Measuring quantities with analogue-digital systems

(Extended Abstract)

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February 21, 2022

Abstract

We consider a model of computation in which a physical device, capable of performing measurements, is coupled with a Turing machine, functioning as an oracle. This interaction, between the machine and the experiment, is mediated by a protocol, that specifies the experimental precision, and a schedule, that clocks the number of machine steps during a call to the physical oracle.

We start by studying the class of sets that can be decided using this hybrid model of computation, in polynomial time, when the schedule is an exponential function and considering three types of experimental precision: infinite, unbounded and finite.

We then introduce Hempel's theory of measurement, which captures the intuitive notion of a measurement procedure, and present a new axiomatization, which, including the concept of the duration of an experiment, recovers Hempel's theory as we allow time to approach infinity. We look at three forms of physical measurement and prove that, in each case, an experimental apparatus and a process can be devised to satisfy this axiomatization.

Finally, we study the physical parameters (regarded as real numbers) that can be measured with a given schedule – the measurable numbers. We consider the case where we allow the schedule to be an arbitrary (time constructible) function and the case where the complexity of the schedule is to be fixed a priori. In this last case, we characterize the real numbers that can be measured with two types of exponential schedules and with a primitive recursive schedule.

Keywords: Digital-analogue computation. Physical oracle. Fundamental measurement. Forms of physical measurement. Measurable numbers. Measurement complexity of a real number.

1. Introduction

In this thesis, following the work from [1], we consider a model of computation, the Smooth Scatter Machine (SmSM), in which a Turing machine is coupled with a physical experiment, the Smooth Scatter Experiment (SmSE), querying it as physical oracle by running the physical experiment. When a particle is shot at a position z and arrives at the right (left) collecting box, we know that the vertex is to the left (right) of z. Performing consecutive firings allows us to determine (or estimate) the position of y. We use the SmSE to boost the computational power of the Turing machine by encoding information that the Turing machine can use in the position of y. Figure 1 contains a schematic depiction of the SmSE.

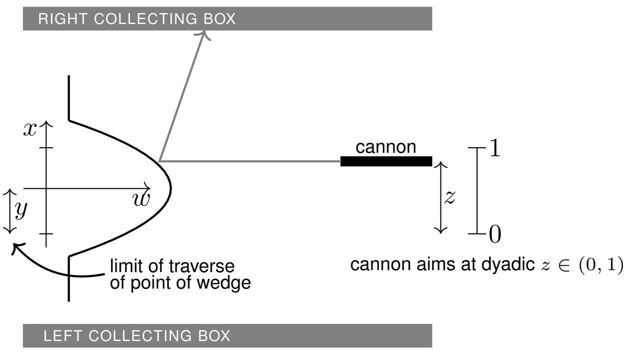

Figure 1: Schematic representation of the SME.

We present the computational power of the SmSM, when polynomial time restrictions are imposed, and see that they can be viewed as ideal technicians, controlling a measurement experiment. We study how to make the leap from comparing object in a domain to being able to assign them a numerical value, presenting the work from Campbell and Jeffreys in [5], to classify fundamental magnitudes, and Hempel's work from [6], regarding the axiomatization of measurement. We then present the work from [2], in which, considering the physical duration of an experiment, a new axiomatization of measurement was introduced, which recovered Hempel's as a limit concept. We propose a different axiomatization of measurement, which arises from performing comparisons using only dyadic rationals, and prove that, in the limit, we can again recover Hempel's notion of a measurement procedure. As a consequence of introducing the concept of time in an experiment, we study the time complexity required to measure a real number. This motivates the definition of a measurable number, as one that can be measured using some time constructible schedule, and the classification of a real number according to the time complexity of the schedule required to measure it.

2. Background

To classify the computational power of the SmSM, when polynomial time restrictions are imposed, we consider the following non-uniform complexity classes.

Definition 2.1. The class $\texttt{P}/\texttt{log}\star$ consists of sets B for which there is a prefix advice function $f \in \texttt{log}$ and a Turing machine M, running in polynomial time, such that, for every word w with $|w| \leq n$, $w \in B$ if and only if M accepts $\langle w, f(|w|)\rangle$.

Definition 2.2. The class $\texttt{BPP}//\texttt{log}\star$ consists of sets B for which, given a prefix advice function $f \in \texttt{log}$, there is a probabilistic Turing machine M, running in polynomial time, and a constant $\gamma < 1/2$, such that, for every word w with $|w| \leq n$, the probability of M rejecting $\langle w, f(|n|)\rangle$, with $w \in B$, or accepting $\langle w, f(|n|)\rangle$, with $w \notin B$, is, at most, γ.

A SmSM combines a digital computation, performed by a Turing machine, and an analogue computation, performed by the SmSE. To invoke the SmSE, the Turing machine writes a word z in the query tape and enters a shooting state: the cannon is aimed at z and an experiment is run. We consider three protocols that rule the data exchange between the Turing machine and the SmSM: *error-free*, if the position of the cannon can be set without any error, *error-prone* with *unbounded precision*, if the error in the position of the cannon can be arbitrarily small (but non-zero), and *error-prone* with *fixed precision*, if the error in the positioning of the cannon is a fixed value. We classify a SmSM according to the protocol with which is its defined.

For this work, we can assume that running the experiment with a query z and unknown vertex position y takes a physical time that is bounded by the following inequalities, for $A, C > 0$:

$$\frac{A}{|y-z|} \leq t(z) \leq \frac{C}{|y-z|}$$

To measure the vertex with the infinite and unbounded precision protocols, we use Algorithm 2.1. To measure the vertex with the fixed precision protocol we use Algorithm 2.2, in which we perform enough firings to estimate the position of the vertex with a given accuracy.

Algorithm 2.1: Measurement algorithm for infinite and unbounded precision.

```
Data: Positive integer l, representing the desired accuracy;
z_0 = 0; z_1 = 1; z = 0;
while z_1 - z_0 > 2^{-l} do
    z = (z_0 + z_1)/2;
    s = Prot_IP(z⌊_l) (resp. Prot_UP(z⌊_l));
    if s == q_r then
        z_1 = z;
    if s == q_l then
        z_0 = z;
    else
        z_0 = z; z_1 = z;
return z;
```

Algorithm 2.2: Measurement algorithm for fixed precision, with an error smaller than 2^{-h}.

```
Data: Positive integer l, representing the desired accuracy;
x = 0; i = 0; ξ = 2^(2l+h);
while i < ξ do
    s = Prot_FP(0.1↓_l);
    if s == q_l then
        c = c + 2;
    if s == q_t then
        c = c + 1;
    i++;
return c/(2ξ);
```

To achieve a characterization of the computational power of the error-free SmSM, we have to make the assumption that it is setup with a schedule $T(k) \in \Omega(2^k)$. For a complete characterization for the error-prone cases, we have to assume that the physical duration of an experiment is given by an expression of the form $t(z) = 1/|y - z|$. Then, depending on the protocol being used, we have the following classification of the computational power of the SmSM, in polynomial time and with an exponential schedule.

	Infinite	Unbounded	Bounded
Lower bound	$\mathtt{P/log\star}$	$\mathtt{BPP//log\star}$	$\mathtt{BPP//log\star}$
Upper bound	$\mathtt{P/log\star}$ Schedule in $\Omega(2^k)$	$\mathtt{BPP//log^2\star}$	$\mathtt{BPP//log^2\star}$
Upper bound Explicit time	–	$\mathtt{BPP//log\star}$	$\mathtt{BPP//log\star}$

Table 1: Computational results.

3. Theory of measurement

3.1. Fundamental measurement

We consider a set $\mathcal{O}$, of physical objects, endowed with some attribute, such as mass or length, and an experimental apparatus, which compares the objects in $\mathcal{O}$ according to that attribute. The following two definitions are found in [6].

Definition 3.1. Let $\mathcal{L}$ and $\mathcal{E}$ be two binary relations:

- $\mathcal{L}$ is $\mathcal{E}$-irreflexive if, for every $a, b \in \mathcal{O}$, if $a\mathcal{E}b$ holds, then $a\mathcal{L}b$ does not hold;
- $\mathcal{L}$ is $\mathcal{E}$-connected if, for every $a, b \in \mathcal{O}$, if $a\mathcal{E}b$ does not hold, then $a\mathcal{L}b$ or $b\mathcal{L}a$ holds;
- $\mathcal{E}$ and $\mathcal{L}$ are a comparative concept, if $\mathcal{E}$ is an equivalence relation and $\mathcal{L}$ is transitive, $\mathcal{E}$-irreflexive, and $\mathcal{E}$-connected.

The relations $\mathcal{L}$ and $\mathcal{E}$ are intended to represent, respectively, that an object has less or the same amount of the chosen attribute than another.

Definition 3.2. Let $\mathcal{L}$ and $\mathcal{E}$ be comparative concepts on the set $\mathcal{O}$ of objects. Suppose there is an experimental apparatus to witness these relations. Then, the map $M : \mathcal{O} \to \mathbb{R}$ is a *measurement map* if the following axioms are verified:

- If $a\mathcal{E}b$ holds, then so does $M(a) = M(b)$;
- if $a\mathcal{L}b$ holds, then so does $M(a) < M(b)$.

Following the idea from [2], we propose a new axiomatization of measurement, which, indexed in a parameter t, representing time, recovers Hempel's notion as time is allowed to increase arbitrarily.

Definition 3.3. Two collections of binary relations $\mathcal{E}_t$ and $\mathcal{L}_t$ determine a *limit* timed comparative concept for the elements of $\mathcal{O}$, if, for every $t > 0$ and $a, b, c \in \mathcal{O}$, the following hold:

- Exactly one of $a\mathcal{E}_t b$, $a\mathcal{L}_t b$, $b\mathcal{L}_t a$ holds;
- $\mathcal{E}_t$ is reflexive and symmetric;
- $\mathcal{L}_t$ is transitive;
- if $a\mathcal{L}_t c$ holds, there is a time T such that, if $a\mathcal{E}_T b$ holds, then $b\mathcal{L}_T c$ holds as well;
- if $a\mathcal{L}_t b$ holds, then there is an order T after which $a\mathcal{L}_{t'} b$ holds, for any $t' \geq T$.

Definition 3.4. Let $\mathcal{E}_t$ and $\mathcal{L}_t$ be limit timed comparative relations on a set $\mathcal{O}$, of objects, witnessed by some experimental apparatus. Then, $M : \mathcal{O} \to \mathbb{R}$ is a *measurement map* if, for any time $t > 0$, whenever $a\mathcal{L}_t b$ holds, so does $M(a) < M(b)$.

Definition 3.5. An apparatus, witnessing a limit timed comparative concept $\mathcal{E}_t$ and $\mathcal{L}_t$, satisfies the *separation property* for a measurement map $M : \mathcal{O} \to \mathbb{R}$ if, for every objects $a, b \in \mathcal{O}$, whenever $M(a) < M(b)$ holds, then so does $a\mathcal{L}_t b$, for some time bound $t > 0$.

Definition 3.6. Given a limit timed comparative concept $\mathcal{E}_t$ and $\mathcal{L}_t$, we define the relations $\mathcal{E}_{lim}$ and $\mathcal{L}_{lim}$ as follows:

- $a\mathcal{E}_{lim} b$ if, for every time bound t, $a\mathcal{E}_t b$;
- $a\mathcal{L}_{lim} b$ if there is a time bound t such that $a\mathcal{L}_t b$.

Proposition 3.7. Suppose there is physical apparatus witnessing a *limit* timed comparative concept $\mathcal{E}_t$ and $\mathcal{L}_t$ and that a measurement map is defined, in the sense of Definition 3.4, such that the apparatus satisfies the separation property. Then, in the sense of Definitions 3.1 and 3.2, $\mathcal{E}_{lim}$ and $\mathcal{L}_{lim}$ are a comparative concept and M is a measurement map.

3.2. Three types of measurement

Consider $\mathcal{O}$ the set of vertices with which we can setup the SmSM. To perform comparisons between vertices $a, b \in \mathcal{O}$, we setup two experiments, one with unknown vertex a and the other with unknown vertex b, comparing each of them by performing experiments with only dyadic rational positions. We will represent these two experiments as a single one: the Two Wedge Smooth Scatter Experiment.

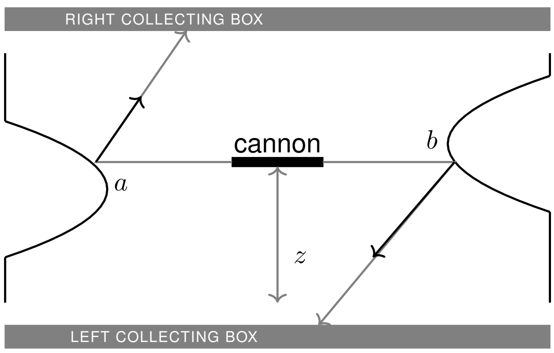

Figure 2: Schematic representation of the TSmSE.

By changing the capabilities of the collecting boxes, we can make it so this experiment perform each of the three from of comparison identified in [4]: *signed comparison*, if both $a\mathcal{L}b$ and $b\mathcal{L}a$ can be tested, *threshold comparison*, if either $a\mathcal{L}b$ or $b\mathcal{L}a$ can be tested, and *vanishing comparison*, if we can only test the predicate ($a\mathcal{L}b$ or $b\mathcal{L}a$).

Proposition 3.8. For signed, threshold and vanishing type comparison,it is possible to define relations $\mathcal{E}_t$ and $\mathcal{L}_t$ which, together with the respective TSmSE and the linear search algorithm, satisfy the axiomatization from Definitions 3.6 and 3.5.

4. Measurable numbers

Following the work from [3], we introduce the class of measurable numbers, presenting some properties of this class. We then use the concept of measurability to classify a real number according to the time complexity of the schedule required to measure it.

Definition 4.1. We say that a real number $y \in (0, 1)$ is *measurable* if there exists a SmSM M, with vertex y and a time constructible schedule T, such that M, knowing $n - 1$ digit of y, outputs its nth digit in time $T(n)$, i.e., without timing out.

We consider the following form of a non-dyadic real $y \in (0,1)$, where $u_1 \geq 0$ and $u_i \geq 1$, for every $i \geq 2$:

$$y = 0 \cdot \underbrace{1 \ldots 1}_{u_1}\underbrace{0 \ldots 0}_{u_2}\underbrace{1 \ldots 1}_{u_3} \ldots \tag{1}$$

Proposition 4.2.

- A non-dyadic $y \in (0,1)$, written according to Pattern 1, is measurable if and only if the sequence u_k is bounded by a computable function.
- A non-dyadic $y \in (0,1)$, whose sequence u_k is bounded by a computable function, can be measured by the Turing machine equipped with the linear search algorithm.

Proposition 4.3. With probability 1, a real numbers in $(0,1)$ is measurable, but there is an uncountable set of non-measurable real numbers.

We study four types of bounds on the sequence u_k, of a real number y: the class of polynomials, of exponential functions, of function in a given layer of the Grzegorczyk hierarchy and of primitive recursive functions. For a real number $y \in (0,1)$, written according to Pattern 1, we summarize the results obtained in Table 2. The first row represents the order of the function bounding u_k; the second row represents the time complexity of the schedule with which we can measure the vertex y.

Bound on u_k	$O(k^m)$	$2^{O(k)}$	PR
Time complexity of $T(k)$	$O(2^{k+O(\lfloor k^{(m-1)/m} \rfloor)})$	$2^{O(k)}$	PR

Table 2: Characterization of measurable numbers with a fixed schedule complexity.

Moreover, if u_k is in the nth level of the Grzegorczyk hierarchy, then y can be measured with a schedule, at most, in that same level. The reverse property is not verified, which we show by giving an elementary schedule with which we can measure a real number, whose expansion is not bounded by any elementary function.

5. Conclusions

This work regarded the measurement of quantities by means of an analogue-digital system. We started by studying the class of sets that can be decided by the SmSM, when polynomial time restrictions are imposed, observing that there is a limitation as to how much a physical experiment can boost the computational power of a Turing machine. Many questions remain unanswered in this topic, such as whether or not it is possible to achieve full characterizations as the one presented in Table 1, without making the previously mentioned assumptions, and if we obtain the same computational classes when we do not restrict the interaction between the Turing machine and the physical world to a measurement experiment with one of the three presented protocols.

We then studied the measurement performed by the error-free SmSM from the perspective of fundamental measurement, presenting a new axiomatization of measurement, with an indexation in a time parameter, and proving that the three forms of physical measurement identified in [4] can be represented by an experiment that will satisfy this axiomatization. As future work in this area, we suggest that the same study be conducted with the error-prone protocols, either by proving that it is possible to satisfy, in the limit, Hempel's axiomatization, or by considering a new axiomatization for measuring with errors.

We concluded with the study of the class of numbers that can be measured by the SmSM, presenting the characterization of this class given in [3]. We then considered sub-classes of measurable numbers that can be measured with a schedule with a given time complexity, achieving the characterizations form Table 2. This work has just begun and there is still a lot to explore, such as the possibility to create a hierarchy of the set of real numbers according to their measurement complexity.

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