Some Generalizations of the Manipulability of Fuzzy Social Choice Functions

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Abstract

This paper presents some generalizations of the Gibbard -Satterthwaite result (1973, 1975) for fuzzy preference relations. It explores the implication of weakening the transitivity condition of the Ben Abdealziz *et al.*'s (2008) fuzzy manipulability concept. For this purpose, the max-min transitivity is replaced by weaker transitivity: the max-$\star$-transitivity, where $\star$ is a t-norm. In addition, the best alternative set concept is addressed in two ways. In the first way, it is defined from the t-norm concept. In the second one, it is defined based on the decomposition of fuzzy weak relations in terms of symmetric and regular components. The achieved results can be viewed as more general than the one presented in Ben Abdelaziz *et al.* (2008) on the strategy-profness of fuzzy social choice functions.

Key words: fuzzy preference relation, manipulability, Gibbard-Satterthwaite result, fuzzy social choice functions

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1 Introduction

The problem of the strategic manipulation of social choice systems has attracted the attention of many researchers since the publication of the Gibbard and Satterthwaite result (1973, 1975) (henceforth G-S). They proved independently that any non-dictatorial social choice system can be subject to strategic manipulation. In others words, there exists a situation where an individual can change the social choice in his favor by revealing a non-sincere preference relation.

However, research on the same problem with fuzzy preferences is rather scarce. In the existing literature, one can cite only two works as generalizations of the G-S result to the fuzzy context. The first work, related to strict fuzzy preferences relations, is that of Tang (1994). The second is the one of Ben Abdelaziz *et al.* (2008) dealing with fuzzy weak preferences satisfying the max-min and the weak max-min transitivity. They considered that the strategic manipulation of a fuzzy social function is possible if two conditions are fulfilled. The first condition is that the sincere social choice function does not belong to the best alternative set of the manipulator. The second one is that there exists a fuzzy relation securing the manipulator an outcome at least as good as the sincere social choice. Thus, three versions of a generalization of the G-S's result are provided.

This paper explores the implication of weakening the transitivity condition for the fuzzy manipulability concept of Ben Abdealziz *et al.* (2008). For this purpose, the max-min transitivity is replaced by weaker transitivity in the above conditions. Indeed, any max-$\star$ transitivity is less restrictive than the max-min transitivity, for any t-norm $\star$ (Jain, 1990). Therefore, in a first step, the concept of choice functions based on t-norms, as proposed by Roubens (1989), is reintroduced here to provide a generalization of a certain type of the Ben Abdelaziz *et al.* manipulability concept. In a second step, we define a best alternative set based on the strict regular component of a fuzzy preference relation. The regular decomposition of a weak max-$\star$ transitive fuzzy preference relation is proposed by Fono and Andjiga (2005) based on a t-conorm. This best alternative set allows us to extent the manipulability concept with max-min transitive fuzzy preference relations in a second way [2]. Thus, another impossibility result on the manipulability fuzzy social choice functions is proven with max-$\star$ transitive fuzzy relations with any t-conorm.

The paper is organized as follows. Section 2 presents the basic concepts related to fuzzy preference relations as well as the fuzzy social choice concept. Section 3 is devoted to the different generalizations on the manipulability and

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[2] A first attempt in this direction was presented at the IEEE-IEMC, Europe 2008 (see Meddeb *et al.*, 2008).

dictatorship for max-min transitive fuzzy preference relations with illustrative examples. Section 4 provides the proof of the both impossibility results on the strategy-proofness of fuzzy social choice functions with $\star$-transitive fuzzy preference relations. The last section outlines concluding remarks and avenues for future research.

2 Concepts: Definitions and notation

This section presents the basic concepts and notation related to fuzzy relations. It introduces also the fuzzy social choice function concept.

2.1 Fuzzy operators and fuzzy binary relations

The basic definitions of fuzzy operators and fuzzy binary relations as well as some of their fundamental properties are introduced here as in Fono and Andjiga (2005).

2.1.1 Fuzzy operators

Definition 1. (t-norm)
A triangular norm (t-norm) is a function $\star : [0,1] \times [0,1] \to [0,1]$ satisfying the following properties for all $x, y, z, u \in [0,1]$:

- $x \star 1 = x$, (boundary conditions)
- $x \star y \leq u \star z$ if $x \leq u$ and $y \leq z$, (monotonicity)
- $x \star y = y \star x$, (commutativity)
- $(x \star y) \star z = x \star (y \star z)$. (associativity)

Definition 2. (t-conorm)
A t-conorm is a function $\oplus : [0,1] \times [0,1] \to [0,1]$ satisfying the following properties for all $x, y, z, u \in [0,1]$:

- $x \oplus 0 = x$, (boundary conditions)
- $x \oplus y \leq u \oplus z$ if $x \leq u$ and $y \leq z$, (monotonicity)
- $x \oplus y = y \oplus x$, (commutativity)
- $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.(associativity)

Definition 3. (quasi-inverse of a t-norm)
Let $\star$ be a continuous t-norm. The quasi-inverse of $\star$ is the internal composition law denoted by $\|$ and defined for all $x, y \in [0, 1]$ as follows:

$$x\|y = \max\{t \in [0, 1], x \star t \leq y\}.$$

Definition 4. (quasi-subtraction of a t-conorm)
Let $\oplus$ be a continuous t-conorm. The quasi-subtraction of $\oplus$ is the internal composition law denoted by $\ominus$ and defined for all $x, y \in [0, 1]$ as follows:

$$x \ominus y = \min\{t \in [0, 1], x \oplus t \geq y\}.$$

Definition 5. (strict t-conorm)
A t-conorm $\oplus$ is strict if for all $x, y \in [0, 1], \forall\, z \in [0, 1[$, with $x < y$, then $x \oplus z < y \oplus z$.

Example 1.

(1) Let $\star_Z$ and $\oplus_Z$ denote the Zadeh's min t-norm and the Zadeh's max t-conorm respectively, i.e., for all $x, y \in [0, 1], x \star_Z y = \min\{x, y\}$ and $x \oplus_Z y = \max\{x, y\}$. The quasi-subtraction of $\oplus_Z$ is defined as follows:
for all $x, y \in [0, 1], x \ominus_Z y = x$, if $x > y$; and 0, otherwise.
(2) Let $\star_L$ and $\oplus_L$ denote the Lukasiewicz's t-norm and the Lukasiewicz's t-conorm respectively, i.e., for all $x, y \in [0, 1], x \star_L y = \max\{0, (x + y - 1)\}$ and $x \oplus_L y = \min\{1, (x + y)\}$. The quasi-subtraction of $\oplus_L$, denoted by $\ominus_L$, is defined as follows:
for all $x, y \in [0, 1], x \ominus_L y = \max\{0, (y - x)\}$.

2.1.2 Fuzzy binary relations

The vagueness of preferences over a given finite set of *alternatives*, $X = \{x, y, z, \ldots\}$, can be modeled by using fuzzy binary relations. These relations can be viewed as fuzzy sets in the two-dimensional cartesian product, $X^2 = X \times X$ with a membership function, R.

Definition 6. (fuzzy binary relation)
A fuzzy binary relation (FBR) on X is a function $R : X^2 \to [0, 1]$.

- R is reflexive, if for all $x \in X, R(x, x) = 1$,
- R is connected, if for all $x, y \in X, R(x, y) + R(y, x) \geq 1$,
- R is a fuzzy weak preference relation $(FWPR)$ if it is reflexive and connected,
- R is max-$\star$-transitive if for all $x, y, z \in X, R(x, z) \geq R(x, y) \star R(y, z)$.

4

Definition 7. ($\star$-fuzzy order)
Let $\star$ be a t-norm. A *FWPR* is $\star$-fuzzy order if it satisfies the max-$\star$-transitivity.

Let $H^\star$ be the set of the $\star$-fuzzy orders such that $\forall x, y \in X, R(x,y) \neq R(y,x)$.

Remark 1. Let R be an *FWPR* and $\star$ a t-norm.

(1) For all $(x, y) \in X^2, R(x, y)$ is the degree to which x is at least as good as y,
(2) R is a crisp binary relation if for all $x, y \in X, R(x, y) \in \{0, 1\}$. Let $\mathcal{H}$ be the set of crisp $\star$-fuzzy orders, known as linear orders.

Example 2.

(1) The max-$\star_Z$-transitivity is known as the min-transitivity, defined as follows,

$$\forall\ x, y, z \in X, R(x, z) \geq \min\{R(x, y), R(y, z)\},$$

(2) The max-$\star_L$-transitivity is called the L-transitivity, defined as follows,

$$\forall\ x, y, z \in X, R(x, z) \geq R(x, y) + R(y, z) - 1.$$

Definition 8. (symmetry and asymmetry)
Let R be a crisp preference relation. R is

(1) symmetric, if for all $x, y \in X, R(x, y) = R(y, x)$,
(2) asymmetric, if for all $x, y \in X, \min\{R(x, y), R(y, x)\} = 0$.

2.2 Fuzzy social choice functions

Consider a finite set of *individuals*, $N = \{1, 2, \ldots, i, \ldots, n\}$, with $|N| \geq 3$. The *social choice* is the chosen alternative in X according to the preferences of all the individuals in N. When individuals express their preferences as *FWPR*s on X, fuzzy social choice functions can be applied to obtain to the social choice.

Definition 9. (fuzzy social choice function)
Let $\mathcal{R}_N = (R_1, R_2, \ldots, R_i, \ldots, R_n)$ be a profile of individuals' preference relations. A fuzzy social choice function (*FSCF*) is a function that associates a single alternative to a profile of individuals' preference relations.

Throughout this paper, it is assumed that individual *FWPR*s belong to a set $H^\star$, where $\star$ is a given t-norm.

Definition 10. (⋆-fuzzy social choice function)
Let ⋆ be a t-norm. A ⋆-fuzzy social choice function (⋆-*FSCF*) is an *FSCF* such that the profiles of individuals' preference relations belong to $(H^\star)^n$.

Example 3. Consider the following illustrative example with $X = \{a, b, c\}$ and $N = \{1, 2, 3\}$. The relations, R_i, for $i \in \{1, 2, 3\}$ belong to $H^\star$, where ⋆ is the Zadeh's min t-norm. These three relations are presented in the following tables.

R_1	a	b	c
a	1	0.7	0.8
b	0.5	1	0.6
c	0.5	0.5	1

R_2	a	b	c
a	1	0.8	0.7
b	0.4	1	0.4
c	0.4	0.6	1

R_3	a	b	c
a	1	0.4	0.7
b	0.8	1	0.7
c	0.4	0.4	1

Now let us apply the arithmetic mean function to obtain the fuzzy social relation R^s.

R^s	a	b	c
a	1	0.63	0.73
b	0.56	1	0.56
c	0.43	0.5	1

Then, we consider $P(X, R^s) = \{x \in X : R^s(x, y) \geq R^s(y, x)\} = \{a\}$ as the social choice. Such an *FSCF* can be viewed as a ⋆-*FSCF*, where ⋆ = min.

Remark 2. Let ⋆ be a t-norm. Since for all $a, b \in [0, 1], \star(a, b) \leq \min\{a, b\}$, the max-min transitivity induces the max-⋆ transitivity and $\mathcal{H} \subseteq \mathcal{H}^{\min} \subseteq \mathcal{H}^\star$.

3 On the manipulability of fuzzy social choice functions

This section begins with a review on the manipulability and dictatorship of *FSCF*s with individual *FPR*s. Then, it introduces two extensions of the presented result.

Ben Abdelaziz et *al.* (2008) are interested in some particular cases of $\star_Z$-*FSCF*s and $\star_L$-*FSCF*s. They considered that an individual has incentives to manipulate a $\star_Z$-*FSCF* if the sincere social choice does not belong to his best alternative set and there exists an *FWPR* that allows him to obtain an outcome at least good as the sincere social choice on the basis of his sincere preference relation. We introduce here only the manipulability of $\star_Z$-*FSCF*s, called the 1-manipulability in Ben Abdelaziz et *al.* (2008), as follows.

Definition 11. (1-manipulability)
Let $\star$ be the Zadeh t-norm and $\nu^\star$ be an $\star$-*FSCF*. The function $\nu^\star$ is 1-manipulable if there exists $m \in N$, $\mathcal{R}_N \in (\mathcal{H}^\star)^n$, and $R'_m \in \mathcal{H}^\star$ such that there exists $x \in X$, such that $R_m(x, \nu_1(\mathcal{R}_N)) > R_m(\nu_1(\mathcal{R}_N), x)$, $\nu_1(\mathcal{R}_N \mid R'_m) \neq \nu_1(\mathcal{R}_N)$, and $R_m(\nu_1(\mathcal{R}_N), \nu_1(\mathcal{R}_N \mid R'_m)) \geq R_m(\nu_1(\mathcal{R}_N \mid R'_m), \nu_1(\mathcal{R}_N))$.

Example 4. Let us reconsider the previous example. Individual 3 can 1-manipulate the $\star$-*FSCF*. Indeed, $R_3(b, a) > R_3(a, b)$. Therefore, he can reveal the non-sincere fuzzy relation R'_3 to obtain b as the social choice.

R'_3	a	b	c
a	1	0.1	0.7
b	1	1	1
c	0.3	0.3	1

Ben Abdelaziz et al. (2008) stated the following impossibility result.

Theorem 1.
Let $\star = \min$ and $\nu^\star$ be $\star$-*FSCF*. If $\nu^\star$ is 1-strategy-proof, then it is 1-dictatorial too.

Remark 3. The above theorem is a generalization of the Gibbard-Satterthwaite's result for crisp linear order on the manipulability of social choice functions.

From the previous example, one can see that the strategic manipulation of a $\star$-*FSCF*, where $\star = \min$, in the sense 1 can be possible in situations where

there exists an individual m with a fuzzy preference relation R_m such that,

($\mathcal{C}_1$) the sincere social choice $\nu^\star(\mathcal{R}_N)$ does not belong to his best alternative
set $P(X, R_m) = \{x \in X \mid \forall\, y \in X, R_i(x,y) \geq R_i(y,x)\}$. $(P(X, R_3) = \{b\})$

($\mathcal{C}_2$) there exists a fuzzy relation R'_m in $\mathcal{H}^\star$ such that $\nu^\star(\mathcal{R}_N \mid R'_m)$ belongs
to his best alternative set $P(\{\nu^\star(\mathcal{R}_N), \nu^\star(\mathcal{R}_N \mid R'_m)\}, R_m)$, $(R_3(b,a) > R_3(a,b))$.

Let us notice that in the case where, for all $x, y \in X, R_i(x,y) \neq R_i(y,x)$, for
all $i \in N$, for all $R_i \in \mathcal{H}^\star$, the best alternative set of an individual $P(X, R)$
contains a unique alternative and the second condition represents a sufficient
condition $(\mathcal{C}'_2)$ for the manipulation of min-$FSCF$ when the sincere social
choice is different to his best alternative. In addition, an $FSCF$ is 1-dictator
if for all situations, the social choice belongs to the dictator's best alternative
set.

The following sections introduce two extensions of the 1-manipulability con-
cept by using both generalizations of the best alternative set under the above
conditions.

3.2 A generalization based on a t-norm

Roubens (1989) defined the choice set as follows

$$C(X, R) = \{x \in X \mid C(x) = \max_{y \in X} C(y)\}$$

where,

$$C(x) = T_1(1 - P(y, x)), \forall\ \ y \in X - \{x\}, P(y, x) = T_2(R(y, x), 1 - R(x, y))$$

and T_1, T_2 are two t-norms.
If $T_2(a, b) = \max\{0, a + b - 1\}$ and $T_1(a, b) = \min\{a, b\}$, then $C(X, R)$ corre-
sponds to $P(R, X) = \{x \in X \mid \forall\, y \in X, R(x, y) \geq R(y, x)\}$.

Since the T_1 t-norm corresponds to the type of the transitivity used with
the best alternative $P(X, R)$, we replace it by the $\star$ t-norm when the max-$\star$
transitive fuzzy preference relations are considered. Thus, for any t-norm $\star$,
the following best alternative set can be introduced as follows.

$$P^\star(R, X) = C(X, R) \text{ where, } T_1 = \star, \quad P(y, x) = \max\{R(y, x) - R(x, y), 0\}$$

Also, the $\star$-manipulability of a $\star$-$FSCF$ can be introduced as follows.

Definition 12. ($\star$-manipulability)
Let $\star$ be a t-norm and $\nu^\star$ be a $\star$-FSCF. $\nu^\star : (H^\star)^n \to X$ is $\star$-manipulable by the individual m at $\mathcal{R}_N \in (H^\star)^n$ via $\overline{R}_m \in H^\star$ if

$$\nu^\star(\mathcal{R}_N) \notin P^\star(X, R_m). \tag{1}$$

$$\nu^\star(\mathcal{R}_N \mid \overline{R}_m)) \in P^\star(\{\nu^\star(\mathcal{R}_N), \nu^\star(\mathcal{R}_N \mid R_m')\}, R_m) \tag{2}$$

Following the same reasoning, the dictatorship of $\star$-FSCF can be defined as follows.

Definition 13. ($\star$-dictatorship)
An individual $d \in N$ is a $\star$-dictator for a $\star$-FSCF $\nu^\star$, if for all $\mathcal{R}_N \in (H^\star)^n$,

$$\nu^\star(\mathcal{R}_N) \in P^\star(X, R_d)$$

When using Definitions 12 and 13, an impossibility result can be stated as follows.

Theorem 2.
Let $\star$ be a t-norm and $\nu^\star$ be $\star$-FSCF. If $\nu^\star$ is $\star$-strategy-proof, then it is $\star$-dictatorial.

The proof of Theorem 2 will be given in Section 4.

3.3 A generalization based on a regular decomposition of fuzzy preference relations

The condition $\mathcal{C}_2'$ can be expressed as follows:

$$\nu_1(\mathcal{R}_N \mid R_m') \in P(X, R_m) \Rightarrow$$

$$R_m(\nu_1(\mathcal{R}_N \mid R_m'), y) - R_m(\nu_1(y, \nu_1(\mathcal{R}_N \mid R_i')) \geq \ 0, \forall \, y \in X$$

$$\Rightarrow \max\{-R_m(\nu_1(\mathcal{R}_N \mid R_m'), y) + R_m(\nu_1(y, \nu_1(\mathcal{R}_N)); 0\} = 0, \ \forall \, y \in X,$$

$$\Rightarrow P_m^L(y, \nu_1(\mathcal{R}_N) \mid R_m')) = 0, \ \ \forall \, y \ \in X,$$

$$\Rightarrow P_m^L(\nu_1(\mathcal{R}_N) \mid R_m'), y) > 0, \ \ \forall \, y \ \in X, \tag{3}$$

where P_m^L is the regular strict component of the fuzzy relation R_m when the t-conorm $\oplus$ is the Lukasiewcz's t-conorm, defined by Fono and Andjiga (2005) as follows.

> *Regular Decomposition.* Let $\oplus$ be a continuous t-conorm, $\ominus$ be its quasi-substraction, and R be an $FWPR$.
> (1) $\forall\, x, y \in X, R(x,y) = P(x,y) \oplus I(x,y)$,
> (2) P is asymmetric and I is symmetric,
> (3) P is simple,
> then $\forall\, x, y \in X$,

$$\begin{cases} I(x,y) = \min\{R(x,y), R(y,x)\}, \\ P \text{ is regular, } i.e., \forall\, x, y \in X, R(x,y) \leq R(y,x) \Rightarrow P(x,y) = 0. \end{cases}$$

> The minimal regular fuzzy strict component P_R associated, where $\oplus$ is defined as follows,

$$\forall\, x, y \in X, P_R(x,y) = R(y,x) \ominus R(x,y)$$

Now, let $\star$ be any t-norm and $\oplus$ be a strict t-conorm or the Zadeh's max t-conorm. We propose a generalization of the set $P(X, R)$ as follows:

$$S(X, R) = \{x \in X | S_R(x) = \max_{y \in X}\{S_R(y)\}\}$$

where, $S_R(y)$ is the cardinality of the subset $\{y \in X \mid P_R(x,y) > 0\}$ and it is called the score of x on the basis of R.

Remark 4. $S(X, R)$ reduces to $P(X, R)$, when $\star$ is the min t-norm and $\oplus$ is the Lukasiewcz's t-conorm, for all $R \in H^\star$.

Based on conditions $\mathcal{C}_1$ and $\mathcal{C}_2$ in terms of the best alternative set $S(X, R)$, the formal definition of manipulability of a $\star$-$FSCF$, as well as the definition of dictatorship and strategy-proofness, can be presented as follows.

Definition 14. (manipulability, dictatorship, and strategy-proofness)
Let $\star$ be a t-norm and $\nu^\star$ be a $\star$-$FSCF$.

(1) The function $\nu^\star$ is manipulable by the individual m at $\mathcal{R}_N \in (H^\star)^n$ via $\overline{R}_m \in H^\star$ if $S_{R_m}(\nu^\star(\mathcal{R}_N \mid \overline{R}_m)) > S_{R_m}(\nu^\star(\mathcal{R}_N))$.
(2) The function $\nu^\star$ is dictatorial if there exists $d \in N$ such that for every $\mathcal{R}_N \in (H^\star)^n$, if $\nu^\star(\mathcal{R}_N) = a$, then $S_{R_d}(a) \geq S_{R_d}(x), \forall x \in X$.

(3) The function $\nu^\star$ is strategy-proof, if $\nu^\star$ is not manipulable.

Example 5. Let us reconsider the previous example. Individual 3 can manipulate the $\star$-*FSCF*. Indeed, $S_{R_3}(b) > S_{R_3}(a)$. Therefore, he can reveal the non-sincere fuzzy relation R_3' to obtain b as the social choice. Let us notice that the generalized manipulability coincides with the 1-manipulability in this example.

Another impossibility result of the manipulability of $\star$-fuzzy social choice functions can be introduced as follows.

Theorem 3.
Let $\star$ be a t-norm and $\nu^\star$ be $\star$-*FSCF*. If $\nu^\star$ is strategy-proof, then it is dictatorial.

The proof of Theorem 3 will be given in Section 4. A second proof for unanimous $\star$-*FSCF*s is also given when *FPR*s satisfy a certain sufficient and necessary condition for the transitivity of their strict regular components (Fono and Andjiga, 2005). It proceeds by induction on the number of individuals as in Sen (2003).

4 Proofs

The proof of Theorem 2 follows the main steps as in Ben Abdelaziz *et al.*, (2008).

4.1 Proof of Theorem 2

Consider a $\star$-strategy-proof $\star$-*FSCF*, $\nu^\star : (\mathcal{H}^\star)^n \to X$.

Let $\nu_1 : (\mathcal{H}^{\min})^n \to X$ be a *1-FSCF* such that for all $\mathcal{R}_N \in (\mathcal{H}^{\min})^n, \nu_1(\mathcal{R}_N) = \nu^\star(\mathcal{R}_N)$. The *1-FSCF* is min-strategy-proof because of the $\star$-strategy-proofness of $\nu^\star$. Thus, according to Theorem 1, ν_1 is *1*-dictatorial. Let individual 1 be the *1*-dictator for ν_1.

In the remaining we show that individual 1 is also $\star$-dictator for $\nu^\star$.

Let $\mathcal{R}_N$ be any profile of individuals' preference relations in $(\mathcal{H}^\star)^n$. Let $x(0)$ be $\nu^\star(\mathcal{R}_N)$. Consider

$$P^\star(R, X) = C(X, R), \text{ where } T_1 = \star, \quad P(y, x) = \max\{R(y, x) - R(x, y), 0\}$$

11

be the best alternative set of individual 1. Next, it will be proved that $x(0)$ belongs to $P^\star(X, R_1)$.

Let $\mathcal{R}'_N \in \mathcal{H}^n$ be a profile of individuals' crisp linear orders such that

– For $i = 1$

$$\begin{cases} R'_i \text{ is a crisp linear order on } P^\star(X, R_1), \\ R'_i(x, y) = 1 \text{ and } R'_i(y, x) = 0, \text{ if } x \in P^\star(X, R_1), y \in X \backslash P^\star(X, R_1), \\ R'_i \text{ is a crisp linear order on } X \backslash P^\star(X, R_1). \end{cases}$$

– For all $i \neq 1$

$$\begin{cases} R'_i \text{ is a crisp linear order on } P^\star(X, R_1), \\ R'_i(x, y) = 1 \text{ and } R'_i(y, x) = 0, \text{ if } x \in P^\star(X, R_1), y \in X \backslash P^\star(X, R_1), \\ R'_i \text{ is a crisp linear order on } X \backslash P^\star(X, R_1). \end{cases}$$

According to Lemma 1 of Ben Abdealziz *et al.* (2008), R'_i is a crisp linear order for all $i \in N$ and $P(X, R'_1) \subseteq P^\star(X, R_1)$.

Suppose that $x(k) = \nu_1(\mathcal{R}_N \mid R'_1, R'_2, \dots, R'_i, \dots, R'_k)$ is the social choice when the k first individuals change their preference relations R_i into R'_i in order to contradict individual 1.

Note that $k \in \{0, 1, \dots, n\}$. If $k = n$, then $x(n) = \nu^\star(\mathcal{R}'_N) = \nu_1(\mathcal{R}'_N)$. Thus, $x(n)$ belongs to $P(X, R'_1)$ because of the dictatorship of ν_1. Therefore, $x(n)$ belongs to $P^\star(X, R_1)$.

Now, suppose that j denotes the least k in $\{0, 1, \dots, i, \dots, n\}$ such that $x(k) \in P^\star(X, R_1)$. To have $x(0)$ in $P^\star(X, R_1)$, it is needed to show that $j = 0$. The proof is made by contradiction. Suppose that $j \geq 1$.

– If $j = 1$, then

$$x(1) = \nu^\star(\mathcal{R}_N \mid R'_1) \in P^\star(X, R_1) \tag{4}$$
$$x(0) = \nu^\star(\mathcal{R}_N) \notin P^\star(X, R_1) \tag{5}$$

Equation (4) implies that $x(1) \in P^\star(X, R_1)$. Consequently, $\nu^\star$ is $\star$-manipulable by individual 1 at $\mathcal{R}_N$.

– If $j > 1$, then

12

$$\begin{cases} x_j = \nu_1(\mathcal{R}_N \mid R'_1, R'_2, \ldots, R'_j) \in P^\star(X, R_1) \\ x_{j-1} = \nu_1(\mathcal{R}_N \mid R'_1, \ldots, R'_i, \ldots, R'_{j-1}) \notin P^\star(X, R_1) \end{cases}$$

Therefore, $R'_j(x(j-1), x(j)) = 1$, and $R'_j(x(j), x(j-1)) = 0$.

And the alternative $x(j-1)$ is ranked before the alternative $x(j)$ since R'_j is a crisp linear order on X. Consider the situation where $(\mathcal{R}_N \mid R'_1, R'_2, \ldots, R'_j)$ is the profile of individuals' preference relations. If individual j declares a fuzzy preference R_j instead of a crisp relation R'_j, then $\nu^\star(\mathcal{R}_N \mid R'_1, R'_2, \ldots, R'_j)$ is changed in his favor. Consequently, $\nu^\star$ is $\star$-manipulable by individual j at $(\mathcal{R}_N \mid R'_1, R'_2, \ldots, R'_j)$.

We conclude that j must be equal to 0. Thus, $x(0) = \nu^\star(\mathcal{R}_N) \in P^\star(X, R_1)$, for any $\mathcal{R}_N \in (\mathcal{H}^\star)^n$. It follows that individual 1 is also a dictator for $\nu^\star$.

4.2   Proof of Theorem 3

When replacing $P^\star(X, R)$ by the set $S(X, R)$ in the above proof, we obtain the one of Theorem 3.

Now, let us focus on the $\star$-$FSCF$s satisfying the unanimity property as follows.

Definition 15. (unanimity)
Let $\nu^\star$ be a $\star$-$FSCF$ and $x \in X$. Let $\mathcal{R}_N$ be a profile such that $S(X, R_i) = \{x\}, \forall\, i \in N$. then $\nu^\star(\mathcal{R}_N) = x$.

For unanimous $FSCF$s, Theorem 3 is a direct consequence of the Lemma 1 and 2 following the induction reasoning as in Sen (2001). They are stated as follows.

Lemma 1.
Let $N = \{1, 2\}$, $\star$ be t-norm and $\nu^\star$ be a $\star$-FSCF. If $\nu^\star$ is strategy-proof, then it is dictatorial.

Lemma 2.
Let $n \geq 3$ and consider the following two statements;
Statement (a): for all k with $k \leq n$, if $\nu^\star : (H^\star)^k \to X$ is strategy-proof, then f is dictatorial.
Statement (b): if $\nu^\star : (H^\star)^n \to X$ is strategy-proof, then $\nu^\star$ is dictatorial.
Statement (a) implies Statement (b).

13

Proof of Lemma 1.
Consider a strategy-proof $\nu^\star$. We have to prove the following statements:

(1) for a given profile $\widetilde{\mathcal{R}}_N = (R_1, R_2)$ the outcome of $\nu^\star$ must be an element of the set $S(X, R_1)$ or the set $S(X, R_2)$.

(2) if the first statement holds for one profile, then it holds for any profile in $(H^\star)^2$.

Let us begin the first statement.

(1) Fix a profile $\widetilde{\mathcal{R}}_N = (R_1, R_2) \in (H^\star)^2$. We prove that if $\nu^\star(\widetilde{\mathcal{R}}_N) \notin S(X, R_1)$, then $\nu^\star(\widetilde{\mathcal{R}}_N) \in S(X, R_2)$.

Suppose that $\nu^\star(R_1, R_2) = c$, $a \in S(X, R_1)$, and $b \in S(X, R_2)$, and c is distinct from a and b. Note that a and b must be distinct from each other, otherwise we immediately contradict the unanimity property. Let $\overline{R}_2$ be a $\star$-fuzzy order such that $b \in S(X, R_1)$, and $\forall\ x \neq a, S_{\overline{R}_2}(b) > S_{\overline{R}_2}(x)$.

Observe that $\nu^\star(R_1, \overline{R}_2)$ can not be equal to b. In fact, if $\nu^\star(R_1, \overline{R}_2) = b$, then $S_{R_2}(\nu^\star(R_1, \overline{R}_2)) > S_{R_2}(\nu^\star(R_1, R_2))$, since $\nu^\star(R_1, \overline{R}_2) = b \in S(X, R_2)$. Thus, $\nu^\star$ is manipulable at $\mathcal{R}_N$ via $\overline{R}_2$. Therefore, $\nu^\star(R_1, \overline{R}_2)$ must be different from b.

Let $\nu^\star(R_1, \overline{R}_2) = x$. Consider that the alternative x is distinct form a and b. We have $S_{\overline{R}_2}(a) > S_{\overline{R}_2}(x)$ and $\nu^\star$ would manipulate at $(R_1, \overline{R}_2)$ via a relation R with S_R corresponding to alternative a. The outcome would then be a because of the unanimity of $\nu^\star$. Therefore, $\nu^\star(R_1, \overline{R}_2) = a$.

Let $\overline{R}_1$ be a $\star$-fuzzy order with $a \in S(X, \overline{R}_1)$ and $\forall\ x \neq a, S_{\overline{R}_2}(b) > S_{\overline{R}_2}(x)$. We must have $\nu^\star(\overline{R}_1, \overline{R}_2) = a$, otherwise individual 1 manipulates at $(\overline{R}_1, \overline{R}_2)$ via $\overline{R}_1$.

Let $\nu^\star(\overline{R}_1, R_2) = x$. If $x = b$, then individual 2 manipulates at $(\overline{R}_1, \overline{R}_2)$ via R_2. If x is distinct from both a and b, then $S_{\overline{R}_1}(x) < S_{\overline{R}_1}(b)$. Therefore, individual 1 will manipulate at $(\overline{R}_1, R_2)$ via a relation R with S_R corresponding to alternative b. Therefore, $x = a$. But, then individual 1 manipulates at $\mathcal{R}_N$ via $\overline{R}_1$. Therefore, $\nu^\star(\widetilde{\mathcal{R}}_N)$ must be in $S(X, R_2)$.

(2) Now, show that if $\nu^\star(\mathcal{R}_N) = x$, with $x \in S(X, R_1)$ or $x \in S(X, R_2)$, for a given $\widetilde{\mathcal{R}}_N$, then it is holds for any $\mathcal{R}_N$.

Let $\mathcal{R}_N$ be a profile where $a \in S(X, R_1)$, $\forall\ x \neq a, S_{\overline{R}_1}(b) > S_{\overline{R}_1}(x)$, and $a \neq b$.

If individual's 2 preference relation is fixed at R_2, observe that the outcome for all profiles where $\forall\ x \neq a, S_{\overline{R}_1}(b) > S_{\overline{R}_1}(x)$, must be a.

14

Otherwise, individual 1 manipulates $(\overline{R}_1, R_2)$ via R_1.

If individual's 1 preference relation is fixed at R_1, observe that the individual 2 can never obtain outcome b by varying R_2. Therefore, according to the first statement, it follows that the outcome must be either a or b.

Consider an arbitrary outcome c distinct from both a and b. Let $\overline{R}_1$ be a $\star$-fuzzy order such that $c \in S(X, R_1)$, and $\forall \ x \neq a, S_{\overline{R}_1}(b) > S_{\overline{R}_1}(x)$. Thus, according to the first statement, it follows that $\nu^\star(\overline{R}_1, R_2)$ is either b or c. However, if it is b, individual 1 would manipulate at $(\overline{R}_1, R_2)$ via R_1. Therefore, the outcome is in $S(X, \overline{R}_1)$.

The proof is completed by showing that the outcome is in $S(X, R_1)$, or $S(X, R_2)$. Pick an arbitrary outcome x distinct from b and c. Consider that $b \in S(X, R_2)$ and $\forall y \neq b, S_{R_2}(x) > S_{R_2}(y)$. Let $\overline{R}_2$ be a $\star$-order, where $x \in S(X, \overline{R}_2)$ and $\forall y \neq x, S_{R_2}(b) > S_{R_2}(y)$. Note that $\nu^\star(\overline{R}_1, \overline{R}_2)$ must be either c or x. But if it is x then individual 2 will manipulate at $(\overline{R}_1, R_2)$ via $\overline{R}_2$. Since x and c were picked arbitrarily, the second statement is established.

Proof of Lemma 2.

Assume that statement (a) holds. Let $\nu^\star$ be strategy-proof $\star$-FSCF $\nu^\star$: $(H^\star)^n \to X$. Define a $\star$-FSCF $\mu : (H^\star)^{n-1} \to X$ as follows.

For all $(R_1, R_3, \ldots, R_n) \in (H^\star)^{n-1}, \mu(R_1, R_3, \ldots, R_n) = \nu^\star(R_1, R_1, R_3, \ldots, R_n)$. Since $\nu^\star$ satisfies unanimity, μ satisfies unanimity too. Note that μ is strategy-proof. Otherwise, $\nu^\star$ is manipulable.

Pick an arbitrary *n-1* individual profile $(R_1, R_3, \ldots, R_n)$ and let $\mu(R_1, R_3, \ldots, R_n) = \nu^\star(R_1, R_1, R_3, \ldots, R_n) = a$. Let $\overline{R}_1$ be an arbitrary $\star$-fuzzy order. Let $\nu^\star(\overline{R}_1, R_1, R_3, \ldots, R_n) = b$ and $\nu^\star(\overline{R}_1, \overline{R}_1, R_3, \ldots, R_n) = \mu(\overline{R}_1, R_3, \ldots, R_n) = c$. Since $\nu^\star$ is strategy-proof, $a \neq b$ implies $S_{R_1}(a) > S_{R_1}(b)$, $c \neq b$ implies $S_{R_1}(b) > S_{R_1}(c)$. This implies $S_{R_1}(a) > S_{R_1}(c)$. Therefore, μ cannot be manipulated by individual 1. Since μ satisfies unanimity and it is strategy-proof, statement (a) implies that μ is dictatorial. There are two cases to consider.

Suppose that the dictator is individual $j \in \{3, \ldots, n\}$. We will prove that j is a dictator for $\nu^\star$.

Pick an arbitrary profile $(R_1, R_2, R_3, \ldots, R_n)$. Let a be in $S(X, R_j)$ and let

$\nu^\star(R_1, R_2, R_3, \ldots, R_n) = b$. Since j is a dictator for μ, individual 1 can change the outcome from b in the profile $(R_1, R_2, R_3, \ldots, R_n)$ to a by announcing R_2.

Since $\nu^\star$ is strategy-proof, we must have $S_{R_1}(b) > S_{R_1}(a)$. Similarly, since $\nu^\star(R_1, R_1, R_3, \ldots, R_n) = a$, we must have $S_{R_1}(a) > S_{R_1}(b)$, or else individual 2 will manipulate at $(R_1, R_1, R_3, \ldots, R_n)$ via R_2. Thus, we have $a = b$. Therefore, $\nu^\star(R_1, R_2, R_3, \ldots, R_n) = a \in S(X, R_j)$. This returns that j dictates in $\nu^\star$.

Finally, we need to consider the case where j is individual 1 in μ. Pick arbitrary $n-2$ individual profile $(R_3, R_4 \ldots, R_n)$. Now define a two individual $\star$-$FSCF$ λ as follows: for all pairs of $\star$-fuzzy orders R_1, R_2, $\lambda(R_1, R_2) = \nu^\star(R_1, R_1, R_3, \ldots, R_n)$.

Since individual 1 is a dictator in μ, it follows that λ satisfies unanimity. Moreover, since $\nu^\star$ is strategy-proof, it follows immediately that λ is strategy-proof too. From step 1, we know that λ is strategy-proof, $i.e.$, λ is dictatorial. In order to complete the proof, we need only to show that the identity of the dictator does not depend on the n-2 profile $(R_3, R_4, \ldots, R_n)$ while 2 is dictator for $(\overline{R}_3, \overline{R}_4, \ldots, \overline{R}_n)$.

Now, progressively change preferences for each individual from 3 through n from the first profile to the second. There must be an individual j for $3 \leq j \leq n$ such that 1 is the dictator in $(\overline{R}_3, \ldots, \overline{R}_{j-1}, R_j, \ldots, R_n)$ while 2 dictates in $(\overline{R}_3, \ldots, \overline{R}_{j-1}, \overline{R}_j, R_{j+1}, \ldots, R_n)$.

Let a and b be such that $S_{R_j}(a) > S_{R_j}(b)$. Pick R_1 and R_2 such that $b = f_{R_1}$ and $a = f_{R_2}$, respectively. Then, $\nu^\star(R_1, R_2, \overline{R}_3, \ldots, \overline{R}_{j-1}, R_j, \ldots, R_n) = b$ while $\nu^\star(R_1, R_2, \overline{R}_3, \ldots, \overline{R}_{j-1}, \overline{R}_j, R_{j+1}, \ldots, R_n) = a$. Thus, j will manipulate at $(R_1, R_2, \overline{R}_3, \ldots, \overline{R}_{j-1}, R_j, R_{j+1}, \ldots, R_n)$ via $\overline{R}_j$. This completes the proof of *Lemma 2*.

5 Conclusion

The paper proposes two generalizations of the fuzzy manipulability and dictatorship of fuzzy social choice functions, by Ben Abdelaziz *et al.* (2008). Starting with max-$\star$ transitive *FPR*s, it considers first a best alternative set based on a the t-norm $\star$. Second, the decomposition of a weak fuzzy individual preference relation into a strict preference relation and an indifference one is used to generate a best alternative set. In both cases, an impossibility result on the strategy-proofness is shown. Future research avenue is to consider other types of fuzzy relation decompositions (*e.g.* De Baets *et al*, 1995).

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