu_{ij}) \\ \text{subject to:} & \pi_i - \pi_j + \gamma_{ij} - \mu_{ij} = c_{ij} \qquad \forall (i,j) \in \mathcal{A} \\ & \gamma_{ij}, \mu_{ij} \geq 0 \qquad \forall (i,j) \in \mathcal{A} \\ & \pi_k \text{ free} \qquad \forall k \in \mathcal{S} \end{array} \tag{3}$$

where $\pi = (\pi_1, \cdots, \pi_i, \cdots, \pi_m)$, $i \in \mathcal{S}$ is the vector of dual variables associated with constraints (1), $\gamma = (\gamma_{i_1j_1}, \cdots, \gamma_{ij}, \cdots, \gamma_{i_nj_n})$, $(i,j) \in \mathcal{A}$ is the vector of dual variables associated with the constraints $x_{ij} \geq l_{ij}$ and $\mu = (\mu_{i_1j_1}, \cdots, \mu_{ij}, \cdots, \mu_{i_nj_n})$, $(i,j) \in \mathcal{A}$ is the vector of dual variables associated with the constraints $x_{ij} \leq u_{ij}$. In matrix notation the dual is

$$\begin{array}{ll} \text{maximize} & b^T\pi + l^T\gamma - u^T\mu \\ \text{subject to:} & A^T\pi + \gamma - \mu = c \\ & \gamma, \mu \geq 0 \\ & \pi \text{ free} \end{array} \tag{4}$$

It can be assumed without loss of generality that $\gamma_{ij} \cdot \mu_{ij} = 0$ for all $(i,j) \in \mathcal{A}$. If both γ_{ij} and μ_{ij} are greater than zero for some arc $(i,j) \in \mathcal{A}$ let $\varepsilon_{ij} = \min\{\gamma_{ij}, \mu_{ij}\}$ and define

$$\begin{array}{rcl} \gamma'_{ij} & = & \gamma_{ij} - \varepsilon_{ij} \quad \geq \quad 0 \\ \mu'_{ij} & = & \mu_{ij} - \varepsilon_{ij} \geq 0. \end{array}$$

Problem (3) remains dual feasible if variables γ_{ij} and μ_{ij} are replaced by $\gamma'_{ij} + \varepsilon_{ij}$ and $\mu'_{ij} + \varepsilon_{ij}$, since $\gamma_{ij} - \mu_{ij} = \gamma'_{ij} - \mu'_{ij}$.

The reduced cost of the arc (i,j) is defined as $\bar{c}_{ij} = c_{ij} - \pi_i + \pi_j - \gamma_{ij} + \mu_{ij}$.

Consider primal and dual feasible solutions x and (π, γ, μ) to the linear programs (1) and (3), respectively. Then they are both optimal to their respective problems if and only if they have the same objective value, i.e.

$$\sum_{(i,j) \in \mathcal{A}} c_{ij}x_{ij} = \sum_{k \in \mathcal{S}} b_k\pi_k + \sum_{(i,j) \in \mathcal{A}} (l_{ij}\gamma_{ij} - u_{ij}\mu_{ij}).$$

Theorem 2.1 (Complementary Slackness Theorem) *Let x^* and (π^*, γ^*, μ^*) be any feasible solutions to the primal (1) and dual (3) problems, respectively. Then they are optimal if and only if*

$$\gamma^*_{ij}(x^*_{ij} - l_{ij}) = 0 \quad \forall (i,j) \in \mathcal{A}$$

and

$$\mu^*_{ij}(x^*_{ij} - u_{ij}) = 0 \quad \forall (i,j) \in \mathcal{A}.$$

This theorem indicates that at least one of the two terms in each expression must be zero. In particular

$$\begin{aligned} x_{ij}^* > l_{ij} &\Rightarrow \gamma_{ij}^* = 0 \\ \gamma_{ij}^* > 0 &\Rightarrow x_{ij}^* = l_{ij} \end{aligned} \tag{5}$$

$$\begin{aligned} x_{ij}^* < u_{ij} &\Rightarrow \mu_{ij}^* = 0 \\ \mu_{ij}^* > 0 &\Rightarrow x_{ij}^* = u_{ij}. \end{aligned} \tag{6}$$

Other statements can also be deduced. If the reduced cost of the arc (i, j) is not equal to zero then the flow in the arc is either l_{ij} or u_{ij}. First we show that

$$\pi_i^* - \pi_j^* < c_{ij} \Rightarrow x_{ij}^* = l_{ij}. \tag{7}$$

We have

$$\begin{aligned} \pi_i^* - \pi_j^* < c_{ij} \quad &\Leftrightarrow \quad \gamma_{ij}^* - \mu_{ij}^* > 0 \\ &\Leftrightarrow \quad \gamma_{ij}^* > \mu_{ij}^* \geq 0 \\ &\Rightarrow \quad \gamma_{ij}^* > 0 \\ &\Rightarrow \quad x_{ij}^* = l_{ij}. \end{aligned}$$

Furthermore

$$\begin{aligned} \pi_i - \pi_j > c_{ij} \quad &\Leftrightarrow \quad \gamma_{ij}^* - \mu_{ij}^* < 0 \\ &\Leftrightarrow \quad \gamma_{ij}^* < \mu_{ij}^* \\ &\Rightarrow \quad \mu_{ij}^* > 0 \\ &\Rightarrow \quad x_{ij}^* = u_{ij} \end{aligned}$$

which shows that

$$\pi_i^* - \pi_j^* > c_{ij} \Rightarrow x_{ij}^* = u_{ij}. \tag{8}$$

Moreover,

$$l_{ij} < x_{ij}^* < u_{ij} \Rightarrow \pi_i^* - \pi_j^* = c_{ij}^*. \tag{9}$$

The bi-criteria linear network flow problem can be stated as follows:

$$\begin{array}{ll} \text{minimize} & f_1(x) = \sum\limits_{(i,j)\in\mathcal{A}} c_{ij}^1 x_{ij} = (c^1)^T x \\ \text{minimize} & f_2(x) = \sum\limits_{(i,j)\in\mathcal{A}} c_{ij}^2 x_{ij} = (c^2)^T x \\ \text{subject to:} & \sum\limits_{(k,j)\in\mathcal{A}} x_{kj} - \sum\limits_{(i,k)\in\mathcal{A}} x_{ik} = b_k \qquad \forall k \in \mathcal{S} \\ & l_{ij} \leq x_{ij} \leq u_{ij} \qquad \forall (i,j) \in \mathcal{A} \end{array} \tag{10}$$

Let

$$X = \Big\{ x \in \mathbb{R}^n : \ x = (x_{i_1 j_1}, x_{i_2 j_2}, \cdots, x_{ij}, \cdots, x_{i_n j_n}), (i,j) \in \mathcal{A}, \\ \sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k, \forall k \in \mathcal{S} \text{ and } l_{ij} \le x_{ij} \le u_{ij} \Big\}$$

and

$$Y = \Big\{ (y_1, y_2) \in \mathbb{R}^2 : y_1 = \sum_{(i,j)\in\mathcal{A}} c^1_{ij} x_{ij}, y_2 = \sum_{(i,j)\in\mathcal{A}} c^2_{ij} x_{ij}, \text{ and } x \in X \Big\}.$$

X and Y are the sets of *feasible solutions* in the *decision space*, $\mathbb{R}^n$, and in the *criterion space*, $\mathbb{R}^2$, respectively.

Definition 2.1 (Efficient solution) *A feasible solution $x \in X$ is* efficient *iff there does not exist another feasible solution $x' \in X$ such that $y' = f(x') \le y = f(x)$ and $y' \neq y$. The set of all efficient solutions will be denoted by X_E.*

Definition 2.2 (Nondominated point) *A point $y \in Y$ is nondominated if there is some efficient solution $x \in X_E$ such that $y = f(x)$.*

A fundamental result in multiobjective linear programming, and in particular the biobjective network flow problem (10) is stated in the following theorem, see e.g. [8].

Theorem 2.2 *A feasible solution $x \in X$ is efficient if and only if there exists a $\lambda \in]0,1[$ such that x minimizes the weighted-sum linear program*

$$\min\{(\lambda(c^1 - c^2) + c^2)^T x : x \in X\}.$$

3 A Primal-dual Network Flow Algortihm

In this section a primal-dual simplex algorithm for finding an optimal solution for the minimum cost network flow (1) with lower and upper bounds is described.

Consider the primal (1) and dual (3) formulation of the minimum cost network flow problem. Let the vector (π', γ', μ') be an initial dual feasible solution. The components, π'_k, γ'_{ij}, and μ'_{ij} of π', γ', and μ', respectively, are such that $\pi'_i - \pi'_j + \gamma'_{ij} - \mu'_{ij} = c_{ij}$ for all $(i,j) \in \mathcal{A}$.

By the complementary slackness property for optimality it is necessary that, if $\pi'_i - \pi'_j \neq c_{ij}$, then $x_{ij} = l_{ij}$ or $x_{ij} = u_{ij}$. Let $\mathcal{A}^=$ be the set of arcs such that

$\pi'_i - \pi'_j = c_{ij}$, that is, the set of arcs with reduced cost zero. For each arc not in $\mathcal{A}^=$ set x_{ij} to l_{ij} or u_{ij} in (1), depending on the sign of $c_{ij} - \pi'_i + \pi'_j$.

Then the reduced primal problem that attempts to find a feasible solution to the (primal) minimum cost flow problem associated with the arcs $\mathcal{A}^=$ becomes

$$\begin{array}{ll} \text{minimize} & z = \sum\limits_{k \in \mathcal{S} \setminus \{1\}} y_k \\ \text{subject to:} & \sum\limits_{(1,j) \in \mathcal{A}^=} x_{1j} - \sum\limits_{(i,1) \in \mathcal{A}^=} x_{i1} - y_1 + (-1)^{t2} y_2 + (-1)^{t3} y_3 + \\ & \qquad + \cdots + (-1)^{tr} y_r + \cdots + (-1)^{tm} y_m = b'_1 \\ & \sum\limits_{(k,j) \in \mathcal{A}^=} x_{kj} - \sum\limits_{(i,k) \in \mathcal{A}^=} x_{ik} - (-1)^{t_k} y_k = b'_k, \;\; k \in \mathcal{S} \setminus \{1\} \\ & l_{ij} \le x_{ij} \le u_{ij}, \;\; \forall (i,j) \in \mathcal{A}^= \\ & y_k \ge 0 \end{array} \tag{11}$$

where

$$t_k = \begin{cases} 1 & \text{if } b'_k \ge 0 \\ 2 & \text{if } b'_k < 0 \end{cases}$$

for $k \in \mathcal{S}$. $y_k, k \in \mathcal{S}$ are artificial variables used to obtain a starting basic solution to the phase I problem and $b'_k, k \in \mathcal{S}$ obtained from b_k adding or subtracting the values l_{ij} and u_{ij} in the right-hand-side according with the replacement of the variables $x_{ij} \notin \mathcal{A}^=$ built in the initial problem.

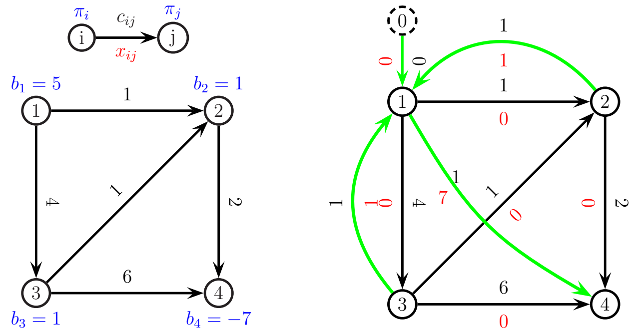
This auxiliary problem is obtained considering the first node as root node and adding to the initial network the following artificial arcs

$$\begin{cases} (k,1) & \text{if } b'_k \ge 0 \\ (1,k) & \text{if } b'_k < 0 \end{cases}$$

for each $k = 2, 3, \cdots, m$ with flow $|b'_k|$ and cost 1 as well as node 0 and the arc $(0,1)$ with both flow and cost 0 (see [6]). For example, suppose we have the network in Figure 1(a), then the network with the artificial arcs is in Figure 1(b).

The optimal objective value, $\hat{z}$ of the reduced primal problem (11) is either $\hat{z} = 0$ or $\hat{z} > 0$. If $\hat{z} = 0$ let $(\hat{x}, \hat{y})$ be an optimal solution to (11), $\hat{x} = \left(\hat{x}_{ij}\right)_{(i,j) \in \mathcal{A}}$ and $\hat{y} = \left(\hat{y}_k\right)_{k \in \mathcal{S}}$. The solution x^* such that $x^*_{ij} = \hat{x}_{ij}$ for all $(i,j) \in \mathcal{A}^=$ and $x^*_{ij} = l_{ij}$ or $x^*_{ij} = u_{ij}$ for the remaining arcs in $\mathcal{A}$, according to the changes made for the auxiliary problem, is a feasible solution to the initial problem (1) since all the artificial variables are zero. Furthermore (π', γ', μ') is a dual feasible solution, and the complementary slackness conditions also hold. So x^* is an optimal solution to (1).

If $\hat{z} > 0$ the current solution is not feasible for the initial problem. In this case a new primal solution that improves the objective can be found or it is necessary to



(a) (b)

Figure 1: (b) is the auxiliary network for solving the minimum cost flow problem (a).

conclude that primal problem is infeasible. The current solution is the best solution for the minimum cost problem associated with the current network $\mathcal{A}^=$. A new arc must be add to this network to obtain a new solution.

Consider the dual of problem (11):

$$
\begin{array}{ll}
\text{maximize} & \sum_{k\in\mathcal{S}} b'_k\pi_k + \sum_{(i,j)\in\mathcal{A}} (l_{ij}\gamma_{ij} - u_{ij}\mu_{ij}) \\
\text{subject to:} & \pi_i - \pi_j + \gamma_{ij} - \mu_{ij} = 0 \quad \forall (i,j)\in\mathcal{A}^= \\
& \pi_1 \geq 0 \\
& (-1)^{t_k}\pi_1 - (-1)^{t_k}\pi_k \leq 1
\end{array}
\tag{12}
$$

Let $(\hat{\pi}, \hat{\gamma}, \hat{\mu})$ be an optimal solution to this problem. By the complementary slackness we obtain that

$$
\hat{\pi}_i - \hat{\pi}_j + \hat{\gamma}_{ij} - \hat{\mu}_{ij} = 0 \text{ for all } (i,j)\in\mathcal{A}^= \tag{13}
$$

$$
\hat{\pi}_i - \hat{\pi}_j < 0 \Rightarrow \hat{x}_{ij} = l_{ij} \tag{14}
$$

$$
\hat{\pi}_i - \hat{\pi}_j > 0 \Rightarrow \hat{x}_{ij} = u_{ij}, (i,j)\in\mathcal{A}^=. \tag{15}
$$

Consider now a new solution (π'', γ'', μ'') for the initial dual problem defined as

$$\pi'' = \pi' + \theta\hat{\pi}$$

$$\gamma'' = \gamma' + \theta\hat{\gamma}$$

$$\mu'' = \mu' + \theta\mu$$

where $\theta > 0$. In this case we have

$$\begin{aligned} c_{ij} - \pi''_i + \pi''_j - \gamma''_{ij} + \mu''_{ij} &= c_{ij} - (\pi'_i + \theta\hat{\pi}_i) + \pi'_j + \theta\hat{\pi}_j - \gamma'_{ij} - \theta\hat{\gamma}_{ij} + \mu'_{ij} + \theta\hat{\mu}_{ij} \\ &= c_{ij} - \pi'_i + \pi'_j - \gamma'_{ij} + \mu'_{ij} - \theta(\hat{\pi}_i - \hat{\pi}_j + \hat{\gamma}_{ij} - \hat{\mu}_{ij}). \end{aligned}$$

This new solution is a dual feasible solution for the initial problem if θ is small enough. There are two cases to examine.

1. If $(i,j) \in \mathcal{A}^=$ then
$$\pi''_i - \pi''_j + \gamma''_{ij} - \mu''_{ij} = c_{ij}$$
since $c_{ij} - \pi'_i + \pi'_j - \gamma'_{ij} + \mu'_{ij} = 0$ and also $\hat{\pi}_i - \hat{\pi}_j + \hat{\gamma}_{ij} - \hat{\mu}_{ij} = 0$.

2. If $(i,j) \notin \mathcal{A}^=$ then
$$\pi''_i - \pi''_j + \gamma''_{ij} - \mu''_{ij} = c_{ij} \Leftrightarrow \hat{\pi}_i - \hat{\pi}_j = \hat{\gamma}_{ij} - \hat{\mu}_{ij}$$
which is always possible since $\hat{\gamma}_{ij} - \hat{\mu}_{ij}$ can be any real number.

The new arc (i,j) to be include in the current network must reduce the objective value, so if $(i,j) \in L$ we must have $\pi_i - \pi_j > 0$ and if $(i,j) \in U$, $\pi_i - \pi_j < 0$. In the first case $c_{ij} - \pi'_i + \pi'_j > 0$ and

$$c_{ij} - \pi''_i + \pi''_j \geq 0 \Leftrightarrow \theta \leq \frac{c_{ij} - \pi'_i + \pi'_j}{\hat{\pi}_i - \hat{\pi}_j}.$$

In the second case, $(i,j) \in U$, $c_{ij} - \pi'_i + \pi'_j < 0$ and

$$c_{ij} - \pi''_i + \pi''_j \leq 0 \Leftrightarrow \theta \leq \frac{c_{ij} - \pi'_i + \pi'_j}{\hat{\pi}_i - \hat{\pi}_j}.$$

This means that if we choose $\theta > 0$ such that

$$\begin{aligned} \theta = \min\Bigg\{ \frac{c_{ij} - \pi'_i + \pi'_j}{\hat{\pi}_i - \hat{\pi}_j} : (i,j) \notin \mathcal{A}^= \text{ such that } (c_{ij} - \pi'_i + \pi'_j > 0 \text{ and } \hat{\pi}_i - \hat{\pi}_j > 0) \\ \text{or } (c_{ij} - \pi'_i + \pi'_j < 0 \text{ and } \hat{\pi}_i - \hat{\pi}_j < 0)\Bigg\} \end{aligned} \quad (16)$$

at least one arc (i,j) not in the current set $\mathcal{A}^=$ will be in the set $\mathcal{A}^=$ of the next iteration. This arc will reduce the objective function value.

In the next iteration the updated set $\mathcal{A}^=$ of all the arcs such that $\pi''_i + \pi''_j = c_{ij}$ is considered and the new restricted problem is solved. Some arcs previously in $\mathcal{A}^=$ may not be in $\mathcal{A}^=$ in this new iteration.

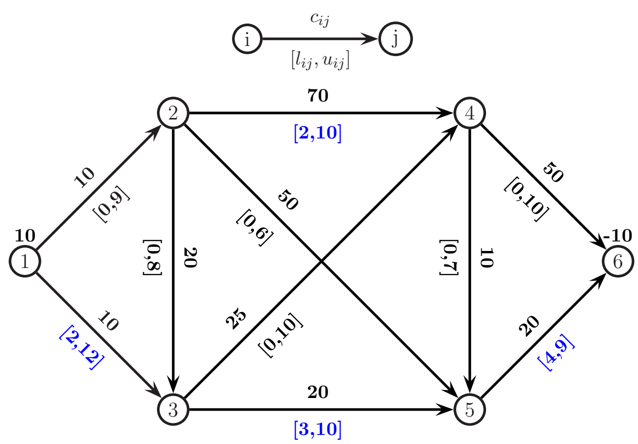


Figure 2: Network flow example.

The foregoing process is continued until either $\hat{z} = 0$, in which case we have an optimal solution for (1), or there are neither arcs in L such that $\hat{\pi}_i - \hat{\pi}_j > 0$ or arcs in U such that $\hat{\pi}_i - \hat{\pi}_j < 0$. In this case the initial problem is not feasible.

Example 3.1.

Consider the minimum network flow problem illustrated in Figure 2.

It. 1: The algorithm starts with an initial feasible solution for the dual problem. A solution with $\pi'_k = 0$ for all nodes k and $\gamma_{ij} = \mu_{ij} = 0$ for all arcs $(i, j) \in \mathcal{A}$ is feasible. It can be easily seen that $c_{ij} - \pi'_i + \pi'_j$ are all positive for all arcs (i, j) in $\mathcal{A}$ (see Table 1) and so all variables $x_{ij} = l_{ij}$. Therefore, the initial restricted problem is

$$
\begin{array}{llllllll}
\text{minimize } y_2 + y_3 + y_4 + y_5 + y_6 & & & & & & \\
\text{subject to:} & y_2 & +y_3 & -y_4 & +y_5 & +y_6 & = 8 \\
 & -y_2 & & & & & = -2 \\
 & & -y_3 & & & & = -1 \\
 & & & y_4 & & & = 2 \\
 & & & & -y_5 & & = -1 \\
 & & & & & -y_6 & = -6 \\
 & \multicolumn{6}{l}{y_k \geq 0, \; k = 2, 3, 4, 5, 6}
\end{array}
$$

This is the minimum cost network flow problem of Figure 7(a). The optimal value of this problem is not zero and, therefore, the optimal solution is not optimal for the initial problem. The dual of the restricted problem has optimal solution $\hat{\pi} = (0, -1, -1, 1, -1, -1)$. The diferences $\pi_i - \pi_j$ for each arc $(i, j) \in \mathcal{A}$ are shown in Table 1. Thus, we have

$$\theta = \min \{10, 10, 5, 25\} = 5$$

and the new dual feasible solution

$$\pi'' = (0, 0, 0, 0, 0, 0) + 5 \times (0, -1, -1, 1, -1, -1) = (0, -5, -5, 5, -5, -5)$$

and we proceed to the next iteration.

It. 2: The arc $(4, 5)$ is added to the current network since it is the only one with minimum ratio $\frac{c_{ij} - \pi'_i + \pi'_j}{\hat{\pi}_i - \hat{\pi}_j}$.

The new restricted problem is depicted in Figure 7(b). This problem has an optimal solution with objective value greater than zero (see Figure 7(c)). The dual of this problem has an optimal solution $\hat{\pi} = (0, -1, -1, 1, 1, -1)$ and $\theta = \min \{5, 5, 20, 10\} = 5$ obtained from the arcs $(1, 2)$ and $(1, 3)$. Thus the new dual solution is

$$\pi''' = 0, -5, -5, 5, -5, -5) + 5 \times (0, -1, -1, 1, 1, -1) = (0, -10, -10, 10, 0, -10)$$

We proceed to the next iteration.

The subsequent iterations are summarized in Table 2. The algorithm ends with $\pi^* = (0, -10, -10, -20, -30, -70)$, $\mathcal{A}^= = \{1, 2), (1, 3), (3, 5), (4, 5), (4, 6), (5, 6)\}$, $x^* = (2, 8, 0, 2, 0, 0, 8, 1, 1, 9)$ and $f(x^*) = 640$.

■

4 A Primal-dual Algorithm for the Bi-criteria Network Flow Problem

In this section a primal-dual algorithm to solve the bi-criteria network flow problem (10) is developed. This algorithm is a direct application of the algorithm of [4] for the bi-criteria minimum cost flow problem.

Table 1: Iteration output.

arc	$c_{ij}-\pi'_i+\pi'_j$	$\hat\pi_i-\hat\pi_j$	$\frac{c_{ij}-\pi'_i+\pi'_j}{\hat\pi_i-\hat\pi_j}$
$(1,2)$	10	1	10
$(1,3)$	10	1	10
$(2,3)$	20	0	
$(2,4)$	70	-2	
$(2,5)$	50	0	
$(3,4)$	25	-2	
$(3,5)$	20	0	
$(4,5)$	10	2	$\frac{10}{2}=5$
$(4,6)$	50	2	$\frac{50}{2}=25$
$(5,6)$	20	0	

a)First iteration

$c_{ij}-\pi''_i+\pi''_j$	$\hat\pi_i-\hat\pi_j$	$\frac{c_{ij}-\pi''_i+\pi''_j}{\hat\pi_i-\hat\pi_j}$
5	1	5
5	1	5
20	0	
80	-2	
50	-2	
35	-2	
20	0	
0	2	
40	2	$\frac{40}{2}=20$
20	0	$\frac{20}{2}=10$

b) Second iteration

arc	$c_{ij}-\pi'''_i+\pi'''_j$	$\hat\pi_i-\hat\pi_j$	$\frac{c_{ij}-\pi'''_i+\pi'''_j}{\hat\pi_i-\hat\pi_j}$
$(1,2)$	0	0	
$(1,3)$	0	0	
$(2,3)$	20	0	
$(2,4)$	90	-1	
$(2,5)$	60	-1	
$(3,4)$	45	-1	
$(3,5)$	30	-1	
$(4,5)$	0	0	
$(4,6)$	30	2	$\frac{30}{2}=15$
$(5,6)$	10	2	$\frac{10}{2}=5$

c)Third iteration

$c_{ij}-\pi^{iv}_i+\pi^{iv}_j$	$\hat\pi_i-\hat\pi_j$	$\frac{c_{ij}-\pi^{iv}_i+\pi^{iv}_j}{\hat\pi_i-\hat\pi_j}$
0	0	
0	0	
20	0	
95	1	95
65	1	65
50	1	50
35	1	35
0	0	
20	0	
0	0	

d) Fourth iteration

arc	$c_{ij}-\pi^{v}_i+\pi^{v}_j$	$\hat\pi_i-\hat\pi_j$	$\frac{c_{ij}-\pi^{v}_i+\pi^{v}_j}{\hat\pi_i-\hat\pi_j}$
$(1,2)$	0	0	
$(1,3)$	0	0	
$(2,3)$	20	0	
$(2,4)$	60	0	
$(2,5)$	30	0	
$(3,4)$	15	0	
$(3,5)$	0	0	
$(4,5)$	0	0	
$(4,6)$	20	1	20
$(5,6)$	0	-1	

e)Fifth iteration

$c_{ij}-\pi^{vi}_i+\pi^{vi}_j$	$\hat\pi_i-\hat\pi_j$	$\frac{c_{ij}-\pi^{vi}_i+\pi^{vi}_j}{\hat\pi_i-\hat\pi_j}$
0		
0		
20		
60		
30		
15		
0		
0		
0		
-30		

f) Sixth iteration

Table 2: Primal-dual results.

Iter.	Dual solution	$\mathcal{A}^=$	Dual restricted solution	θ
1	π=(0,0,0,0,0,0)	{}	$\hat{\pi}$=(0,-1,-1,1,-1,-1)	5
2	π=(0,-5,-5,5,-5,-5)	{(4,5)}	$\hat{\pi}$=(0,-1,-1,1,1,-1)	5
3	π=(0,-10,-10,10,0,-10)	{(1,2), (1,3), (4,5)}	$\hat{\pi}$=(0,0,0,1,1,-1)	5
4	π=(0,-10,-10,15,5,-15)	{(1,2), (1,3), (4,5), (5,6)}	$\hat{\pi}$=(0,0,0,0,-1,-1)	35
5	π=(0,-10,-10,-20,-30,-50)	{(1,2), (1,3), (3,5), (4,5), (5,6)}	$\hat{\pi}$=(0,0,0,0,0,-1)	20
6	π=(0,-10,-10,-20,-30,-70)	{(1,2), (1,3), (3,5), (4,5), (4,6), (5,6)}		

Finding all efficient solutions of (10) is equivalent to finding all optimal solutions of network flow problems of the form

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\mathcal{A}} \Big(\lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}\Big)x_{ij} \\ \text{subject to:} & \sum_{(k,j)\in\mathcal{A}} x_{kj} - \sum_{(i,k)\in\mathcal{A}} x_{ik} = b_k \qquad \forall k \in \mathcal{S} \\ & l_{ij} \leq x_{ij} \leq u_{ij} \qquad \forall (i,j) \in \mathcal{A} \end{array} \tag{17}$$

for all $\lambda \in]0,1[$ (Theorem 2.2).

It is well known that there exists a finite partition of the interval $]0,1[$ such that an efficient solution of (10) is associated with one and only one set in the partition (see [9]).

The algorithm of [4] solves problem (10) by applying the primal-dual algorithm to (17). The dual feasible solutions constructed in the algorithm do not depend on the λ parameter. Moreover, the sequence of problems solved correspond to a partition of the interval $]0,1[$.

Consider problem (17). The dual is

$$\begin{array}{ll} \text{maximize} & \sum_{k\in\mathcal{S}} b_k\pi_k + \sum_{(i,j)\in\mathcal{A}} \Big(l_{ij}\gamma_{ij} - u_{ij}\mu_{ij}\Big) \\ \text{subject to:} & \pi_i - \pi_j + \gamma_{ij} - \mu_{ij} = \lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}, \quad \forall (i,j) \in \mathcal{A} \\ & \gamma_{ij}, \mu_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A}. \end{array} \tag{18}$$

Let $\pi'(\lambda)$ be be an initial feasible solution to (18) and $\Lambda'_{q'}, q' = 1,2,\cdots,r'$ be a partition of $]0,1[$ such that for $q' = 1,2,\cdots,r'$ there exists $\mathcal{A}^=_{q'} \subset \mathcal{A}$ such that

- for all $\lambda \in \Lambda_{q'}$,

  - for all $(i,j) \in \mathcal{A}^{=}_{q'}$, $\pi'_i(\lambda) - \pi'_j(\lambda) = \lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}$ and
  - for all $(i,j) \notin \mathcal{A}^{=}_{q'}$, $\pi'_i(\lambda) - \pi'_j(\lambda) \neq \lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}$

- for all $\lambda \notin \Lambda_{q'}$ there is $(i,j) \in \mathcal{A}^{=}_{q'}$ $\pi'_i(\lambda) - \pi'_j(\lambda) \neq \lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}$

Consequently, for $\lambda \in \Lambda_{q'}$ and $(i,j) \notin \mathcal{A}^{=}_{q'}$, $x_{ij} = l_{ij}$ or $x_{ij} = u_{ij}$. Let $\mathcal{A}^{<}_{q'}$ be the set of arcs (i,j) such that $x_{ij} = l_{ij}$, i.e., such that $\pi'_i(\lambda) - \pi'_j(\lambda) < \lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}$ and $\mathcal{A}^{>}_{q'}$ be the set of arcs (i,j) such that $x_{ij} = u_{ij}$, i.e., such that $\pi'_i(\lambda) - \pi'_j(\lambda) > \lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij}$.

For each interval $\Lambda_{q'}$ we have the restricted primal problem

$$
\begin{array}{ll}
\text{minimize} & z = \sum_{k \in \mathcal{S} \setminus \{1\}} y_k \\
\text{subject to:} & \sum_{(1,j) \in \mathcal{A}^{=}_{q'}} x_{1j} - \sum_{(i,1) \in \mathcal{A}^{=}_{q'}} x_{i1} - y_1 + (-1)^{t2} y_2 + (-1)^{t3} y_3 + \\
& \qquad + \cdots + (-1)^{tr} y_r + \cdots + (-1)^{tm} y_m = b'_1 \\
& \sum_{(k,j) \in \mathcal{A}^{=}_{q'}} x_{kj} - \sum_{(i,k) \in \mathcal{A}^{=}_{q'}} x_{ik} - (-1)^{t_k} y_k = b'_k, \quad k \in \mathcal{S} \setminus \{1\} \\
& l_{ij} \leq x_{ij} \leq u_{ij}, \quad \forall (i,j) \in \mathcal{A}^{=}_{q'} \\
& y_k \geq 0.
\end{array}
\qquad (RP(\mathcal{A})^{=})
$$

If the optimal objective value z^* of $(RP(\mathcal{A})^{=})$ is zero, its optimal solution $\hat{x}$ is optimal for (17) for any $\lambda \in \Lambda_{q'}$ as in the case of the single objective primal-dual algorithm. If $z^* > 0$ we can formulate the dual $(DRP(\mathcal{A}^{=}))$.

$$
\begin{array}{ll}
\text{maximize} & \sum_{k \in \mathcal{S}} b'_k \pi_k + \sum_{(i,j) \in \mathcal{A}} (l_{ij} \gamma_{ij} - u_{ij} \mu_{ij}) \\
\text{subject to:} & \pi_i - \pi_j + \gamma_{ij} - \mu_{ij} = 0 \quad \forall (i,j) \in \mathcal{A}^{=} \\
& \pi_1 \geq 0 \\
& (-1)^{t_k} \pi_1 - (-1)^{t_k} \pi_k \leq 1.
\end{array}
\qquad (DRP(\mathcal{A}^{=}))
$$

Let $(\hat{\pi}(\lambda), \hat{\gamma}(\lambda), \hat{\mu}(\lambda))$ be an optimal solution of $(DRP(\mathcal{A}^{=}))$. A new solution for the dual initial (18) is

$$
\begin{array}{rcl}
\pi''(\lambda) & = & \pi' + \theta(\lambda) \hat{\pi} \\
\gamma''(\lambda) & = & \gamma' + \theta(\lambda) \hat{\gamma} \\
\mu''(\lambda) & = & \mu' + \theta(\lambda) \mu
\end{array}
$$

where $\theta(\lambda) > 0$ is defined by

$$\theta(\lambda) = \min \Big\{ \frac{\lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij} - \pi'_i + \pi'_j}{\hat{\pi}_i - \hat{\pi}_j} :$$
$$\hat{\pi}_i - \hat{\pi}_j > 0 \text{ and } c^1_{ij} - c^2_{ij} + c^2_{ij} - \pi'_i + \pi'_j > 0 \text{ or}$$
$$\hat{\pi}_i - \hat{\pi}_j < 0 \text{ and } c^1_{ij} - c^2_{ij} + c^2_{ij} - \pi'_i + \pi'_j < 0 \Big\}.$$

Notice that $\theta(\lambda)$ might have different values since the quotient $\frac{\lambda(c^1_{ij} - c^2_{ij}) + c^2_{ij} - \pi'_i + \pi'_j}{\hat{\pi}_i - \hat{\pi}_j}$ depends on the λ value. The interval $\Lambda_{q'}$ is partitioned in intervals $\Lambda_{q'q''}, q'' = 1, 2, \cdots, r''$. The algorithm continues with the dual solution $\pi''(\lambda)$ and an interval $\Lambda_{q'q''}$.

Algorithm 1: *Primal-dual bi-criteria network flow algorithm*

1. *Choose* $\pi'(\lambda)$*, an initial value of the vector* π *in problem (18) and compute the partition* $\Lambda'_{q'}$*,* $q' = 1, 2, \cdots, r'$ *of the interval* $]0, 1[$.
2. *Compute the set* $\mathfrak{L} = \big\{(\mathcal{A}^{<}_{q'}, \mathcal{A}^{>}_{q'}),\ q' = 1, 2, \cdots, r'\big\}$.
3. *While* $(\mathfrak{L} \neq \emptyset)$ *do*

   Choose $\big(\mathcal{A}^{<}_{q'}, \mathcal{A}^{>}_{q'}\big) \in \mathfrak{L}$ *and solve* $RP(\mathcal{A})^{=}$.

   (a) *If the optimal value is* 0 *then its solution is an efficient solution to (10). Set* $\mathfrak{L} = \mathfrak{L} \backslash \big(\mathcal{A}^{<}_{q'}, \mathcal{A}^{>}_{q'}\big)$.

   (b) *Else solve* $DRP(\mathcal{A}^{=})$ *and let* $(\hat{\pi}, \hat{\gamma}, \hat{\mu})$ *be an optimal solution.*

      i. *If there is no arc* (i, j) *such that* $c_{ij} - \pi'_i + \pi'_j > 0$ *and* $\hat{\pi}_i - \hat{\pi}_j > 0$ *or* $c_{ij} - \pi'_i + \pi'_j < 0$ *and* $\hat{\pi}_i - \hat{\pi}_j < 0$ $P(\lambda)$ *is infeasible. STOP.*

      ii. *Else compute the partition* $\Delta_{q'q''}$ *of* $\Delta_{q'}$*,* $q'' = 1, 2, \cdots, r''$ *and set* $\mathfrak{L} = \mathfrak{L} \backslash \big(\mathcal{A}^{<}_{q'}, \mathcal{A}^{>}_{q'}\big) \cup \big\{\big(\mathcal{A}^{<}_{q'q''}, \mathcal{A}^{>}_{q'q''}\big) : q'' = 1, 2, \cdots, r''\big\}$.

5 An Improvement of the Algorithm of [4]

The primal-dual algorithm as it has been presented in Section 4 leads to a partition of the interval]0,1[into more sub-intervals than necessary. As a consequence the same solutions are found repeatedly. In this section we use the knowledge of each extreme solution (an extreme solution is an efficient solution such that $f(x)$ is an extreme point of Y) to find a partition of the interval]0,1[in $\Lambda_q, q = 1, 2, \cdots, r$ with one subset for each extreme nondominated point.

The primal-dual algorithm divide the interval $]0, 1[$ successively according to the tree in Figure 3.

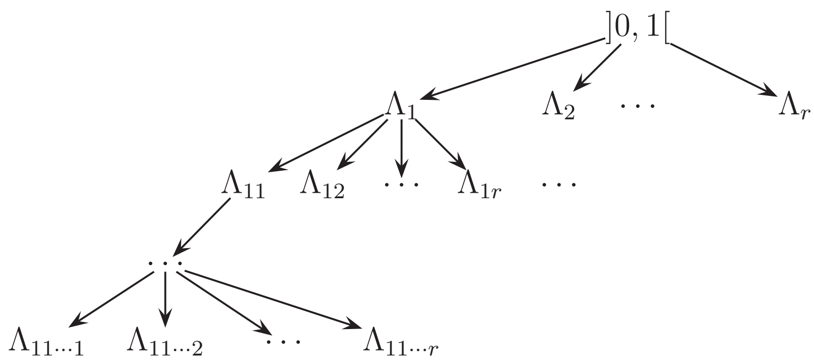


Figure 3: Interval division scheme.

Using a depth-first search for the choice of the interval Λ_q the first extreme point x^* obtained is an optimal solution for the problem (17) with $\lambda = 0 + \varepsilon$, where ε is a sufficiently small positive number. It is known that the optimal Spanning Tree Structure (STS) associated with this solution has reduced costs $\bar{c}_{ij} = c_{ij} - \pi_i + \pi_j, (i, j) \in \mathcal{A}$. Both c_{ij}, π_i and π_j are linear functions in λ. Thus $\bar{c}_{ij}$ are also linear functions

$$\begin{aligned} \bar{c}_{ij} &= a_{ij}\lambda + b_{ij} \geq 0 \qquad \text{for all } (i,j) \in L \\ \bar{c}_{ij} &= a_{ij}\lambda + b_{ij} \leq 0 \qquad \text{for all } (i,j) \in U. \end{aligned}$$

where a_{ij} and b_{ij} are real numbers.

Let $\lambda' \in]0, 1[$ be such that

$$\lambda' = \max\Big\{\lambda : \bar{c}_{ij} \geq 0, \forall (i,j) \in L \text{ and } \bar{c}_{ij} \leq 0, \forall (i,j) \in U\Big\}.$$

The solution x^* is an optimal solution for the problem (17) for all $\lambda \in]0, \lambda']$. Therefore, in the second iteration of the primal-dual algorithm all the intervals $\Lambda_q \subset]0, \lambda']$ can be removed from further analysis since the same solution x^* would be obtained.

6 An Illustrative Example

Consider the bicriteria network flow problem which is sketched in Figure 4. Mathematically it becomes

$$
\begin{array}{rl}
\text{minimize} & f_1(x) = 3x_{12} + 8x_{13} + 5x_{23} + 3x_{24} + 2x_{34} + 10x_{35} + x_{45} \\
\text{minimize} & f_2(x) = 5x_{12} + x_{13} + 5x_{23} + 9x_{24} + 7x_{34} + 2x_{35} + 4x_{45}
\end{array}
$$

$$
\text{subject to:} \quad
\begin{array}{cccccccrl}
x_{12} & +x_{13} & & & & & & = & 10 \\
-x_{12} & & +x_{23} & +x_{24} & & & & = & 0 \\
 & -x_{13} & -x_{23} & & +x_{34} & +x_{35} & & = & 0 \\
 & & & -x_{24} & -x_{34} & & +x_{45} & = & 0 \\
 & & & & & -x_{35} & -x_{45} & = & -10
\end{array}
\tag{19}
$$

$$
\begin{array}{l}
x_{12} \leq 10, x_{13} \leq 5, x_{23} \leq 4, x_{24} \leq 7, x_{34} \leq 8, x_{35} \leq 6, \\
x_{45} \leq 8 \\
x_{12}, x_{13}, x_{23}, x_{24}, x_{34}, x_{35}, x_{45} \geq 0.
\end{array}
\tag{20}
$$

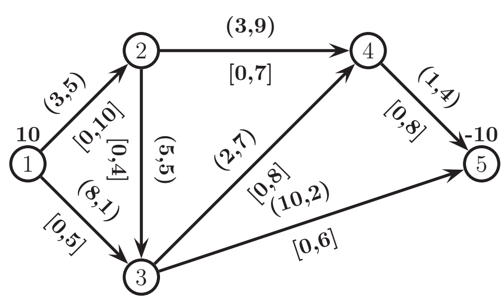


Figure 4: Bi-criteria network flow problem.

Let $\lambda \in]0, 1[$ and consider the parametric problem

$$
\begin{array}{ll}
\text{minimize} & (-2\lambda+5)x_{12}+(7\lambda+1)x_{13}+5x_{23}+(-6\lambda+9)x_{24}+ \\
& \quad +(-5\lambda+7)x_{34}+(8\lambda+2)x_{35}+(-3\lambda+4)x_{45} \\
\text{subject to:} & \begin{array}{ccccccccl}
x_{12} & +x_{13} & & & & & & = & 10 \\
-x_{12} & & +x_{23} & +x_{24} & & & & = & 0 \\
& -x_{13} & -x_{23} & & +x_{34} & +x_{35} & & = & 0 \\
& & & -x_{24} & -x_{34} & & +x_{45} & = & 0 \\
& & & & & -x_{35} & -x_{45} & = & -10
\end{array} \\
& x_{12}\le 10, x_{13}\le 5, x_{23}\le 4, x_{24}\le 7, x_{34}\le 8, x_{35}\le 6, \\
& x_{45}\le 8 \\
& x_{12}, x_{13}, x_{23}, x_{24}, x_{34}, x_{35}, x_{45}\ge 0.
\end{array}
\tag{21}
$$

Let $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ be the dual variables associated with constraints (19) and the dual variables $\mu_{12}, \mu_{13}, \cdots, \mu_{45}$ associated with constraints (20). Then the dual problem is defined as follows

$$
\begin{array}{ll}
\text{maximize} & 10\pi_1-10\pi_5-10\mu_{12}-5\mu_{13}-4\mu_{23}-7\mu_{24}-8\mu_{34}-6\mu_{35}-8\mu_{45} \\
\text{subject to:} & \begin{array}{ccccccll}
\pi_1 & -\pi_2 & & & & -\mu_{12} & \le & -2\lambda+5 \\
\pi_1 & & -\pi_3 & & & -\mu_{13} & \le & 7\lambda+1 \\
& \pi_2 & -\pi_3 & & & -\mu_{23} & \le & 5 \\
& \pi_2 & & -\pi_4 & & -\mu_{24} & \le & -6\lambda+9 \\
& & \pi_3 & -\pi_4 & & -\mu_{34} & \le & -5\lambda+7 \\
& & \pi_3 & & -\pi_5 & -\mu_{35} & \le & 8\lambda+2 \\
& & & \pi_4 & -\pi_5 & -\mu_{45} & \le & -3\lambda+4
\end{array} \\
& \mu_{12}, \mu_{13}, \mu_{23}, \mu_{24}, \mu_{34}, \mu_{35}, \mu_{45}\ge 0.
\end{array}
\tag{22}
$$

Solving problem (21) by using primal-dual algorithm of Section 4 the efficient STSs and the intervals Λ_q are outlined in Figure 5. The algorithm starts with $\lambda \in]0,1[$. It first splits this interval into two subintervals $]0, \frac{2}{11}]$ and $[\frac{2}{11}, 1[$. If the interval $]0, \frac{2}{11}]$ is first considered two new subintervals are obtained $]0, \frac{2}{22}]$ and $[\frac{2}{22}, \frac{2}{11}]$.

Taking the interval $]0, \frac{2}{22}]$ the first efficient extreme point for the bi-criteria problem is achieved

$$x' = (x_{12}=5, x_{13}=5, x_{23}=1, x_{24}=4, x_{34}=0, x_{35}=6, x_{45}=4).$$

The STS associated with this solution is in Figure 6 with dual variables π and reduced costs for the first and second criteria $\bar{c}^1_{ij}$ and $\bar{c}^2_{ij}$, respectively. This STS

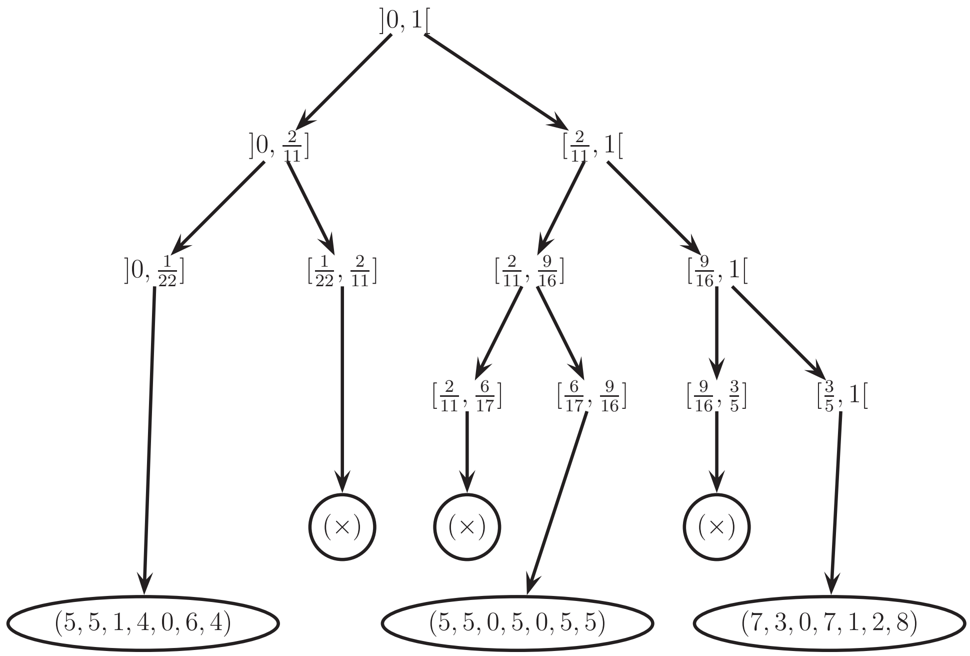


Figure 5: Interval division scheme and solutions.

remains an optimal STS for problem (21) while $\lambda \in]0, \lambda']$ where

$$\lambda' = \min_{(i,j)\in J'} \frac{-\bar{c}^2_{ij}}{\bar{c}^1_{ij} - \bar{c}^2_{ij}}$$

with

$$J' = \{(i,j) \in L : \bar{c}^1_{ij} < 0 \text{ and } \bar{c}^2_{ij} \geq 0\} \cup \{(i,j) \in U : \bar{c}^1_{ij} > 0 \text{ and } \bar{c}^2_{ij} \leq 0\}.$$

So $\lambda' = \frac{6}{11+6} = \frac{6}{17}$ and solution x' remains optimal for (21) for all $\lambda \in]0, \frac{6}{17}]$. This knowledge avoids the search of efficient extreme points in both intervals $[\frac{1}{22}, \frac{2}{11}]$ and $[\frac{2}{11}, \frac{6}{17}]$.

Next, problem (21) is solved for $(\frac{6}{17}, \frac{9}{16})$. The optimal solution is $x'' = (5, 5, 0, 5, 0, 5, 5)$. Computing the reduced costs for the STS associated with this optimal solution we conclude that this is also an optimal solution for the interval $[\frac{9}{16}, \frac{3}{5}]$. The last efficient extreme point $x''' = (7, 3, 0, 7, 1, 2, 8)$ is obtained when $\lambda \in [\frac{3}{5}, 1[$.

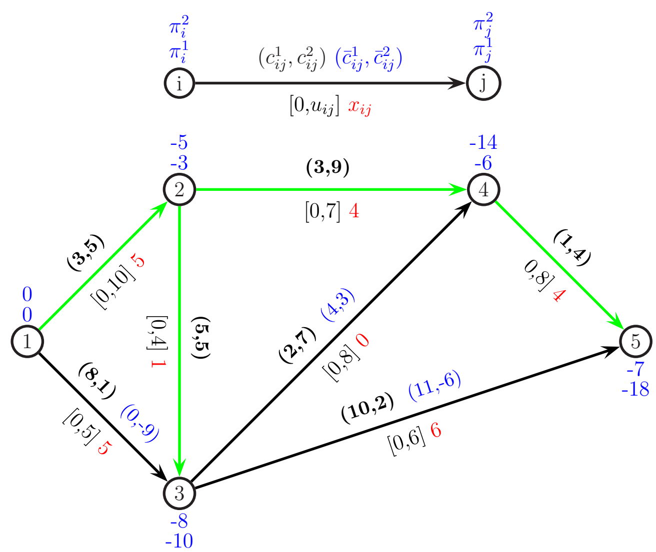


Figure 6: Efficient solution and reduced costs to problem of Figure 4

.

Table 3: Problem parameters.

Problem	Nodes	Arcs	Sources	Sinks	Supply	Skeleton Arcs with max cost 1	Skeleton Arcs with max cost 2	Capacity
150-179	10	40	5	4	100	20	30	0-20
250-279	20	100	7	5	200	30	30	0-30
350-379	30	300	8	12	300	25	25	0-30
450-479	40	600	12	14	400	20	30	0-40
550-579	50	1000	15	15	600	25	25	0-40
650-679	60	1400	20	20	800	20	20	0-40

7 Computational Experiments

Two versions of the primal-dual simplex algorithm for the minimum cost bi-criteria network flow problem have been implemented using the C programming language. The first version is the implementation of algorithm 1 and the second is the same algorithm with the modification made in Section 5. The computer used for the experiments is equipped with an Intel Pentium processor 2.13GHz with 1GB of RAM, and runs under OS X operating system.

Several instances (30 of each type) of the minimum cost bi-criteria network flow problem were generated using the *NETGEN* network generator after some changes for this particular problem. The objective function coefficients were randomly generated (uniform distribution) from the integer set $\{0, 1, 2, \cdots, 100\}$. All arcs are capacitated with minimum value of 0. The remaining parameters of the generated problems are tabled in Table 3. The designations are the same used in NETGEN (See [7]). The number of nodes and arcs, the number of nondominated extreme points and the average CPU time for both algorithms are shown in Table 4. It can be seen that the average CPU time using the second algorithm is smaller than that of the first algorithm. The improvement is about a factor of 6 to 7.

Table 4: Numerical results.

Problem	Nondominated extreme points (average)	CPU Time (sec.) 1st version	2nd version
150-179	7.77	0.10	0.02
250-279	29.63	0.65	0.14
350-379	85.67	7.08	1.02
450-479	126.77	28.40	4.18
550-579	104.52	139.23	18.27
650-679	164.32	317.17	50.35
Average	86.45	82.10	12.33

8 Conclusions

In this paper we have developed a primal-dual simplex algorithm for the biobjective network flow problem. Our numerical results show that the algorithm can solve medium size instances in reasonable time. In the future we plan to compare our algorithm with other algorithms for the same problem, e.g. a parametric network simplex algorithm. Another topic of research is the investigation of the integer bicriteria flow problem. In contrast to the single objective case, this poses considerable difficulties, because nondominated points in the interior of Y exist. The proposed algorithm can be used in phase 1 of a 2 phase approach to this problem.

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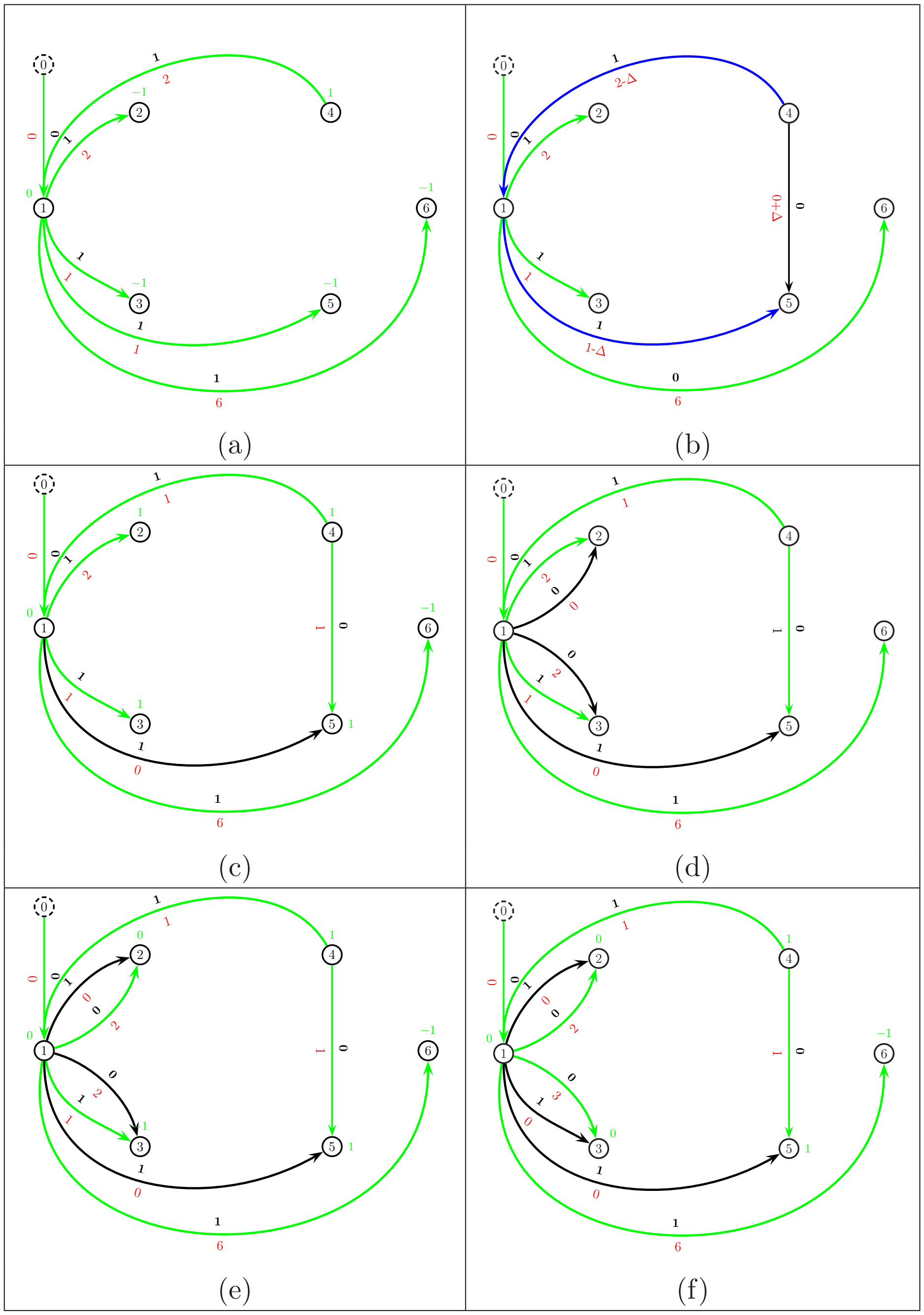

Figure 7:

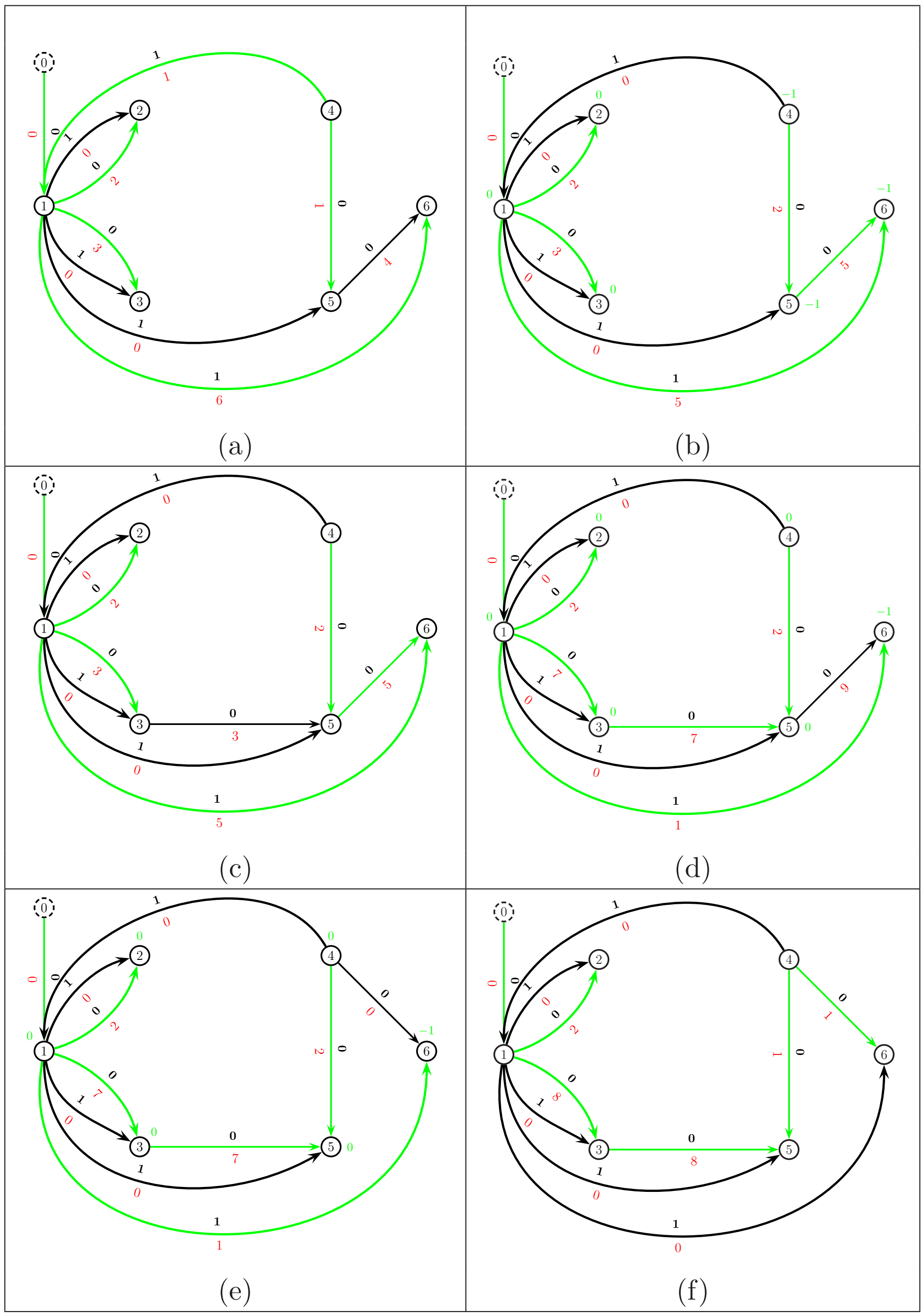

Figure 8: