ELECTRE METHODS WITH INTERACTION BETWEEN CRITERIA: AN EXTENSION OF THE CONCORDANCE INDEX

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Contents

MÉTHODES ELECTRE AVEC INTERACTION ENTRE CRITÈRES : UNE GÉNÉRALISATION DE L'INDICE DE CONCORDANCE

RÉSUMÉ

Cet article est consacré à une généralisation de l'indice de concordance global pour les méthodes ELECTRE. Une telle généralisation a été conçue pour prendre en compte l'interaction entre critères. Trois types d'interaction ont été considérés : auto-renforcement, auto-affaiblissement et antagonisme. Dans des situations de décision réelles, il est raisonnable de considérer l'interaction entre un petit nombre de paires de critères. Afin que le nouvel indice de concordance prenne correctement en compte ces types d'interactions, diverses conditions de frontière, de monotonicité et de continuité ont été imposées. On démontre que l'indice généralisée pend en compte de façon satisfaisante les trois types d'interaction (ou dépendance entre critères), tout d'abord en présence de quasi-critères puis en présence de pseudo-critères.

Mots-clés : Aide Multicritère à la Décision, Méthodes de Surclassement, Interaction entre Critères.

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ABSTRACT

This paper is devoted to an extension of the comprehensive concordance index of ELECTRE methods. Such an extension have been considered to take into account the interaction between criteria. Three types of interaction has been considered, self-strengthening, self-weakening, and antagonism. In real-world decision-making situations is reasonable to consider the interaction between a small number of pairs of criteria. So that the new concordance index takes correctly into account these types of interactions, various conditions, boundary, monotonicity, and continuity have been imposed. We demonstrate that the generalized index takes into account in a satisfactory way the three types of interaction (or dependencies between criteria), first of all in presence of quasi criteria then in presence of pseudo criteria.

Key-words: Multiple Criteria Analysis, Outranking Methods, Interaction between Criteria.

1 Introduction

In this paper it is assumed that a coherent set or family F of n criteria has been built to support a comprehensive preference model for comparing the actions of a set A in a decision aiding perspective. Almost all the current well-known outranking multiple criteria methods (see Part III in Figueira et al., 2005) assume that F is composed of independent criteria (for an exception see Greco and Figueira, 2003).

It is important to point out that the interest of using outranking methods is to take into account purely ordinal scales (Martel and Roy, 2005) keeping thus their concrete verbal meaning, without any kind of conversion of the original scales into abstract ones with the same range (this conversion does not imply necessarily a loss of meaning of the abstract scales). The same does not occur with, for example, MACBETH (Bana e Costa and Vansnick, 1994 and Bana e Costa et al., 2005) and fuzzy integrals based methods (Grabisch, 1997 and Grabisch and Labreuche, 2005).

The concept of independence is rather a complex notion (see Roy, 2006 and Chap. 2 in Roy and Bouyssou, 1993). This paper only deals with the extension of the concordance notion, as it has been designed for ELECTRE methods (see Figueira et al, 2005), to three particular types of interaction designated in this paper by self-strengthening, self-weakening, and antagonism.

This paper is organized as follows. Section 2 provides two illustrative examples for a better understanding of the three types of interactions that may occur in real-world decision-making situations. Section 3 is devoted to the elementary concepts, definitions, and notation; the comprehensive concordance index is defined as well as the fundamental properties of this index. Section 4 is consecrated to the definition of the three types of interaction considered. Section 5 presents the extension of the concordance index, starting from the simplest case, when only quasi criteria are considered and then the extension for the pseudo criteria model. Finally, a section is devoted to outline the main conclusions of the paper.

2 Illustrative Examples

This section presents two illustrative examples that shed some light on the concrete meaning of the three types of interaction this paper deals with. When describing the criteria we will assign [min] to the criteria to be minimized and [max] to the criteria to be maximized.

2.1 Choosing the site for constructing a new hotel

This example is related to the selection of a site for building a new hotel in a given city where a company is not present yet. For aiding to make the "best" decision, a family of four criteria has been built,

g_1: land purchasing and construction costs (investment costs) [min];

g_2: operating rates (variable costs) [min];

g_3: the image of the ward of the town for clients (image) [max];

g_4: easiness in recruiting personnel (recruitment) [max].

The first two criteria are of financial nature while the remaining are qualitative (purely ordinal). As for the criteria of financial nature they can be quantitative, but very often the value attached to each site is not precisely known. The scales are not purely ordinal, but if we want to use indifference and preference thresholds for modelling the imprecision, then this scales cannot be considered as interval scales (Martel and Roy, 2005).

Suppose that a relative weight characterizing adequately, from the decision-maker point of view, the individual role that each one of the four criteria should play could have been defined when making the abstraction of the possible interactions that can exist with one of the three remaining criteria (as it can be shown later on, the SRF procedure can, for example, be used with such an objective). Consider now two sites a and b and take into account the individual weights when there is only one criterion in favor of a over b. In other words, when doing the abstraction of all interactions between criteria, only the considered criterion is in favor of a over b. But, as soon as the two criteria form a coalition where both are in favor of a over b, the relative weight of this coalition become stronger than the summation of the individual weights the investment and the variable costs. Consequently, an interaction weight or coefficient should be introduced to express this reinforcement. Therefore, when comparing sites a and b a reinforcement (called also strengthening) effect occurs. On the contrary, the criteria image and recruitment lead to the reverse effect. If these two criteria are both in favor of a over b, then there is a weakening effect of the summation of the two individual weights. The conjoint weights are too much strong and it is necessary to weaken their summation.

2.2 Choosing a new digital camera model

A manufacturer wants to introduce a new digital camera model in the market. To support his/her decision the following criteria have been considered,

g_1: purchasing costs (cost) [min];

g_2: weakness (fragility) [min];

g_3: buttons user friendliness (workability) [max];

g_4: quality of the image (image) [max];

g_5: aesthetics [max];

g_6: volume [min];

g_7: weight [min].

The problem consists of choosing the "best" digital camera among the available models. It is supposed that the individual weights are defined as in the previous example.

Consider two digital camera models, a and b; assume that a is better than b on fragility but not on cost; model a is less fragile, but more expensive no matter on what happens with the remaining criteria. The weights of fragility and cost are important either if they are or not in favor of a over b. Therefore, these weights play well their role *vis-a-vis* the remaining criteria weights. In other words, as soon as a is better than b on fragility, one admits that the weights are adequate. In this case, one of the criteria is not in favor of a over b while the other is. But if both were in favor of a over b, the individual weights were good too. When a is less fragile than b, one supposes that the individual weights of cost and fragility play well the role we want when both are in favor of a over b.

Now, what happens if we consider model c that is less expensive than model d, but more fragile? This loss of fragility makes the weight of cost too much strong. In this situation fragility degrades the weight of cost where there is no fragility. The presence of a bad performance in fragility should decrease the weight that one granted to cost when considering a model that is not fragile. In other words, an interaction coefficient strictly greater than zero have to be considered. There is a deterioration of the weight of cost due to the fact of a surplus in fragility. The previous hypothesis is not true in the reverse sense. The supplement of the cost does not depend on fragility. There is an antagonism in a sense, but not in the two senses simultaneously.

How can this case be taken into account? This is a new situation of interaction. Therefore, we shall examine this case. It is about an interaction between the cost (which is in favor of c over d) and fragility which is in favor of d over c. It is therefore there that an interaction occurs.

3 Concepts: definitions and notation

This section is only devoted to to some elementary concepts, definition, and the notation used in the rest of this paper. As for the key concepts and the main features concerning ELECTRE methods (the context in which they are relevant, modelling with an outranking relation, their structure, the role of criteria, and how to account for imperfect knowledge) see Figueira et al. (2005). A comprehensive treatment of ELECTRE methods may be found in Roy and Bouyssou (1993) and Vincke (1992). Much of the theory developed on this field is presented in these books.

3.1 Basic data

The basic data of a multiple criteria problem is composed of a set or family of coherent criteria, a set of actions, and an evaluation matrix. Let,

- $F = \{g_1, \ldots, g_i, \ldots, g_n\}$ denote a coherent set or family of *criteria*; for the sake of simplicity we shall use also F as the set of criteria indices (the same will apply later on for subsets of F);

- $A = \{a, b, c, \ldots\}$ denote a finite set of *actions* with cardinality m;

- $g_i(a) \in E_i$ denote the *evaluation* of action a on criterion g_i, for all $a \in A$ and $i \in$, where E_i is the scale associated to criterion g_i (no restriction is imposed to the scale type)

In what follows it is assumed that all the criteria as to be maximized, which is not a restrictive assumption.

3.2 Binary relations

When comparing two actions a and b, the following comprehensive binary relations can be defined on the set A. For a couple $(a,b) \in A$ let,

- P denote the *strict preference* relation; aPb means that "a is strictly preferred to b";
- I denote the *indifference* relation; aIb means that "a is indifferent to b";
- Q denote the *weak preference* relation; aQb means that "a is weakly preferred to b, which expresses hesitation between indifference (I) and preference (P);
- S denote the *outranking* relation; aSb means that "a outranks b" or more precisely that "a is at least as good as b".

For a given criterion g_i, the same interpretation of the above binary relations is valid, but now these relations are called partial binary relations, P_i, I_i, Q_i, and S_i, respectively.

3.3 The notion of pseudo criterion

The concept of pseudo criterion is based on the definition of two preference parameters, called thresholds. Let

- $q_i(g_i(a))$ denotes the *indifference threshold* for criterion g_i, for all $a \in A$ and $i \in F$;
- $p_i(g_i(a))$ denotes the *preference threshold* for criterion g_i, for all $a \in A$ and $i \in F$.

such that $p_i(g_i(a)) \geq q_i(g_i(a))$, for all $i \in E_i$.

Definition 1 (pseudo criterion). *A pseudo criterion is a function g_i associated with the two threshold functions $q_i(g_i(a))$ and $p_i(g_i(a))$ satisfying the following condition, for all $a \in A$ (Roy, 1991, 1996): $g_i(a) + p_i(g_i(a))$ and $g_i(a) + q_i(g_i(a))$ are non-decreasing monotone function of $g_i(a)$.*

By definition, for all couples $(a,b) \in A$ with $g_i(a) \geq g_i(b)$,

$$
\begin{aligned}
aI_ib &\quad \Leftrightarrow g_i(a) \leq g_i(b) + q_i(g_i(b)); \\
aQ_ib &\quad \Leftrightarrow g_i(b) + q_i(g_i(b)) < g_i(a) \leq g_i(b) + p_i(g_i(b)); \\
aP_ib &\quad \Leftrightarrow g_i(b) + p_i(g_i(b)) < g_i(a).
\end{aligned}
$$

Definition 2 (quasi criterion). *If, $q_i(g_i(a)) = p_i(g_i(a))$, for all $a \in A$, then g_i is called a quasi criterion. It is a particular case of a pseudo criterion which is also considered in the rest of the paper. For a quasi criterion there is no ambiguity zone, that is, weak preference Q_i.*

3.4 The criteria weights and the concordance index

In ELECTRE methods, the *relative importance coefficients* attached to the criteria refer to *intrinsic weights*. For a given criterion g_i, the weight k_i can be interpreted as its voting power when it contributes to the majority which is in favor of an outranking; it is not a substitution weight. For more details about the question on how to attribute numerical values to the parameters which must reflect the relative importance of criteria, see Figueira and Roy (2002), Mousseau (1993, 1995) and Roy and Mousseau (1996).

ELECTRE Multiple Criteria Aggregation Procedures (MCAPs) are based on a *concordance index* $c(a,b)$ which is used both to validate the assertion "a outranks b" and/or to give a measure of the credibility of such an assertion. The concordance index can be defined as follows,

$$c(a,b) = \sum_{i \in C(aSb)} \frac{k_i}{K}, \quad \text{where } K = \sum_{i \in F} k_i \tag{1}$$

where, $C(aSb)$ represents the coalition of criteria in favor of the assertion "a outranks b", when F if composed of quasi criteria.

When F contains at least a pseudo criterion, this index should be rewritten in the following way,

$$c(a,b) = \sum_{i \in F} \frac{k_i}{K} c_i(a,b) \tag{2}$$

where,

$$c_i(a,b) = \begin{cases} 1, & \text{if} \quad g_i(a) + q_i(g_i(a)) \geq g_i(b), \ (aS_ib), \\ \dfrac{g_i(a) + p_i\big(g_i(a)\big) - g_i(b)}{p_i\big(g_i(a)\big) - q_i\big(g_i(a)\big)}, & \text{if} \quad g_i(a) + q_i(g_i(a)) < g_i(b) \leq g_i(a) + p_i(g_i(a)), \ (bQ_ia), \\ 0, & \text{if} \quad g_i(a) + p_i(g_i(a)) < g_i(b), \ (bP_ia). \end{cases} \tag{3}$$

It is easy to see that when F is composed of quasi criteria, index (2) becomes (1).

Let $\bar{C}(bPa)$ denote the complement of $C(bPa)$. It should be remarked that when F comprises only quasi criteria $\bar{C}(bPa) = C(aSb)$; if F is composed of at least one pseudo criterion $\bar{C}(bPa) = C(aSb) \cup C(bQa)$. In both cases this set represents the coalition of all the criteria which are not strongly opposed to the assertion aSb (let us recall that bQa is not a strong opposition).

3.5 Properties of $c(a,b)$

The following properties of $c(a,b)$ hold for all couples $(a,b) \in A$,

> **Boundary conditions**: $0 \leq c(a,b) \leq 1$.
>
> **Monotonicity**: *$c(a,b)$ is a monotonous non-decreasing function of $\Delta_i = g_i(a) - g_i(b)$, for all $i \in F$.*
>
> **Continuity**: *if $p_i(g_i(a)) > q_i(g_i(a))$, for all $i \in F$ and $a \in A$, then $c(a,b)$ is a continuous function of both $g_i(a)$ and $g_i(b)$.*

The proof of the boundary conditions is obvious. The proof of monotonicity is based on the fact that, for each i, $c_i(a,b)$ has the same property. Continuity is not valid for quasi criteria. The proof for the case of pseudo criteria is also based on the fact that, for each i, $c_i(a,b)$ has the same property.

4 Types of interactions considered

The above formulae (1) and (2) do not take into account any type of dependence which may exists between the considered criteria. Very often this is justified because we deal with a structural dependence related to some different points concerning distinct stakeholders (Roy and Bouyssou, 1993). Moreover, for the sake of the clarity, the coherent family of criteria must be defined in such a way that other types of dependence should be reduced as much as possible (see, for example Bisdorff, 2001). It is also necessary to completely remove dependencies derived from dispersion or in the sense of the utility classical approach. Consequently, from a practical point of view, the dependencies that need to be really taken into account are not numerous and in general they concern only pairs of criteria. As for triple of criteria, quadruple of criteria and so on, it would complicate too much their effective consideration within a process of decision aiding because their formulation should deal with so many difficulties of interpretation and comprehension that would vanish their expected added value (see Roy, 2006).

Therefore, we consider the cases where the only dependencies between criteria which deserve to be taken into account in MCAPs are related to interactions between pairs of criteria. In this paper we are interested in the situations in which the interactions can be modelled by one of the three types that we shall present hereafter. It consists of proper modifications of formulae (1) and (2). The conditions in which these modifications take place are related to the type of the considered interaction. Similar types of interactions can be found in Greco and Figueira (2003).

4.1 Self-strengthening

If criteria g_i and g_j are both strongly, or even weakly, in favor of the assertion *aSb* (more precisely, $g_i, g_j \in \bar{C}(bPa)$), their contribution to the concordance index must be larger than the summation

$k_i + k_j$, because the two weights k_i and k_j are those that represent the contribution of each one of the two criteria to the concordance index when the other criterion is not in favor of *aSb*. We suppose that the effect of the conjoint presence of g_i and g_j among the criteria in favor of the assertion *aSb*, can be modelled by a self-strengthen coefficient $k_{ij} > 0$ that must intervene algebraically in $c(a,b)$. See the interaction between g_1 and g_2 in Section 2.1. Note that $k_{ij} = k_{ji}$.

4.2 Self-weakening

If criteria g_i and g_j are both strongly, or even weakly, in favor of the assertion *aSb* (more precisely, $g_i, g_j \in \bar{C}(bPa)$, we consider that their contribution to the concordance index must be smaller than the summation of the weights $k_i + k_j$ (the two weights k_i and k_j are those that represent the contribution of each one of the two criteria to the concordance index when the other criterion is not in favor of *aSb*). We suppose that this effect can be modelled by means of one self-weakening coefficient $k_{ij} < 0$ that must intervene algebraically in $c(a,b)$. See the interaction between g_3 and g_4 in Section 2.1. Note that $k_{ij} = k_{ji}$.

4.3 Antagonism

When criterion g_i is strongly or weakly in favor of the assertion *aSb* and criterion g_h is against this assertion, we take into account that the contribution of criterion g_i to the concordance index must be smaller than the weight k_i considered in case that g_h does not belong to $C(bPa)$. We suppose that this effect can be modelled with the introduction of an antagonism coefficient $k'_{ih} > 0$, that must intervene in $c(a,b)$. See Section 2.2. Let us remark that the presence of an antagonism coefficient $k'_{ih} > 0$ is compatible with both the absence of antagonism in the reverse direction ($k'_{hi} = 0$) and with the presence of a positive coefficient $k'_{hi} > 0$.

4.4 Practical aspects

In this section we examine the question of how to assign values to the weights of criteria and to the interaction coefficients.

1. In a first step, it is important to start by assigning values to the individual weights, k_i for all $i \in F$. As indicated in the first above example (see Section 2.1), these weights are those that characterize the relative importance of criterion g_i, when it belongs to $\bar{C}(bPa)$ in absence of any other criteria susceptible to interact with g_i. SRF software (Figueira and Roy, 2002) can again be used to elicit those weights under the condition of focusing the attention on the previous stipulated requirement.

2. Then, in a second step, it is necessary to identify,

   - the pairs $\{i, j\}$ for which it seems necessary to take into account a self-strengthening or self-weakening interaction effect;

- the couples (i,h) for which it seems necessary to take into account an antagonism interaction effect.

  (Note that $\{i,j\} = \{j,i\}$, while $(i,h) \neq (h,i)$)

Antagonism is based on a contraposition of one criterion g_i towards another criterion g_h. In this context the direction is important and therefore the contraposition of g_h towards g_i does not imply a possible contraposition of g_i towards g_h. Anyway, we suppose that if there are both self-strengthening and self-weakening effects with respect to the pair $\{g_i,g_h\}$, then there is no antagonism neither of g_h against g_i nor of g_i against g_h. In practice the number of interaction pairs or couples of criteria is rather very small.

3. A third step comes and it concerns the assignment of a value for each interaction coefficient k_{ij} and k'_{ih} for, respectively, the pairs $\{i,j\}$ and the couples (i,h) identified in Step 2. How these coefficients intervene in $c(a,b)$? When dealing with a family of quasi-criteria, the interaction coefficients are algorithmically added to $k_i + k_j$ or $k_i + k_h$. Thus,

   - The greater the absolute value of the coefficients the stronger the impact of the interaction;
   - To assign a value of the interaction coefficients, it is necessary to examine along with the decision-maker the variation that should be assigned to the wight of the coalition formed by the two criteria when the occurs:
     - the passage from $k_i + k_j$ to $k_i + k_j + k_{ij}$
     - the passage from $k_i + k_j$ to $k_i + k_h + k'_{ih}$

   Let us start by the interaction of Types 1 and 2, we should reflect and inquire ourselves about the importance we should assign to the self-strengthening or to the self-weakening effect taken into account by k_{ij} when both criteria g_i and g_j contribute to validate the assertion aSb. This should allow to attribute a value,

$$k_i + k_j + k_{ij}$$

   that we think it is adequate for modelling interactions of the above mentioned types. The interaction weights k_{ij} are immediately deduced. When proceeding the same way as in the previous step we should be able to assign a value to the antagonism coefficients, k'_{ih}.

   As an illustrative example assume that for the case in Section 2.1 we obtained with SRF, $k_1 = 5$ and $k_2 = 4$. We can think that it would be adequate to get $k_1 + k_2 + k_{12} = 12$; thus $k_{12} = 3$. Let us now take the criteria g_3 and g_4 and assume that $k_3 = 3$ and $k_4 = 2$; if we think that it would be adequate to have $k_3 + k_4 + k_{34} = 4$, then $k_{34} = -1$. As for the antagonism interaction coefficients it is easy do derive an illustrative example from Section 2.2.

Remark 1. *For a given pair $\{i,j\}$, it is not possible to have simultaneously self-strengthening and self-weakening because there is only a value for $k_{ij} = k_{ji}$.*

Remark 2. *We also assume that for a pair* $\{i,h\}$ *it is impossible to have simultaneously* $k'_{ih} \neq 0$ *and* $k'_{hi} \neq 0$.

4. Finally, a fourth step is devoted to check the consistency of the interaction coefficients with, if necessary, the possibility of decreasing the values of certain $k_{ij} < 0$ or k'_{ih}. The above approach for getting the weights and the interaction coefficients requires to impose a constraint called, the *non-negative net balance condition* of the interactions.

   Condition 1 (non-negative net balance). *For all* $i \in F$,

   $$\left(k_i\right) - \Big(\sum_{i,j:k_{ij}<0} |k_{ij}| + \sum_h k'_{ih}\Big) \geq 0$$

   It expresses the following requirement: As soon as a criterion g_i becomes a member of $\bar{C}(bPa)$, the conjunction of all the interactions susceptible to reduce the contribution of this criterion to value of $\bar{C}(bPa)$ cannot, in the worst case (only if the whole set of pairs such that $k_{ij} < 0$ and $k'_{ih} \neq 0$ intervene in the concordance coalition), lead to a negative contribution of g_i.

5 Extensions of the concordance index

This section is devoted to the definition of the concordance index, first when F is composed of quasi criteria, and then when at least one criterion is a pseudo criterion.

Before presenting the formulae it is useful to introduce the following additional notation. Let,

- $L(a,b)$ denote the set of all pairs $\{i, j\}$ such that $i, j \in \bar{C}(bPa)$;
- $O(a,b)$ denote the set of all couples (i,h) such that $i \in \bar{C}(bPa)$ and $h \in C(bPa)$.

5.1 The quasi criterion model

Let us recall that a quasi criterion is a pseudo criterion such that $q_i(g_i(a)) = p_i(g_i(a))$, for all $a \in A$.

5.1.1 Definition of $c(a,b)$

The comprehensive concordance index, when F is composed of quasi criteria, is defined as follows,

$$c(a,b) = \frac{1}{K}\Big(\sum_{i \in \bar{C}(bPa)} k_i + \sum_{i,j \in L(a,b)} k_{ij} - \sum_{i,h \in O(a,b)} k'_{ih}\Big) \tag{4}$$

The coefficient K should be defined to ensure the validity of the boundary conditions,

$$0 \leq c(a,b) \leq 1$$

Let us consider separately the two inequalities,

1. $c(a,b) \geq 0$

   This inequality derives from the definition of $c(a,b)$ and the non-negative net balance condition; it is fulfilled for every K. The proof is obvious.

2. $c(a,b) \leq 1$

   Two cases have to be considered,

   (a) $\bar{C}(bPa) = F$ (all the criteria belong to the concordant coalition)
   It represents *unanimity* and the index must be equal to one,

   $$c(a,b) = 1$$

   Since unanimity leads to the absence of antagonism interactions $c(a,b)$ can be rewritten as follows,

   $$c(a,b) = \frac{1}{K}\Big(\sum_{i \in F} k_i + \sum_{\{i,j\}} k_{ij}\Big) = 1 \tag{5}$$

   This result implies that,

   $$K = \sum_{i \in F} k_i + \sum_{\{i,j\}} k_{ij}$$

   (b) $\bar{C}(bPa) \neq F$ (at least one criterion belongs to $C(bPa)$)
   In the previous case, the antagonism coefficients were not present. As soon as these coefficients appear in $c(a,b)$ it leads to $c(a,b) < 1$.

Remark 3. *If F is composed of quasi criteria, the function $c(a,b)$ presents a discontinuity when $g_i(a) + q_i(g_i(a))$ becomes strictly lower than $g_i(b)$. In the case of pseudo criteria, $p_i(g_i(a)) > q_i(g_i(a))$, for all $i \in F$ and $a \in A$, this discontinuity will not occurs.*

5.1.2 Fundamental theorem

Before introducing the main result it is important to establish the following lemma.

Lemma 1. *For all $(a,b) \in A$ and for all $f \in F$, $c(a,b)$ defined as in (4) is a non-decreasing function of Δ_f.*

Proof.
The proof of this lemma is based on the fact that, if the difference Δ_f decreases, either $c(a,b)$ remains constant or it decreases. Two cases should be considered.

1. Criterion f belongs to $C(bPa)$.
   If f belongs to $C(bPa)$ it cannot belong to $\bar{C}(bPa)$. Consequently, the pair $\{i,f\}$ will not pertain to $L(a,b)$. The decreasing of Δ_f does not affect neither the first nor the second summations. Whatever, it will occurs with the existence or not of couples $(i,f) \in O(a,b)$, the decreasing of Δ_f has no influence on the third summation. Consequently, $c(a,b)$ remains constant.
2. Criterion f belongs to $\bar{C}(bPa)$.

   Two subcases have to be considered,

   (a) Criterion f stills remain in $\bar{C}(bPa)$.
   The decreasing of Δ_f will not move f from $\bar{C}(bPa)$. Hence, the three summations will not be affected. Then, $c(a,b)$ remains constant too.

   (b) Criterion f moves to $C(bPa)$.
   The decreasing of Δ_f moves f from $\bar{C}(bPa)$ to $C(bPa)$. This moving has some implications on the result. The new value of $c(a,b)$ will become,

   $$c(a,b)^{New} = c(a,b)^{Old} - \frac{1}{K}\Big(k_f + \sum_{f,j \in L(a,b)} k_{fj} - \sum_{f,h \in O(a,b)} k'_{fh}\Big)$$

   The quantity in between big parenthesis is necessarily non-negative according to the non-negative net balance condition. Consequently, $c(a,b)$ cannot increase.

The proof is complete.

□

The fundamental result is established in the following theorem.

Theorem 1. *Monotonicity and boundary conditions hold for $c(a,b)$ defined as in formula (4).*

Proof.
Lemma 1 proves monotonicity. Let us now prove the boundary conditions,

1. $c(a,b) \geq 0$
   If $\bar{C}(bPa) = \varnothing$, then $c(a,b) = 0$. Suppose that we could have $c(a,b) < 0$. This implies that at least one criterion f does not belong to $\bar{C}(bPa) \neq \varnothing$. Consider that exists at least one criterion in $\bar{C}(bPa)$. If for all f in $\bar{C}(bPa)$, Δ_f is forced to decrease till $\bar{C}(bPa) = \varnothing$, then $c(a,b)$ cannot increase. Contradiction!
2. $c(a,b) \leq 1$
   From condition (5), $c(a,b) = 1$, since $\bar{C}(bPa) = F$. Suppose that we could have $c(a,b) > 1$. This implies that at least one criterion f does not belong to $\bar{C}(bPa)$. Consider that there exists at least one criterion that does not belong to $\bar{C}(bPa)$. If for all f, Δ_f is forced to increase till f becomes an element of $\bar{C}(bPa)$, then $c(a,b)$ cannot decrease. Contradiction!

The proof is now complete.

□

5.2 The pseudo criterion model

When dealing with a pseudo criterion, g_i, an ambiguity zone should be taken into account, for all $(a,b) \in A$,

$$g_i(a) + q_i(g_i(a)) < g_i(b) \leq g_i(a) + p_i(g_i(a))$$

5.2.1 Definition of $c(a,b)$

The definition of $c(a,b)$ can be stated in the following manner,

$$c(a,b) = \frac{1}{K}\Big(\sum_{i \in \bar{C}(bPa)} c_i(a,b)k_i + \sum_{i,j \in L(a,b)} Z(c_i(a,b), c_j(a,b))k_{ij} - \sum_{i,h \in O(a,b)} Z(c_i(a,b), c_h(b,a))k'_{ih}\Big) \tag{6}$$

Function $Z(.,.)$ in formula (6) is used to capture the effects in the ambiguity zone.

Remark 4. *For the sake of clarity and simplicity, the same function $Z(.,.)$ is used in both, the second and the third summations. There is no reason a priori to use a different function $Z'(.,.)$ on one of the two mentioned summations; it is, however, possible to use different functions.*

Let $x = c_i(a,b)$ and $y = c_j(a,b)$ or $y = c_h(b,a)$. Consequently, $x, y \in [0,1]$. Function $Z(x,y)$ is used to get the reduction coefficients for k_{ij}/K and k'_{ih}/K when, at least one of the arguments of $Z(x,y)$ is within the range $]0,1[$.

What are the properties of $Z(x,y)$ to guarantee the coherence of formula (6)?

Extreme value conditions: When leaving the ambiguity zones $c(a,b)$ should regain the form presented in formula (4). Thus, $Z(1,1) = 1$ and $Z(x,0) = Z(0,y) = 0$.

Symmetry: From the fact that $k_{ij} = k_{ji}$ then $Z(x,y) = Z(y,x)$.

Monotonicity: When the ambiguity diminishes the effect due to the interaction cannot increase. Then $Z(x,y)$ is a *non-decreasing monotone function* of both arguments x and y.

Marginal impact condition: When the ambiguity diminishes we pass from x to $x+w$, the relative marginal impact of the interactions is bounded from below,

$$\frac{1}{w}\Big(Z(x+w,y) - Z(x,y)\Big) \leq 1 \quad x, y, w, x+w \in [0,1]$$

We will see the interest of this condition in the proof of Lemma 2.

Continuity: Formula (2) is a continuous function of $g(a)$ and $g_i(b)$ when $p_i(g_i(a)) > q_i(g_i(a))$, for all $a \in A$. If we want to preserve continuity then it is necessary that $Z(x,y)$ is a *continuous function* of each argument.

Boundary condition: For preserving the non-negative net balance condition, it is necessary that $Z(x,y) \leq \min\{x,y\}$.

In order to generalize the above boundary condition when at least one of the partial concordance indices $c_i(a,b)$, $c_j(a,b)$, or $c_h(b,a)$ is strictly lower than 1 we must consider the worst case, where all the self-weakening and antagonism interactions occur for g_i, which leads to impose the extended net balance condition,

Condition 2 (extended non-negative net balance). *For all $i \in F$,*

$$\Big(c_i(a,b)k_i\Big) - \Big(\sum_{i,j:k_{ij}<0} Z(c_i(a,b),c_j(a,b))|k_{ij}| + \sum_h Z(c_i(a,b),c_h(b,a))k'_{ih}\Big) \geq 0$$

(it was assumed that $k_{ij}k'_{ij} = 0$)

To get this condition fulfilled for all the possible cases, it is *necessary* and *sufficient* that

$$c_i(a,b) \geq Z(c_i(a,b),c_j(a,b))$$

and

$$c_i(a,b) \geq Z(c_i(a,b),c_h(a,b))$$

In other words, $x \geq Z(x,y)$. And according to the symmetry of $Z(x,y)$, $Z(x,y) \leq \min\{x,y\}$.

Among the multiple forms that can be chosen for $Z(x,y)$, we only present two of them which have an intuitive and meaningful interpretation.

$Z(x,y) = \min\{x,y\}$;

$Z(x,y) = xy$.

If x and/or y are equal to 1, both formulae are equivalent. But, when x and y are both different from 1, that is, when the two interacting criteria belong to the ambiguity zone, then the impact of the interaction is weaker with xy than with $\min\{x,y\}$. Choosing the $\min\{x,y\}$ formula means that the reduction coefficient is not influenced by what happens it the other ambiguity zone. For these reasons the formula xy seems preferable to $\min\{x,y\}$.

5.2.2 Extension of the fundamental theorem

This section presents an extension of the previous results when F is composed of at least a pseudo criteria. The proofs are similar to the ones provided for Lemma 1 and Theorem 1.

Lemma 2. *For all $(a,b) \in A$ and for all $f \in F$, $c(a,b)$ defined as in (6) is a non-decreasing function of Δ_f.*

Proof.
The proof of this lemma is also based on the fact that if the difference Δ_f decreases, either $c(a,b)$ remains constant or it decreases. Two cases have to be considered.

1. Criterion f belongs to $C(bPa)$.
   For the same reasons as in the absence of pseudo criteria, diminishing Δ_f does not affect neither the first nor the second of the three summations in (6). The same hold for the third summation when there is no couple $(i,f) \in O(a,b)$. If there exist couples $(i,f) \in O(a,b)$, then decreasing Δ_f leads to $c_f(b,a) = 1$; the third summation will not change. Consequently, $c(a,b)$ remains constant.
2. Criterion f belongs to $\bar{C}(bPa)$.
   Now, three subcases have to be considered.

   (a) Criterion f belongs to $C(aSb)$.
   The decreasing of Δ_f does not moves f; it remains thus in $C(aSb)$. More precisely, the decreasing of Δ_f will not make any change in the three components of $c(a,b)$. This index remains constant.

   (b) Criterion f belongs to $C(bQa)$.
   After decreasing Δ_f, criterion f stills remain in $C(bQa)$, either because it belonged to this coalition before or because it moved to $C(bQa)$ due to the decreasing of Δ_f. All the summations are affected. Let us suppose that $c_f(a,b)$ changes its new value and becomes $c_f(a,b)+\Delta$, with $\Delta > 0$.

$$c(a,b)^{New} = c(a,b)^{Old} -$$

$$-\frac{1}{K}\Big(\Delta k_f + \sum_{j\in\bar{C}(a,b)} \Big(Z(c_f(a,b)+\Delta, c_j(a,b)) - Z(c_f(a,b), c_j(a,b))\Big)k_{fj} -$$

$$-\sum_{h\in C(bPa)} \Big(Z(c_f(a,b)+\Delta, c_h(b,a)) - Z(c_f(a,b), c_h(b,a))\Big)k'_{fh}\Big)$$

   Let us denote by $\Delta c(a,b)$ the quantity in between big parenthesis. From the impact marginal condition we obtain,

$$Z(c_f(a,b)+\Delta, c_j(a,b)) - Z(c_f(a,b), c_j(a,b)) \leq \Delta$$

   which leads to the following inequality,

$$\Delta k_f + \sum_{j\in\bar{C}(a,b):k_{fj}<0} \Big(Z(c_f(a,b)+\Delta, c_j(a,b)) - Z(c_f(a,b), c_j(a,b))\Big)k_{fj} -$$

$$-\sum_{h\in C(bPa)} \Big(Z(c_f(a,b)+\Delta, c_h(b,a)) - Z(c_f(a,b), c_h(b,a))\Big)k'_{fh}$$

$$\geq$$

$$\Big(k_j + \sum_{j \in \bar{C}(a,b): k_{fj} < 0} k_{fj} - \sum_{h \in C(bPa)} k'_{fh}\Big)\Delta$$

which is a non-negative quantity, from the non-negative net balance condition and the fact that $\Delta > 0$. And now we can established that the following quantity is also non-negative,

$$\Delta k_f + \sum_{j \in \bar{C}(a,b): k_{fj} < 0} \Big(Z(c_f(a,b) + \Delta, c_j(a,b)) - Z(c_f(a,b), c_j(a,b))\Big) k_{fj} -$$

$$- \sum_{h \in C(bPa)} \Big(Z(c_f(a,b) + \Delta, c_h(b,a)) - Z(c_f(a,b), c_h(b,a))\Big) k'_{fh} \geq 0$$

and we can conclude that $c(a,b)$ cannot increase.

(c) Criterion f moves to $C(bPa)$.
The decreasing of Δ_f will move f to $C(bPa)$. In such a case, $c_f(a,b)k_f$ can no more be found in the expression of $c(a,b)^{New}$. If there are j such that $f, j \in L(a,b)$, then the terms $Z(c_f(a,b), c_j(a,b))k_{fj}$ will be removed from the second summation. If there are i such that $i, f \in O(a,b)$, then $Z(c_i(a,b), c_f(b,a))k'_{if}$ will be introduced in the third summation. The new value of $c(a,b)$, $c(a,b)^{New}$ is equal to $c(a,b)^{Old}$ minus a certain quantity; it is calculated as follows,

$$c(a,b)^{New} = c(a,b)^{Old} -$$

$$-\frac{1}{K}\Big(c_f(a,b)k_f + \sum_{f,j \in L(a,b)} Z(c_f(a,b), c_j(a,b))k_{fj} - \sum_{i,f \in O(a,b)} Z(c_i(a,b), c_f(b,a))k'_{if}\Big)$$

Now, it remains to prove that the quantity between big parenthesis, denoted $\Delta c(a,b)$, is non-negative.

$$\Delta c(a,b) \geq \frac{1}{K}\Big(c_f(a,b)k_f + \sum_{f,j \in L(a,b): k_{fj} < 0} Z(c_f(a,b), c_j(a,b))|k_{fj}| - \sum_{i,f \in O(a,b)} Z(c_i(a,b), c_f(b,a))k'_{if}\Big)$$

which is guarantee by the fact that $Z(x,y) \leq \min\{x,y\}$. Consequently, $c(a,b)$ cannot increase.

The proof is thus complete.

□

Now the fundamental result can be established.

Theorem 2. *Boundary conditions, monotonicity, and continuity hold for $c(a,b)$ defined as in formula (6).*

Proof.
Lemma 2 establishes monotonicity. Boundary conditions hold when considering pseudo criteria. And continuity derives from the fact that,

1. the functions $c_f(a,b)$, $Z(x,y)$ are continuous, and
2. the conditions $c_f(a,b) = 0$ if $g_f(a) + q_f(g_f(a)) - g_f(g_f(b)) = 0$ and $Z(0,y) = Z(x,0) = 0$ guarantees continuity when a criterion becomes member or is removed from one of the following sets, $\bar{C}(a,b)$, $L(a,b)$, or $O(a,b)$.

The proof is thus complete for the general case.

□

6 Conclusion

In this paper we introduced three types of interaction that allow modelling a large number of dependence situations in real-world decision-making problems. We showed how to take into account these types of interaction in the concordance index used within the ELECTRE methods framework. Formula (2) can be simply replaced by (6) in all of ELECTRE methods.

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References

[1] C. Bana e Costa, J-M. De Corte, and J-C. Vansnick. On the mathematical foundations of MACBETH. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: The State of the Art Surveys*, pages 409–442. Springer Science+Business Media, Inc., New York.

[2] C. Bana e Costa and J. Vansnick. MACBETH - An interactive path towards the construction of cardinal value functions. *International Transactions in Operational Research*, 1:489–500, 1994.

[3] R. Bisdorff. Logical foundation of multicriteria preference aggregation. In D. Bouyssou, E. Jacquet-Lagrèze, P. Perny, R. Słowiński, D. Vanderpooten, and Ph. Vincke, editors, *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, pages 379–403. Kluwer Academic Publishers, Dordrecht, 2001.

[4] J. Figueira, S. Greco, and M. Ehrgott, editors. *Multiple Criteria Decision Analysis: The State of the Art Surveys*. Springer Science+Business Media, Inc., New York, 2005.

[5] J. Figueira, V. Mousseau, and B. Roy. ELECTRE Methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: The State of the Art Surveys*, pages 133–162. Springer Science+Business Media, Inc., New York.

[6] J. Figueira and B. Roy. Determining the weights of criteria in the ELECTRE type methods with a revised Simos' procedure. *European Journal of Operational Research*, 139:317–326, 2002.

[7] M. Grabisch. $k-$order additive discrete fuzzy measures and their representatives. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[8] M. Grabisch and C. Labreuche. Fuzzy measures and integrals in MCDA. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: The State of the Art Surveys*, pages 563–608. Springer Science+Business Media, Inc., New York.

[9] S. Greco and J. Figueira. Dealing with interaction between bi-polar multiple criteria preferences in outranking methods. Research Report 11-2003, INESC-Coimbra, Portugal, 2003.

[10] J-M. Martel and B. Roy. Analyse de la signifiance de diverses procédures d'agrégation multicritère. *INFOR*, 43(3), 2005. (revised version to appear INFOR, August 2006).

[11] V. Mousseau. *Problèmes liés à l'évaluation de l'importance relative des critères en aide multicritère à la décision : Réflexions théoriques, expérimentations et implémentations informatiques*. PhD thesis, Université Paris-Dauphine, 1993.

[12] V. Mousseau. Eliciting information concerning the relative importance of criteria. In P. Pardalos, Y. Siskos, and C. Zopounidis, editors, *Advances in Multicriteria Analysis*, Nonconvex Optimization and its Applications, pages 17–43. Kluwer Academic Publishers, Dordrecht, 1995.

[13] B. Roy. The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31:49–73, 1991.

[14] B. Roy. *Multicriteria Methodology for Decision Aiding*. Nonconvex Optimization and its Applications. Kluwer Academic Publishers, Dordrecht, 1996.

[15] B. Roy. A propos de la signification des dépendances entre critères : Quelle place et quels modèles de prise en compte pour l'aide à la décision ? 2006. To appear in the Cahiers du LAMSADE Series.

[16] B. Roy and D. Bouyssou. *Aide Multicritère à la Décision : Méthodes et Cas*. Economica, Paris, 1993.

[17] B. Roy and V. Mousseau. A theoretical framework for analysing the notion of relative importance of criteria. *Journal of Multi-Criteria Decision Analysis*, 5:145–159, 1996.

[18] Ph. Vincke. *Multicriteria Decision-Aid*. Wiley & Sons, Chichester, 1992.