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Optimal Boundary Control of The Time Dependent Stokes Equations with Mixed Boundary Conditions

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Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

Acknowledgments

I would like to thank to my professor Jorge Tiago for the help that he gave me on the construction of this thesis, by giving some references and comments on how the work should evolve, but always letting me be free in the choice of how to do it.

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Resumo

No presente trabalho, propomos uma possível formulação teórica para o estudo do controlo ótimo nos dados na fronteira para a classe das equações de Stokes com condições de fronteira mistas.
Fornecemos também uma explicação teórica do problema e dos espaços funcionais usados de forma a deduzir (i) o problema de controlo ótimo está bem posto, e (ii) para fornecer as condições necessárias de otimalidade de primeira ordem, que no nosso caso será também uma condição suficiente.
Consideramos custos quadráticos do tipo de velocity-tracking, de vorticidade ou então uma soma dos dois com diferentes pesos.
Para o processo de otimização um método de descida é utilizado. São apresentados resultados numéricos (2D) para vários casos, num domínio que pretende simular a bifurcação arterial que se forma após um bypass.

Palavras-chave : Equações de Stokes com condições de fronteira mistas; problema não estacionário; elementos Finitos; controlo ótimo na fronteira; equações diferenciais parciais; dinâmica de fluídos

Abstract

In the present work er propose a possible framework for a optimal boundary control problem, for the time dependent Stokes mixed boundary conditions.
We provide a theoretical framework to address (i) the well posedness analysis for the optimal control problem related to this system and (ii) the derivation of a system of first-order optimality conditions.
We consider the minimization of quadratic cost (e.g., tracking or vorticity) functionals of the velocity. A descent method is then applied for numerical optimization.
Numerical results are shown for (2D) simulation of an arterial bifurcation after a bypass is done.

Keywords: Optimal boundary control; partial differential equations; fluid dynamics; Time- dependent Stokes equations with Dirichlet-Neumann boundary conditions;finite element method

1 Introduction

Optimal control problems (OCP's) in the framework of partial differential equations, is a research field with challenging theoretical questions, for mathematical analysis and definition of numerical efficient algorithms, which has a large range of applicability on engineering, as for instance, in aerodynamics. In the present work, we will focus on doing the minimization of quadratic cost functionals, or a combinations of quadratic cost functionals (see for example [15; 4; 24; 11; 12; 9]), associated to a non-stationary, incompressible fluid dynamic problem, with mixed boundary conditions. To be more precise, we are interested in minimizing a particular collection of cost functionals, which are constrained to be evaluated at solutions (in some sense) to the following mixed boundary time-dependent Stokes equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \nabla p - \mu \Delta y = f \ , \text{ in } \Omega_T \ , (1) \\ \nabla \cdot y = 0 \ , \text{ in } \Omega_T \ , (2) \\ y = g \ , \text{ in } \Gamma_1 \times (0,T) \ , (3) \\ y = 0 \ , \text{ in } \Gamma_w \times (0,T) \ , (4) \\ \mu \nabla y \cdot \mathbf{n} - p\mathbf{n} = 0 \ , \text{ in } \Gamma_{out} \times (0,T) \ , (5) \\ y(0,.) = y_0 \ , (6) \end{cases} \tag{1}$$

where Ω and it's boundary parts are illustrated in (1) (we denote by $\Gamma_1 = \Gamma_{in} \cup \Gamma_c$), T is a fixed terminal time and $Q = \Omega_T = \Omega \times (0,T)$. The components (1) and (2) of this system of equations describe the behavior of an imcompressible fluid with low viscosity in the domain Ω. The condition (3) is the imposed Dirichlet condition, and (4) is the no-slip condition and is physically interpreted as saying that the fluid aggregated to the wall has no movement. This condition in the framework of blood flow may also be interpreted as saying that the walls have no movement. Lastly, the condition (5) is an artificial condition, that is usually used in this type of problems, which appears from a simplification for the variational formulation that appears when integration by parts is done in (1) of (1).

One possible application of this theoretical problem, may be the study of the control of the blood flow in an arterial bifurcation, after a bypass is done[1], with the goal of knowing which velocity profile must exist on Γ_c in such a way that the vorticity of the fluid is minimized, or the proximity (in some norm which is usually the L^2-norm) to some target velocity, that we know to be representative of a well-behaved blood flow. Also is very common, and we do that in here, to add a penalization parameter, to the cost functional, which in our case is a quadratic functional depending on the control.

Boundary control problems in the context of fluid dynamics, isn't a new subject, and a large amount of theory and research has already been done, see for example the book [11] where the author shows a collection of typical problems in boundary optimal control. However, as far as we know, the type of boundary conditions considered in the present work is not yet fully studied, at least on the framework of a time dependent problem. Also, notice that in our case, we do not ask the boundary control function u, to satisfy an integral condition $\int_{\Gamma_c} u \cdot \boldsymbol{n} ds = 0$.

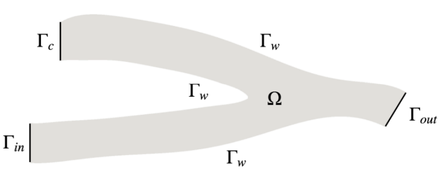


Figure 1: Omega Domain.

In what follows, we discuss some of the references that can be found in the literature, which were relevant for our work.

The stationary version of our problem, was studied in [24]. In that article, the authors considered a similar domain as the one illustrated in figure (1), and they were also interested in minimizing the vorticity and target velocity functionals, by controlling the input velocities in Γ_c, where this

[1]The bypass being the upper part of the bifurcation, and the obstructed the bottom one of the image (1).

functionals were evaluated on a collection of pairs (y, u) (u being the control and y the velocity) which are restricted to satisfy in some weak sense the stationary Navier-Stokes problem in Ω, given by,

$$\begin{cases} -\mu\Delta y + (y\cdot\nabla)y + \nabla p = 0 \text{ in } \Omega \\ \nabla\cdot y = 0 \text{ in } \Omega \\ y|_{\Gamma_c} = u \\ y|_{\Gamma_w} = 0 \\ y|_{\Gamma_{in}} = v^{in} \\ (\partial_n y - p\boldsymbol{n})|_{\Gamma_{out}} = 0 \end{cases} \tag{2}$$

As we can see, this problem is very similar to ours, but, however, by introducing the time dependence we fall into a problem which is not easily adapted by the work done in that paper. From there, and the examples in [11] which are about boundary control problems, we also took the idea of using, for our weak formulation, velocities which do not necessary have null trace in $\Gamma_c \cup \Gamma_{in}$. This type of functions play an important role in the first order conditions for optimality.

In [15] a boundary control problem in a time-dependent framework is considered, but in that case, the control is done in the whole boundary,

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = 0 \text{ in } Q \\ -\nabla\cdot y = 0 \text{ in } Q \\ y = Bu \text{ in } \Sigma \\ y(0,\cdot) = y_0 \text{ in } \Omega \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. In that article, the method used to arrive to the first order optimality conditions were very useful to our problem. The paper also provides, some guide steps for which functional spaces we should look for in this type of problems, in order to define a proper weak formulation for our problem.

Lastly, we refer to the article of [12] were they also address a minimization problem for a velocity target functional, associated with a time-dependent Navier-Stokes equation,

$$\begin{cases} \dfrac{\partial y}{\partial t} + (y\cdot\nabla)y + \nabla p = 0 \text{ in } \Omega\times(0,T) \\ \nabla\cdot y = 0 \text{ in } \Omega\times(0,T) \\ y = g \text{ in } \Gamma_c\times(0,T) \\ y = 0 \text{ in } (\Gamma\setminus\Gamma_c)\times(0,T) \\ y(0,\cdot) = y_0 \text{ in } \Omega \end{cases} \tag{4}$$

However, as can be seen, the problem has not a mixed boundary condition,and therefore, since the solution must satisfy the equation $\nabla\cdot v = 0$ and the boundary value in $\Gamma\setminus\Gamma_c$ is null, we must have, by a compatibility condition, that

$$\int_{\Gamma_c} g\cdot\boldsymbol{n}ds = 0 \tag{5}$$

where $\boldsymbol{n}$ is the unitary normal to the boundary in Γ_c. This restriction is not realistic in our case since it is possible to have a control with non zero boundary flux, as for example a parabolic inflow in Γ_c.

Other works were also seen such as [4; 11; 9; 7] and were important for the present work.

This work is organized as follows. In section-2 we give a motivation for the weak formulation and define the functional setting for our weak problem. This part is very important not only for the state equation but also for the first order necessary conditions of optimality. In section-3 we analyze the well-posedness of the state equation, by using the classical Galerkin method for a linear parabolic PDE. In section-4 we give a proof of the existence of an optimal solution for the minimization problem and also we deduce the first order optimality conditions. In section-5 we turn our attention to the discretization of the problem and lastly in section-6 we present some numerical results, for a velocity tracking functional, with observations on the whole domain, and also in some particular areas of the domain, in order to also study the influence of the observations zone on efficiency of the numerical method to reach the target flow. A vorticity functional and a combination of the two is also considered. On those simulations, we considered two cases for the input and fixed velocity v_{in}: (i) the case with the total obstruction $v_{in} = 0$, and the case of a strongly decreasing function with a high value on the center of Γ_{in}, in order to modulate a type of almost total obstruction.

2 Functional Setting and Preliminary Results

In this section we start by giving a motivation for the definition of the functional spaces that we will use in the weak formulation. After, we show a possible way of construction lifting operators, for the Dirichlet boundary data, a norm that yields an Hilbertian structure for the set of boundary functions on the control zone, and the set of admissible initial conditions. We close this chapter with some important notes.

2.1 Motivation for the Functional Spaces

In classical terms, a solution of the problem (1) is a pair (y, p) that satisfies the equation, and such that $y \in C^{2,1}(\overline{\Omega_T})$, where the superscript indices correspond to the order of the derivatives in x and in t, by this order, and $p \in C^1(\overline{\Omega_T})$.
However this formulation is too strong. For instance, if we consider $f \in L^2(0,T; L^2(\Omega))$ (see the subsection (7.3) of the appendix for the definition of this spaces), which is the case for many of the applications, we have that in general f would be a discontinuous, both in the time and space variables, and in this case , there is no hope in finding a classical solution u that satisfies (1), since then we would get that the second space derivatives of u are equal to a discontinuous function, and therefore y cannot be in $C^{2,1}(\overline{\Omega})$.
Therefore we need to weaken the concept of solution to (1), in order to proof the well-posedness for a more variability of forms of the equation (1). This corresponds to consider solutions in a bigger space, which must contained the set of the classical ones. In some cases using regularity improving results, which can be seen, for instance, in the books, [8] or [21], we can proof that, in fact, the founded weak solution, is also a classical solution. These results are build under some regularity assumptions on the force term f, the boundary conditions, the initial condition and also on the regularity of the boundary $\partial\Omega$, and we will talk about them on the chapter of the solution regularity, but not in a deep way.

In order to accomplish the goal of weakening the concept of solution to (1), we will use the classical Sobolev spaces (see the chapter (7.2) of the appendix for a quick review and the notation used) or in more detail [8; 1; 32; 33; 22], the $L^p(0,T;X)$ spaces where X is a Banach space (appendix section (7.3)) or in detail [8; 32; 33], and vector-valued (see appendix section or scalar distributions (7.4)) or in detail [32; 33].

As usual we will denote by $\mathscr{D}(\Omega) = \{\varphi : \varphi \in C_0^\infty(\Omega)\}$ the set of test functions whose dual space $\mathscr{D}(\Omega)^*$, is called the set of distributions on Ω. Also to set notations, we will denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product, by $((\cdot,\cdot))$ the $H_0^1(\Omega)$ inner product, and lastly by $\langle f, \varphi\rangle_{X^*,X}$ the duality product between a functional $f \in X^*$ and an element of X.
Let us now introduce a space that is fundamental for our construction (see [2] pag.3)

$$\mathscr{E}(\overline{\Omega}) = \{u \in C^\infty(\overline{\Omega})^2 : \nabla \cdot u = 0 \text{ , } supp(u) \cap \Gamma_D{}^2 = \emptyset\}$$

where $supp(u)$ is the support of u, i.e, is the biggest closed set where the function u is different from zero. Notice that the functions of $\mathscr{E}(\overline{\Omega})$, which is not an empty set, are characterized by having null divergence and null trace in the Dirichlet boundary part of $\partial\Omega$. $\mathscr{E}(\overline{\Omega})$ has also functions which

[2] $\Gamma_D = \Gamma_c \cup \Gamma_{in} \cup \Gamma_c$.

have non-null traces on $\partial\Omega \setminus \Gamma_D$. By this fact, this set is not contained in $\mathscr{D}(\Omega)$, but, and this is very important, $\mathscr{E}(\overline{\Omega})$ contains the set $\mathscr{V} = \{\varphi \in \mathscr{D}(\Omega)^2 : \nabla \cdot \varphi = 0\}$.[3]

Now we introduce a formal motivation to the weak formulation, and therefore of the used functional spaces.
We start by supposing that every term of 1-(1) is sufficient regular, both in time and in space[4], and multiply that equation by $\varphi \in \mathscr{E}(\overline{\Omega})$ followed by integration in the space variables. We get, by using Green formulas, which are valid if we assume enough regularity (see [10]), and the fact $\varphi|_{\Gamma_D} = 0$, that

$$\left(\frac{\partial u}{\partial t}, \varphi\right) - \mu(\Delta u, \varphi) + (\nabla p, \varphi) = (f, \varphi) \Leftrightarrow$$
$$\left(\frac{\partial u}{\partial t}, \varphi\right) + \mu(\nabla u, \nabla \varphi) = (f, \varphi) + \int_{\Gamma_{out}} (\mu \nabla u \cdot n - p\boldsymbol{n})\varphi$$

From the do-nothing condition (equation (5) of the system (1)), the last term on the right hand side is zero, from where we get that for every $\varphi \in \mathscr{E}(\overline{\Omega})$,

$$\left(\frac{\partial u}{\partial t}(t), \varphi\right) + \mu(\nabla u(t), \nabla \varphi) = (f(t), \varphi) \text{ for all } t \in [0, T] \tag{6}$$

where we suppressed the space dependence.
Notice now that if, $\varphi \in H^1_{\Gamma_D}(\Omega)^2 := \{u \in H^1(\Omega)^2 : \widehat{\tau_D}(u) = 0\}$, where $\widehat{\tau_D}$ is the trace operator $\widehat{\tau_D} : H^1(\Omega) \to L^2(\Gamma_D)$ (for the definition and properties of this operators see the appendix section (7.2)), the above argument still makes sense (see [10]).
This motivates the definition of the space

$$\widetilde{V} = \overline{\mathscr{E}(\overline{\Omega})}^{H^1(\Omega)} \tag{7}$$

Now since clearly $\widetilde{V} \subset H^1(\Omega)^2$, and is constituted by functions with null trace in Γ_D, with $|\Gamma_D| > 0$, by theorem 7.3, the space $\widetilde{V}$ equipped with the inner product of $H^1_0(\Omega)$ is a Hilbert space.
Until now, we only weaken the space of test functions and not the solution u. Suppose now that u is C^1 in time (in particular we can fix $t = t_0 \in [0, T]$) but only $H^1(\Omega)^2$ in the space coordinates, that is, u can be seen as a vector valued function $u : [0, T] \to H^1(\Omega)^2$ which is C^1 in the variable t. In this case all the terms in the expression (6) still makes sense. But again , assuming classical diferentiability for the solution u, can be restrictive, since as said above, we are interested in analyzing a problem with a force term of the form $f \in L^2(0, T; X)$, X being a Banach space, which does only have L^2 regularity in time.
So we want our solution u to also have at least L^2 regularity in time[5], but in this cases we need to clarify the meaning of the time derivative. This can be done by using the notion of derivative in the sense of vector-valued distribution (see appendix section (7.5)).
Notice now that from the no slip condition in Γ_w for almost every time, we are looking for a solution, which has null trace on the lateral boundary $\Sigma_w = \Gamma_w \times (0, T)$.
Therefore, it makes sense to introduce the space $L^2(0, T; \mathbb{V})$ where,

$$\mathbb{V} := \{u \in H^1(\Omega)^2 : \nabla \cdot u = 0, \text{ and } \widehat{\tau_w}(u) = 0\} \tag{8}$$

with $\widehat{\tau_w} : H^1(\Omega)^2 \to L^2(\Gamma_w)$ being the trace operator on the subset Γ_w of Γ, which is linear and continuous as can be seen in the appendix section (7.2).
Let us see, that $\mathbb{V}$ is a Hilbert space, when equipped with the $H^1_0(\Omega)^2$ inner product.
Since the operator $\widehat{\tau_w} : H^1(\Omega)^2 \to L^2(\Gamma_w)$ is a continuous operator, when the spaces $H^1(\Omega)^2$ and $L^2(\Gamma_w)$ are equipped with the usual norms, $\widehat{\tau_w}$ has a closed kernel. Moreover, since the divergence operator div : $H^1(\Omega)^2 \to L^2(\Omega)^2$ is also continuous and linear, we get that $\mathbb{V}$ is the intersection of two closed sets, and therefore is a closed subset of $H^1(\Omega)^2$ for the $H^1(\Omega)^2$ norm. By the fact that every closed subset of a complete space, is also complete, we get that $\mathbb{V}$ is complete. Notice

[3]This inclusion is fundamental for the application of the DE Rham's theorem, which is used to the construction of a pressure field.
[4]This regularity assumptions allows us to fix a time $t \in [0, T]$.
[5]The choice of L^2 regularity can be explain by the fact that this space as a Hilbert structure, what can be more easily work with, then only a Banach space.

now, that from the Poincaré's inequality (theorem 7.3), since Ω is bounded and the functions of $\mathbb{V}$ have null trace in a subset of $\partial\Omega$ with positive Lebesgue measure, the H^1 norm is equivalent to the H_0^1 norm in the subspace $\mathbb{V}$. So the two norms are equivalent, and since $\mathbb{V}$ is complete for the H^1 norm is also complete for the H_0^1 norm.
In conclusion $\mathbb{V}$ is a Hilbert space for the inner product of $H_0^1(\Omega)$.

In this context, motivated by (6), we are looking for a solution $u(t) \in L^2(0,T;\mathbb{V})$ and a time derivative $u'(t)$ also in $L^2(0,T;\mathbb{V})$ such that

$$(u'(t), v) + \mu(\nabla u(t), \nabla v) = (f(t), v) \text{ a.e } t \in (0,T) \text{ and for all } v \in \widetilde{V} \tag{9}$$

Notice however, that if $u(t) \in L^2(\mathbb{V})$ and $f(t) \in L^2(0,T;L^2(\Omega)^2)$ then from (9) we get that

$$(\frac{\partial u}{\partial t}, v) = (f(t), v) - \mu(\nabla u, \nabla v) \ , \forall v \in \widetilde{V} \text{ a.e } t \in (0,T) \tag{10}$$

and the expression on the right defines a continuous functional in $\widetilde{V}$ for almost every $t \in (0,T)$. Therefore the term on the left $u'(t)$ should also be in $\widetilde{V}^*$ for almost every $t \in (0,T)$.
We conclude that $u'(t)$ should be in the bigger space $L^2(0,T;\widetilde{V}^*)$, and we arrive to the variational formulation

$$\langle u'(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} + \mu((u(t), v)) = (f(t), v) \text{ a.e } t \in (0,T) \text{ and for all } v \in \widetilde{V} \tag{11}$$

The above reasoning, leads us to the introduction of the space

$$\widetilde{\mathbb{W}}(0,T) := \{u \in L^2(0,T;\mathbb{V}) : u'(t) \in L^2(0,T;\widetilde{V}^*)\}$$

on which we define the norm

$$||y||_{\mathbb{W}(0,T)} := \left(||y||^2_{L^2(\mathbb{V})} + ||y_t||^2_{L^2(\tilde{V}^*)}\right)^{1/2}$$

It is also useful to introduce the following time dependent Sobolev spaces,

$$\mathbb{W}(0,T) := \{u \in L^2(0,T;\mathbb{V}) : u'(t) \in L^2(0,T;\mathbb{V}^*)\}$$

with norm

$$||y||_{\mathbb{W}(0,T)} = \left(||y||^2_{L^2(\mathbb{V})} + ||y'||^2_{L^2(\mathbb{V}^*)}\right)^{1/2}$$

which is induced by the inner product $(u,v)_{\mathbb{W}(0,T)} = \int_0^T (y(t), u(t))_{\mathbb{V}} dt + \int_0^T (y'(t), u'(t))_{\mathbb{V}^*} dt$ and the space

$$W(0,T) := \{u \in L^2(0,T;\widetilde{V}) : u'(t) \in L^2(0,T;\widetilde{V}^*)\}$$

with the norm with the norm

$$||y||_{W(0,T)} = \left(||y||^2_{L^2(\tilde{V})} + ||y'||^2_{L^2(\tilde{V}^*)}\right)^{1/2}$$

which is induced by the inner product $(u,v)_{W(0,T)} = \int_0^T (y(t), u(t))_{\tilde{V}} dt + \int_0^T (y'(t), u'(t))_{\tilde{V}^*} dt$.
It is possible to proof by using the theorem 7.6, that $\mathbb{W}(0,T)$ and $W(0,T)$ are Hilbert spaces for the inner products introduce above. The introduction of this time-dependent spaces is by now not intuitive. However we can antecede their justification, by saying that is from the space $\mathbb{W}(0,T)$ that we will get the a proper lifting for the Dirichlet data, and the space $W(0,T)$ is the space from where we will get a variational solution, for our variational problem to be define later.
It is possible to proof a certain time regularity for the functions in $\mathbb{W}(0,T)$ and $W(0,T)$, which will be useful for the weak formulation for the present problem, but first one needs to introduce another two spaces , $\widetilde{H}$ and $\mathbb{H}$ given by

$$\widetilde{H} = \overline{\mathscr{E}(\overline{\Omega})}^{L^2(\Omega)}$$

and

$$\mathbb{H} = \overline{\{v \in H^1(\Omega)^2 : \nabla \cdot v = 0 \text{ and } \widehat{\tau_w}(v) = 0\}}^{L^2} = \overline{\mathbb{V}}^{L^2}$$

and we give to $\widetilde{H}, \mathbb{H}$ the inner product of $L^2(\Omega)$ which gives to these spaces a Hilbertian structure. By definition is clear that $\mathbb{H}$ contains the set $\mathbb{V}$. To see that we also have $\widetilde{V} \subset \widetilde{H}$, notice that since $H^1(\Omega)^2 \subset L^2(\Omega)^2$, by choosing a function $y \in \widetilde{V} \subset H^1(\Omega)^2$, y satisfies

$$||y - y_n||_{H^1(\Omega)} \to 0$$

for a certain sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathscr{E}(\overline{\Omega})$. Now from the fact that for every $x \in H^1(\Omega)^2$, $||x||_{L^2(\Omega)} \leq ||x||_{H^1(\Omega)}$, we also have that

$$||y - y_n||_{L^2(\Omega)} \leq ||y - y_n||_{H^1(\Omega)} \to 0$$

and therefore y is also in $\overline{\mathscr{E}(\overline{\Omega})}^{L^2} = \widetilde{H}$.
Now we proof a density result.

Lemma 2.1. *The set $\widetilde{V}$ is dense in $\widetilde{H}$.*

Proof: For this proof we start by introducing the stationary version of the problem (1) with homogeneous Dirichlet data on $\Gamma_D = \Gamma_w \cup \Gamma_{in} \cup \Gamma_c$, and with $f \in \widetilde{H}$

$$\begin{cases} -\mu\Delta u + \nabla P = f \text{ in } \Omega \\ \nabla \cdot u = 0 \text{ in } \Omega \\ u = 0 \text{ in } \Gamma_D \\ \mu\nabla u \cdot \mathbf{n} - p\mathbf{n} = 0 \text{ in } \Gamma_N \end{cases} \tag{12}$$

The weak formulation of (12), using the space $\widetilde{V}$, is given by,

$$\text{Find } u \in \widetilde{V} \text{ , such that } \mu((u, v)) = (f, v) \text{ , for all } v \in \widetilde{V} \tag{13}$$

Notice that the application $((.,.))$ is an inner product in $\widetilde{V}$, and therefore, is a bi-linear, bounded and coercive application on $\widetilde{V}$. Thus by the Hilbertian structure of $\widetilde{V}$, and the Lax-Milgram theorem (see [31]), the Stokes solver operator $\Lambda : \widetilde{H} \to \widetilde{V}$ is well defined, linear and a bounded operator.
Now, since[6] by theorem 7.2, $H^1(\Omega)^2 \hookrightarrow\hookrightarrow L^2(\Omega)^2$, we also have that $\widetilde{V} \hookrightarrow\hookrightarrow \widetilde{H}$. In fact, let $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{V}$ be a bounded a sequence, and let $I : H^1(\Omega) \to L^2(\Omega)$ be the compact embedding operator. Thus, since $\widetilde{V} \subset H^1(\Omega)$ and $I(\widetilde{V}) \subset \widetilde{H}$, the sequence $\{I(u_n)\}_{n\in\mathbb{N}}$ is contained in $\widetilde{H}$ and by the compactness of I, has a subsequence which is convergent for an element of $L^2(\Omega)$. But since $\widetilde{H}$ is closed in $L^2(\Omega)$, the subsequence converges for an element in $\widetilde{H}$, what yields $\widetilde{V} \hookrightarrow\hookrightarrow \widetilde{H}$. Let $\widetilde{I} : \widetilde{V} \to \widetilde{H}$ be the compact injection.
Now, since the operator $\Lambda : \widetilde{H} \to \widetilde{V}$ is continuous, the operator $\widetilde{\Lambda} : \widetilde{H} \to \widetilde{H}$ given by $\widetilde{\Lambda} = I \circ \Lambda$, is compact, since is the composition of a compact operator with a continuous one.
This operator is also positive[7] and self-adjoint. In fact, for arbitrary $f_1, f_2 \in \widetilde{H}$, let $\Lambda(f_1) = u_1$ and $\Lambda(f_2) = u_2$

$$(\widetilde{\Lambda} f_1, f_2) = (u_1, f_2) = \mu((u_1, u_2)) = (f_1, u_2) = (\widetilde{\Lambda} f_2, f_1) \text{ , for all} f_1, f_2 \in \widetilde{H}$$

and

$$(\widetilde{\Lambda} f_1, f_1) = \mu((u_1, u_1)) \geq C||u_1||^2_{\widetilde{V}}$$

where C is the coercive constant of $((.,.))$. Notice also that the strict positiveness implies that $\widetilde{\Lambda}$ is injective.

From the above properties we can conclude, by the spectral theorem for self-adjoint, compact

[6] In the case of two Banach spaces $X \subset Y$, we denote by $X \hookrightarrow\hookrightarrow Y$ the fact that the injection $I : X \to Y$, given by $I(x) = x$ is a compact and linear operator.
[7] An operator $T : X \to X$, with X a Hilbert space, is a positive application if $(Tx, x) \geq 0$ for all $x \in X$

and positive operators, that exist a family $\{\lambda_1, \lambda_2, ...\}$ of eigenvalues, which satisfy $\lambda_i > 0$ for all $i \in \mathbb{N}$ and $\lambda_i \to 0$ when $i \to \infty$, and that also exists an orthonormal family of vectors in $\widetilde{H}$ $\{u_1, u_2, ...\}$, such that $\widetilde{\Lambda}$ can be written as

$$\tilde{\Lambda}(u) = \sum_n \lambda_n (u, u_n) u_n \ , \forall u \in \widetilde{H} \tag{14}$$

Now, since $\widetilde{\Lambda}(\widetilde{H}) = \widetilde{V} \subset \widetilde{H}$, the orthonormal vectors $\{u_n\}$ are all in $\widetilde{V}$, and they define a Hilbertean base for $\overline{Im(\widetilde{\Lambda})} = \overline{\widetilde{V}}$, and since $\overline{\widetilde{V}} = Ker(\widetilde{\Lambda})^{\perp} = \{0\}^{\perp} = \widetilde{H}$, we conclude that $\widetilde{V}$ is dense in $\widetilde{H}$ as we wanted to show. □

Now, from definition $\mathbb{V}$ is dense in $\mathbb{H}$, since $\mathbb{H}$ is the closure of $\mathbb{V}$ in the L^2-norm, which is the norm of $\mathbb{H}$, and also, from lemma 2.1 that $\widetilde{V}$ is dense in $\widetilde{H}$. Therefore, we have the Gelfand's[8] triples

$$\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}^* \qquad\qquad \tilde{V} \hookrightarrow \tilde{H} \hookrightarrow \tilde{V}^*$$

what leads us to the conclusion, using the theorem 7.7, that $\mathbb{W}(0,T) \hookrightarrow C([0,T], \mathbb{H})$ and $W(0,T) \hookrightarrow C([0,T]; \tilde{H})$, i.e , any function in $\mathbb{W}(0,T)$ is a continuous function from $[0,T]$ to $\mathbb{H}$ eventually after a change in a set of measure zero, and the same for $W(0,T)$.
Now, since $\widetilde{H} \subset \mathbb{H}$ and they both have the same norm, we have that $\widetilde{H} \hookrightarrow \mathbb{H}$. This yields that $W(0,T) \hookrightarrow C([0,T]; \mathbb{H})$.
A very important consequence of this type of embedding is that an analogue integration by parts formula is valid (see [30]).

Lemma 2.2. *Let $t \in (0,T]$.*

For every $y(t), p(t) \in \mathbb{W}(0,T)$

$$\int_0^t \langle y'(s), p(s) \rangle_{\mathbb{V}^*, \mathbb{V}} ds = \big(y(t), p(t)\big) - \big(y(0), p(0)\big) - \int_0^t \langle y(s), p'(s) \rangle_{\mathbb{V}, \mathbb{V}^*} ds \tag{15}$$

And similarly, for every $y(t), p(t) \in W(0,T)$

$$\int_0^t \langle y'(s), p(s) \rangle_{\widetilde{V}^*, \widetilde{V}} ds = \big(y(t), p(t)\big) - \big(y(0), p(0)\big) - \int_0^t \langle y(s), p'(s) \rangle_{\widetilde{V}, \widetilde{V}^*} ds$$

Notice that in this formulas $(.,.)_{\widetilde{H}} = (.,.)_{\mathbb{H}} = (.,.)$ and so we do not distinguish them, denoting all simply by $(.,.)$

We take here the chance, to make an observation, that clarifies some steps in the proof of theorem (3.1). The equation (11) can be written in a equivalent integral formulation, which is sometimes more convenient to work with, given by

$$\int_0^T \langle u'(t), v(t) \rangle_{\widetilde{V}^*, \widetilde{V}} + \int_0^T \mu(\nabla u(t), \nabla v(t)) = \int_0^T \langle f(t), v(t) \rangle_{\widetilde{V}^*, \widetilde{V}} \ , \text{for all } v \in L^2(0,T; \widetilde{V}) \tag{16}$$

or

$$\langle u'(t), v \rangle_{\widetilde{V}^*, \widetilde{V}} + \mu(\nabla u(t), \nabla v) = \langle f(t), v \rangle_{\widetilde{V}^*, \widetilde{V}} \text{ a.e. } t \in (0,T) \ , \text{and for all } v \in \widetilde{V} \tag{17}$$

Let us proof that in fact the formulation (16) is equivalent to (17).

Lemma 2.3. *The formulation (16) is equivalent to (17)*

Proof: Suppose that $u \in \widetilde{\mathbb{W}}(0,T)$ satisfies the formulation (17). If we proof that

$$\langle u'(t), v(t) \rangle_{\widetilde{V}^*, \widetilde{V}} + \mu(\nabla u(t), \nabla v(t)) = \langle f(t), v(t) \rangle_{\widetilde{V}^*, \widetilde{V}} \text{ a.e. } t \in (0,T) \ , \text{and for all } v \in L^2(0,T; \widetilde{V}) \tag{18}$$

[8]For a Gelfand's triple V needs to be densely and continuously inject in H.

then, since $f \in L^2(0,T;L^2(\Omega))$, which can be seen as $f \in L^2(0,T;\widetilde{V}^*)$, and from the fact that $u \in \mathbb{W}(0,T)$, the terms above are all in $L^1(0,T)$ and thus (18) can be integrated from 0 to T to obtain (16).
We just need to proof that (18) is valid, what can be seen by using step functions.
Let $v(t) = \sum_{i=1}^{n} 1_{E_i}(t)v_i$, with E_i being Lebesgue mensurable sets in $[0,T]$ and $v_i \in \widetilde{V}$, then from (17) and the linearity of the operators involved,

$$\langle u'(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} + \mu(\nabla u(t), v(t)) - \langle f(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} =$$
$$= \sum_{i=1}^{n} 1_{E_i}(t)\big(\langle u'(t), v_i\rangle_{\widetilde{V}^*,\widetilde{V}} + \mu(\nabla u(t), v_i) - \langle f(t), v_i\rangle_{\widetilde{V}^*,\widetilde{V}}\big) = 0 \text{ , for a.e. } t \in (0,T)$$

Now from definition, every function $v(t) \in L^2(0,T;\widetilde{V})$ is the limit of a step function sequence $v_n(t)$ such that $v_n(t) \to v(t)$ a.e in t. Then, due to continuity, and since (18) is valid for step functions, it is also valid in every $v(t) \in L^2(0,T;\widetilde{V})$.

Suppose now, that $u(t) \in \widetilde{\mathbb{W}}(0,T)$ satisfies (16) and we want to see that (17) is also valid. By contradiction let us suppose that exists a $z \in \widetilde{V}$ and a Lebesgue mensurable set E with non zero measure such that taking $v = z$ in (17) the difference of the terms is, without lost of generality, positive, almost everywhere in E. Then taking the function $v(t) = 1_E(t)z$ we get that

$$\int_0^T \langle u'(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} + \int_0^T \mu(\nabla u(t), \nabla v(t)) - \int_0^T \langle f(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} > 0$$

and (16) fails, what is a contradiction. □

Now, since we do not have an homogeneous Dirichlet condition, we will, at some point, use the lifting technique, in order to obtain an appropriate variational problem. As we mentioned in the introduction, another way of addressing this type of boundary condition, is by introducing some weak form of boundary conditions as is done for example in [24]. However in the present work we prefer to use the lifting technique.
The liftings that we need to use must satisfy some conditions, such has the null divergence condition for almost every time, and this consequently will affect the type of boundary controls that we can use. In fact we will only be permitted to use boundary controls, for which we know that there exists a lifting for that boundary data, that satisfies the incompressible condition.
First we need to define the set of admissible trace functions for our problem, since not every function in[9] $L^2(0,T;H^{1/2}(\partial\Omega))$ is appropriate, because may not satisfy the flux condition (a consequence of the fluid being imcompressible). A way of dealing with this problem is to first define an appropriate lifting space, and then define the set of admissible traces functions as the set formed by the traces of the functions in that lifting space. This approach have same limitations, from the point of view of applications, since we do not know à priori which functions are in those set of admissible traces.
This is a major difference from the problems with only Dirichlet boundary control, since the controls $g \in L^2(0,T,H^{1/2}(\partial\Omega))$ in that case must necessary satisfy

$$\int_{\partial\Omega} g \cdot \boldsymbol{n} ds = 0 \text{ for almost every } t \in (0,T)$$

and in practice this can be known à priori.
Also in some references [11; 12], the type of boundary controls which are admissible, are for example of the form

$$\mathscr{T} = \Big\{ u \in L^2\Big(0,T;H^{1/2}\big(\Gamma_{in}\big)\Big) : \int_{\Gamma_{in}} u(t) \cdot \mathbf{n} ds = 0 \text{ for almost every } t \in (0,T) \Big\}$$

This spaces are not considered in this work, because they became unrealistic in the modulation of , for instance, the blood flow in a artery, as we mentioned in the introduction, since we may

[9]The space $H^{1/2}(\Omega)$ can be defined as the image of the trace operator $\widehat{\tau} : H^1(\Omega) \to L^2(\partial\Omega)$. Therefore $H^{1/2}(\Omega) \subset L^2(\partial\Omega)$.

consider the case of parabolic inflow.
In what follows, we make a construction of the admissible set of controls for Σ_c, velocity inputs for Σ_{in} and the constructions of the liftings operators.

2.2 Lifting and Admissible Boundary Functions

2.2.1 Lifting L_c and Admissible Boundary Functions for Σ_c

We want to treat the input velocity y_{in}, and the control velocity y_c in a separated way, since in the control problem, we will fix y_{in}, and make the control on the boundary function y_c. To do this we decoupled these two functions on the boundary, treating them as functions from different spaces, and then we add them, to obtain a function that verifies the Dirichlet boundary conditions.

We begin by introducing the following trace operators,

$$\widehat{\tau_{in}} : L^2(H^1(\Omega)^2) \to L^2(\Sigma_{in}) \qquad \widehat{\tau_c} : L^2(H^1(\Omega)^2) \to L^2(\Sigma_c) \qquad \widehat{\tau_D} : L^2(H^1(\Omega)^2) \to L^2(\Sigma_D)$$

where $\Sigma_{in} = \Gamma_{in} \times (0,T), \Sigma_w = \Gamma_w \times (0,T)$ and $\Sigma_c = \Gamma_c \times (0,T)$. These applications are linear, and, in the case that the spaces involved have the usual norms, they are also continuous .

Now we define the set

$$\mathbb{W}_c(0,T) := \{u \in \mathbb{W}(0,T) : \widehat{\tau}_{in}(u) = 0\}$$

Notice that, $\mathbb{W}_c(0,T)$ is composed by the functions of $\mathbb{W}(0,T)$ which have null trace on the segment Σ_{in} of the lateral boundary Σ. Also recall, that the functions in $\mathbb{W}(0,T)$ have by definition a null trace in Σ_w and they are null divergent for almost every instant.
We set the admissible boundary functions in Σ_c to be the set $\mathscr{T}_c = \widehat{\tau}_c(\mathbb{W}_c(0,T))$, and we define the restriction of $\widehat{\tau}_c$ to $\mathbb{W}_c(0,T)$ by

$$\tau_c(u) : \mathbb{W}_c(0,T) \to \mathscr{T}_c \qquad\qquad \tau_c(u) = \hat{\tau}_c(u) \text{ for all } u \in \mathbb{W}_c(0,T)$$

We want the space $\mathscr{T}_c$ to have a complete norm induced by an inner product[10]. Also this norm must allow, the linear lifting operator $L_c : \mathscr{T}_c \to \mathbb{W}_c(0,T)$, (which we still need to define) and the trace operator τ_c, to be continuous.
This requests can be fulfilled as we show in the following lemmas 2.4-2.9.

Lemma 2.4. *There exists a lifting operator, that we will call L_c, that maps boundary functions $y \in \mathscr{T}_c$ to elements of $\mathbb{W}_c(0,T)$, with the property that (and that is why is called a lifting)*

$$\tau_c\big(L_c(y)\big) = y \text{ for every } y \in \mathscr{T}_c \tag{19}$$

Proof: We start by admitting that the norm of the image space of $\widehat{\tau}_{in}$, $L^2(\Sigma_{in})$, is the L^2 norm, in order to guarantee that the trace operator $\widehat{\tau}_{in} : \mathbb{W}(0,T) \to L^2(\Sigma_{in})$ is linear and *continuous*, and therefore has a closed Kernel in $\mathbb{W}(0,T)$.
Then, the vector space $\mathbb{W}_c(0,T)$ is a closed subspace of the Hilbert space $\mathbb{W}(0,T)$, and therefore, is also a Hilbert space, when equipped with the same inner product of $\mathbb{W}(0,T)$.
Now, we assume (H1) that the norm of the space $\mathscr{T}_c \subset L^2(\Sigma_c)$ is the L^2 norm, what again, leads to the continuity of $\tau_c : \mathbb{W}_c(0,T) \to \mathscr{T}_c$. The hypotheses (H1) will only be used in order to construct an operator $L_c : \mathscr{T}_c \to W_c(0,T)$ that verifies

$$L_c(y) = g \in \mathbb{W}_c(0,T) \text{ such that } \tau_c(g) = y$$

[10]For the control problem is important that the test space is reflexive. This is verified if it is a Hilbert space.

and this operator is well defined independently of (H1), being this hypotheses only temporary. Now with (H1), the quotient space $\mathbb{W}_c(0,T)/Ker(\tau_c)$ is a Banach space for the quotient norm [11]

$$||[g]||_{\mathbb{W}_c(0,T)/Ker(\tau_c)} = \inf_{u\in\mathbb{W}_c(0,T):\tau_c u=\tau_c g} ||u||_{\mathbb{W}(0,T)} \tag{20}$$

where we denote $[g]$ the class of $\mathbb{W}_c(0,T)/Ker(\tau_c)$ with $g \in \mathbb{W}_c(0,T)$ as it's representative. Observe that from the definition of this norm,

$$||[g]||_{\mathbb{W}_c(0,T)/Ker(\tau_c)} = \inf_{u\in\mathbb{W}_c(0,T):\tau_c u=\tau_c g} ||u||_{\mathbb{W}(0,T)} = \inf_{u\in\mathbb{W}_c(0,T):\tau_c u=\tau_c g} ||u-0||_{\mathbb{W}(0,T)}$$

which tell us that the infimum if it is attained, is attained by the optimal approximation of 0 by elements of the affine space $S = \{u \in \mathbb{W}_c(0,T) : \tau_c u = \tau_c g\} \subset \mathbb{W}_c(0,T)$. It is possible to proof that the optimal approximation exists and it is unique. To see that we enunciate the next auxiliary lemma (see [20]).

Lemma 2.5 (Optimal Approximation)**.** *Let X be a Hilbert space and $A \subset X$ a subset which is complete (close), convex and non-empty. Then for every $x \in X$ there exists an unique element $a_0 \in A$ such that*

$$||x-a_0||_X = \inf_{a\in A} ||x-a||_X$$

The affine space S is clearly non-empty. It is also convex, since for every $\lambda \in [0,1]$ and for every $u, v \in S$, by the linearity of τ_c,

$$\begin{aligned}\tau_c\big(\lambda u + (1-\lambda)v\big) &= \lambda\tau_c(u) + (1-\lambda)\tau_c(v)\\ &= \lambda\tau_c(g) + (1-\lambda)\tau_c(g)\\ &= \tau_c(g)\end{aligned}$$

and thus also $\lambda u + (1-\lambda)v \in S$.

To see that S is closed, let $\{u_n\}_{n\in\mathbb{N}} \subset S$ be a convergent sequence to an element $u \in \mathbb{W}_c(0,T)$, and we want to see that $u \in S$. By the continuity of the trace τ_c (which is the case if H1 is valid),

$$\lim_{n\to\infty} \tau_c(u_n) = \tau_c(\lim_{n\to\infty} u_n) = \tau_c(u)$$

On the other hand, the sequence $\tau_c(u_n) = \tau_c(g)$ for every $n \in \mathbb{N}$, by definition of S. Then

$$\tau_c(g) = \lim_{n\to\infty} \tau_c(u_n) = \tau_c(u)$$

and thus $u \in S$.
Therefore since $\mathbb{W}_c(0,T)$ is a Hilbert space, from lemma 2.5 we conclude that the infimum of (20) is attained by a unique element in S. Now, let y be an arbitrary element of $\mathscr{T}_c$, then by definition of this space, exists at least one element $g_y \in \mathbb{W}_c(0,T)$ such that $\tau_c(g_y) = y$, and we can define the affine set $S_y = \{u \in \mathbb{W}_c(0,T) : \tau_c(u) = \tau_c(g_y) = y\}$. Moreover, this afine set has the same properties as the set S above, being non-empty, convex and closed. Therefore, by lemma 2.5, for every $y \in \mathscr{T}_c$ exists a unique $\widetilde{g}_y \in S_y \subset \mathbb{W}_c(0,T)$ such that

$$||\widetilde{g}_y||_{\mathbb{W}_c(0,T)} = \inf_{u\in S_y} ||u||_{\mathbb{W}_c(0,T)} \qquad\qquad \tau_c(\widetilde{g}_y) = y$$

This suggests the definition of the following lifting operator $L_c : \mathscr{T}_c \to \mathbb{W}_c(0,T)$, given by

$$\mathscr{T}_c \ni y \mapsto \widetilde{g} \in \mathbb{W}_c(0,T)$$

where $\widetilde{g}$ is the (unique) element of S_y that attains the infimum of the quotient norm

$$||\widetilde{g}||_{\mathbb{W}(0,T)} = \inf_{u\in S_y} ||u||_{\mathbb{W}(0,T)}$$

[11] In [20] we have the result that, the quotient space X/M, of a Banach space X, is complete for the quotient norm, iff the set M is closed.

where S_y is defined above.
In this way we have a well defined operator $L_c : \mathscr{T} \to \mathbb{W}_c(0,T)$ that satisfies, the lifting property, as we wanted to show. $\square$

Let us proof that L_c is linear.

Lemma 2.6. *The lifting operator L_c is linear.*

Proof: Suppose that $\alpha \in \mathbb{R}$ and $y \in \mathscr{T}_c$, and let $g = L_c(y)$. We want to proof that $L_c(\alpha y) = \alpha g$. Let $v \in S_{\alpha y}$ be the element that attains the infimum, (as we saw this element exists and it is unique). Then

$$\begin{aligned}
||v||_{\mathbb{W}(0,T)} &= \inf_{u \in \mathbb{W}_c(0,T):\tau_c(u)=\alpha y} ||u||_{\mathbb{W}(0,T)} = \inf_{u \in \mathbb{W}_c(0,T):\tau_c(\frac{u}{\alpha})=y} ||u||_{\mathbb{W}(0,T)} \\
&= \inf_{w \in \mathbb{W}_c(0,T):\tau_c(w)=y} ||\alpha w||_{\mathbb{W}(0,T)}, \quad \textbf{make the subs. } w = \frac{u}{\alpha} \\
&= |\alpha| \inf_{w \in \mathbb{W}_c(0,T):\tau_c(w)=y} ||w||_{\mathbb{W}(0,T)} \\
&= |\alpha| ||g||_{\mathbb{W}(0,T)} = ||\alpha g||_{\mathbb{W}(0,T)}
\end{aligned}$$

Thus, since by the linearity of τ_c, we have that $\tau_c(\alpha g) = \alpha \tau_{in}(g) = \alpha y$. But as we saw, αg is also a minimizer of the quotient norm, therefore by uniqueness $v = \alpha g$, and so, $L(\alpha y) = v = \alpha g = \alpha L(y)$.

Before finishing the proof of the linearity, we make an observation. Let y be an element of $\mathscr{T}_c$ and $g = L_c(y)$, then

$$||g||_{\mathbb{W}(0,T)} = \inf_{u \in \mathbb{W}_c(0,T):\tau_c(u)=y} ||u||_{\mathbb{W}(0,T)} = \inf_{m \in ker(\tau_c)} ||g - m||_{\mathbb{W}(0,T)}$$

Let us see that the infimum are equal. First they are both achieved since, the left infimum was already analyzed, and the right hand side infimum is obtained by the optimal approximation of g by elements of $\text{Ker}(\tau_c)$, which is a closed subspace of $\mathbb{W}_c(0,T)$, and thus lemma 2.5 guarantees the existence and uniqueness of the infimum element.
By simplification we denote

$$I = \inf_{u \in \mathbb{W}_c(0,T):\tau_c(u)=y} ||u||_{\mathbb{W}(0,T)} \qquad \qquad \inf_{m \in ker(\tau_c)} ||g - m||_{\mathbb{W}(0,T)} = J$$

We want to see that $J = I$. Suppose by contradiction that $I > J$, and let $g - m$ (with $m \in Ker(\tau_c)$) be the unique element whose norm is equal to J. Choosing $u = g - m$, we get that $u \in \mathbb{W}_c(0,T)$, $\tau_c(u) = y$, and

$$I \leq ||u||_{\mathbb{W}(0,T)} = J$$

which is a contradiction, and therefore $I \leq J$. Using the same reasoning we conclude that $I \geq J$, and thus $J = I$.

Now since

$$||g||_{\mathbb{W}(0,T)} = \inf_{u \in \mathbb{W}_c(0,T):\tau_c(u)=y} ||u||_{\mathbb{W}(0,T)} = \inf_{m \in ker(\tau_c)} ||g - m||_{\mathbb{W}(0,T)}$$

we conclude that choosing $m = 0$ we obtain exactly $||g||$ which is the infimum, and since this m is unique by lemma 2.5, we can say that the zero is the optimal approximation of g by elements of $\ker(\tau_c)$.
Now we enunciate the next lemma, which is valid only for closed subspaces (see [20]).

Lemma 2.7. *Let X be a Hilbert space, $x \in X$ and $A \subset X$, a closed subspace of X. Then, the following affirmations are equivalent,*

$$(i) a_0 \textit{ is the best approximation of } x \textit{ by elements of } A$$

$$(ii) a_0 \in A \textit{ and } (x - a_0) \perp A$$

In our case, since $Ker(\tau_c)$ is a closed subspace of the Hilbert space $\mathbb{W}_c(0,T)$, lemma 2.7 is valid. Now since, zero is the best approximation of g by elements of $Ker(\tau_c)$ we have by lemma 2.7-(ii) that

$$g \perp m \text{ for every } m \in Ker(\tau_c) \tag{21}$$

Returning to the poof of the linearity of the lifting L_c, let $y_1, y_2 \in \mathscr{T}_c$ have the liftings $g_1 = L_c(y_1)$ and $g_2 = L_c(y_2)$, also let $v = L_c(y_1 + y_2)$. We want to proof that $v = g_1 + g_2$, what finishes the proof of the linearity of L_c.

Suppose by contradiction, that $v \neq g_1+g_2$, then since $\tau_c(v) = \tau_c(g_1+g_2)$ we have that $v = g_1+g_2+m$ where $m \in Ker(\tau_c)$ is non null.

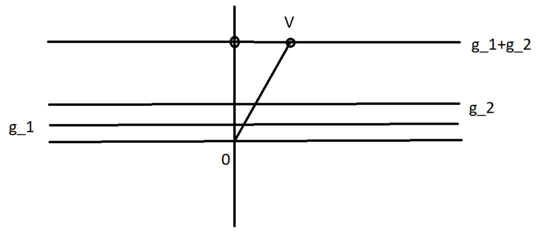


Figure 2: Illustration of the reasoning.

Notice now, that since g_1 and g_2 are minimizers, they are orthogonal to the kernel of τ_c. Therefore $g_1 + g_2 \perp m$ and by the Pythagoras's theorem we have that

$$||v||^2_{\mathbb{W}(0,T)} = ||m||^2_{\mathbb{W}(0,T)} + ||g_1 + g_2||^2_{\mathbb{W}(0,T)}$$

and since, by hypotheses $m \neq 0$, we arrive to a contradiction since $||g_1 + g_2||_{\mathbb{W}(0,T)} < ||v||_{\mathbb{W}(0,T)}$ and v is the element that attains the infimum.
Therefore, we must have $L(y_1 + y_2) = v = g_1 + g_2 = L_c(y_1) + L_c(y_2)$. □

We introduce in $\mathscr{T}_c$ the norm[12], given for every $y \in \mathscr{T}_c$, by

$$||y||_{\mathscr{T}_c} = ||L_c(y)||_{\mathbb{W}(0,T)} \tag{22}$$

We have the following continuity result.

Lemma 2.8. *The operators* $\tau_c : \mathbb{W}_c(0,T) \to \Big(\mathscr{T}_c, ||\cdot||_{\mathscr{T}}\Big)$ *and* $L_c : \Big(\mathscr{T}_c, ||\cdot||_{\mathscr{T}}\Big) \to \mathbb{W}_c(0,T)$*, where* $||\cdot||_{\mathscr{T}_c}$ *is the norm in (22), are continuous.*

Proof: The trace operator $\tau_c : \mathbb{W}_c(0,T) \to \Big(\mathscr{T}_c, ||\cdot||_{\mathscr{T}_c}\Big)$ is still continuous, since it is linear, and for every $g \in \mathbb{W}_c(0,T)$

$$||\tau_c(g)||_{\mathscr{T}_c} = ||L_c(\tau_c(g))||_{\mathbb{W}(0,T)} = \inf_{u \in \mathbb{W}_c(0,T):\tau_c(u)=\tau_c(g)} ||u||_{\mathbb{W}(0,T)} \leq ||g||_{\mathbb{W}(0,T)}$$

where we used the fact that by construction of L_c,

$$||L_c(\tau_c(g))||_{\mathbb{W}(0,T)} = \inf_{u \in \mathbb{W}_c(0,T):\tau_c(u)=\tau_c(g)} ||u||_{\mathbb{W}(0,T)}$$

and the fact that the space $\mathbb{W}_c(0,T)$ is not modified by the hypotheses (H1)[13], and thus the element g is an element of the set $S_g = \{u \in \mathbb{W}_c(0,T) : \tau_c(u) = \tau_c(g)\}$ which is the same set where

[12] We have to verify that this application is in fact a norm, what is done in the lemma 2.9.
[13] Notice that the space $\mathbb{W}_c(0,T)$ is defined using $\widehat{\tau_{in}}$ and thus is independent of the hypotheses (H1) .

the infimum that defines the norm of the lifting result is calculated. Thus, with this norm for $\mathscr{T}_c$ we still guarantee the continuity of the trace operator τ_c.

The lifting operator L_c is also continuous, since it is linear and bounded

$$||L(g)||_{\mathbb{W}_c(0,T)} = ||L(g)||_{\mathbb{W}(0,T)} = ||g||_{\mathscr{T}_c} \leq ||g||_{\mathscr{T}_c}$$

□

Regarding the wanted Hilbertian structure for $\mathscr{T}_c$, we have the following result.

Lemma 2.9. *The space $(\mathscr{T}_c, ||\cdot||_{\mathscr{T}_c})$ is a Hilbert space, for the inner product given by:*

for every $g_1, g_2 \in \mathscr{T}_c$

$$\big(g_1, g_2\big)_{\mathscr{T}_c} = (L_c(g_1), L_c(g_2))_{\mathbb{W}(0,T)}$$

Proof: Our first step is to see that, in fact, $||\cdot||_{\mathscr{T}_c}$ is a norm. This is a simple consequence of the lifting operator being linear. In fact, for every $g_1, g_2 \in \mathscr{T}_c$ and $\alpha \in \mathbb{R}$,

$$(i)\ ,\ ||g_1||_{\mathscr{T}_c} = ||L_c(g_1)||_{\mathbb{W}(0,T)} \geq 0$$

$$(ii) \text{ if } ||g_1||_{\mathscr{T}_c} = ||L_c(g_1)||_{\mathbb{W}(0,T)} = 0 \Leftrightarrow L_c(g_1) = 0 \text{ in } \mathbb{W}_c(0,T) \text{ and thus } \tau_c\big(L_c(g_1)\big) = 0$$

$$(iii)\ ||\alpha g_1||_{\mathscr{T}_c} = ||L_c(\alpha g_1)||_{\mathbb{W}(0,T)} = ||\alpha L_c(g_1)||_{\mathbb{W}(0,T)} = |\alpha|||L_c(g_1)||_{\mathbb{W}(0,T)} = |\alpha|||g_1||_{\mathscr{T}_c}$$

(iv)

$$\begin{aligned}||g_1 + g_2||_{\mathscr{T}_c} &= ||L_c(g_1 + g_2)||_{\mathbb{W}(0,T)} = ||L_c(g_1) + L_c(g_2)||_{\mathbb{W}(0,T)} \\ &\leq ||L_c(g_1)||_{\mathbb{W}(0,T)} + ||L_c(g_2)||_{\mathbb{W}(0,T)} = ||g_1||_{\mathscr{T}_c} + ||g_2||_{\mathscr{T}_c}\end{aligned}$$

Thus it is proofed that $||\cdot||_{\mathscr{T}_c}$ is a norm in $\mathscr{T}_c$.

Now let us see that this norm makes the space complete. Let $\{g_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{T}_c$ and, from the continuity of the lifting operator we get that the image sequence $y_n = L_c(g_n)$ is a also a Cauchy sequence in $\mathbb{W}_c(0,T)$, since

$$||y_n - y_m||_{\mathbb{W}(0,T)} = ||L_c(g_n) - L_c(g_m)||_{\mathbb{W}(0,T)} = ||g_n - g_m||_{\mathscr{T}_c}$$

Thus, the sequence $\{y_n\}_{n\in\mathbb{N}}$ has limit $y \in \mathbb{W}_c(0,T)$ (since $\mathbb{W}_c(0,T)$ is complete). Using the continuity of the trace operator τ_c (which we saw that is continuous when $\mathscr{T}_c$ has this norm) we have

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \tau_c(y_n) = \tau_c\big(\lim_{n\to\infty} y_n\big) = \tau_c(y) \in \mathscr{T}_c$$

and therefore every Cauchy sequence in $\mathscr{T}_c$ is convergent.

Lastly, we observe that we can induce the norm $||\cdot||_{\mathscr{T}_c}$, by the inner product

$$(g_1, g_2)_{\mathscr{T}_c} = (L_c(g_1), L_c(g_2))_{\mathbb{W}_c(0,T)} = (L_c(g_1), L_c(g_2))_{\mathbb{W}(0,T)}$$

since the inner product of $\mathbb{W}_c(0,T)$ is the same as the original space. Notice that $||g_1||^2_{\mathscr{T}_c} = (g_1, g_1)_{\mathscr{T}_c} = (L_c(g_1), L_c(g_1))_{\mathbb{W}(0,T)} = ||L_c(g_1)||^2_{\mathbb{W}(0,T)}$, and therefore this inner product induces in fact the norm which is complete.
To see that the application $(\cdot,\cdot)_{\mathscr{T}_c}$ is in fact an inner product in $\mathscr{T}_c$ is a simple consequence of the linearity of L_c and the properties of the original inner product $(\cdot,\cdot)_{\mathbb{W}(0,T)}$. □

2.2.2 Lifting L_{in} and Admissible Boundary Functions for Σ_{in}

Now we focus in constructing an appropriate trace space for the input velocities function in Σ_{in}. We define the set

$$\mathbb{W}_{in}(0,T) := \{u \in \mathbb{W}(0,T) : \widehat{\tau}_c(u) = 0\}$$

where the operator $\widehat{\tau_c} : \mathbb{W}(0,T) \to \big(L^2(\Sigma_c), ||\cdot||_{L^2(\Sigma_c)}\big)$ is the original trace operator. Notice that, since $\widehat{\tau_c}$ is continuous, the space $\mathbb{W}_{in}(0,T)$ is a Hilbert space.
In the above construction, the map $\widehat{\tau}_{in} \in \mathscr{L}(\mathbb{W}(0,T), L^2(\Sigma_{in}))$ was continuous since we used the L^2-norm in the image space. Now we define the restriction of this map to the set $\mathbb{W}_{in}$ and the respective image space to be

$$\tau_{in} : \mathbb{W}_{in}(0,T) \to \mathscr{T}_{in} \qquad\qquad \widehat{\tau_{in}}(u) = \tau_{in}(u) \text{ for all } u \in \mathbb{W}_{in}(0,T)$$

By now $\mathscr{T}_{in}$ does not yet have a norm. However we will need to introduce a norm in this vector space, that guarantees the existence of a linear and continuous lifting $L_{in} : \mathscr{T}_{in} \to \mathbb{W}_{in}(0,T)$. In this case, we do not need the space $\mathscr{T}_{in}$ to have an hilbertian structure, since the only property that we really need is an estimate of the lifting with respect to the traces.
Using the same method, as was used to obtain the lifting L_c, we can construct a linear and continuous lifting $L_{in} : (\mathscr{T}_{in}, ||\cdot||_{\mathscr{T}_{in}}) \to \mathbb{W}_{in}(0,T)$ where for every $g \in \mathscr{T}_{in}$,

$$||g||_{\mathscr{T}_{in}} = ||L_{in}(g)||_{\mathbb{W}(0,T)} \tag{23}$$

We close this section about the traces spaces with the following proposition that is a simple consequence of the above constructions.

Proposition 2.1. *For every $y_c \in \mathscr{T}_c$ and $y_{in} \in \mathscr{T}_{in}$, we can always find an element of $u \in \mathbb{W}(0,T)$, that satisfies*

$$u\big|_{\Sigma_c} = y_c$$
$$u\big|_{\Sigma_{in}} = y_{in}$$
$$u\big|_{\Sigma_w} = 0$$

in the trace sense.

Proof: Given $y_{in} \in \mathscr{T}_{in}$ and $y_c \in \mathscr{T}$, by using the lifting operators, we can define the object $u = L_c(y_c) + L_{in}(y_{in})$, which is a function in $\mathbb{W}(0,T)$.
Now, since $\mathbb{W}(0,T) \subset L^2(H^1(\Omega))$, the function u has a well defined trace $f = \widehat{\tau_{\Sigma_D}}(u) \in L^2(\Sigma_D)$ and we want to see that the restriction of this trace function to the sets Σ_{in}, Σ_c and Σ_w coincide with the trace imposed conditions, y_{in}, y_c and zero respectively.
The restriction of f to Σ_{in} is given by

$$\chi_{\Sigma_{in}} f = \chi_{\Sigma_{in}} \circ \chi_{\Sigma_D} \circ \widehat{\tau_\Sigma}(u) = \chi_{\Sigma_{in}} \circ \widehat{\tau_\Sigma}(u) = \widehat{\tau_{in}}(u) = \widehat{\tau_{in}}(u_{in}) + \widehat{\tau_{in}}(u_c) = \widehat{\tau_{in}}(u_{in})$$

where we used the fact that $\chi_{\Sigma_{in}} \circ \chi_{\Sigma_D} = \chi_{\Sigma_{in}}$ because $\Sigma_{in} \subset \Sigma_D$, the definition of $\widehat{\tau_{in}}$ (see section 7.2), it's linearity and the fact that $\widehat{\tau_{in}}(u_c) = 0$ since $u_c \in \mathbb{W}_c(0,T)$.
Also from the definition of τ_{in} and the fact that $u_{in} \in \mathbb{W}_{in}(0,T)$ we have that

$$\chi_{\Sigma_{in}} f = \widehat{\tau_{in}}(u_{in}) = \tau_{in}(u_{in}) = \tau_{in}(L_{in}(y_{in})) = y_{in}$$

Therefore, the function $u \in \mathbb{W}(0,T)$ has a trace, that when restricted to the boundary part Σ_{in}, coincides with the imposed trace data y_{in}.
The same reasoning leads us to the conclusion that, we also have $u\big|_{\Sigma_c} = y_c$, in the trace sense.
For the part Σ_w of the lateral boundary, we need to verify that u has zero trace. From the definition of $\mathbb{W}(0,T)$, we get that $\widehat{\tau_w}(u(t)) = 0$ for almost every $t \in (0,T)$ and therefore the trace of u restricted to that part of the boundary is null. □

2.3 Admissible Initial Conditions

Let us now discuss the admissible initial conditions. We want the initial condition $u_0 \in \mathbb{H}$ to have compatible information on the Dirichlet segment of the boundary, with the boundary information of the liftings $g_{in} = L_{in}(y_{in})$, $g_c = L_c(y_c)$ where $y_{in} \in \mathscr{T}_{in}$ and $y_c \in \mathscr{T}_c$. That is, we want that,

$$u_0 - \Big(L_{in}(y_{in})(0) + L_c(y_c)(0)\Big) \in \widetilde{H} \tag{24}$$

Let us see that the condition (24) makes sense. In the following arguments we use the notation $\mathbb{W}_{in}(0,T) \oplus \mathbb{W}_c(0,T)$ to denote the set of functions $g \in \mathbb{W}(0,T)$ that may be written (may not be uniquely) in he form $g = g_{in} + g_c$ where $g_{in} \in \mathbb{W}_{in}(0,T)$ and $g_c \in \mathbb{W}_c(0,T)$.
Since the functions y_{in}, y_c, that impose the Dirichlet boundary conditions, belong to $\mathscr{T}_{in}, \mathscr{T}_c$ respectively, we get that the sum of the lifting $\tilde{g} = L_{in}(y_{in}) + L_c(y_c)$ is in $\mathbb{W}_{in}(0,T) \oplus \mathbb{W}_c(0,T) \subset \mathbb{W}(0,T) \hookrightarrow C([0,T];\mathbb{H})$, and therefore we can talk in the point evaluation $\tilde{g}(0)$.
Now we define the following observation operator $\tau_0^t : \mathbb{W}(0,T) \to \mathbb{H}$, given by the following chain of compositions,

$$\mathbb{W}(0,T) \ni g \xrightarrow{\hookrightarrow C([0,T];\mathbb{H})} g \xrightarrow{\tau_0} g(0)$$

where $\tau_0 : C([0;T];\mathbb{H}) \to \mathbb{H}$, is the observation operator on the instant $t = 0$. τ_0 is linear and bounded, and therefore continuous. Thus, since we have a chain of composition of linear and continuous operators, τ_0^t is also continuous and linear.

Definition 2.1 (Space $\mathscr{H}$). *We define the set of admissible initial conditions as*

$$\mathscr{H} = \tau_0^t\big(L_c(\mathscr{T}_c)\big) \oplus \tau_0^t\big(L_{in}(\mathscr{T}_{in})\big)$$

where L_c, L_{in} are the above lifting operators.

We close this section with the following result.

Proposition 2.2. *$\mathscr{H} \neq \emptyset$, and for every $u_0 \in \mathscr{H}$, there exists at least two functions $y_{in} \in \mathscr{T}_{in}$ and $y_c \in \mathscr{T}_c$, whose liftings, satisfies the compatibility condition with the given u_0.*

Proof: Let u_0 be an arbitrary element of $\mathscr{H}$. Then, by definition, exists y_{in}, y_c belonging to $\mathscr{T}_{in}, \mathscr{T}_c$, respectively, such that

$$u_0 = \tau_0^t\big(L_c(y_c)\big) + \tau_0^t\big(L_{in}(y_{in})\big)$$

and therefore, the liftings of the boundary data y_{in}, y_c satisfy the initial compatibility condition, since

$$u_0 - (L_c(y_c))(0) - (L_{in}(y_{in}))(0) = 0 \implies u_0 - (L_c(y_c))(0) - (L_{in}(y_{in}))(0) \in \widetilde{H}$$

□

When dealing with the control problem, we fix $y_{in} \in \mathscr{T}_{in}$, $u_0 \in \mathscr{H}$, and in this case the control y_c must be chosen from a proper set in such a way that the initial condition compatibility is satisfied.

2.4 Chapter Final Notes

Note (1): We have that $\mathbb{W}(0,T) \hookrightarrow \widetilde{\mathbb{W}}(0,T)$ To see that this is possible, (see footnote) we need to show that the injection of $\mathbb{W}(0,T)$ in $\widetilde{\mathbb{W}}(0,T)$ is continuous.

Let u be an element of $\mathbb{W}(0,T)$, then $u \in L^2(\mathbb{V})$ and the derivative $u' \in L^2(\mathbb{V}^*)$. We define the application $F : L^2(\widetilde{V}) \to \mathbb{R}$ by

$$F(v) = \int_0^T \langle u'(t), v(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt \ , \ \forall v \in L^2(\widetilde{V})$$

and the duality product makes sense, since $\widetilde{V} \hookrightarrow \mathbb{V}$.
This application is linear since, for every $\alpha, \beta \in \mathbb{R}$ and $v, z \in L^2(\tilde{V})$

$$\begin{aligned} F(\alpha v + \beta z) &= \int_0^T \langle u'(t), \alpha v(t) + \beta z(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt \\ &= \alpha \int_0^T \langle u'(t), v(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt + \beta \int_0^T \langle u'(t), z(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt \\ &= \alpha F(v) + \beta F(z) \end{aligned}$$

and also bounded

$$\begin{aligned}|F(v)| &= |\langle u', v\rangle_{L(\mathbb{V}^*),L^2(\mathbb{V})}| \\ &\leq ||u'||_{L^2(\mathbb{V}^*)}||v||_{L^2(\mathbb{V})} \\ &= ||u'||_{L^2(\mathbb{V}^*)}||v||_{L^2(\widetilde{V}^*)}\end{aligned}$$

where we used the fact that $L^2(\widetilde{V}) \hookrightarrow L^2(\mathbb{V})$, which cames from the fact that $\widetilde{V} \hookrightarrow \mathbb{V}$. Therefore F is an element of $L^2(\widetilde{V})^*$, a space that, from theorem (7.5), is isometrically isomorphic to $L^2(\widetilde{V}^*)$, therefore, exists a unique $f \in L^2(\widetilde{V}^*)$ such that for all $v \in L^2(\widetilde{V})$,

$$F(v) = \int_0^T \Big\langle f(t), v(t)\Big\rangle_{\widetilde{V}^*,\widetilde{V}} dt$$

Notice that from the isometrically isomorphism, $||f||_{L^2(\widetilde{V}^*)} = ||F||_{L^2(\widetilde{V})^*} \leq ||u'||_{L^2(\mathbb{V}^*)}$

Thus, $u \in L^2(\mathbb{V})$ and $u'|_{\widetilde{V}} = f \in L^2(\widetilde{V}^*)$, so $u \in \widetilde{\mathbb{W}}(0,T)$ and we have the estimate

$$||u||_{\widetilde{\mathbb{W}}(0,T)} \leq ||u||_{\mathbb{W}(0,T)}$$

Now that $\widetilde{u}$ is seen as an element of $\widetilde{\mathbb{W}}(0,T)$ the sum $u = \widetilde{u} + w$ is well defined and is an element of $\widetilde{\mathbb{W}}(0,T)$.

Note (2): We do not have, for the space $\widetilde{\mathbb{W}}(0,T)$, where our solution is suppose to be, the integration by parts formula. This seems to be a problem since we need it for some proofs. But we can at least say that for a special set of functions we have this integration by parts formula.
In fact, the solution, will be obtained by a decomposition process such that, where $\widetilde{\mathbb{W}}(0,T) \ni u = \widetilde{u} + w$, with $\widetilde{u} \in \mathbb{W}(0,T)$ seen as an element in $\widetilde{\mathbb{W}}(0,T)$ and $w \in W(0,T)$.
Now, to use integration by parts, we need to use as test functions a special set of functions, that for our proofs is sufficient. Consider the function $v(t) = V\varphi(t)$ where $V \in \widetilde{V}$ and φ is at least a C^1 function in $[0,1]$. From the results of the appendix (derivative) we conclude that the weak derivative of $v(t)$ coincides with the strong derivative which is given by $v'(t) = V\varphi'(t)$.
Thus we can conclude that, $v(t) \in \mathbb{W}(0,T)$ and $v(t) \in W(0,T)$ at the same time. In fact, from the fact that the weak derivative coincides with the weak derivative, we get that $v(t), v'(t) \in L^2(\widetilde{V})$, and thus the derivative u' is also in $L^2(\widetilde{V}^*)$.
Also, $v(t), v'(t)$ are in $L^2(\mathbb{V})$, since $L^2(\widetilde{V}) \hookrightarrow L^2(\mathbb{V})$, and from this we get that $v'(t) \in L^2(\mathbb{V}^*)$. Let us see, how we can use integration by parts. Let $u \in \widetilde{\mathbb{W}}(0,T)$ be of the form $u = \widetilde{u} + w$ where $w \in W(0,T)$ and $\widetilde{u} \in \mathbb{W}(0,T)$ seen as an element in $\widetilde{\mathbb{W}}(0,T)$. Then for a function $v(t)$ of the above form we have

$$\begin{aligned}\int_0^T \langle u', V\varphi(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt &= \int_0^T \langle \widetilde{u}', V\varphi(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt + \int_0^T \langle w', V\varphi(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt \\ &= \int_0^T \langle \widetilde{u}', V\varphi(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt + \int_0^T \langle w', V\varphi(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt\end{aligned}$$

where in the last equality is due to the fact that

$$\int_0^T \langle \widetilde{u}'(t), V\varphi(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt = F(V\varphi(t)) = \int_0^T \langle \widetilde{u}'(t), V\varphi(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt$$

where the map F was defined on the note-1.
Now, since, as seen above, $v(t) \in W(0,T) \cap \mathbb{W}(0,T)$ and $\widetilde{u} \in \mathbb{W}(0,T), w \in W(0,T)$, now we can use integration by parts and obtain,

$$\begin{aligned}\int_0^T \langle \widetilde{u}', V\varphi(t)\rangle_{\mathbb{V}^*,\mathbb{V}} dt + \int_0^T \langle w', V\varphi(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt &= -\int_0^T \langle \widetilde{u}, V\varphi'(t)\rangle_{\mathbb{V},\mathbb{V}^*} dt + \big(\widetilde{u}(T), \varphi(T)V\big) - \big(\tilde{u}(0), \varphi(0)V\big) \\ &\quad - \int_0^T \langle w, V\varphi'(t)\rangle_{\widetilde{V},\widetilde{V}^*} dt + \big(w(T), \varphi(T)V\big) - \big(w(0), \varphi(0)V\big) \\ &= -\int_0^T \langle \widetilde{u}, V\varphi'(t)\rangle_{\mathbb{V},\mathbb{V}^*} dt - \int_0^T \langle w, V\varphi'(t)\rangle_{\widetilde{V},\widetilde{V}^*} dt \\ &\quad + \big(u(T), \varphi(T)V\big) - \big(u(0), \varphi(0)V\big)\end{aligned}$$

2.5 Functional Spaces

We close this section with a conclusion with the functional spaces that are used in this work. We decided to add this page in order to facilitate the reading process.

$$\mathscr{E}(\overline{\Omega}) := \{u \in C^\infty(\overline{\Omega})^2 : \nabla \cdot u = 0, supp(u) \cap \Gamma_D = \emptyset\} \tag{25}$$

$$\widetilde{V} := \overline{\mathscr{E}(\overline{\Omega})}^{H^1} \qquad ||u||_{\widetilde{V}} = ||\nabla u||_{L^2(\Omega)} \tag{26}$$

$$\widetilde{H} := \overline{\mathscr{E}(\overline{\Omega})}^{L^2} \qquad ||u||_{\widetilde{H}} = ||u||_{L^2(\Omega)} \tag{27}$$

$$\mathbb{V} := \left\{u \in H^1(\Omega)^2 : \nabla \cdot u = 0, \text{ and } \widehat{\tau_w}(u) = 0\right\} \qquad ||u||_{\mathbb{V}} = ||\nabla u||_{L^2(\Omega)} \tag{28}$$

$$\mathbb{H} := \overline{\left\{u \in H^1(\Omega)^2 : \nabla \cdot u = 0, \text{ and } \widehat{\tau_w}(u) = 0\right\}}^{L^2} \qquad ||u||_{\mathbb{H}} = ||u||_{L^2(\Omega)} \tag{29}$$

$$W(0,T) := \left\{u \in L^2(\widetilde{V}) : u' \in L^2(\widetilde{V}^*)\right\} \qquad ||u||_{W(0,T)} = \left(||u||^2_{L^2(\widetilde{V})} + ||u'||^2_{L^2(\widetilde{V}^*)}\right)^{1/2} \tag{30}$$

$$\mathbb{W}(0,T) := \left\{u \in L^2(\mathbb{V}) : u' \in L^2(\mathbb{V}^*)\right\} \qquad ||u||_{\mathbb{W}(0,T)} = \left(||u||^2_{L^2(\mathbb{V})} + ||u'||^2_{L^2(\mathbb{V}^*)}\right)^{1/2} \tag{31}$$

$$\widetilde{\mathbb{W}}(0,T) := \left\{u \in L^2(\mathbb{V}) : u' \in L^2(\widetilde{V}^*)\right\} \qquad ||u||_{\widetilde{\mathbb{W}}(0,T)} = \left(||u||^2_{L^2(\mathbb{V})} + ||u'||^2_{L^2(\widetilde{V}^*)}\right)^{1/2} \tag{32}$$

$$W(0,T) \hookrightarrow C([0,T];\widetilde{H}) \qquad \mathbb{W}(0,T) \hookrightarrow C([0,T];\mathbb{H}) \qquad \text{both with the max norm} \tag{33}$$

$$\mathbb{W}_c(0,T) := \{u \in \mathbb{W}(0,T) : \widehat{\tau_{in}}(u) = 0\} \qquad \text{same norm as } \mathbb{W}(0,T) \tag{34}$$

$$\mathbb{W}_{in}(0,T) := \{u \in \mathbb{W}(0,T) : \widehat{\tau_c}(u) = 0\} \qquad \text{same norm as } \mathbb{W}(0,T) \tag{35}$$

$$\mathscr{T}_c := \tau_c(\mathbb{W}_c(0,T)) \qquad ||u||_{\mathscr{T}_c} = ||L_c(u)||_{\mathbb{W}(0,T)} \tag{36}$$

$$\mathscr{T}_{in} := \tau_{in}(\mathbb{W}_{in}(0,T)) \qquad ||u||_{\mathscr{T}_{in}} = ||L_{in}(u)||_{\mathbb{W}(0,T)} \tag{37}$$

$$\mathscr{H} := \tau_0^t\Big(L_c(\mathscr{T}_c)\Big) \oplus \tau_0^t\Big(L_{in}(\mathscr{T}_{in})\Big) \qquad \text{same norm as } \mathbb{H} \tag{38}$$

3 Analysis of the Forward Problem

In this section we start by introducing a weak formulation for the problem (1), followed by the study of it's well posedness which is the main goal of this chapter.

For a better exposition we introduce the following operators,

$$\widetilde{a} : \widetilde{V} \times \widetilde{V} \to \mathbb{R} \qquad (u,v) \mapsto \mu(\nabla u, \nabla v)$$

and the same operation but with a different domain

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \qquad (u,v) \mapsto \mu(\nabla u, \nabla v)$$

This two operators are continuous, since they are linear, and bounded.
In fact, for every $\varphi, \phi \in \widetilde{V}$

$$|\widetilde{a}(\varphi,\phi)| \leq \mu||\nabla\varphi||||\nabla\phi|| = \mu||\varphi||_{\tilde{V}}||\phi||_{\tilde{V}}$$

and for every $u, v \in H^1(\Omega)^2$

$$|a(u,v)| \leq \mu||\nabla u||||\nabla v|| \leq \mu||u||_{H^1(\Omega)^2}||v||_{H^1(\Omega)^2}$$

We also define the following linear applications that are induced by these two bi-linear maps,

$$\mathscr{A} : \widetilde{V} \to \widetilde{V}^* \qquad u \mapsto \mathscr{A}u \in \widetilde{V}^* \text{ defined by } \langle \mathscr{A}u, \varphi\rangle_{\widetilde{V}^*,\widetilde{V}} = \widetilde{a}(u,v)\ ,\ \forall v \in \widetilde{V} \tag{39}$$

and similarly

$$\mathbb{A} : H^1(\Omega)^2 \to (H^1(\Omega)^2)^* \qquad u \mapsto \mathbb{A}u \in (H^1(\Omega)^2)^* \text{ defined by}$$

$$\langle \mathbb{A}u, \varphi\rangle_{(H^1(\Omega)^2)^*, H^1(\Omega)^2} = a(u,v)\ ,\ \forall v \in H^1(\Omega)^2 \tag{40}$$

Notice that if $\widetilde{u}$ is an element of $\widetilde{V}$, $\widetilde{u}$ is then, also in $H^1(\Omega)^2$ and we have

$$\widetilde{a}(\widetilde{u},\widetilde{v}) = a(\widetilde{u},\widetilde{v})\ ,\ \forall\ \widetilde{v} \in \widetilde{V} \tag{41}$$

Thus, $\widetilde{a}$ may be interpreted as an restriction of a to the subset $\widetilde{V} \subset H^1(\Omega)^2$. Notice however that the norm used in the space $\widetilde{V}$ is not induced by the norm in the bigger space $H^1(\Omega)^2$, which may lead to some confusion if we had denoted the operators $a, \widetilde{a}$ with the same letter, since they continuity constants are not the same.
The expression (41) has an important consequence. Suppose that $\widetilde{u} \in \widetilde{V}$ is a function that for every $\widetilde{v} \in \widetilde{V}$ has the value

$$\widetilde{a}(\widetilde{u},\widetilde{v}) = 0\ ,\ \forall\ \widetilde{v} \in \widetilde{V}$$

Therefore by (41) we have also that $a(\widetilde{u},\widetilde{v}) = \langle \mathbb{A}\widetilde{u},\widetilde{v}\rangle_{H^{1*},H^1} = 0$ for all $\widetilde{v} \in \widetilde{V}$. Now, since $\mathscr{V} \subset \widetilde{V}$ and $\mathbb{A}\widetilde{u} \in H^1(\Omega)^*$, we may use the theorem 3.2 for the construction of the pressure field. This is the main reason why we introduce the operator $\mathbb{A} : H^1 \to H^{1*} \subset H^{-1}$, since, if we had instead, worked only with $\mathscr{A} : \widetilde{V} \to \widetilde{V}^*$ the theorem 3.2 will stop to be valid, because we do not have $\widetilde{V} \subset H^{-1}$.

We also denote by $\mathscr{A}u(t)$ the functional in $L^2(\widetilde{V}^*)$ given by

$$\langle \mathscr{A}u(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} = \int_0^T \widetilde{a}(u(t),v(t))dt = \int_0^T \mu(\nabla u(t),\nabla v(t))dt$$

and similarly we denote by $\mathbb{A}u(t)$ the functional in $L^2([H^1(\Omega)^2]^*)$ defined by

$$\langle \mathbb{A}u(t), v(t)\rangle_{L^2(H^{1*}),L^2(H^1)} = \int_0^T a(u(t),v(t))dt = \int_0^T \mu(\nabla u(t),\nabla v(t))dt$$

We have the following lemma.

Lemma 3.1. *For $u(t) \in L^2(\widetilde{V})$ we have that $\mathscr{A}u(t)$ is in $L^2(\widetilde{V}^*)$. In the case of $u(t) \in L^2(H^1(\Omega)^2)$ we have that $\mathscr{A}u(t) \in L^2(\mathbb{V}^*)$ or in $L^2(\widetilde{V}^*)$ by restricting the evaluation to $\mathbb{V}$ or $\widetilde{V}$ respectively.*

Lastly we have the following estimates

$$||\mathscr{A}u(t)||_{L^2(\widetilde{V})^*} \leq ||\mathscr{A}||_{\mathscr{L}(L^2(\widetilde{V}),L^2(\widetilde{V}^*))}||u||_{L^2(\widetilde{V})} \text{ for all } u \in L^2(\widetilde{V}) \tag{42}$$

$$||\mathbb{A}u(t)||_{L^2(\tilde{V}^*)} \leq ||\mathbb{A}||_{\mathscr{L}(H^1,H^{1*})}C_1||u(t)||_{L^2(H^1)} \text{ for all } u \in L^2(H^1(\Omega)) \tag{43}$$

$$||\mathbb{A}u(t)||_{L^2(\mathbb{V}^*)} \leq ||\mathbb{A}||_{\mathscr{L}(H^1,H^{1*})}C_2||u(t)||_{L^2(H^1)} \text{ for all } u \in L^2(H^1(\Omega)) \tag{44}$$

where C_1, C_2 denote the Poincaré's constants.

Proof: Let us see that in fact $\mathscr{A}u(t)$ is in $L^2(\widetilde{V}^*)$.
The linearity is simple, therefore we only focus on the boundeness property. Let φ be an element of $L^2(\widetilde{V})$, then

$$\begin{aligned}\Big|\langle \mathscr{A}u(t),\varphi(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})}\Big| &\leq \int_0^T ||\mathscr{A}u(t)||_{\widetilde{V}^*}||\varphi(t)||_{\widetilde{V}}dt\\ &\leq \int_0^T ||\mathscr{A}||_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)}||u(t)||_{\widetilde{V}}||\varphi(t)||_{\widetilde{V}}dt\\ &\leq ||\mathscr{A}||_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)}\Big(\int_0^T ||u(t)||^2_{\widetilde{V}}dt\Big)^{1/2}\Big(\int_0^T ||\varphi(t)||^2_{\widetilde{V}}dt\Big)^{1/2}\end{aligned}$$

and the above inequality implies

$$||\mathscr{A}u(t)||_{L^2(\widetilde{V})^*} \leq ||\mathscr{A}||_{\mathscr{L}(L^2(\widetilde{V}),L^2(\widetilde{V}^*))}||u||_{L^2(\widetilde{V})}$$

which is the boundness of $\mathscr{A}u(t)$ and the estimate (42).
Using the same argument as above we can also conclude that $\mathbb{A}u(t) \in L^2([H^1(\Omega)]^*)$ and that, for every $\varphi \in L^2(H^1(\Omega))$

$$\begin{aligned}\Big|\langle \mathbb{A}u(t),\varphi(t)\rangle_{L^2((H^1)^*),L^2(H^1)}\Big| &\leq \int_0^T ||\mathbb{A}u(t)||_{(H^1)*}||\varphi(t)||_{H^1}dt\\ &\leq \int_0^T ||\mathbb{A}||_{\mathscr{L}(H^1,(H^1)^*)}||u(t)||_{H^1}||\varphi(t)||_{H^1}dt\\ &\leq ||\mathbb{A}||_{\mathscr{L}(H^1,(H^1)^*)}\Big(\int_0^T ||u(t)||^2_{H^1}dt\Big)^{1/2}\Big(\int_0^T ||\varphi(t)||^2_{H^1}dt\Big)^{1/2}\end{aligned}$$

what gives $||\mathbb{A}u(t)||_{L^2(H^{1*})} \leq ||\mathbb{A}||_{\mathscr{L}(H^1,H^{1*})}||u(t)||_{L^2(H^1)}$. Now, if we restrict the evaluation of $\mathbb{A}u(t)$ to $\widetilde{V}$ or $\mathbb{V}$, and denote this operator by the same letter, the above estimations change if we consider the norms of $\widetilde{V}$ or $\mathbb{V}$ instead of the H^1-norm.

$$\begin{aligned}||\mathbb{A}u(t)||_{L^2(\tilde{V}^*)} &\leq ||\mathbb{A}||_{\mathscr{L}(H^1,H^{1*})}C_1||u(t)||_{L^2(H^1)}\\ ||\mathbb{A}u(t)||_{L^2(\mathbb{V}^*)} &\leq ||\mathbb{A}||_{\mathscr{L}(H^1,H^{1*})}C_2||u(t)||_{L^2(H^1)}\end{aligned}$$

For these estimates, we used the fact that, for example, for a $u(t) \in L^2(H^1)$, and for every $v(t) \in L^2(\widetilde{V})$

$$\begin{aligned}\Big|\int_0^T \langle \mathbb{A}u(t),v(t)\rangle_{(H^1)^*,H^1}dt\Big| &\leq \int_0^T ||\mathbb{A}u(t)||_{(H^1)^*}||v(t)||_{H^1}dt\\ &\leq \int_0^T ||\mathbb{A}||_{\mathscr{L}(H^{1*},H^1)}||u(t)||_{H^1}||v(t)||_{H^1}dt\\ &\leq ||\mathbb{A}||_{\mathscr{L}(H^{1*},H^1)}\sqrt{(1+c_p^2)}\int_0^T ||u(t)||_{H^1}||v(t)||_{\widetilde{V}}dt\\ &\leq ||\mathbb{A}||_{\mathscr{L}(H^{1*},H^1)}\sqrt{(1+c_p^2)}||u||_{L^2(H^1)}||v||_{L^2(\widetilde{V})}\end{aligned}$$

where in the last step we used the Cauchy-Schwarz inequality. □

The fact that for a $u(t) \in L^2(H^1)$, we have $\mathbb{A}u(t)\Big|_{\widetilde{V}} \in L^2(\widetilde{V}^*)$ is important for the weak solution definition, since we need the lifting of the Dirichlet data to be mapped, via $\mathbb{A}$, to a functional in $L^2(\widetilde{V}^*)$ when restricted to $\widetilde{V}$.

3.1 Weak Form and Variational Form

To simplify, we will say that we are in conditions of hypotheses-2 (H2) if we have a $f(t) \in L^2(0,T;L^2(\Omega))$, $y_{in} \in \mathscr{T}_{in}$ $y_c \in \mathscr{T}_c$ and an initial condition $y_0 \in \mathscr{H}$ which is compatible with

the Dirichlet data.
In the case (H2) is valid, we know from propositions 2.1 and 2.2, that there exists liftings of y_{in}, y_c denoted by, $\widetilde{g}_{in}$ and $\widetilde{g}_c$ respectively, and that the function $\mathbb{W}(0,T) \ni \widetilde{y} = \widetilde{g}_{in} + \widetilde{g}_c$ satisfies the Dirichlet conditions, and the compatibility conditions with the initial condition

$$y_0 - \widetilde{y}(0) \in \widetilde{H}$$

We are now in conditions to give the definition of the weak solution of (1).

Definition 3.1. *[Weak solution] Suppose that the hypotheses (H2) is valid. We say that $y \in \widetilde{\mathbb{W}}(0,T)$ is a weak solution to the problem (1) if it satisfies*

$$\begin{cases} \int_0^T \langle y'(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt + \int_0^T a(y(t), v(t))dt = \int_0^T (f(t), v(t))dt \text{ for all } v \in L^2(0,T;\widetilde{V}) \\ y(0) = y_0 \\ \widehat{\tau_{in}}(y) = y_{in} \\ \widehat{\tau_c}(y) = y_c \end{cases} \tag{45}$$

The initial condition $y(0) = y_0$, at first sight may no make sense, since we do not have, at least as far as we know, that $\widetilde{\mathbb{W}}(0,T) \hookrightarrow C([0,T],\mathbb{H})$. However in this particular case, since y_0 is chosen as an element of $\mathscr{H}$ and not arbitrarily in $\mathbb{H}$, and, the solution is obtained by a decomposition process which yields a specif form that we will specify bellow , we can in fact see by the proofs to come, that this condition is always satisfied, and therefore this conditions makes sense in à posteriori analysis.

The system of the definition 3.1 has the asymmetric problem, that we are seeking for a solution in $\widetilde{\mathbb{W}}(0,T) \subset L^2(\mathbb{V})$ and using test functions in $L^2(\widetilde{V})$. To avoid this we use the lifting technique which allows to write the problem in definition 3.1 in the equivalent form

$$\text{Find } w \in W(0,T) \text{ such that } \begin{cases} \int_0^T \langle w'(t), \varphi\rangle_{\widetilde{V}^*,\widetilde{V}} dt + \int_0^T \widetilde{a}(w(t), \varphi)dt = \int_0^T \langle L(t), \varphi\rangle_{\widetilde{V}^*,\widetilde{V}} dt \ , \forall \varphi \in L^2(\widetilde{V}) \\ w(0) = y_0 - \widetilde{y}(0) \end{cases} \tag{46}$$

where the operator $L(t) \in L^2(0,T;\widetilde{V}^*)$ (see lemma 3.2) is given by

$$\langle L(t), \varphi\rangle_{\widetilde{V}^*,\widetilde{V}} = (f(t), \varphi) - \langle \widetilde{y}'(t), \varphi\rangle_{\widetilde{V}^*,\widetilde{V}} - \mu(\nabla\widetilde{y}(t), \nabla\varphi)\forall \ , v \in \widetilde{V} \text{ a.e. } t \in (0,T) \tag{47}$$

Problem (46) is a better variational problem then the one in definition 3.1, since this variational problem has the good property, that we are seeking the solution in $L^2(\widetilde{V})$, the same space of the test functions.
Before proving the equivalence between the problems (46) and (45), let us see that in fact $L(t)$ belongs to the space $L^2(\widetilde{V}^*)$.

Lemma 3.2. *In the case (H2) is satisfied, we have that the operator L defined in (47) is in $L^2(\widetilde{V^*})$.*

Proof:

$$|\langle L(t), v\rangle_{\widetilde{V}^*,\widetilde{V}}| \leq ||f(t)||_{L^2} C_p ||v||_{\widetilde{V}} + ||\nabla\widetilde{y}(t)||_{L^2} ||\nabla v||_{L^2} + ||\widetilde{y}'(t)||_{\widetilde{V}^*} ||v||_{\widetilde{V}}$$

therefore for almost every $t \in (0,T)$

$$||L(t)||_{\widetilde{V}} \leq C_p ||f(t)||_{L^2} + \mu||\widetilde{y}(t)||_{\widetilde{V}} + ||\widetilde{y}'(t)||_{\widetilde{V}^*}$$

From the Minkoski's inequality if $f, g \in L^2(0,T)$ then

$$\int_0^T |f+g|^2 dt \leq \int_0^T f^2 dt + \int_0^T g^2 dt$$

Therefore the $L^2(\widetilde{V}^*)$ norm of $L(t)$ is given by

$$\begin{aligned}\int_0^T ||L(t)||_{\widetilde{V}}^2 &\le \int_0^T \Big(||f(t)||_{L^2} + \mu||\widetilde{y}(t)||_{H_0^1} + ||\widetilde{y}'(t)||_{\widetilde{V}^*}\Big)^2 dt \\ &\le \int_0^T ||f(t)||^2 dt + \mu \int_0^T ||\widetilde{y}(t)||_{H_0^1}^2 dt + \int_0^T ||\widetilde{y}'(t)||_{\widetilde{V}^*}^2 dt \text{ **by Mink.' ine. 2 times**} \\ &\le ||f||_{L^2(L^2)}^2 + \max\{1,\mu\}||\widetilde{y}||_{\mathbb{W}(0,T)}^2\end{aligned}$$

□

The initial condition in (46) makes sense, since the space $W(0,T) \hookrightarrow C([0,T],\widetilde{H})$, and by hypotheses (H2), the object $y_0 - \widetilde{y}(0) \in \widetilde{H}$. In the following lemma $\widetilde{y}$ denotes the lifting of the Dirichlet data, which is given by the sum of $L_{in}(y_{in}) + L_c(y_c)$, where $y_{in} \in \mathscr{T}_{in}$ and $y_c \in \mathscr{T}_c$.

Lemma 3.3. *The problems (46) and (45) are equivalent in the sense that, if $y \in \widetilde{\mathbb{W}}(0,T)$ is a solution of (45) then $y - \widetilde{y} = w$ is a solution of (46), and on the other hand if $w \in W(0,T)$ is a solution of (46) then the function $y = \widetilde{y} + w$ is a solution of the problem (45).*

Proof: Suppose that y is a solution of (45). Then the difference $y - \widetilde{y}$ is in $w(0,T)$ since both functions have the same trace on Σ_D. We define $w = y - \widetilde{y}$. Then we have that

$$\begin{aligned}&w \in W(0,T)\\ &w(0) = y(0) - \widetilde{y}(0) = y_0 - \widetilde{y}(0)\\ &\int_0^T \langle w'(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} + \widetilde{a}(w(t),v(t))dt = \int_0^T \langle L(t),\varphi\rangle_{\widetilde{V}^*,\widetilde{V}}\end{aligned}$$

and therefore w is a solution to (46).

On the other direction, let $w(t) \in W(0,T)$ be a solution of (46). Then the function $y = \widetilde{y} + w$ is in $\widetilde{\mathbb{W}}$ and satisfies

$$\begin{aligned}&y(0) = w(0) + \widetilde{y}(0) = y_0 - \widetilde{y}(0) + \widetilde{y}(0) = y_0\\ &\int_0^T \langle y(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} + a(y(t),v(t))dt = \int_0^T \langle f(t),\varphi\rangle_{\widetilde{V}^*,\widetilde{V}}\end{aligned}$$

and therefore u is a solution of (45). □

The problem (46) can also be written in the form, *Find $w \in W(0,T)$ such that*

$$\begin{cases}\langle w'(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} + \widetilde{a}(w(t),v(t))dt = \langle L(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} \forall v \in \widetilde{V} a.e.t \in (0,T)\\ w(0) = y_0 - \widetilde{y}(0)\end{cases} \tag{48}$$

in a more compact form by

$$\begin{cases}w'(t) + \mathscr{A}w(t) = L(t) \text{ in } L^2(\widetilde{V}^*)\\ w(0) = y_0 - \widetilde{y}(0)\end{cases} \tag{49}$$

or even in the form

$$\begin{cases}\langle w'(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} + \langle \mathscr{A}w(t), v\rangle_{V^*,V} = \langle L(t), v\rangle_{\widetilde{V}^*,\widetilde{V}}\ , \forall v \in \widetilde{V} \text{ and a.e. in } (0,T)\\ w(0) = y_0 - \widetilde{y}(0)\end{cases} \tag{50}$$

By the section (7.4) on the appendix, the problem (50) can be also written by using the continuous extension of the map $(\cdot, i\cdot)_{\widetilde{H}}$ to the space $\widetilde{V}^* \times \widetilde{V}$,

$$\begin{cases}(w'(t), v) + (\mathscr{A}w(t), v) = (L(t), v)\ , \forall v \in \widetilde{V} \text{ and a.e. in } (0,T)\\ w(0) = y_0 - \widetilde{y}(0)\end{cases} \tag{51}$$

3.2 Well Posedness of the Variational Problem

Before proceeding to the proof of existence and uniqueness of solution to the problem (46) (or equivalently (49)-(51)), we proof a simple result.

Lemma 3.4. *The operator $\widetilde{a} : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ is $\widetilde{V}$- elliptic, i.e, exists a $\alpha > 0$ such that*

$$\widetilde{a}(u,u) \geq \alpha ||u||^2_{\widetilde{V}} \ , \forall u \in \widetilde{V}$$

Proof: For every $u \in \widetilde{V}$ we have by the Poicaré's inequality that, $||\nabla u||^2 \geq C_p^2 ||u||^2$, therefore

$$\widetilde{a}(u,u) = \mu ||\nabla u||^2 = \mu ||u||^2_{\widetilde{V}}$$

□

We have the following well posedness result which is guided by [29] and [33].

Theorem 3.1. *If assumptions (H2) are verified, then exists a unique solution $g \in W(0,T)$ to (46).*
Moreover, the solution satisfies the estimate,

$$||g||_{W(0,T)} \leq K\big(||g_0||^2 + \int_0^T ||L(t)||^2_{\widetilde{V}^*}\big)$$

where L is the functional define by (47), and $g_0 = y_0 - \widetilde{y}(0)$.

Proof: (i) **Existence:**
For the proof of the existence we use the Galerkin method, which provides a constructive form of obtaining the solution
We start by observing that $\widetilde{V}$ is a separable Hilbert space, therefore exists a set of linearly independent vectors $\{v_j\}_{j \in \mathbb{N}}$ whose linear span is dense in $\widetilde{V}$, i.e, $\overline{\{\sum_{j=1}^{m} \alpha_j v_j : \text{ m is finite } \}} = \widetilde{V}$.
The set $\{v_j\}_{j \in \mathbb{N}}$ is also linear independent in the set $\widetilde{H}$ and with the Gram-Schmidt orthonormalization process[14] we can obtain a new set (which we also denote by $\{v_j\}_{j \in \mathbb{N}}$) of orthonormal elements in inner product of $\widetilde{H}$. By the footnote, we conclude that this new vectors, having the same span, they have also the same closure in the $\widetilde{V}$ norm, and therefore their span is again dense in $\widetilde{V}$.

Now for $m \in \mathbb{N}$ fixed, we define the ansatz,

$$g_m(t) = \sum_{i=1}^{m} g_{im}(t) v_i$$

where the coefficients functions g_{im} are unknown scalar functions defined on [0,T]. We impose that the function $g_m(t)$ satisfies the system of equations,

$$\big(\frac{d}{dt} g_m(t), v_j\big) + \mu(\nabla g_m(t), \nabla v_j) = \langle L(t), v_j \rangle \text{ for } j \in \{1,..,m\} \text{ and a.e. } t \in (0,T) \qquad (52)$$

The system (52) is incomplete, we need to introduce the initial condition. We want the solution $g \in W(0,T)$ to satisfy $g(0) = g_0$ in $\widetilde{H}$. Therefore we impose that our "approximate solution" g_m has an initial condition that converges, in the $\widetilde{H}$ norm, to the imposed initial condition, i.e,

$$g_m(0) = \sum_{j=1}^{m} g_{jm}(0) v_j = \sum_{j=1}^{m} \xi_{jm} v_j \to g_0 \text{ in } \widetilde{H} \text{ as } m \to \infty \qquad (53)$$

[14]By applying this process we obtain a new set of linear independent vectors v_j whose span is the same as the original vectors.

Let us see how to determine a possible set of values $\{\xi_{jm}\}$ such that the condition (53) is satisfied. By intuition, we will try to define the sequence of initial values as the coefficients of the orthogonal projection (using the $\widetilde{H}$ inner product) of the element g_0 in the space $\mathscr{L}\big(\{v_1, v_2, ...v_m\}\big)$. Therefore we need to see that the sequence

$$g_m(0) = \sum_{j=1}^{m} (g_0, v_j) v_j \to g_0 \text{ in } \widetilde{H}$$

We start by seeing that, the point g_0 is an accumulation point of the set $\{\sum_{j=1}^{m} \alpha_j v_j : \text{ m is finite}\}$. Let $\epsilon > 0$ be arbitrary, then by the density of $\widetilde{V}$ in $\widetilde{H}$ exists a $y \in \widetilde{V}$ such (the $\widetilde{H}$ norm is the L^2 norm)

$$||g_0 - y||_{\tilde{H}} < \frac{\epsilon}{2} \tag{54}$$

Also since $y \in \widetilde{V}$, exists a element of the span $\mathscr{L}\big(\{v_j\}_{j\in\mathbb{N}}\big)$ such that

$$\frac{1}{C_p} ||y - \sum_{j=1}^{m} \alpha_j v_j||_{\tilde{V}} < \frac{\epsilon}{2} \tag{55}$$

and since $|| \cdot ||_{\widetilde{V}} \geq C_p || \cdot ||_{L^2}$
By the triangular inequality

$$||g_0 - \sum_{j=1}^{m} \alpha_j v_j|| \leq \epsilon \tag{56}$$

Since the value of $\epsilon > 0$ may be arbitrarily small we conclude that g_0 has the representation

$$g_0 = \sum_{j=1}^{\infty} \alpha_j v_j \tag{57}$$

Now since the vectors v_j are orthonormal in $\widetilde{H}$, we can explicit calculate the coefficients α_j by using the inner product,

$$\big(g_0, v_j\big) = \big(\sum_{i=1}^{\infty} \alpha_i v_i, v_j\big) = \alpha_j \big(v_j, v_j\big) = \alpha_j$$

Using the Bessel's inequality we have that

$$\sum_{i=1}^{\infty} |(g_0, v_j)|^2 \leq ||g_0||^2 \tag{58}$$

and therefore using again the orthonormality of the vectors v_j in the inner product of $\widetilde{H}$, we obtain the important estimate

$$||g_m(0)||^2 = \sum_{i=1} |(g_0, v_j)|^2 \leq \sum_{i=1}^{\infty} |(g_0, v_j)|^2 \leq ||g_0||^2 \tag{59}$$

and moreover

$$||g_0 - g_m(0)|| = ||\sum_{i=1}^{\infty} v_j(g_0, v_j) - \sum_{i=1}^{m} v_j(g_0, v_j)|| = ||\sum_{i=m+1}^{\infty} |(g_0, v_j)|| \to 0 \text{ when } m \to \infty$$

Thus to the system (52) we add the condition

$$g_{jm}(0) = \alpha_j = (g_0, v_j) \text{ for } j \in \{1, ..., m\} \tag{60}$$

Using the expression which defines g_m on the system (52) and (60) we obtain the following differential system for the coefficients $g_{im}(t)$,

$$\begin{cases} \sum_{i=1}^m (v_i, v_j) g'_{im}(t) + \sum_{i=1}^m a(v_i, v_j) g_{im}(t) = \langle L(t), v_j \rangle \text{ a.e t } \in (0,T) \\ g_{im}(0) = (g_0, v_i) \end{cases} \tag{61}$$

Notice now that, since the vectors $\{v_i\}$ are orthonormal in $\widetilde{H}$ we have that the matrix $(v_i, v_j)_{ij}$ is the identity, and so the system (61) is equivalent to

$$\begin{cases} g'_{im}(t) + \sum_{j=1}^m \alpha_{i,j} g_{im}(t) = \langle L(t), v_j \rangle \text{ a.e t } \in (0,T) \\ g_{im}(0) = (g_0, v_i) \end{cases} \tag{62}$$

for $i \in \{1, 2, ..., m\}$, where $\alpha_{ij} = a(v_i, v_j)$.
From a result of [9], the unique vector solution of the system (61), $g_m = (g_{1m}, ..., g_{mm}) \in [H^1(\Omega)]^m$ has absolute continuous functions as it's components.

Now if we multiply (52) by $g_{jm}(t)$ and add the result from $j = 1$ to $j = m$ we obtain in almost every $t \in (0,T)$ the expression

$$\big(\frac{d}{dt} g_m(t), g_m(t)\big) + a(g_m(t), g_m(t)) = \langle L(t), g_m(t) \rangle_{\widetilde{V}^*, \widetilde{V}} \tag{63}$$

Notice that de time derivative in (63) makes sense, since the time coefficients $g_{jm}(t)$ are absolute continuous functions, and so $\frac{d}{dt} g_m(t)$ exists almost everywhere in $(0,T)$.

Estimates for g_m

As mentioned before the proof, the function $L(t)$ belongs to $L^2(\tilde{V}^*)$. This fact tells us that the function $\langle L(t), v_j \rangle_{\widetilde{V}^*, \widetilde{V}}$ is square integrable since

$$\int_0^T |\langle L(t), v_j \rangle_{\widetilde{V}^*, \widetilde{V}}|^2 dt \leq \int_0^T ||L(t)||^2_{\widetilde{V}^*} ||v_j||^2 dt < \infty$$

using the Holder's inequality and the fact that the set $(0,T)$ is bounded, we also have that $\langle L(t), v_j \rangle_{\widetilde{V}^*, \widetilde{V}}$ is in $L^1(0,T)$.
Thus we can integrate the term $\langle L(t), v \rangle_{\widetilde{V}^*, \widetilde{V}}$. Also, from the fact that $\langle L(t), v_j \rangle_{\widetilde{V}^*, \widetilde{V}}$ is square integrable we get that the coefficients $g_{im}(t), g'_{im}(t)$ from (62) are also in $L^2(0,T)$ what leads us to $g_m, g'_m \in L^2(0,T; \widetilde{V})$.

Now for an arbitrary but fixed $\tau \in (0,T]$, we have, using integration by parts, the identity

$$\int_0^\tau \big(g'_m(t), g_m(t)\big) dt = \frac{1}{2}\big(||g_m(\tau)||^2 - ||g_m(0)||^2\big)$$

Thus, integrating (63) over $[0, \tau]$ yields

$$\frac{1}{2}||g_m(\tau)||^2 + \int_0^\tau a(g_m(t), g_m(t)) dt = \frac{1}{2}||g_m(0)||^2 + \int_0^\tau \langle L(t), g_m(t) \rangle_{\widetilde{V}^*, \widetilde{V}} dt \tag{64}$$

Using the $\widetilde{V}$-coercivity of the application $a(.,.)$ the integral term with $a(.,.)$ in (64) is non-negative. Also the integral with the term $\langle L(t), g_m(t) \rangle_{\widetilde{V}^*, \widetilde{V}}$ can be estimated by using the Cauchy-Schwarz and Young inequalities.

$$\begin{aligned} \int_0^T \langle L(t), g_m(t) \rangle_{\widetilde{V}^*, \widetilde{V}} dt &\leq \int_0^T |\langle L(t), g_m(t) \rangle_{\widetilde{V}^*, \widetilde{V}}| dt \\ &\leq \int_0^T ||L(t)||_{\widetilde{V}^*} ||g_m(t)||_{\widetilde{V}} dt \ , \ \textbf{Cauchy-Schwarz} \\ &\leq \int_0^T \frac{1}{2\epsilon} ||L(t)||^2_{\widetilde{V}^*} + \int_0^T \frac{\epsilon}{2} ||g_m(t)||^2_{\widetilde{V}} \ , \ \textbf{Young} \end{aligned}$$

This leads us, using (64), the $\widetilde{V}$-coercivity of a and the Bessel's inequality (58) , to the following,

$$\begin{aligned}||g_m(\tau)||^2 + \int_0^\tau 2\mu||g_m(t)||^2_{\widetilde{V}}dt &\le ||g_m(\tau)||^2 + \int_0^\tau 2a(g_m(t), g_m(t))dy \\ &= ||g_m(0)||^2 + 2\int_0^\tau \langle L(t), g_m(t)\rangle_{\widetilde{V}^*,\widetilde{V}}dt \\ &\le ||g_0||^2 + \int_0^\tau \frac{1}{\epsilon}||L(t)||^2_{\widetilde{V}^*} + \int_0^\tau \epsilon||g_m(t)||^2_{\widetilde{V}^*}\end{aligned}$$

and choosing $\epsilon = 2\mu$,

$$||g_m(\tau)||^2 \le ||y_0||^2 + \int_0^\tau \frac{1}{2\mu}||L(t)||^2_{\widetilde{V}^*} \le ||g_0||^2_{\widetilde{H}} + \int_0^T \frac{1}{2\mu}||L(t)||^2_{\widetilde{V}^*} \tag{65}$$

(65) says in particular that the sequence $\{g_m\}_{m\in\mathbb{N}}$ is a bounded sequence in $L^\infty(0,T;\tilde{H})$
If instead we choose $\epsilon = \mu$ we get

$$||g_m(\tau)||^2 + \int_0^T \mu||g_m(t)||^2_{\widetilde{V}}dt \le ||g_0||^2 + \int_0^\tau \frac{1}{2\mu}||L(t)||^2_{\widetilde{V}^*} \le ||g_0||^2_{\widetilde{H}} + \int_0^T \frac{1}{\mu}||L(t)||^2_{\widetilde{V}^*} \tag{66}$$

what implies that $\{g_m\}_{m\in\mathbb{N}}$ is also a bounded sequence in $L^2(0,T;\widetilde{V})$.

Passage to the limit

Since the sequence $\{g_m\}_{m\in\mathbb{N}}$ is by (65) bounded in $L^\infty(0,T;\widetilde{H})$ we know from the Banach-Alaoglu-Bourbaki theorem ([5] pág. 66 theorem 3.16), that exists a subsequence $\{g_{m_k}\} \subset \{g_m\}$ which converges in the weak* topology to a $u \in L^\infty(0,T;\widetilde{H})$. Note now that by the identification $\widetilde{H} = \widetilde{H}^*$, (which is possible since $\widetilde{H}$ is an Hilbert space) we have $L^\infty(0,T;\widetilde{H}) = \big(L^1(0,T;\widetilde{H})\big)^*$.
Thus $g_{m_k} \xrightarrow{w^*} u$ implies that for every $v \in L^1(0,T;\widetilde{H})$

$$\int_0^T \big(g_{m_k} - u, v\big)dt \to 0 \tag{67}$$

when $k \to \infty$.
On the other hand, from the fact that $L^2(0,T;\widetilde{V})$ is a Hilbert space, this implies that $L^2(0,T;\widetilde{V})$ is a reflexive space, and thus every bounded sequence in this space has a subsequence which converges weakly for some $z \in L^2(0,T;\widetilde{V})$.
So taking a subsubsequence $g_\nu \subset g_{m_k}$ we have $g_\nu \xrightarrow{w} z$ in $L^2(0,T;\widetilde{V})$.
Written in another way when $\nu \to \infty$,

$$\int_0^T \big(g_\nu - z, v\big)dt \to 0 \ , \ \forall v \in L^2(0,T;\tilde{V}) \tag{68}$$

or even in another way

$$\int_0^T \big\langle g_\nu - z, v\big\rangle_{\tilde{V},\tilde{V}^*}dt \to 0 \ , \ \forall v \in L^2(0,T;\tilde{V}^*) \tag{69}$$

From the density of $\widetilde{V}$ in $\widetilde{H}$ we have

$$\langle f, v\rangle_{\widetilde{V}^*,\widetilde{V}} = (f,v)_{\widetilde{H}} \ , \ \forall f \in \widetilde{H}, \forall v \in \widetilde{V}$$

and thus from (69) and the fact that $\widetilde{H}^* \subset \widetilde{V}^*$

$$\int_0^T \big(g_\nu - z, v\big)dt \to 0 \ , \ \forall v \in L^2(0,T;\widetilde{H}) \tag{70}$$

Notice now that by the Holder's inequality $L^2(0,T;\widetilde{H}) \subset L^1(0,T;\widetilde{H})$, then by (67) we have

$$\int_0^T \big(g_\nu, v\big)dt \to \int_0^T \big(u, v\big)dt$$

and from (70)

$$\int_0^T (g_\nu, v)dt \to \int_0^T (z, v)dt$$

thus

$$\int_0^T (z, v)dt = \int_0^T (u, v)dt$$

for each $v \in L^2(\widetilde{H})$.
So we have that $u = z \in L^2(0, T; \widetilde{V}) \cap L^\infty(0, T; \widetilde{H})$.

We also have that $g_\nu(0) \to g_0$ strongly in $\widetilde{H}$. To see this, observe that from (66) we can get

$$||g_m(\tau)||^2_{\widetilde{H}} \leq ||g_0||^2_{\widetilde{H}} + \int_0^\tau \frac{1}{\mu}||L(t)||^2_{\widetilde{V}^*} \tag{71}$$

Now since $g_m(t)$ is a continuous function in relation to the variable t, we get

$$||g_m||_{C([0,T];\widetilde{H})} := \max_{t\in[0,T]} ||g_m(t)|| \leq \big(||g_0||^2_{\widetilde{V}} + \int_0^T \frac{1}{\epsilon}||L(t)||^2_{\widetilde{V}^*}\big)^{1/2} = \widetilde{K} \tag{72}$$

Now we see that for every $t \in [0, T]$ and every $m \in \mathbb{N}$ $||g_m(t)||^2_{\widetilde{H}} \leq \widetilde{K}^2$ which is equivalent to say

$$\sum_{i=1}^m |g_{im}(t)|^2 \leq \widetilde{K}^2 \text{ , } \forall t \in [0, T] \text{ and } \forall m \in \mathbb{N}$$

This yields the strong convergence of $g_\nu(0)$ to g_0 in $L^2(\Omega)$.
In fact,

$$||g_\nu(0) - g_0|| = ||\sum_{i=1}^\nu g_i^\nu(0)v_i - \sum_{i=1}^\infty (g_0, v_i)v_i|| = ||\sum_{i=\nu+1}^\infty g_i(0)v_i|| \to 0$$

Consider now $j \in \mathbb{N}$ arbitrary but fixed, and let $\mathbb{N} \ni \nu > j$. From (52)

$$\big(\frac{d}{dt}g_m(t), v_j\big) + \mu(\nabla g_m(t), v_j) = \langle L(t), v_j\rangle \text{ para } j \in \{1, .., m\}$$

choosing $\varphi \in C^1[0, T]$ such that $\varphi(T) = 0$ and multiplying the above equation by φ we get

$$\big(\frac{d}{dt}g_m(t), \phi_j\big) + \mu(\nabla g_m(t), \nabla\phi_j) = \langle L(t), \phi_j\rangle \text{ para } j \in \{1, .., m\} \tag{73}$$

where $\phi_j(t) = \varphi(t)v_j$.
Integrating the equation (73) from 0 to T, and using the formulas of integration by parts, which are valid in this context (see [11]), we arrive at

$$\int_0^T \big(\frac{d}{dt}g_m(t), \phi_j\big)dt + \int_0^T \mu(\nabla g_m(t), \phi_j)dt = \int_0^T \langle L(t), \phi_j\rangle dt \Leftrightarrow$$
$$-\int_0^T \big(g_m(t), \frac{d}{dt}\phi_j\big)dt + \int_0^T \mu(\nabla g_m(t), \nabla\phi_j)dt = \int_0^T \langle L(t), \nabla\phi_j\rangle dt + \big(g_\nu(0), \varphi_j(0)\big)$$

Now notice that from the regularity of ϕ_j, this element is in $L^2(0, T; \widetilde{V})$.
Taking the limit $\nu \to \infty$ we arrive at, using the weak convergences and the strong convergences seen above,

$$-\int_0^T \big(z(t), \frac{d}{dt}\phi_j\big)dt + \int_0^T \mu(\nabla z(t), \nabla\phi_j)dt = \int_0^T \langle L(t), \phi_j\rangle dt + \big(z(0), \varphi_j(0)\big) \tag{74}$$

The expression (74) is also valid in $\mathscr{D}((0, T))$, and thus we get that

$$\frac{d}{dt}(z(t), v_j) + a(z(t), v_j) = \langle L(t), v_j\rangle_{\tilde{V}^*,\tilde{V}} \tag{75}$$

in the sense of the scalar distributions.
On the other hand, since $j \in \mathbb{N}$ was chosen in an arbitrary way, (74) is valid for every V_j with $j \in \mathbb{N}$, which is a dense set of $\widetilde{V}$.
Thus we have that (74) is valid in $\widetilde{V}$ and we arrive at

$$\frac{d}{dt}\langle z(t), v\rangle = \langle L(t) - \mathscr{A}z(t), v\rangle \text{ , for all } v \in \widetilde{V}$$

But, this from lemma 7.9 is equivalent to say that

$$L(t) - \mathscr{A}z(t) = z'(t)$$

where $z'(t)$ is the derivative in the sense of the vector valued distributions, and since $L(t) - \mathscr{A}z(t) \in L^2(0,T;\widetilde{V}^*)$ we also have that $z'(t) \in L^2(0,T;\widetilde{V}^*)$. So $z(t) \in W(0,T)$.
It remains to check if the initial condition $z(0) = g_0$ is satisfied.
To see this, we choose a function $\varphi \in C^1([0,T])$ such that $\varphi(T) = 0$. With this differentiable function we define a function in $L^2(\widetilde{V})$ by $v(t) = \varphi(t)v$ where $v \in \widetilde{V}$. From an appendix result, we know that the function $v(t)$ has the weak derivative $v'(t) = \varphi'(t)v$ which is again in $L^2(\widetilde{V})$ and therefore is also in $L^2(\widetilde{V}^*)$.
Now since $z(t) \in W(0,T)$ is the solution of $z'(t) + \mathscr{A}z(t) = L(t)$ in $L^2(\widetilde{V}^*)$, we have that, using integration by parts

$$\int_0^T \langle z'(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt + \int_0^T \langle \mathscr{A}z(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} = \int_0^T \langle L(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt \Leftrightarrow$$

$$\Leftrightarrow -\int_0^T \langle z(t), v'(t)\rangle_{\widetilde{V},\widetilde{V}^*} dt + \int_0^T \langle \mathscr{A}z(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} = (z(0), v(0)) + \int_0^T \langle L(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt$$

Now from the construction of $z(t)$ we also have that for the same $v(t)$ as above

$$-\int_0^T \langle z(t), v'(t)\rangle_{\widetilde{V},\widetilde{V}^*} dt + \int_0^T \langle \mathscr{A}z(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} = (g_0, v(0)) + \int_0^T \langle L(t), v(t)\rangle_{\widetilde{V}^*,\widetilde{V}} dt$$

Therefore we conclude that

$$(g_0 - z(0), v(0)) = (g_0 - z(0), v)\varphi(0)$$

ans choosing $\varphi \in C^1([0,1])$ which has $\varphi(0) \neq 0$ we arrive at

$$(g_0 - z(0), v) = 0 \text{ , for all } v \in \widetilde{V}$$

and since the inner product is a continuous application and $\widetilde{V}$ is dense in $\widetilde{H}$ we get that $g_0 = z(0)$, that is, our solution candidate, $z(t)$ satisfies the initial condition.

Uniqueness

Consider the homogeneous version of (49), i.e, $L = 0$ and $g_0 = 0$. If this problem has only the null solution, then the uniqueness is proven.
Writing the system (49) with a solution y as a test function (notice that y exists from the existence argument seen above) we get

$$\Big(\frac{d}{dt}y, y\Big) + \mu(\nabla y, \nabla y) = 0 \Leftrightarrow \frac{d}{dt}||y(t)||^2 \leq -\mu||y(t)||^2_{\widetilde{V}} \leq 0$$

Thus $y(t)$ is a non-increasing function, therefore $||y(t)|| \leq ||y(0)|| = 0$, and this leads to $y(t) = 0$ for $t \in [0,T]$.

Estimate for the solution:

We saw that $g_\nu \xrightarrow{w} z$ in the sense of $L^2(0,T;\widetilde{V})$, and since the operator $\displaystyle\int_0^T ||g(t)||^2_{\widetilde{V}} dt$ is a norm

in that space, is convex and continuous, and thus is weakly lower semicontinuous.
Then, from (appendix) we have that

$$\int_0^T ||g(t)||^2_{\widetilde{V}} dt \leq \liminf_{\nu \to \infty} \int_0^T ||g_\nu(t)||^2_{\widetilde{V}} dt \leq C\Big(||g_0||^2 + \int_0^T ||L(t)||^2_{\widetilde{V}^*} dt\Big)$$

As we saw our solution z, assumes $z'(t) = L(t) - \mathscr{A}z(t)$ in $L^2(0,T;\widetilde{V}^*)$ and we have that, using triangular inequality

$$||z'(t)||^2_{\widetilde{V}^*} \leq \big(||L(t)||^2_{\widetilde{V}^*} + ||\mathscr{A}||^2_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)}||z(t)||^2_{\widetilde{V}}\big)^2$$

and using Young's Inequality, we get choosing $\epsilon = 1$.

$$||z'(t)||^2_{\widetilde{V}^*} \leq 2||L(t)||^2_{\widetilde{V}^*} + 2||\mathscr{A}||^2||z(t)||^2_{\widetilde{V}}$$

Then we have the following chain of inequalities

$$\begin{aligned}\int_0^T ||z'(t)||^2_{\widetilde{V}^*} dt &\leq 2\int_0^T ||L(t)||^2_{\widetilde{V}^*} dt + 2||\mathscr{A}||^2_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)} \int_0^T ||z(t)||^2_{\widetilde{V}} dt \\ &\leq 2\int_0^T ||L(t)||^2_{\widetilde{V}^*} dt + 2C||\mathscr{A}||^2_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)}\Big(||g_0||^2 + \int_0^T ||L(t)||^2_{\widetilde{V}^*} dt\Big) \\ &\leq 2(1 + ||\mathscr{A}||^2_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)}C)\Big(||g_0||^2 + \int_0^T ||L(t)||^2_{\widetilde{V}^*} dt\Big)\end{aligned}$$

and we call $\widetilde{K} = 2(1 + ||\mathscr{A}||^2_{\mathscr{L}(\widetilde{V},\widetilde{V}^*)}C)$.
Thus we can see that the solution g satisfies the estimate

$$||z||^2_{W(0,T)} \leq K\big(||g_0||^2 + ||L(t)||^2_{L^2(0,T;\widetilde{V}^*)}\big) \tag{76}$$

where $K = \widetilde{K} + 1$. □

Theorem 3.1 gives us the unique solution to the variational problem (46), which by construction is an element of $W(0,T)$ that satisfies,

$$w'(t) + \mathscr{A}w(t) = L(t) \Leftrightarrow w'(t) + \mathscr{A}w(t) = f(t) - \widetilde{y}'(t) - \mathbb{A}\widetilde{y}(t) \text{ , in } L^2(0,T;\widetilde{V}^*)$$

where $\mathbb{A}\widetilde{y}(t)$ is restricted to $\widetilde{V}$.
Thus the function $y(t) = \widetilde{y}(t) + w(t)$, is solution to the equation

$$y'(t) + \mathscr{A}w(t) + \mathbb{A}\widetilde{y}(t) = f(t) \Leftrightarrow y'(t) + \mathbb{A}y(t) = f(t) \text{ , in } L^2(0,T;\widetilde{V}^*)$$

where we used the fact that $\langle \mathscr{A}w(t), \varphi\rangle_{\widetilde{V}^*,\widetilde{V}} = \langle \mathbb{A}w(t), \varphi\rangle_{(H^1)^*,H^1}$. Since as was seen in (note 1) $u \in C([0,T];\mathbb{H})$, in particular $y(0)$ is well defined. Moreover, since by construction $w(0) = y_0 - \widetilde{y}(0)$ we get that $y(0) = y_0$, and the initial condition is satisfied.
Also, again by construction, the Dirichlet boundary condition are satisfied in the sense of the traces, since we have that

$$w\big|_{\Sigma_D} = 0 \text{ and } \widetilde{y}\big|_{\Sigma_D} = g_D \implies \widetilde{y} + w\big|_{\Sigma_D} = g_D$$

where

$$g_D = \begin{cases} y_{in} \text{ in } \Sigma_{in} \\ y_c \text{ in } \Sigma_c \\ 0 \text{ in } \Sigma_w \end{cases} \tag{77}$$

Thus we have just proved that the problem (1) has a weak solution in the sense of definition 3.1.
To see the uniqueness, if we choose another lifting $\widetilde{y}_1$, and determine another variational solution w_1 such that $y_1 = \widetilde{y}_1 + w_1$is also a weak solution of (1), we get that $y - y_1 \in W(0,T)$, because

both y, y_1 satisfy the same Dirichlet boundary condition on Σ_D. Also $y - y_1$ is a solution to the homogeneous problem

$$\begin{cases} (y-y_1)'(t) + \mathbb{A}(y-y_1)(t) = 0 \text{ in } L^2(0,T;\widetilde{V}^*) \\ (y-y_1)(0) = 0 \end{cases}$$

Notice that, since $(y-y_1)(t) \in L^2(\widetilde{V})$, and $\mathbb{A}(y-y_1)(t)$ has, in the above expression, the evaluation restricted to $\widetilde{V}$, we have that for all $v \in \widetilde{V}$ and for almost every $t \in (0,T)$,

$$\langle \mathbb{A}(y-y_1)(t), \varphi\rangle_{(H^1)^*,H^1} = a((y-y_1)(t), v) = \widetilde{a}((y-y_1)(t), v) = \langle \mathscr{A}(y-y_1)(t), v\rangle_{\widetilde{V}^*,\widetilde{V}}$$

and therefore the system is equivalent to

$$\begin{cases} (y-y_1)'(t) + \mathscr{A}(y-y_1)(t) = 0 \text{ in } L^2(0,T;\widetilde{V}^*) \\ (y-y_1)(0) = 0 \end{cases}$$

By the uniqueness proved in the theorem 3.1, this system has only the zero solution, and we conclude that $y = y_1$.

The following estimate result is important for the proof of the existence of an optimal solution in control chapter.

Lemma 3.5. *There exists a positive constant* $\widetilde{C}'$ *such that*

$$||y||_{C([0,T];\mathbb{H})} + ||y||_{\widetilde{\mathbb{W}}(0,T)} \le \widetilde{C}'\Big(||y_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + ||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c}\Big)^{1/2} \tag{78}$$

Proof: We start by observing that[15]

$$||y||^2_{\widetilde{\mathbb{W}}(0,T)} \le 2\Big(||\widetilde{y}||^2_{\mathbb{W}(0,T)} + ||w||^2_{W(0,T)}\Big) \tag{79}$$

To see this, we have the following chain of inequalities

$$\begin{aligned} ||y||^2_{\widetilde{\mathbb{W}}(0,T)} &= ||\widetilde{y} + w||^2_{\widetilde{\mathbb{W}}(0,T)} = ||\widetilde{y} + w||^2_{L^2(\mathbb{V})} + ||\widetilde{y}' + w'||^2_{L^2(\widetilde{V}^*)} \\ &= \int_0^T ||\widetilde{y} + w||^2_{\mathbb{V}} dt + \int_0^T ||\widetilde{y}' + w'||^2_{\widetilde{V}^*} dt \\ &\le 2\int_0^T ||\widetilde{y}||^2_{\mathbb{V}} dt + 2\int_0^T ||w||^2_{\mathbb{V}} dt + 2\int_0^T ||\widetilde{y}'||^2_{\widetilde{V}^*} dt + 2\int_0^T ||w'||^2_{\widetilde{V}^*} dt \\ &= 2||\widetilde{y}||^2_{\widetilde{\mathbb{W}}(0,T)} + 2||w||^2_{W(0,T)} \\ &\le 2||\widetilde{y}||^2_{\mathbb{W}(0,T)} + 2||w||^2_{W(0,T)} \end{aligned}$$

where we used the fact that $||w||^2_{\mathbb{V}} = ||w||^2_{\widetilde{V}}$, the Young's inequality with $\epsilon = 1$ (see the lemma 7.1 and post commentary).

Now, we have from the estimates for the liftings of the Dirichlet's data

$$||\tilde{y}||^2_{\mathbb{W}(0,T)} = ||L_{in}(y_{in}) + L_c(y_c)||^2_{\mathbb{W}(0,T)} \le 2||L_{in}(y_{in})||^2_{\mathbb{W}(0,T)} + 2||L_c(y_c)||^2_{\mathbb{W}(0,T)} = 2||y_{in}||^2_{\mathscr{T}_{in}} + 2||y_c||^2_{\mathscr{T}_c} \tag{80}$$

To estimate the term $||w||^2_{W(0,T)}$ we use (76).
Thus,

$$||w||^2_{W(0,T)} \le K\big(||g_0||^2 + ||L(t)||^2_{L^2(0,T;\widetilde{V}^*)}\big) \tag{81}$$

where $g_0 = y_0 - \widetilde{y}(0)$.

[15]Notice that $\mathbb{W}(0,T) \hookrightarrow \widetilde{\mathbb{W}}(0,T)$.

From (79), and the above estimates we get that

$$
\begin{aligned}
||y||^2_{\widetilde{\mathbb{W}}(0,T)} &\leq 2\Big(||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c} + K\big(||g_0||^2 + ||f||^2_{L^2(0,T\widetilde{V}^*)} + ||\widetilde{y}'||^2_{L^2(0,T;\widetilde{V}^*)} \\
&\quad + ||\mathscr{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}||\widetilde{y}||^2_{L^2(0,T;\mathbb{V})}\big)\Big) \\
&\leq 2\Big(||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c} + K\big(||g_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + \max\{1, ||\mathscr{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}\}||\widetilde{y}||^2_{\widetilde{\mathbb{W}}(0,T)}\big)\Big) \\
&\leq 2\Big(||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c} + K\big(||g_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + \max\{1, ||\mathscr{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}\}||\widetilde{y}||^2_{\mathbb{W}(0,T)}\big)\Big) \\
&\leq 2\Big(||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c} + K\big(||g_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + \max\{1, ||\mathscr{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}\}||g||^2_{\mathscr{T}_D}\big)\Big) \\
&\leq \widetilde{K}\Big(||g_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + ||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c}\Big) \\
&\leq \widetilde{K}\Big(||y_0||^2 + ||\widetilde{y}(0)||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + ||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c}\Big)
\end{aligned}
$$

Observe now that

$$||\widetilde{y}(0)||^2 \leq ||\widetilde{y}||^2_{C([0,T];\mathbb{H})} \leq c||\widetilde{y}||^2_{\mathbb{W}(0,T)} \leq 2c\big(||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c}\big)$$

and thus we arrive at the estimate

$$||y||_{\widetilde{\mathbb{W}}(0,T)} \leq \widetilde{K}'\Big(||y_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + ||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c}\Big)^{1/2} \tag{82}$$

Now, by the continuous embeddings $\mathbb{W}(0,T) \hookrightarrow C([0,T];\mathbb{H})$ and $W(0,T) \hookrightarrow C([0,T];\mathbb{H})$ exist constants $c_1, c_2 > 0$ such that

$$
\begin{aligned}
||y||^2_{C([0,T];\mathbb{H})} &\leq 2\Big(||\widetilde{y}||^2_{C([0,T];\mathbb{H})} + ||w||^2_{C([0,T];\mathbb{H})}\Big) \\
&\leq 2\Big(c_1^2||\widetilde{y}||^2_{\mathbb{W}(0,T)} + c_2^2||w||^2_{W(0,T)}\Big)
\end{aligned}
$$

and using the estimate (82) we get

$$||y||_{C([0,T];\mathbb{H})} + ||y||_{\widetilde{\mathbb{W}}(0,T)} \leq \widetilde{C}'\Big(||y_0||^2 + ||f||^2_{L^2(0,T;\widetilde{V}^*)} + ||y_{in}||^2_{\mathscr{T}_{in}} + ||y_c||^2_{\mathscr{T}_c}\Big)^{1/2} \tag{83}$$

□

3.3 Pressure Recover

Let us now see that with a velocity $y \in \widetilde{\widetilde{\mathbb{W}}}(0,T)$ we can construct a pressure field, such that the Stokes equation is satisfied in the distribution sense.
For that we need to use the next two results which are from [29].

Theorem 3.2. *Let Ω be an open set of $\mathbb{R}^n$, and let $f = \{f_1, f_2, ...f_n\}$ be a vector such that each component $f_i \in \mathscr{D}'(\Omega)$ for $i \in \{1, ..., n\}$. Then a necessary and sufficient condition for*

$$f = \nabla p \ , \ \text{ for some } p \in \mathscr{D}'(\Omega)$$

is that $\langle f, \varphi\rangle = 0$ for every $\varphi \in \mathscr{V}(\Omega)$.

The next theorem is also from ([29] pag. 14) and gives more regularity to the distribution p obtain from theorem 3.2, under certain conditions.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz and bounded set. If :*
(i) p is a distribution which has all the derivatives $D_i p \in L^2(\Omega)$ then $p \in L^2(\Omega)$ and

$$||p||_{L^2_0(\Omega)} \leq c(\Omega)||\nabla p||_{L^2(\Omega)}$$

(ii) p is a distribution which has all the derivatives $D_i p \in H^{-1}(\Omega)$ then $p \in L^2(\Omega)$ and

$$||p||_{L^2_0(\Omega)} \leq c(\Omega)||\nabla p||_{H^{-1}(\Omega)}$$

Let $y(t) \in \widetilde{\mathbb{W}}(0,T)$ be a weak solution of (1) in the sense of our definition 3.1. Thus, y satisfies $y'(t) + \mathbb{A}y(t) = f(t)$ in $L^2(0,T;\widetilde{V}^*)$, which is equivalent to

$$\langle y'(t) + \mathbb{A}y(t) - f(t), v\rangle = 0 \text{ for almost every } t \in (0,T) \text{ and } \forall v \in \widetilde{V}$$

Integrating this expression from 0 to $t \in (0,T]$ we get

$$\int_0^t \langle y'(s), v\rangle_{\widetilde{V}^*,\widetilde{V}} ds + \langle \mathbb{A}Y(t) - F(t), v\rangle = 0 \text{ for every } t \in (0,T)\ , \forall v \in \widetilde{V} \tag{84}$$

where $Y(t) = \displaystyle\int_0^t u(s)ds$ belonging to $C([0,T];\mathbb{V})$ and $F(t) = \displaystyle\int_0^t f(s)ds$ belonging to $C([0,T];L^2(\Omega)^2)$
.
In (84), the integral on the left can be evaluated using integration by parts. In fact by note-2, we are in the special case of $\varphi(t) = 1$ which is a C^1 function in $[0,T]$, with the particularity that $\varphi' = 0$. Thus we have

$$\int_0^t \langle y'(s), v\rangle_{\widetilde{V}^*,\widetilde{V}} ds = (y(t) - y(0), v) \text{ for every } t \in [0,T], \forall v \in \widetilde{V}$$

Thus we arrive at the functional

$$y(t) - y(0) + \mathbb{A}Y(t) - F(t)$$

which is well defined in $[H_0^1(\Omega)^2]^*$, since for each $t \in [0,T]$, $F(t)$ is an element in $L^2(\Omega)^2$ and therefore in $[H_0^1(\Omega)^2]^*$, $\mathbb{A}Y(t)$ is an element in $(H^1(\Omega)^2)^* \subset H^{-1}(\Omega)^2$ and $(y(t) - y_0) \in \mathbb{H} \subset L^2(\Omega)^2$. Thus, this functional can be seen as an element of $H^{-1}(\Omega)^2$ for each $t \in [0,T]$. Moreover since this functional has the property,

$$\langle y(t) - y_0 + \mathbb{A}Y(t) - F(t), v\rangle = 0 \text{ for all } v \in \widetilde{V}$$

and $\mathscr{V} \subset \widetilde{V}$, we conclude from theorems 3.2-3.3 the existence of a function $P(t) \in L^2(\Omega)$ for each $t \in [0,T]$, such that

$$y(t) - y_0 + \mathbb{A}Y(t) + \nabla P(t) = F(t) \text{ for each } t \in [0,T] \tag{85}$$

Notice now that, when $\mathbb{A}y$ is restricted to $\widetilde{V}$ it gives the same result as $\langle -\Delta y, v\rangle$.
Since $y(t) - y_0 + \mathbb{A}Y(t) - F(t) \in C([0,T], H^{-1}(\Omega))$ we get from the gradient isomorphism that $P(t) \in C([0,T], L_0^2(\Omega))$.
This regularity in $P(t)$ is sufficient to let us differentiate (85) in time, in the sense of the distributions in Q, to give us

$$y'(t) - \mu\Delta y(t) + \nabla p(t) = f(t) \text{ in } Q \tag{86}$$

being this equation understood in the sense of the distributions in Q.

Suppose now, that we have more regularity on y and on the pressure p, for instance, $y \in \widetilde{\mathbb{W}}(0,T) \cap L^2(0,T;H^2(\Omega)^2)$ and $p \in L^2(0,T;H^1(\Omega)^2)$. Thus since

$$\mu(\nabla y(t), \nabla v) + \langle y'(t) - f(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} = 0 \text{ for all } v \in \widetilde{V} \text{ and a.e } t \in (0,T)$$

substituting in this expression f by $y'(t) - \mu\Delta y(t) + \nabla p(t)$ we obtain

$$\langle \mu\Delta y(t) - \nabla p(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} + \mu(\nabla y(t), \nabla v) \text{ for all } v \in \widetilde{V} \text{ and a.e } t \in (0,T)$$

and using integration by parts [10] we get

$$\begin{aligned} 0 = \langle \mu\Delta y(t) - \nabla p(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} + \mu(\nabla y(t), \nabla v) &= \int_\Gamma \big(\mu\nabla y \cdot \mathbf{n} - p\mathbf{n}\big)v dt \\ &= \int_{\Gamma_{out}} \big(\mu\nabla y \cdot \mathbf{n} - p\mathbf{n}\big)v dt \end{aligned}$$

for every $v \in \widetilde{V}$ and a.e $t \in (0,T)$.

3.4 Regularity

This section is dedicated to the regularity of the weak solution of problem (1). In our case we are dealing with a second order parabolic operator

$$Pu = u_t - a^{ij} D_{ij} u \tag{87}$$

where in this case $a^{ij} = \delta_{ij}$ to give the Laplacian. In fact the operator P is parabolic since we have (Evans Definition for a parabolic operator) since exists a constant $\theta > 0$ such that,

$$\sum_{i,j=1}^{2} a^{i,j}(x,t)\xi_i\xi_j \geq \theta|\xi|^2 \text{ for all } (x,t) \in Q, \xi \in \mathbb{R}^2$$

Our method for a solution construction gave us a function $y(t,x)$ which is in $L^2(0,T;H^1(\Omega))$.
Is natural to question if the weak solution y has more regularity both in space and time.
With respect to higher regularity for the space variables, one can proves (see Lieberman theorem 6.6) the following result,

Theorem 3.4. *Let y be the weak solution of the second order parabolic equation $Py = f$ where $f \in L^2(0,T;L^2(\Omega))$, if the coefficients of the operator P satisfy*

$$||Da^{i,j}||_{L^\infty} \leq K$$

for a positive constant k. Then for any subset $\Omega' \subset\subset \Omega$, we have D^2y and y_t in $L^2(0,T;L^2(\Omega))$. Moreover, exists a constant C not depending in y such that

$$||D^2y||_{L^2(\Omega')} + ||y||_{L^2(\Omega')} \leq C\big(||Dy||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\big)$$

The theorem (3.4) yields that $y(t,\cdot)$ is locally in H^2 for almost every $t \in (0,T)$.
To extend this type of regularity to the all domain Ω, that is to have $y(t,\cdot) \in H^2(\Omega)$ for almost every t, every consulted book on PDEs solutions regularity asks the domain to have a smooth regularity. This is not our case and therefore we only hope to have interior regularity.

Another possible way of improving the solution' s regularity is done in the book [3]. However the author in those results are using the semi-group theory and interpolation results, a topic that we do not discuss in the present work.

4 Optimal Control Problem

In this section we turn our attention to our boundary control problem, on the Ω domain. We start by giving a proper definition of our minimization problem, followed by analyzing if it has at least one solution. We close this chapter with a deduction of the first order optimality conditions.

4.1 Preliminary Definitions for the Minimization Problem

We are interested in analyzing a similar optimal problem as what was done in [24], but with the differences that in the present work, we consider only the Stokes equations, and we add the time dependence.
We fix some input velocity in Σ_{in}, an initial condition y_0, which both must be in some appropriate spaces (see chapter 2.2 and 2.3 for their definition), and a force term $f \in L^2(L^2(\Omega))$. In this framework we want to minimize a cost functional, which will be defined shortly, with evaluation constrained to the pairs (y,u), where y is the weak solution in the sense of the definition 3.1, which depends on u, a boundary velocity (that also should be in a proper space U that we will define

bellow) that we control, in order to minimize the cost.
Now we specify, which type of cost functionals are the ones that interested us and the space where we will do the control.

We are interested in minimizing cost functionals of the form

$$J(y,u) = \alpha_1 J_1(y) + \alpha_2 J_2(y) + J_3(u) \tag{88}$$

where, for the functional J_1 we are interested in the two following cases

$$J_1 : L^2(\mathbb{V}) \to \mathbb{R} \qquad J_1(y) = \frac{1}{2}\int_0^T ||y(t) - y_d(t)||^2_{\mathbb{V}} dt \text{ or } J_1(y) = \frac{1}{2}\int_0^T ||\nabla \times y(t)||^2 dt$$

with $y_d \in L^2(\mathbb{V})$, and where, J_2, J_3 are given by

$$J_2 : \mathbb{H} \to \mathbb{R} \qquad J_2(y) = \frac{1}{2}||y - y_d||^2_{\mathbb{H}} \tag{89}$$

with y_d a fixed element of $\mathbb{H}$

$$J_3 : U \to \mathbb{R} \qquad J_3(u) = \frac{\tau}{2}||u||^2_U \tag{90}$$

with $\tau \geq 0$[16].
In what follows, to simplify, we assume that $\alpha_1 = \alpha_2 = 1$, but the results are unchanged by that assumption.

Now, we construct the framework for our minimization problem.
Consider that we have an initial condition $y_0 \in \mathscr{H}$, a fixed input boundary velocity $y_{in} \in \mathscr{T}_{in}$, with lifting $L_{in}(y_{in})$. Notice that this input velocity must be chosen not in arbitrary way, but in the set

$$\mathscr{U}^{y_0}_{in} := \{y_{in} \in \mathscr{T}_{in} : \exists y_c \in \mathscr{T}_c \text{ such that } L_c(y_c) + L_{in}(y_{in}) - y_0 \in \tilde{H}\}$$

Recall that by choosing y_0 from $\mathscr{H}$ we have by lemma 2.2 that the set $\mathscr{U}^{y_0}_{in}$ is not empty.

Suppose now that, U is a Hilbert space[17], for the inner product $(\cdot,\cdot)_U$, and let $B \in \mathscr{L}(U, \mathscr{T}_c)$ be a given operator,[18] such that there exists an object $u_0 \in U$ and a closed subspace $U_0 \subset U$ that,

$$\Big(L_c\big(B(u_0)\big)\Big)(0) - (y_0 - L_{in}(y_{in})(0) \in \widetilde{H} \qquad \Big(L_c\big(B\big(u_1\big)\big)\Big) \in \widetilde{H} \text{ for all } u_1 \in U_0$$

Then, we define the affine space $\widetilde{U} = u_0 + U_0$, which is closed (since U_0 is closed), convex, and has also the property that $B(\widetilde{U})$ is a set of admissible velocities.
In fact, by the linearity of B and L_c we have that, for every $x \in \widetilde{U}$, since x has the form $x = u_0 + \widetilde{u}$ with $u_0 \in U_0$,

$$L_c(B(x)) = L_c(B(u_0) + B(\widetilde{u})) = L_c(B(u_0)) + L_c(B(\widetilde{u}))$$

and therefore

$$L_c(B(x))(0) - \big(y_0 - L_{in}(y_{in})(0)\big) = L_c(B(u_0)) + L_c(B(\widetilde{u})) - \big(y_0 - L_{in}(y_{in})(0)\big) \in \widetilde{H}$$

Therefore, the afine space $\widetilde{U}$, is composed by the elements of U, with the property that they are mapped, via B, to a boundary velocity in $\mathscr{T}_c$, and whose lifting, together with the lifting y_{in}, verify the compatible property with the initial data y_0. Thus, the control must be done in $\widetilde{U}$.

To simplify the notation, we will denote for a given $y_c \in \mathscr{T}_c$, $L_c(y_c)$ by $\widetilde{y_c}$, and we do this only for L_c leaving the notation for L_{in} unchanged.

---

[16]In our study cases we will always consider the case $\tau > 0$.
[17]I from here that cames the motivation to define a norm in $\mathscr{T}_c$ such that this space is a Hilbert Space.
[18]We use the notation $L(X,Y)$ to denote the set of all the linear and continuous operators from X to Y.

Now, consider the following variational problem. Given $L_c(B(u)) = \widetilde{Bu} \in \mathbb{W}_c(0,T)$ find $w \in W(0,T)$ such that

$$\begin{cases} \langle w'(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} + \mu(\nabla w(t), \nabla v) = \langle F_{in}(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} - \mu(\nabla \widetilde{Bu}, \nabla v) - \langle \widetilde{Bu}', v\rangle_{\widetilde{V}^*,\widetilde{V}} \ , \forall v \in \widetilde{V} \text{ and a.e. } t \in (0,T) \\ w(0) = (y_0 - L_{in}(y_{in})(0)) - \widetilde{Bu}(0) \end{cases} \tag{91}$$

where $F_{in}(t)$ is the functional given by,

$$\langle F_{in}(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} = (f(t), v) - a\big(L_{in}(y_{in})(t), v\big) - \langle L_{in}(y_{in})'(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} \ , \forall v \in \widetilde{V} \text{ and a.e. } t \in (0,T)$$

which may be written as

$$\langle F_{in}(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} = (f(t), v) - \langle \mathbb{A}(L_{in}(y_{in}))(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} - \langle L_{in}(y_{in})'(t), v\rangle_{\widetilde{V}^*,\widetilde{V}} \ , \forall v \in \widetilde{V} \text{ and a.e. } t \in (0,T)$$

Notice that, if $w(t) \in W(0,T)$ is the (unique by the theorem 3.1) solution to (91) then, $\widetilde{Bu} + w$ is the (unique) solution of

$$\begin{cases} (w + \widetilde{Bu})'(t) + \mathbb{A}(w + \widetilde{Bu})(t) = F_{in}(t) \text{ in } L^2(\widetilde{V}^*) \\ (w + \widetilde{Bu})(0) = y_0 - L_{in}(y_{in})(0) \end{cases}$$

and has the traces

$$\begin{cases} 0 \text{ in } \Sigma_{in} \\ Bu \text{ in } \Sigma_c \\ 0 \text{ in } \Sigma_w \end{cases}$$

Notice also that if we sum $L_{in}(y_{in})$ to $\widetilde{Bu} + w$, we obtain a function $y = L_{in}(_{in}) + \widetilde{Bu} + w$ such that, y is the weak solution of (1) in the sense of our definition 3.1, and has the traces,

$$\begin{cases} y_{in} \text{ in } \Sigma_{in} \\ Bu \text{ in } \Sigma_c \\ 0 \text{ in } \Sigma_w \end{cases}$$

and also satisfies the initial condition $y(0) = y_0$.

Thus by defining the state equation $e : W(0,T) \times \widetilde{U} \to L^2(\widetilde{V}^*) \times \widetilde{H}$ given by

$$\begin{aligned} e_1(w,u) &= \langle w'(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \int_0^T \mu(\nabla w(t), \nabla v(t))dt - \langle F_{in}(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \\ &+ \int_0^T \mu(\nabla \widetilde{Bu}(t), \nabla v(t))dt + \langle \widetilde{Bu}'(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \ , \forall v \in L^2(\widetilde{V}) \\ e_2(w,u) &= L_{in}(y_{in})(0) + \widetilde{Bu}(0) + w(0) - y_0 \end{aligned}$$

we consider the following minimization problem

$$\begin{cases} \min J\Big(L_{in}(g_{in}) + \widetilde{Bu} + w, \big(L_{in}(y_{in}) + \widetilde{Bu} + w\big)(0), u\Big) \\ e(w,u) = 0 \text{ for } u \in \widetilde{U} \end{cases} \tag{92}$$

Let us now illustrate two possible cases for the definition of U.
Example 1: consider $U = \mathscr{T}_c$, that from our construction is a Hilbert space when equipped with the norm $||\cdot||_{\mathscr{T}_c}$, and we set $B \in \mathscr{L}(\mathscr{T}_c, \mathscr{T}_c)$ as the identity map. In this case, we choose $\widetilde{u}$ as one element in $\mathscr{T}_c$ that satisfies

$$L_c(\widetilde{u})(0) - \big(y_0 - L_{in}(y_{in})(0)\big) \in \widetilde{H}$$

(such $\widetilde{u}$ exists by lemma 2.2), and the subspace U_0 is defined as the set $U_0 := \{u \in \mathscr{T}_c = U : L_c(u)(0) \in \widetilde{H}\}$. This is the same as saying that U_0 is the pre-image of $\widetilde{H}$, via the map $\tau_0^t \circ L_c : \mathscr{T}_c \to \mathbb{H}$. Since this composition map is continuous, and the set $\widetilde{H}$ is closed in $\mathbb{H}$, we obtain that U_0 is a closed[19] subset of $\mathscr{T}_c$.

Example 2: consider the space $U = \mathbb{W}_c(0,T)$ and $B = \tau_c : \mathbb{W}_c(0,T) \to \mathscr{T}_c$ which as we have seen is linear and continuous.
In this case, we choose u_0 as an element in $U = \mathbb{W}_c(0,T)$ such that $L_c(Bu_0)(0) - \big(y_0 - L_{in}(y_{in})\big) \in \widetilde{H}$, and for the subspace U_0 we choose the set

$$U_0 := \{u \in \mathbb{W}_c(0,T) : \tau_0^t \circ L_c(\tau_c(u)) \in \widetilde{H}\}$$

which is the pre-image of the closed set $\widetilde{H}$ by the map $\tau_0^t \circ L_c \circ \tau_c : \mathbb{W}_c(0,T) \to \mathbb{H}$, which is continuous, and so U_0 is a closed subset of U. In this work since we are interested in a boundary control, we will focus on the first example.

4.2 Existence of an Optimal Solution

Now we focus on proofing that the problem (92) as at least one solution. For that, we first proof two lemmas.

Lemma 4.1. *The functionals J_1, J_2, J_3 all satisfy the property of boundness from below, and they are all weak lower semi-continuous .*

Proof: The boundness from bellow is simple since this functionals assume only non-negative values. To see the weak lower semi-continuity we will prove that these operators are convex and continuous.
Before recall the lemma 7.2 of the appendix. Now, since the norm is a convex application and the function $[0,\infty[\ni x \mapsto x^2$ is convex and increasing, we may conclude that the composition $||\cdot||^2$ is always a convex map.
Therefore the maps J_3, J_2 defined above, are convex. They are also continuous since they are the composition of continuous applications.
For the first example of J_1, we also may use the above argument to conclude the continuity and convexity, but for the case when J_1 is given by

$$J_1(y) = \frac{1}{2}\int_0^T ||\nabla \times y(t)||^2 dt$$

we need to be more careful.
First notice that in two dimensions the curl of a vector field u, is given by

$$\nabla \times u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Therefore the curl operator $\nabla\times : \mathbb{V} \to L^2(\Omega)$ is well defined and is linear[20]. Now, we have that for $\lambda \in [0,1]$ and $u, v \in L^2(\mathbb{V})$,

$$||\nabla \times \big[\lambda u + (1-\lambda)v\big]||^2 = ||\lambda\nabla \times u + (1-\lambda)\nabla \times v||^2_{L^2(\mathbb{V})} \le \lambda||\nabla \times u||^2 + (1-\lambda)||\nabla \times v||^2$$

where in the first step was used the linearity of $\nabla\times$, and in the second inequality, the convexity of the norm squared. Thus the operator is convex. To see the continuity we only need to see that the operator $\nabla\times : L^2(\mathbb{V}) \to L^2(L^2(\Omega))$ is bounded, since we already know it is linear. To see the boundness, let u be an element of $L^2(\mathbb{V})$, then for almost every $y \in (0,T)$,

$$\begin{aligned}||\nabla \times u(t)||^2 &= ||\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}||^2 \le 2\Big(||\frac{\partial u_1}{\partial x_2}||^2 + ||\frac{\partial u_2}{\partial x_1}||^2\Big)\\ &\le 2\Big(||\frac{\partial u_1}{\partial x_1}||^2 + ||\frac{\partial u_1}{\partial x_2}||^2 + ||\frac{\partial u_2}{\partial x_1}||^2 + ||\frac{\partial u_2}{\partial x_2}||^2\Big)\\ &= 2||\nabla u(u)||^2 = 2||u(t)||^2_{\mathbb{V}}\end{aligned}$$

[19]Notice that $\mathbb{H}$ has the L^2 norm and by definition $\widetilde{H}$ is closed subset of $L^2(\Omega)$. Since $\widetilde{H} \subset \mathbb{H}$, we also have that $\tilde{H}$ is a closed subset of $\mathbb{H}$.

[20]The linearity is a simple consequence of the linearity of the derivation.

Therefore

$$\int_0^T ||\nabla \times u(t)||^2 dt \le 2\int_0^T ||u(t)||^2_{\mathbb{V}} dt = 2||u||^2_{L^2(\mathbb{V})}$$

□

Lemma 4.2. *The cost functional J defined in (88) is weakly lower semi-continuous.*

Proof: We introduce the following operators to make the argument more precise . We define,

$$\begin{aligned}
I^{\mathbb{V}}_{W(0,T)} &\to L^2(\mathbb{V}) & W(0,T) \ni y(t) &\mapsto y(t) \in L^2(\mathbb{V})\\
I^{\mathbb{V}}_{\mathbb{W}(0,T)} &\to L^2(\mathbb{V}) & \mathbb{W}(0,T) \ni y(t) &\mapsto y(t) \in L^2(\mathbb{V})\\
I^{T,\mathbb{H}}_{\mathbb{W}(0,T)} &\to \mathbb{H} & \mathbb{W}(0,T) \ni y(t) &\mapsto y(T) \in \mathbb{H}\\
I^{T,\mathbb{H}}_{W(0,T)} &\to \mathbb{H} & W(0,T) \ni y(t) &\mapsto y(T) \in \mathbb{H}
\end{aligned}$$

The above operators are linear and continuous. Now we introduce the operator $I : W(0,T) \times \widetilde{U} \to L^2(\mathbb{V}) \times \mathbb{H} \times \widetilde{U}$, and for simplification of notation, we will denote $L^2(\mathbb{V}) \times \mathbb{H} \times \widetilde{U}$ by X, given by, for $y \in W(0,T)$ and $u \in \widetilde{U}$,

$$\begin{aligned}
I(y,u) =& \Big(I^{\mathbb{V}}_{W(0,T)}(y) + I^{\mathbb{V}}_{\mathbb{W}(0,T)}\big(L_c(B(u))\big) + I^{\mathbb{V}}_{\mathbb{W}(0,T)}\big(L_{in}(y_{in})\big), I^{T,\mathbb{H}}_{W(0,T)}(y) +\\
& I^{T,\mathbb{H}}_{\mathbb{W}(0,T)}\big(L_c(B(u))\big) + I^{T,\mathbb{H}}_{W(0,T)}\big(L_{in}(y_{in})\big), u\Big)
\end{aligned}$$

which may be also written in a not too heavy form,

$$I(y,u) = \Big(y(t) + L_c(B(u))(t) + L_{in}(y_{in})(t), y(T) + L_c(B(u))(T) + L_{in}(y_{in})(T), u\Big)$$

where the functions are seen in the appropriate spaces in order to have the vector image in X. The operator I is not linear, is only affine linear, unless the term $L_{in}(y_{in}) = 0$. However we have the following important property. Given $\lambda \in [0,1]$ and $(y,u),(w,v) \in W(0,T) \times \widetilde{U}$,

$$\begin{aligned}
I\Big(\lambda(y,u) + (1-\lambda)(w,v)\Big) &= I\Big(\big(\lambda y + (1-\lambda)w, \lambda u + (1-\lambda)v\big)\Big)\\
&= \Big(I^{\mathbb{V}}_{W(0,T)}\big(\lambda y + (1-\lambda)w\big) + I^{\mathbb{V}}_{\mathbb{W}(0,T)}\big(L_c(B(\lambda u + (1-\lambda)v))\big)\\
&+ I^{\mathbb{V}}_{\mathbb{W}(0,T)}\big(L_{in}(y_{in})\big), I^{T,\mathbb{H}}_{W(0,T)}\big(\lambda y + (1-\lambda)w\big) + I^{T,\mathbb{H}}_{\mathbb{W}(0,T)}\big(L_c(B(\lambda u + (1-\lambda)v))\big)\\
&+ L_{in}(y_{in}), \lambda u + (1-\lambda)v\Big)\\
&= \lambda I(y,u) + (1-\lambda)I(w,v)
\end{aligned}$$

where we use the linearity of the maps, and the decomposition $L_{in}(y_{in}) = \lambda L_{in}(y_{in}) + (1-\lambda)L_{in}(y_{in})$ where $\lambda \in [0,1]$.
The operator I is also continuous, when we consider the spaces with the Cartesian norm. In fact, let $(y_n,u_n)_{n\in\mathbb{N}}$ be a converging sequence to (y,u) in $W(0,T) \times \widetilde{U}$, with the Cartesian norm, that is, we have that

$$||(y_n,u_n) - (y,u)||_{W(0,T)\times\widetilde{U}} = \sqrt{||y_n - y||^2_{W(0,T)} + ||u_n - u||^2_{\widetilde{U}}}$$

what in particular implies that

$$||y_n - y||^2_{W(0,T)} \to 0 \qquad\qquad ||u_n - u||^2_{\widetilde{U}} \to 0$$

Then we have

$$
\begin{aligned}
||I(y_n,u_n)-I(y,u)||_X^2 =&\Big|\Big|\Big(y_n+\widetilde{B(u_n)}+L_{in}(y_{in}),y_n(T)+\widetilde{B(u_n)}(T)+L_{in}(y_{in})(T),u_n\Big)\\
&-\Big(y+\widetilde{B(u)}+L_{in}(y_{in}),y(T)+\widetilde{B(u)}(T)+L_{in}(y_{in})(T),u\Big)\Big|\Big|_X^2\\
&=\Big(\big|\big|y_n-y+\widetilde{B(u_n)}-\widetilde{B(u)}\big|\big|^2_{L^2(\mathbb{V})}+\big|\big|y_n(T)-y(T)+\widetilde{B(u_n)}(T)-\widetilde{B(u)}(T)\big|\big|^2_{\mathbb{H}}+\\
&+\big|\big|u_n-u\big|\big|^2_{\widetilde{U}}\Big)\\
&\leq 2\Big(||y_n-y||^2_{L^2(\mathbb{V})}+||\widetilde{B(u_n)}-\widetilde{B(u)}||^2_{L^2(\mathbb{V})}+||y_n(T)-y(T)||^2_{\mathbb{H}}+\\
&+||\widetilde{B(u_n)}(T)-\widetilde{B(u)}(T)||^2_{\mathbb{H}}+||u_n-u||^2_{\widetilde{U}}\Big)
\end{aligned}
$$

where in the last step we used the Young's inequality with $\epsilon=1$. Writing the above expression with the operators in evidence becomes,

$$
\begin{aligned}
||I(y_n,u_n)-I(y,u)||_X^2\leq 2\Big(&||I^{\mathbb{V}}_{W(0,T)}(y_n-y)||^2_{L^2(\mathbb{V})}+||I^{\mathbb{V}}_{\mathbb{W}(0,T)}\big(\widetilde{B(u_n)}-\widetilde{B(u)}\big)||^2_{L^2(\mathbb{V})}+||I^{T,\mathbb{H}}_{W(0,T)}(y_n-y)||^2_{\mathbb{H}}+\\
&+||I^{T,\mathbb{H}}_{\mathbb{W}(0,T)}\big(\widetilde{B(u_n)}-\widetilde{B(u)}\big)||^2_{\mathbb{H}}+||u_n-u||^2_{\widetilde{U}}\Big)
\end{aligned}
$$

and since the operators $I^{\mathbb{V}}_{W(0,T)}, I^{\mathbb{V}}_{\mathbb{W}(0,T)}, I^{T,\mathbb{V}}_{W(0,T)}$ and $I^{T,\mathbb{V}}_{\mathbb{W}(0,T)}$ are bounded and linear we get,

$$
\begin{aligned}
||I(y_n,u_n)-I(y,u)||_X^2\leq 2\Big(&||I^{\mathbb{V}}_{W(0,T)}||^2_{\mathscr{L}(W(0,T),L^2(\mathbb{V}))}||(y_n-y)||^2_{W(0,T)}+\\
&+||I^{\mathbb{V}}_{\mathbb{W}(0,T)}||^2_{\mathscr{L}(\mathbb{W}(0,T),L^2(\mathbb{V}))}||L_c(B)||^2_{\mathscr{L}(\mathbb{W}(0,T),\widetilde{U})}||u_n-u||^2_{\widetilde{U}}+\\
&+||I^{T,\mathbb{H}}_{W(0,T)}||^2_{\mathscr{L}(W(0,T),\mathbb{H})}||y_n-y||^2_{W(0,T)}\\
&+||I^{T,\mathbb{H}}_{\mathbb{W}(0,T)}||^2_{\mathscr{L}(\mathbb{W}(0,T),\mathbb{H})}||L_c(B)||^2_{\mathscr{L}(\mathbb{W}(0,T),\widetilde{U})}||u_n-u||^2_{\widetilde{U}}+||u_n-u||^2_{\widetilde{U}}\Big)\\
&\to 0
\end{aligned}
$$

since $||y_n-y||_{W(0,T)}\to 0$ and $||u_n-u||_{\widetilde{U}}\to 0$. Therefore I is continuous and affine linear. This fact yields that the sequence (y_n,u_n) in $W(0,T)\times\widetilde{U}$ that weakly converges to (y,u) is transformed, via the map I, in a sequence $(I_1(y_n,u_n),I_2(y_n,u_n),I_3(y_n,u_n))$ which converges weakly to $(I_1(y,u),I_2(y,u),I_3(y,u))$ in X (see appendix commentary after property (R5)).
With this, the cost functional $J:W(0,T)\times\widetilde{U}\to\mathbb{R}$ is given by

$$J(y,u)=J_1(Proj_1(I(y,u)))+J_2(Proj_2(I(y,u)))+J_3(Proj_3(I(y,u)))$$

where the Projection operators $Proj_1, Proj_2, Proj_3$ are the projections of the components of the image vector $I(y,u)$. These projections are linear, and continuous with respect to the norms that we defined above. Notice that if $(y_n,u_n)\xrightarrow{w}(y,u)$ in $W(0,T)\times\widetilde{U}$, by the continuity and affine linearity of I, together with the linearity and continuity of projections we get that

$$Proj_1(I(y_n,u_n))\xrightarrow{w}Proj_1(I(y,u))\text{ in }L^2(\mathbb{V})\qquad Proj_2(I(y_n,u_n))\xrightarrow{w}Proj_2(I(y,u))\text{ in }\mathbb{H}$$
$$Proj_3(I(y_n,u_n))\xrightarrow{w}Proj_3(I(y,u))\text{ in }\widetilde{U}$$

Now since from lemma 4.1, the functionals J_1,J_2,J_3 are weakly lower semi-continuous, we have by (R3) from appendix that

$$
\begin{aligned}
J(y,u)&=J_1(Proj_1(I(y,u)))+J_2(Proj_2(I(y,u)))+Proj_3(I(y,u))\\
&\leq\liminf n\to\infty J_1(Proj_1(I(y_n,u_n)))+J_2(Proj_2(I(y_n,u_n)))+J_3(Proj_3(I(y_n,u_n)))\\
&=\liminf n\to\infty J(y_n,u_n)
\end{aligned}
$$

□

We are now in conditions of giving a proof of existence of an optimal solution to (92).

Theorem 4.1. *Given, $f\in L^2(L^2(\Omega))$, $u_0\in\mathscr{H}$, an input velocity $g_{in}\in\mathscr{U}^{u_0}_{in}$ with the lifting given by $L_{in}(g_{in})$. Let U be the Hilbert space mentioned above, with the affine subset $\widetilde{U}$ having the properties also mentioned above. Then the problem (92) has at least one solution.*

Proof: Since the functional J is the sum of functionals which are bounded from bellow, J is also bounded from bellow. Thus, exists a minimizing sequence $\{u_n\} \subset \widetilde{U}$. Since J_3 is radially unbounded, that is,

$$J_3(u) \to \infty \text{ when } ||u||_U \to \infty$$

we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{U}$ is bounded in U. Let $y_n = \widetilde{Bu_n} + w_n$ where w_n is the unique solution of $e(w_n, u_n) = 0$. From the estimates (83) we have that the sequence $\{y_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathbb{W}}(0,T)$ has the estimate (notice that this solution has zero trace in $\mathscr{T}_{in}$)

$$||y_n||_{C([0,T];\mathbb{H})} + ||y_n||_{\widetilde{\mathbb{W}}(0,T)} \leq C'\Big(||u_0 - L_{in}(y_{in})||^2 + ||F_{in}||^2_{L^2(\widetilde{V}^*)} + ||Bu_n||^2_{\mathscr{T}_c}\Big)$$

Since $||u_n||_U \leq M$ for some $M \in \mathbb{R}$ and B is bounded we conclude that the sequence $\{y_n\}$ is also bounded in $\widetilde{\mathbb{W}}(0,T)$ and in $C([0,T];\mathbb{H})$. Again, since $y_n = w_n + \widetilde{Bu_n}$, we conclude that $\{w_n\}$ is also bounded in $\widetilde{\mathbb{W}}(0,T)$. But the norm of w in $W(0,T)$ coincides with the norm $\widetilde{\mathbb{W}}(0,T)$ and therefore w_n is bounded in $W(0,T)$. Consequently, by a method of successive subsequences we conclude that there exists $u \in U$ and $w \in W(0,T)$ such that

$$u_n \xrightarrow{w} u \text{ in } U \qquad w_n \xrightarrow{w} w \text{ in } W(0,T) \qquad w_n(0) \xrightarrow{w} w(0) \text{ in } \tilde{H} \qquad w_n(T) \xrightarrow{w} w(T) \text{ in } \tilde{H} \tag{93}$$

The first to weak convergences are consequence of (R5 of the appendix). For the final two, the weak convergence is consequence the following composition chain of continuous maps,

$$W(0,T) \xrightarrow{\hookrightarrow C([0,T];\widetilde{H})} C([0,T];\widetilde{H}) \xrightarrow{\text{evaluation on t}} \widetilde{H}$$

$$v \in W(0,T) \to v \in C([0,T];\widetilde{H}) \to v(t) \in \widetilde{H}$$

Since this composition is again continuous and linear, it conserves the weak convergence by (R2). We have that the weak convergence also occurs in $\mathbb{H}$ since $\widetilde{H} \hookrightarrow \mathbb{H}$.
The expression (93) has also as consequence that

$$w_n \xrightarrow{w} w \text{ in } L^2(\widetilde{V}) \qquad\qquad w'_n \xrightarrow{w} w' \text{ in } L^2(\widetilde{V}^*) \tag{94}$$

It is also possible to proof that in fact we have that w_n converges to w strongly in $L^2(\widetilde{H})$ (see [15]), but this result is not necessary for our case.

Now on the other hand, since the lifting operator $L_c : \mathscr{T}_c \to \mathbb{W}_c(0,T)$ (denoted by the tilde symbol) and the operator $B \in \mathscr{L}(U, \mathscr{T}_c)$ are continuous, their composition is again linear and continuous, and conserves the weak convergence of u_n in U for the space $\mathbb{W}_c(0,T)$ (if it necessary by taking another subsequences) and thus

$$\widetilde{Bu_n} \xrightarrow{w} \widetilde{Bu} \text{ in } \mathbb{W}_c(0,T) \qquad \widetilde{Bu_n}(0) \xrightarrow{w} \widetilde{Bu}(0) \text{ in } \mathbb{H} \qquad \widetilde{Bu_n}(T) \xrightarrow{w} \widetilde{Bu}(T) \text{ in } \mathbb{H} \tag{95}$$

Again this implies that

$$\widetilde{Bu_n} \xrightarrow{w} \widetilde{Bu} \text{ in } L^2(\mathbb{V}) \qquad\qquad \widetilde{Bu_n}' \xrightarrow{w} \widetilde{Bu}' \text{ in } L^2(\widetilde{V}^*) \tag{96}$$

Let us see that the weak limit (w,u) found is admissible, that is, $e(w,u) = 0$ and $u \in \widetilde{U}$. u is clearly in $\widetilde{U}$ since this afine subspace is close for the weak convergence, and $u_n \in \widetilde{U}$ for every $n \in \mathbb{N}$. To see that $e(w,u) = 0$, first recall (R1 of the appendix). For every $v \in L^2(\widetilde{V})$ we have

$$\begin{aligned}
\langle e_1(w_n,u_n), v\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} &= \langle w'_n(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \int_0^T \mu(\nabla w_n(t), \nabla v(t))dt - \langle F_{in}(t), v\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \\
&+ \mu(\nabla \widetilde{Bu_n}(t), \nabla v(t)) + \langle \widetilde{Bu_n}'(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \\
&= \langle w'_n(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \langle \mathscr{A}(w_n)(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} - \langle F_{in}(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \\
&+ \langle \mathbb{A}(\widetilde{Bu_n})(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \langle \widetilde{Bu_n}'(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \\
&= 0
\end{aligned}$$

for each $n \in \mathbb{N}$, thus the sequence $e_1(w_n, u_n)$ converges strongly to zero in $L^2(\widetilde{V}^*)$ and thus also weakly. On the other hand since the operators $\mathscr{A}, \mathbb{A}$ are continuous and linear (they conserve the weak convergence), by the weak convergence of (94), (96) we have that the limit is also

$$
\begin{aligned}
0 = \lim_{n\to\infty} \langle e_1(w_n, u_n), v\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} =& \langle w'(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \langle \mathscr{A} w(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \\
& - \langle F_{in}(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \langle \mathbb{A}\big(\widetilde{Bu}\big)(t), v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \\
& + \langle \widetilde{Bu}', v(t)\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})}
\end{aligned}
$$

Thus, since $v \in L^2(\widetilde{V})$ was chosen arbitrary, we have $e_1(w, u) = 0$ as we wanted to see.

For the convergence of $e_2(w_n, u_n) = L_{in}(y_{in})(0) + \widetilde{Bu_n}(0) + w_n(0) - u_0 = 0$ for every $n \in \mathbb{N}$ we again have strong convergence of this sequence to zero in $\widetilde{H}$. On the other hand by the weak convergences of (93), (96) we have that

$$\lim_{n\to\infty} e_2(w_n, u_n) = L_{in}(y_{in})(0) + \widetilde{Bu}(0) + w(0) - u_0$$

and thus also $e_2(w, u) = 0$.

Then the pair (w, u) is admissible. Notice now that w_n also converges weakly to w in $L^2(\mathbb{V})$, a consequence of the continuous embedding $L^2(\widetilde{V}) \hookrightarrow L^2(\mathbb{V})$. Therefore the sequence $w_n + \widetilde{Bu_n} + L_{in}(y_{in}) \xrightarrow{w} w + \widetilde{Bu} + L_{in}(y_{in})$ in $L^2(\mathbb{V})$, and $w_n(T) + \widetilde{Bu_n}(T) + L_{in}(y_{in})(T) \xrightarrow{w} w(T) + \widetilde{Bu}(T) + L_{in}(y_{in})(T)$ in $\mathbb{H}$, (notice that $W(0,T) \hookrightarrow C([0,T]; \mathbb{H})$). Then invoking the weakly lower semicontinuous property of J we get that

$$
\begin{aligned}
J(w + \widetilde{Bu} + L_{in}(y_{in}), w(T) + \widetilde{Bu}(T) + \mathrm{L}_{in}(y_{in})(T), u) \leq& \liminf_{n\to\infty} J(w_n + \widetilde{Bu_n} + L_{in}(y_{in}), w_n(T) + \\
& \widetilde{Bu_n}(T) + L_{in}(y_{in})(T), u_n) \\
& = \inf
\end{aligned}
$$

and therefore the problem (92) has at least one solution. □

4.3 First Order Necessary Conditions

For that we start by recalling some classical results from optimization problems with PDE's that we took from [16]. Suppose that we are interested in analysing the following minimization problem,

$$\min_{w\in W} J(w) \text{ such that } w \in \mathscr{C} \tag{97}$$

where W is a general Banach space, with $\mathscr{C}$ being a non-empty, convex and closed subset of W. We have the following result which permits to characterize the optimal solutions for the problem (97).

Proposition 4.1. *Let W be a Banach space, $\mathscr{C} \subset W$ a non-empty, convex and closed subset. Let $J : V \to \mathbb{R}$ where V is a open neighborhood of $\mathscr{C}$. If $\overline{w} \in \mathscr{C}$ is a solution of (97) and J is Gâteaux differentiable at $\overline{w}$, the following optimality condition holds,*

$$\overline{w} \in \mathscr{C} \ , \ \langle J'(\overline{w}), w - \overline{w}\rangle_{W^*,W} \geq 0 \ , \ \forall w \in \mathscr{C} \tag{98}$$

If additionally, the cost functional J is convex, the condition (98) is also a sufficient condition, to $\overline{w}$ be a optimal solution of (97)

Proof: Let $w \in \mathscr{C}$ be arbitrary. We define the function $W(t) = \overline{w} + (1-t)w = tw + (1-t)\overline{w}$ and therefore, by the convexity of $\mathscr{C}$, for each $t \in [0,1]$ $W(t) \in \mathscr{C}$. Since $\overline{w}$ is an optimal solution we have,

$$J(\overline{w} + (w - \overline{w})t) - J(\overline{w}) \geq 0 \text{ for all } t \in [0,1]$$

Also since J is Gâteaux differentiable at $\overline{w}$, we may conclude that

$$0 \leq \lim_{t \to 0^+} \frac{J(\overline{w} + (w - \overline{w})t) - J(\overline{w})}{t} = \langle J'(\overline{w}), w - \overline{w} \rangle_{W^*,W}$$

This inequality is valid for every $w \in \mathscr{C}$, since w was initially chosen as being arbitrary.

Suppose now that J is convex and that $\overline{w} \in \mathscr{C}$ satisfies (98). Then since J is convex,

$$J(\overline{w} + (w - \overline{w})t) \leq tJ(w) + (1-t)J(\overline{W}) \Leftrightarrow J(\overline{w} + (w - \overline{w})t) - J(\overline{w}) \leq t\big(J(w) - J(\overline{w})\big)$$

and therefore, for each $t \in]0,1]$ we have

$$\langle J'(\overline{w}), w - \overline{w} \rangle_{W^*,W} \leftarrow_{t \to p^+} \frac{J(\overline{w} + (w - \overline{w})t) - J(\overline{w})}{t} \leq J(w) - J(\overline{w})$$

and since, by hypotheses (98) holds, we get that,

$$J(w) - J(\overline{w}) \geq 0 \text{ for all } w \in \mathscr{C}$$

and thus, $\overline{w}$ is an optimal solution of (97) □

The result of the proposition (4.1) is quite general, and we are interested in a particular case of minimization problems that may be written as

$$\min_{(y,u) \in Y \times U} J(y,u) \text{ such that } e(y,u) = 0 \text{ and } u \in U_{ad} \tag{99}$$

where J is a functional cost to minimize, which depends on the control $u \in U_{ad}$ and the state equation y, which is a solution of the equation $e(y,u) = 0$.
In order to obtain well-known results for this type of problems, we will assume the following assumption,

Hypotheses 3 (H3):

(i) U_{ad} is a non empty, convex and closed set.
(ii) The applications $J : Y \times U \to \mathbb{R}$ and $e : Y \times U \to Z$ are Fréchet differentiable, and the spaces Y, U, Z are Banach spaces.
(iii) For all $u \in V$, where V is an open neighborhood of U_{ad}, the equation $e(y,u) = 0$ has a unique solution $y = y(u)$.
(iv) The operator $e_y(y(u), u) \in \mathscr{L}(Y, Z)$, has a bounded inverse for every $u \in V$.

Notice that with this assumptions, we may conclude that the solution map $U_{ad} \ni u \mapsto y(u)$ is, by the Implicit theorem function, is locally Fréchet differentiable. Is also common, in the case (H3) is valid, to introduce the reduce cost functional,

$$\hat{J} : U_{ad} \to \mathbb{R} \qquad \hat{J}(u) := J(y(u), u)$$

and the initial minimization problem (99) is equivalent to

$$\min_{u \in U} \hat{J}(u) \text{ such that } u \in U_{ad} \tag{100}$$

Using the proposition (4.1) we have the following result.

Proposition 4.2. *Suppose that the assumption (3) are satisfied. If $\overline{u} \in U_{ad}$ is a local solution to (100) the $\overline{u}$ satisfies the variational inequality,*

$$\overline{u} \in U_{ad} \text{ and } \langle \hat{J}'(\overline{u}), u - \overline{u} \rangle_{U^*,U} \geq 0 \text{ for all } u \in U_{ad} \tag{101}$$

wE have the following.

Proposition 4.3. *The hypotheses (H3) are satisfied.*

Proof: The proof is done in steps.

(i) The admissible set $\widetilde{U}$ is clearly non-empty. It is also closed since $\widetilde{U} = \widetilde{u} + U_0$ and U_0 is closed. To see that is convex, is enough to recall that $\widetilde{U}$ is an affine space.

(ii) To proof that the cost functional, J, is Fréchet differentiable, we have the following two lemmas

Lemma 4.3. *The functionals J_1, J_2, J_3 are Fréchet differentiable.*

Proof: We start by recalling that if H is a Hilbert space, the function $f : H \to \mathbb{R}$ given by, $H \ni x \to ||x||_H^2$ is Fréchet differentiable, and the derivative at the point x calculated in the direction h is given by

$$f'(x)h = 2\langle x, h\rangle_H$$

In fact, since in a Hilbert space $||x||_H^2 = \langle x, x\rangle_H$ we have that,

$$\begin{aligned} f(x+h) - f(x) &= \langle x+h, x+h\rangle_H - \langle x, x\rangle_H \\ &= 2\langle x, h\rangle_H + \langle h, h\rangle_H \end{aligned}$$

and since the term $\langle h, h\rangle_H = O(||h||_H^2)$ we have the result.

With this simple result for the derivative of the square norm function in a Hilbert space, we can conclude that our examples for the J_2, J_3 functionals are differentiable. In the case of J_1 be given by the velocity tracking functional,

$$J_1 : L^2(\mathbb{V}) \to \mathbb{R} \qquad J_1(y) = \frac{1}{2}\int_0^T ||y(t) - y_d(t)||_{\mathbb{V}}^2 dt = \frac{1}{2}||y - y_d||_{L^2(\mathbb{V})}^2$$

we may also apply the above result, since the space $L^2(\mathbb{V})$ is a Hilbert space. However, for the case of vorticity functional

$$J_1 : L^2(\mathbb{V}) \to \mathbb{R} \qquad J_1(y) = \frac{1}{2}\int_0^T ||\nabla \times y(t)||^2 dt$$

we need to use the chain rule, since this functional is given by the composition of $||\cdot||_{L^2(\mathbb{V})}^2 \circ \nabla\times$. The curl operator $\nabla\times$ is linear, therefore the derivative is the operator itself, and we get

$$J_1'(y)h = \int_0^T \big(\nabla \times y(t)\big) \cdot \big(\nabla \times h(t)\big) dt$$

□

The following lemma is important also to see the application of the chain rule, since it will be used in the derivation of the reduce cost functional.

Lemma 4.4. *The cost functional J is Fréchet differentiable with the derivative given at the point (y, u), in the direction (w, v), by*

$$J'(y,u)(w,v) = \langle J_1'(y,u), w + \widetilde{B(v)}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \Big(J_2'(y,u), w(T) + \widetilde{B(v)}(T)\Big) + \langle J_3'(y,u), v\rangle_U$$

Proof: This could be done by seeing that J is a sum of three Fréchet differentiable functions. As an example we will proof the F-differentiability for the function $\widetilde{J}_1 : W(0,T) \times \widetilde{U} \to \mathbb{R}$, given by

$$\widetilde{J}_1(y,u) = J_1\Big(Proj_1\big(I(y,u)\big)\Big)$$

which is sufficient, since the differentiability of the other terms is completely analogous.
Let (w, v) be in $W(0,T) \times U_0$ a direction for the derivative calculation[21]. From the chain rule we have (which is valid since all the terms in the composition are F-differentiable),

$$\widetilde{J}_1'(y,u)(w,v) = J_1'(Proj_1(I(y,u))) \circ Proj_1'(I(y,u)) \circ I'(y,u)(w,v)$$

[21]Notice that v must be chosen from U_0 is order to have $u + v$ still in $\widetilde{U}$.

Now, since the operator $Proj_1 : X \to L^2(\mathbb{V})$ is linear and bounded, the Fréchet derivative of this operator is itself. For the operator I, as we saw, this operator is not linear but affine linear, and therefore, the Fréchet derivative is given only by the linear part. More precisely, the derivative loses the constant term that depends on $L_{in}(y_{in})$,

$$I'(y,u)(w,v) = (w + \widetilde{B(v)}, y(T) + \widetilde{B(v)}(T), v)$$

Now, using the fact that $\widetilde{J}_1$ is a functional we may write the derivative as

$$\langle J_1'(y,u), Proj_1\big((w + \widetilde{B(v)}, y(T) + \widetilde{B(v)}(T), v)\big)\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} = \langle J_1'(y,u), w + \widetilde{B(v)}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})}$$

where in the last inequality w and $\widetilde{B(v)}$ are being seen as elements of $L^2(\mathbb{V})$.

A similar argument holds for the derivatives of J_2 and J_3 and therefore we conclude that J is Fréchet differentiable with derivative at the point (y,u) in the direction (w,v) given by,

$$J'(y,u)(w,v) = \langle J_1'(y,u), w + \widetilde{B(v)}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \Big(J_2'(y,u), w(T) + \widetilde{B(v)}(T)\Big) + \langle J_3'(y,u), v\rangle_U$$

□

In the case of the Fréchet differentiability of the state equation we have.

Lemma 4.5. *The state equation is Fréchet differentiable.*

Proof: Suppose that $(y,s) \in W(0,T) \times U_0$, and again, notice that s needs to be in U_0 in order to the sum $u+s$ still be an element of $\widetilde{U}$, and let us calculate the difference

$$\begin{aligned} e_1(w+y,u+s) - e_1(w,u) &= w' + y' + \mathscr{A}(w+y) - F_{in} + \mathbb{A}\big(\widetilde{B(u+s)}\big) + \widetilde{B(u+s)}' - w' - \mathscr{A}(w) \\ &\quad + F_{in} - \mathbb{A}(\widetilde{Bu}) - \widetilde{Bu}' \\ &= y' + \mathscr{A}(y) + \mathbb{A}(\widetilde{B(u+s)}) - \mathbb{A}(\widetilde{Bu}) - (\widetilde{B(u+s)})' - \widetilde{Bu}' \\ &= y' + \mathscr{A}(y) + \mathbb{A}(\widetilde{Bs}) + (\widetilde{Bs})' \\ &= (y + \widetilde{Bs})' + \mathbb{A}(y + \widetilde{Bs}) \end{aligned}$$

Therefore the map $F(w,u) : W(0,T) \times U_0 \to L^2(\tilde{V}^*)$ given by

$$F(w,u)(y,s) = (y + \widetilde{Bs})' + \mathbb{A}(y + \widetilde{Bs})$$

is our candidate to the Fréchet derivative of the map e_1. To finish we only need to see that $F(w,u)$ belongs to $L(W(0,T) \times U_0, L^2(\widetilde{V}^*))$. The operator is clearly linear since for any $(y_1,s_1),(y_2,s_2) \in W(0,T) \times U_0$ and $\alpha, \beta \in \mathbb{R}$ we have that

$$\begin{aligned} F(w,u)(\alpha(y_1,s_1) + \beta(y_2,s_2)) &= F(w,u)(\alpha y_1 + \beta y_2, \alpha s_1 + \beta s_2) \\ &= (\alpha y_1 + \beta y_2 + B(\widetilde{\alpha s_1 + \beta s_2}))' + \mathbb{A}(\alpha y_1 + \beta y_2 + B(\widetilde{\alpha s_1 + \beta s_2})) \\ &= \alpha(y_1 + \widetilde{Bs_1})' + \alpha\mathbb{A}(y_1 + \widetilde{Bs_1}) + \beta(y_2 + \widetilde{Bs_2})' + \beta\mathbb{A}(y_2 + \widetilde{Bs_2}) \\ &= \alpha F(w,u)(y_1,s_1) + \beta F(w,u)(y_2,s_2) \end{aligned}$$

For the boundness we have, that for every $(y,s) \in W(0,T) \times U_0$

$$\begin{aligned} ||F(w,u)(y,s)||^2_{L^2(\tilde{V}^*)} &= ||(y + \widetilde{Bs})' + \mathbb{A}(y + \widetilde{Bs})||^2_{L^2(\tilde{V}^*)} \\ &\leq 2||(y + \widetilde{Bs})'||^2_{L^2(\tilde{V}^*)} + 2||\mathbb{A}(y + \widetilde{Bs})||^2_{L^2(\tilde{V}^*)} \\ &\leq 2||(y + \widetilde{Bs})'||^2_{L^2(\tilde{V}^*)} + 2||\mathbb{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}(1 + c_p^2)||y + \widetilde{Bs}||^2_{L^2(H^1)} \\ &\leq 4\max(||\mathbb{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}(1 + c_p^2)^3, 1)\big(||y'||^2_{L^2(\tilde{V}^*)} + ||y||^2_{L^2(\mathbb{V})}\big) \\ &\quad + 4\max(||\mathbb{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}(1 + c_p^2)^3, 1)\big(||\widetilde{Bs}'||^2_{L^2(\tilde{V}^*)} + ||\widetilde{Bs}||^2_{L^2(\mathbb{V})}\big) \\ &= 4\max(||\mathbb{A}||^2_{\mathscr{L}(H^1,(H^1)^*)}(1 + c_p^2)^3, 1)\Big(||y||^2_{W(0,T)} + ||\widetilde{Bs}||^2_{\mathbb{W}(0,T)}\Big) \end{aligned}$$

Now by definition we have

$$||\widetilde{Bs}||^2_{\mathbb{W}(0,T)} = ||Bs||^2_{\mathscr{T}_c} \le ||B||^2_{L(U,\mathscr{T}_c)}||s||^2_U$$

Therefore, exists a constant $K > 0$ such that

$$||F(w,u)(y,s)||_{L^2(\widetilde{V}^*)} \le K\Big(||y||^2_{W(0,T)} + ||s||^2_U\Big)^{1/2}$$

Now we focus on the derivative of the map e_2. Again let $(y,s) \in W(0,T) \times U_0$, and let us calculate the difference

$$\begin{aligned} e_2(w+y,u+s) - e_2(y,s) &= L_{in}(g_{in})(0)) + \widetilde{B(u+s)}(0) + (w+y)(0) - u_0 - L_{in}(g_{in})(0)) - \widetilde{Bu}(0) - w(0) + u_0 \\ &= y(0) + \widetilde{Bs}(0) \end{aligned}$$

where we used the linearity of the evaluation map $\tau_0^t : C([0,T],\mathbb{H}) \to \mathbb{H}$, the map B and the Lifting L_c.

Thus our candidate to the Fréchet derivative of e_2 is the map $F(w,u) : W(0,T) \times U_0 \to \tilde{H}$ given by $F(w,u)(y,s) = y(0) + \widetilde{Bs}(0)$. We just need to see that this operator is in $L(W(0,T) \times U_0, \widetilde{H})$. The linearity is simple,

$$\begin{aligned} F_2(w,u)\big(\alpha(y_1,s_1) + \beta(y_2,s_2)\big) &= F_2(w,u)\big(\alpha y_1 + \beta y_2, \alpha s_1 + \beta s_2\big) \\ &= (\alpha y_1 + \beta y_2)(0)B\widetilde{(\alpha s_1 + \beta s_2)}(0) \\ &= \alpha\big(y_1(0) + \widetilde{Bs_1}(0)\big)\beta\big(y_2(0) + \widetilde{Bs_2}(0)\big) \\ &= \alpha F_2(w,u)(y_1,s_1) + \beta F_2(w,u)(y_2,s_2) \end{aligned}$$

For the boundness

$$\begin{aligned} ||F_2(w,u)(y,s)||^2_{\mathbb{H}} &= ||y(0)\widetilde{Bs}(0)||^2_{\widetilde{H}} \le 2||y(0)||^2_{\widetilde{H}} + 2||\widetilde{Bs}(0)||^2_{\mathbb{H}} \\ &\le 2||y||^2_{C([0,T];\tilde{H})} + 2||\widetilde{Bs}||^2_{C([0,T];\mathbb{H})} \\ &\le 2c^2\Big(||y||^2_{W(0,T)} + ||\widetilde{Bs}||^2_{\mathbb{W}(0,T)}\Big) \\ &\le 2c^2\Big(||y||^2_{W(0,T)} + ||B||^2_{\mathscr{L}(U,\mathscr{T}_c)}||s||^2_U\Big) \\ &= K\Big(||y||^2_{W(0,T)} + ||s||^2_U\Big) \end{aligned}$$

where $c = \max\{c_1, c_2\}$, being c_1, c_2 the constants of the embeddings $W(0,T) \hookrightarrow C([0,T];\tilde{H})$ and $\mathbb{W}(0,T) \hookrightarrow C([0,T];\mathbb{H})$ respectively. Therefore the derivative $e_x^2(w,u)(y,s) = y(0) + \widetilde{Bs}(0)$. □

(iii) This assumption in (H3) is valid by the theorem 3.1.

(iv) For this last verification we will calculate the Fréchet derivatives since they are useful later. Notice that since the Fréchet derivative of $e : W(0,T) \times \widetilde{U} \to L^2(\widetilde{V}^*) \times \widetilde{H}$ exists, so does exists too the partial derivatives $e_w : W(0,T) \to L^2(\widetilde{V}^*) \times \widetilde{H}$ and $e_u : U_0 \to L^2(\widetilde{V}^*) \times \widetilde{H}$. The partial derivative in order to u, at $u \in \widetilde{U}$, in the direction $\delta u \in U_0$, is given by

$$e_u(w,u)(\delta u) = \Big(\widetilde{B(\delta u)}' + \mathbb{A}\widetilde{B\delta u}, \widetilde{B(\delta u)}(0)\Big) \text{ in } L^2(\widetilde{V}^*) \times \widetilde{H}$$

For the partial derivative in order to w we have,

$$e_w(w,u)(y) = \big(y' + \mathscr{A}(y), y(0)\big) \tag{102}$$

what corresponds to the equation

$$e_w(w,u)(y) = (f, v_0) \text{ for } (f, v_0) \in L^2(\widetilde{V}^*) \times \widetilde{H} \tag{103}$$

which has a unique solution $y \in W(0,T)$, using the proof of theorem 3.1.
Notice that the operator $e_w(w(u),u) : W(0,T) \to L^2(\widetilde{V}^*) \times \widetilde{H}$ is an isomorphism. In fact, using

the same reasoning as the one done in the existence theorem 3.1 we can conclude that, for every $f \in L^2(\widetilde{V}^*)$ and $v \in \widetilde{H}$ the variational problem (103) has a unique solution $y \in W(0,T)$, and therefore $e_w(w(u),u)$ is a bijection. The theorem 3.1 also provides a estimate for the solution, and thus, the operator e_w is also bounded. Then, using the Banach's open map theorem we conclude that both $e_w(w(u),u)$ and $e_w(w(y),u)^{-1}$ are continuous. □

It is also useful to consider the adjoint equation

$$e_w(w,u)^* y = g \text{ in } W^*(T,0) \text{ for } y \in L^2(\widetilde{V}) \times \widetilde{H} \tag{104}$$

where we have identified $L^2(\widetilde{V}^*)^* = L^2(\widetilde{V})$ (see appendix theorem 7.5) and $\widetilde{H}$ with it's dual.

For the variational formulation of the adjoint equation (104), we suppose that $g \in W^*(0,T)$ has the special form

$$\langle g, \phi\rangle_{W^*,W} = \langle g_1, \phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + (g_0, \phi(T))$$

where $g_1 \in L^2(\widetilde{V}^*)$ and $g_0 \in \widetilde{H}$. In this case, the variational formulation (104) is given by, for every $\phi \in W(0,T)$

$$\begin{cases} \langle -y', \phi\rangle_{L^2(\tilde{V}^*),L^2(\tilde{V})} + \mu \displaystyle\int_0^T (\nabla y(t), \nabla\phi(t))dt = \langle g_1, \phi\rangle_{L^2(\tilde{V}^*),L^2(\tilde{V})} \\ y(T) = g_0 \end{cases} \tag{105}$$

As seen in proposition 4.3 $e_w(w(u),u) : W(0,T) \to L^2(\widetilde{V}^*) \times \widetilde{H}$, has a bounded inverse. Again by proposition 4.3 the map $e : W(0,T) \times \widetilde{U} \to L^2(\widetilde{V}^*) \times \widetilde{H}$ is F-differentiable in $W(0,T) \times \widetilde{U}$. Therefore, by the application of the Implicit Function Theorem (see [16]), given $u \in \widetilde{U}$ we know that locally, the map $\widetilde{U} \ni u \mapsto w(u)$ is F-differentiable, and the derivative may be calculated by

$$w'(u) = -e_w\big((w(u),u\big)^{-1} e_u\big(w(u),u\big) \tag{106}$$

Thus, for every direction $\delta u \in U_0$ we have

$$w'(u)\delta u = -e_w\big((w(u),u\big)^{-1} e_u\big(w(u),u\big)\delta u \in W(0,T)$$

The reduce cost functional is given by

$$\hat{J} : \widetilde{U} \to \mathbb{R}$$

$$\widetilde{U} \ni u \mapsto J(w(u) + L_{in}(y_{in}) + \widetilde{Bu}, w(u)(T) + L_{in}(y_{in})(T) + \widetilde{Bu}(T), u)$$

It is important to know the evaluation of the derivative of the reduce cost functional on a direction $\delta u \in U_0$, since we know that a necessary condition for a point $u \in \widetilde{U}$ to be an optimal solution is, by proposition (4.2)

$$\langle \hat{J}'(u), v - u\rangle_{U^*,U} \geq 0 \ , \ \forall v \in \widetilde{U}$$

Using the chain rule we have (we denote $y(u) = w(u) + L_{in}(y_{in}) + \widetilde{Bu}$),

$$\begin{aligned} \langle \hat{J}'(u), \delta u\rangle_{U^*,U} &= \langle J_1'(y(u)), y'(u)\delta u\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(u)(T)), y'(u)\delta u(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U \\ &= \langle J_1'(y(u)), w'(u)\delta u\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_1'(y(u)), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(u)(T)), w'(u)\delta u(T)\rangle_{\mathbb{H}} \\ &+ \langle J_2'(y(u)(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U \end{aligned}$$

Now, notice that $w'(u)\delta u$ belongs to $W(0,T)$. Moreover, $J_1'(y(u)) \in L^2(\mathbb{V}^*)$ and therefore the restriction of this functional to the subset $L^2(\widetilde{V})$ of $L^2(\mathbb{V})$ defines a functional in $L^2(\widetilde{V}^*)$ denoted with the same letter. On the other hand this restriction is also an element in W^* since it defines an element in that space, via the mapping

$$\langle J_1'(y(u)), \phi\rangle_{W^*,W} = \langle J_1'(y(u)), \phi\rangle_{L^2(\widetilde{V}^*),\widetilde{V}}$$

We do the same reasoning for the functional $J_2'(y(u)(T))$. This operator is in $\mathbb{H}^* = \mathbb{H}$ and thus it's evaluation can be restricted to the closed subset $\widetilde{H}$ of $\mathbb{H}$ to give rise to a functional in $\widetilde{H}^* = \widetilde{H}$. In the same way this induces a functional in w^* via the mapping,

$$\langle J_2'(y(u)(T)), \phi\rangle_{W^*,W} = \big(J_2'(y(u)(T)), \phi(T)\big)$$

Thus we have

$$\langle \hat{J}'(u), \delta u\rangle_{U^*,U} = \langle J_1'(y(u)), w'(u)\delta u\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \langle J_1'(y(u)), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(u)(T)), w'(u)\delta u(T)\rangle_{\widetilde{H}}$$
$$+ \langle J_2'(y(u)(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U$$

which can be written as

$$\langle \hat{J}'(u), \delta u\rangle_{U^*,U} = \langle J_1'(y(u)), w'(u)\delta u\rangle_{W^*,W} + \langle J_1'(y(u)), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(u)(T)), w'(u)\delta u(T)\rangle_{W^*,W}$$
$$+ \langle J_2'(y(u)(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U$$

Recall now that we have the derivative $w'(u) = -e_w(w(u),u)^{-1}e_u(w(u),u)$, where the operators $e_w(w(u),u) : W(0,T) \to L^2(\widetilde{V}^*) \times \widetilde{H}$ and $e_u(w(u),u) : \widetilde{U} \to L^2(\widetilde{V}^*) \times \widetilde{H}$. Thus, the adjoint of the derivative $w'(u)$ is the operator

$$w'(u)^* = -e_u(w(u),u)^* e_w(w(u),u)^{-*} \tag{107}$$

where $e_w(w(u),u)^{-*} : W^* \to L^2(\widetilde{V}) \times \widetilde{H}$ and $e_u(w(u),u)^* : L^2(\widetilde{V}) \times \widetilde{H} \to U^* = U$, and therefore we have

$$\langle \hat{J}'(u), \delta u\rangle_{U^*,U} = \langle -e_u(w(u),u)^* e_w(w(u),u)^{-*} J_1'(y(u)), \delta u\rangle_U + \langle J_1'(y(u)), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})}$$
$$+ \langle -e_u(w(u),u)^* e_w(w(u),u)^{-*} J_2'(y(u)(T)), \delta u\rangle_U + \langle J_2'(y(u)(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U$$

Defining the $\Phi(y(u)) = J_1'(y(u)) + J_2'(y(u)(T))$ this defines an element in W^* via the mapping

$$\langle \Phi, \phi\rangle_{W^*,W} = \langle J_1'(y(u)), \phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \langle J_2'(y(u)(T)), \phi(T)\rangle_{\widetilde{H}} \text{ for every } \phi \in W(0,T)$$

Introducing $\lambda = -e_w(w(u),u)^{-*}\Phi(y(u)) \in L^2(\widetilde{V}) \times \widetilde{H}$ we obtain

$$\langle \hat{J}'(u), \delta u\rangle_{U^*,U} = \langle e_u(w(u),u)^*\lambda, \delta u\rangle_U + \langle J_1'(y(u)), \widetilde{B\delta u}\rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \langle J_2'(y(u)(T)), \widetilde{B\delta u}(T)\rangle_{\mathbb{H}} + \langle J_3'(u), \delta u\rangle_U$$

The equation

$$\lambda = -e_w(w(u),u)^{-*}\Phi(y(u)) \text{ in } L^2(\widetilde{V}) \times \widetilde{H}$$

is equivalent to solving the adjoint equation

$$e_w(w(u),u)^*\lambda = -\Phi(y(u)) \text{ in } W(0,T)^* \tag{108}$$

We have the following lemma.

Lemma 4.6. *The equation (108) has a unique solution $\lambda = (\lambda_1, \lambda_0) \in L^2(\widetilde{V}) \times \widetilde{H}$. Moreover λ_1 is the (unique) solution in $W(0,T)$ of the variational problem*

$$\begin{cases} \langle -\lambda_1', \phi\rangle_{L^2(\tilde{V}^*),L^2(\tilde{V})} + \mu \displaystyle\int_0^T (\nabla\lambda_1(t), \nabla\phi(t))dt = -\langle J_1'(y(u)), \phi\rangle_{L^2(\tilde{V}^*),L^2(\tilde{V})} \text{ for all } \phi \in W(0,T) \\ \lambda_1(T) = -J_2'(y(u)(T)) \end{cases} \tag{109}$$

and $\lambda_0 = \lambda_1(0)$.

Proof: First notice that the operator $e_w(w(u),u) : W(0,T) \to L^2(\widetilde{V}^*) \times \widetilde{H}$ is an isomorphism. In fact, using the same reasoning as the one done in the existence theorem 3.1 we can conclude that, for every $f \in L^2(\widetilde{V}^*)$ and $v \in \widetilde{H}$ the variational problem (105) has a unique solution $g \in W(0,T)$, and therefore $e_w(w(u),u)$ is a bijection. The theorem 3.1 also provides a estimate for the solution, and thus the operator e_w is also bounded. Then, using the Banach's open map theorem we conclude that both $e_w(w(u),u)$ and $e_w(w(y),u)^{-1}$ are continuous.
Now using a classical results from functional Analysis (see the lemma 7.5 of the appendix) we may also conclude that the adjoint operator $e_w(w(u),u)^* : L^2(\widetilde{V}) \times \widetilde{H} \to W^*(0,T)$ is an isomorphism.

Since the operator e_w is a bijection from $W(0,T)$ to $L^2(\widetilde{V}) \times \widetilde{H}$, using the lemma 7.5, that the image of e_w^* is closed in $W(0,T)^*$ and is given by

$$Im(e_w*) = \big(Ker(e_w)\big)^\perp = \big(\{0\}\big)^\perp = W(0,T)^*$$

what is simply the surjectivity.

On the other hand, since $Ker(e_w^*) = (Im(e_w))^\perp = (Y)^\perp = 0$ in Y^*, what is the injectivity. We also have that $||e_w||_{\mathscr{L}(W(0,T),L^2(\widetilde{V}^*)\times\widetilde{H})} = ||e_w^*||_{\mathscr{L}(L^2(\widetilde{V})\times\widetilde{H},W(0,T)^*)}$, and thus using again the Banach's open map theorem we have that $e_w(w(u),u)^*$ has a bounded inverse.

With this, we can conclude that the adjoint equation (108) has a unique solution, $\lambda \in L^2(\widetilde{V}) \times \widetilde{H}$, that is, exists an unique $\lambda = (\lambda_1, \lambda_0)$ with $\lambda_1 \in L^2(\widetilde{V})$ and $\lambda_0 \in \widetilde{H}$ such that they satisfy the equation

$$\langle \lambda_1, e_w^1(w(u),u)\phi\rangle_{L^2(\widetilde{V}),L^2(\widetilde{V}^*)} + \langle \lambda_0, e_w^2(w(u),u)\phi\rangle_{\widetilde{H}} = -\langle J_1'(y(u)),\phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} - \langle J_2'(y(u)(T)),\phi\rangle_{\widetilde{H}}$$

for all $\phi \in W(0,T)$. This, by the definition of the partial derivatives e_w^1, e_w^2, is equivalent to $\lambda = (\lambda_1, \lambda_0)$ be solution of the variational equation

$$\langle \lambda_1, \phi'\rangle_{L^2(\widetilde{V}),L^2(\widetilde{V}^*)} + \mu\int_0^T (\nabla\lambda_1(t),\nabla\phi(t))dt + \langle\lambda_0,\phi(0)\rangle_{\widetilde{H}} = -\langle J_1'(y(u)),\phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} - \langle J_2'(y(u)(T)),\phi\rangle_{\widetilde{H}}$$

for all $\phi \in W(0,T)$, which may be written as

$$\int_0^T \langle\lambda_1(t),\phi'(t)\rangle_{\widetilde{V},\widetilde{V}^*}dt + \mu\int_0^T (\nabla\lambda_1(t),\nabla\phi(t))dt + \langle\lambda_0,\phi(0)\rangle_{\widetilde{H}} = -\langle J_1'(y(u)),\phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} - \\ - \langle J_2'(y(u)(T)),\phi\rangle_{\widetilde{H}} \text{ for all } \phi \in W(0,T)$$

Suppose now that the function λ_1 is in $W(0,T)$, a fact that we will soon see, it is true. Therefore the integration by parts formula is valid, and the above equation is equivalent to

$$\int_0^T -\langle\lambda_1'(t),\phi(t)\rangle_{\widetilde{V},\widetilde{V}^*}dt + \mu\int_0^T (\nabla\lambda_1(t),\nabla\phi(t))dt + \langle\lambda_0 - \lambda_1(0),\phi(0)\rangle_{\widetilde{H}} = -\langle J_1'(y(u)),\phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} - \\ - \langle J_2'(y(u)(T)) - \lambda_1(T),\phi(T)\rangle_{\widetilde{H}} \text{ for all } \phi \in W(0,T)$$

If we now, restrict ϕ to be only in the space $C_c^\infty\big((0,T);\widetilde{V}\big) \subset W(0,T)$ we get, since the values of $\phi(0) = 0 = \phi(T)$,

$$\int_0^T -\langle\lambda_1'(t),\phi(t)\rangle_{\widetilde{V},\widetilde{V}^*}dt + \mu\int_0^T (\nabla\lambda_1(t),\nabla\phi(t))dt = -\langle J_1'(y(u)),\phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \text{ for all } \phi \in C_c^\infty\big((0,T);\widetilde{V}\big)$$

and since $C_c^\infty\big((0,T);\widetilde{V}\big)$ is dense in $L^2(\widetilde{V})$ and the applications in the above expression are continuous, we have that

$$\int_0^T -\langle\lambda_1'(t),\phi(t)\rangle_{\widetilde{V},\widetilde{V}^*}dt + \mu\int_0^T (\nabla\lambda_1(t),\nabla\phi(t))dt = -\langle J_1'(y(u)),\phi\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \text{ for all } \phi \in L^2(\widetilde{V}) \tag{110}$$

In particular (110) is valid in $W(0,T) \subset L^2(\widetilde{V})$ and this implies that

$$\langle\lambda_0 - \lambda_1(0),\phi(0)\rangle_{\widetilde{H}} + \langle J_2'(y(u))(T) + \lambda_1(T),\phi(T)\rangle_{\widetilde{H}} = 0 \tag{111}$$

what yields

$$\lambda_0 = \lambda_1(0) \qquad\qquad \lambda_1(T) = -J_2'(y(u)(T)) \tag{112}$$

With this we conclude that, if instead of looking for a λ_1 in the larger space $L^2(\widetilde{V})$, we restrict ourselves to searching for a solution $\lambda_1 \in W(0,T)$, we arrive at a an equation for λ_1, which is,

$$\begin{cases} \langle -\lambda_1',\phi\rangle_{L^2(\tilde{V}^*),L^2(\tilde{V})} + \mu\displaystyle\int_0^T (\nabla\lambda_1(t),\nabla\phi(t))dt = -\langle J_1'(y(u)),\phi\rangle_{L^2(\tilde{V}^*),L^2(\tilde{V})} \text{ for all } \phi \in W(0,T) \\ \lambda_1(T) = -J_2(y(u)(T)) \end{cases} \tag{113}$$

an equation that has a unique solution $\widetilde{\lambda}_1 \in W(0,T)$. Therefore we know that exists a $\widetilde{\lambda}_1 \in W(0,T)$ that by construction, satisfies the adjoint equation for the term λ_1. Since, we have seen that the solution of the adjoint equation is unique, we must have $\lambda_1 = \widetilde{\lambda}_1$. Moreover, by the above calculations, the second term of the Lagrange's multiplier satisfies $\lambda_0 = \lambda_1(0)$. □

To resume we have the following result which characterizes a optimal solution.

Proposition 4.4. *Let $(\overline{w}, \overline{u})$ be a optimal solution to the problem (92). Then exists an adjoint state $\lambda = (\lambda_1, \lambda_0) \in L^2(\tilde{V}) \times \tilde{H}$ such that (we denote by $\overline{y}$ the function , $\overline{w} + \widetilde{B(\overline{u})} + L_{in}(g_{in})$*

$$\begin{cases} e(\overline{w},\overline{u}) = e(w(\overline{u}),\overline{u}) = 0 \ , \quad \textbf{\textit{State equation}} \\ \\ \textbf{\textit{Adj. eq.}} \quad \begin{cases} \langle -\lambda_1', \phi \rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} + \mu \int_0^T (\nabla\lambda_1(t), \nabla\phi(t))dt = -\langle J_1'(\overline{y}), \phi \rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} \ \textit{for all } \phi \in W(0,T) \\ \lambda_1(T) = -J_2'(\overline{y}(T)) \end{cases} \\ \\ \overline{u} \in \widetilde{U} \ \textit{and} \ \langle \hat{J}'(\overline{u}), u - \overline{u} \rangle_{U^*,U} = \langle e_u^*(w(\overline{u}),\overline{u})\lambda, u - \overline{u} \rangle_U + \langle J_1'(\overline{y}), L_c B(u-\overline{u}) \rangle_{L^2(\mathbb{V}^*),L^2(\mathbb{V})} + \\ + \langle J_2'(\overline{y}(T)), L_c B(u-\overline{u})(T) \rangle_{\mathbb{H}} + \langle J_3'(\overline{u}), u - \overline{u} \rangle_U \geq 0 \ \textit{for all } u \in \widetilde{U} \ , \quad \textbf{\textit{Variational Inequality}} \end{cases}$$

5 Numerical Implementation

5.1 Discrete Concepts and Results

Since our cost functional is constrained to be evaluated in the pairs $(y(u), u) \in \widetilde{\mathbb{W}}(0,T) \times \widetilde{U}$ where $y(u)$ is the weak solution in the sense of the definition 3.1, which depends on the control u, our first step is to discretize the state equation in order to, given a $u \in \widetilde{U}$, obtain an approximate solution for the forward problem.
By construction, if $w(u)$ satisfies the state equation, i.e., $e(w(u), u) = 0$, then the function $y(u) = \widetilde{B(u)} + w(u) + L_{in}(y_{in})$, where $y_{in} \in \mathscr{T}_{in}$ is a given trace function, is a solution to the weak problem of definition (3.1), where again the data u_0, f are prescribed. Therefore by solving the state equation we are in fact solving the weak formulation (3.1).

Now, solving the state equation numerically is not simple, since we are using as test functions free-divergence functions, which are not trivial to implement. However we can avoid this problem by using another problem from which the weak formulation (3.1) is the reduce form.
To introduce that we look again to the problem (1) and we introduce a weak formulation where we do not use free divergence functions. The payback of this procedure is that, by doing this, we will introduce another variable to our problem, the pressure P.

As done before, we will assume that the hypotheses (H2) are satisfied. Therefore the variational formulation for the strong form where we do not use free divergence functions is given by the following definition.

Definition 5.1. *Suppose that H_2 is valid. We say that $y(t) \in L^2(H^1(\Omega)^2)$ with $y'(t) \in L^2([H^1(\Omega)^*]^2)$ is a weak solution for the problem with pressure, if exists and a pressure field $p \in L^2(L^2(\Omega))$ such* *that* (y,p) *satisfy*

$$\begin{cases} \displaystyle\int_0^T \langle y'(t), v(t)\rangle_{(H^1)^*,H^1} dt + \mu \int_0^T \big(\nabla y(t), \nabla v(t)\big) dt - \int_0^T \Big(p(t), \big(\nabla \cdot v(t)\big)\Big) dt = \int_0^T \big(f(t), v(t)\big) dt \\ \text{for all } v \in L^2(H^1_{\Gamma_D}(\Omega)^2) \\ \displaystyle\int_0^T \Big(\big(\nabla \cdot y(t), q(t)\Big) dt = 0 \text{ for all } q(t) \in L^2(L^2(\Omega)) \\ y = y_{in} \text{ in } \Sigma_{in} \\ y = y_c \text{ in } \Sigma_c \\ y = 0 \text{ in } \Sigma_w \\ y(0) = y_0 \end{cases} \tag{114}$$

Let us see that the problem (114) makes sense.
Since the function $y(t)$ is in the set $\{u(t) \in L^2(H^1(\Omega)) : u'(t) \in L^2(H^1(\Omega)^*)\}$, and $H^1(\Omega)$ is dense in $L^2(\Omega)$ (by the fact that Ω is a Lipschitz set) we conclude that $y(t)$ is, after making a change in a set of measure zero, a continuous function, that is, $u(t) \in C\big([0,T]; L^2(\Omega)\big)$. Therefore, the initial condition $y_0 = y(0)$ in the problem (114) makes sense, since $y_0 \in \mathscr{H} \subset L^2(\Omega)$.
The boundary conditions are to be understood in the trace sense, and finally the integrals in the first and second equation of (114) are also well defined, by choosing the functions in the spaces mentioned.

We have the following result.

Lemma 5.1. *If* $y(t)$ *is a solution in the sense of the definition 5.1, then is also the solution in the sense of the definition 3.1.*

Proof: We start by observing that, by choosing as a test function, in the second equation $q(t) = \nabla \cdot y(t)$, which is possible since,

$$||\nabla \cdot y||_{L^2(L^2(\Omega))} = \int_0^T ||\nabla \cdot y(t)||^2_{L^2(\Omega)} dt \leq 2 \int_0^T ||\nabla y(t)||^2_{L^2(\Omega)} \leq 2||y||_{L^2(H^1(\Omega))}$$

we get that,

$$\int_0^T ||\nabla \cdot y(t)||^2_{L^2(\Omega)} dt = 0$$

and so, the function $\nabla \cdot y(t) = 0$ (zero in the space $L^2(\Omega)$) , for almost every $t \in (0,T)$. Therefore, for a.e. $t \in (0,T)$ the function $y(t)$ has null divergence that is $y(t) \in \{v(t) \in L^2(H^1(\Omega)) : \nabla \cdot v(t) = 0\}$, and since satisfies the boundary conditions $y|_{\Sigma_w} = 0$ in the trace sense, we also have that for a.e. $t \in (0,T)$, $y(t)|_{\Gamma_w} = 0$, and thus $y(t) \in L^2(\mathbb{V})$.

In the first equation, since the space $\widetilde{V} \subset H^1_{\Gamma_D}(\Omega)$, we may restrict the equation only to the test functions $v(t) \in L^2(\widetilde{V})$, and therefore, since this functions have null divergence, the integral term with the pressure disappears, what yields,

$$\int_0^T \langle y'(t), v(t)\rangle_{(H^1)^*,H^1} dt + \mu \int_0^T \big(\nabla y(t), \nabla v(t)\big) dt = \int_0^T \big(f(t), v(t)\big) dt \text{ for all } v \in L^2(H^1_{\Gamma_D}(\Omega)^2) \tag{115}$$

Now since the derivative $y'(t)$ belongs to the space $L^2(H^1(\Omega)^*)$, we may define the restriction operator

$$\langle Ry', v\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})} = \langle y', v\rangle_{L^2((H^1)^*),L^2(H^1)} = \int_0^T \langle y'(t), v(t)\rangle_{(H^1)^*,H^1} dt$$

which is the restriction of the derivative operator $y'(t)$ to the set $L^2(\widetilde{V}) \subset L^2(H^1(\Omega))$. This operator is also continuous when the space $\widetilde{V}$ is equipped with the norm of $\widetilde{V}$.

In fact, since the norm of $H^1(\Omega)$ is equivalent to the norm $||\cdot||_{\widetilde{V}}$ we get, by the continuity of $y'(t)$ that, for all $v(t) \in L^2(\widetilde{V})$,

$$|\langle Ry', v\rangle_{L^2(\widetilde{V}^*),L^2(\widetilde{V})}| \leq \Big(\int_0^T ||y'(t)||^2_{(H^1)^*}dt\Big)^{1/2} \times \Big(\int_0^T ||v(t)||^2_{H^1}dt\Big)^{1/2} \leq ||y'(t)||_{L^2((H^1)^*)}C_p||v||_{L^2(\widetilde{V})}$$

where C_p is a Poincaré constant.
Therefore the derivative $y'(t)$ has a continuous restriction to the space $L^2(\widetilde{V})$ (equipped with it's norm) that we denote by the same symbol $y'(t)$. To conclude, since $y(t) \in L^2(\mathbb{V})$ and the derivative is also in $L^2(\widetilde{V})$, this implies by definition that , the solution of (114) $y(t)$ is in the space $\widetilde{\mathbb{W}}(0,T)$. Moreover it also satisfies the boundary conditions and equations of the definition 3.1 and therefore To resume, $y(t)$ is a solution to the problem in the definition 3.1. □

Now we turn to temporal discretization, and enunciate the following result which gives us a consistency result for the time derivative approximation by finite differences.

Lemma 5.2. *Let v, v_t, v_{tt} be function in $L^2(t_n, t_{n+1}; L^2(\Omega))$ the we have,*

$$||\partial_t v_{n+1} - \frac{v_{n+1} - v_n}{\Delta t}||^2_{L^2(\Omega)} \leq \Delta t||u_{tt}||^2_{L^2(t_n,t_{n+1};L^2(\Omega))}$$

Proof: We start by calculating the weak-time derivative of the function $(t - t_n)u_t$.
So, let $\varphi(t)$ be a function of $C_0^\infty(t_n.t_{n+1})$, and we calculate

$$\int_{t_n}^{t_{n+1}} \varphi'(s)(s - t_n)u_t(s)ds \tag{116}$$

Notice that $v(t) = \varphi(t)(t - t_n)$ is also in $C_0^\infty(t_n, t_{n+1})$, thus

$$\int_{t_n}^{t_{n+1}} \varphi'(s)(s - t_n)u_t(s)ds = \int_{t_n}^{t_{n+1}} v'(s)(s - t_n)u_t(s)ds - \int_{t_n}^{t_{n+1}} \varphi'(s)u_t(s)ds$$
$$= -\int_{t_n}^{t_{n+1}} \varphi(s)(s - t_n)u_{tt}(s) + u_t(s)\varphi(s)ds$$

that is we have that the weak time derivative of the function $f(t) = (t - t_n)u_t(t)$ is $f'(t) = (t - t_n)u_{tt}(t) + u_t(t)$ wich is in $L^2(t_n, t_{t_n+1}; L^2(\Omega))$.
Now by (7.9) we have that, by using the above equation

$$(t_{n+1} - t_n)u_t(t_{n+1}) = \int_{t_n}^{t_{n+1}} \big((s - t_n)u_t(s)\big)'ds =$$
$$= \int_{t_n}^{t_{n+1}} \big((s - t_n)u_{tt}(s)\big) + \int_{t_n}^{t_{n+1}} u_t(s)ds = \int_{t_n}^{t_{n+1}} \big((s - t_n)u_{tt}(s)\big) + u_{n+1} - u_n$$

which may be written as,

$$u_t(t_{n+1}) - \frac{u_{n+1} - u_n}{\Delta t} = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (s - t_n)u_{tt}(s)ds \tag{117}$$

Therefore we have

$$||u_t(t_{n+1}) - \frac{u_{n+1} - u_n}{\Delta t}||^2_{L^2(\Omega)} = \frac{1}{\Delta t^2}||\int_{t_n}^{t_{n+1}} (s - t_n)u_{tt}(s)ds||^2_{L^2(\Omega)}$$
$$\leq \frac{1}{\Delta t^2}\Big(\int_{t_n}^{t_{n+1}} ||(s - t_n)u_{tt}(s)||^2_{L^2(\Omega)}ds\Big)^2$$
$$= \frac{1}{\Delta t^2}\Big(\int_{t_n}^{t_{n+1}} |(s - t_n)|||u_{tt}(s)||_{L^2(\Omega)}ds\Big)^2$$
$$\leq \frac{1}{\Delta t^2}\Big(\big(\int_{t_n}^{t_{n+1}} |(s - t_n)|^2ds\big)^{1/2}\big(\int_{t_n}^{t_{n+1}} ||u_{tt}(s)||_{L^2(\Omega)}ds\big)^{1/2}\Big)^2$$
$$= \frac{\Delta t}{3}||u_{tt}||^2_{L^2(t_n,t_{n+1}:L^2(\Omega))}$$

□

Now we will focus on space discretization. Since in the fully discretization method, we will solve in each time step a steady Stokes problem, it is important to recall some concepts and results that will be useful analysis of the numerical method.

Therefore we start by recalling some concepts for the general abstract saddle point problem (this results are form [31]), which is given by finding $(u, v) \in V \times Q$ (real Hilbert spaces) such that

$$\begin{cases} Au + B'p = f \text{ in } V' \\ Bu = r \text{ in } Q' \end{cases} \tag{118}$$

where the operators A and B are given by

$$A : V \to V' \text{ , defined by } \langle Au, v\rangle_{V',V} = a(u, v) \quad B : V \to Q' \text{ , defined by } \langle Bu, q\rangle_{V',V} = b(u, q)$$

and where $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ are two bi-linear and and bounded applications. We say that the problem (118) is well posed if the application

$$\Psi : V \times Q \to V' \times Q' \text{ given by } \Psi(u, p) = (Au + B'p, Bu) = (f, r) \tag{119}$$

is an isomorphism from $V \times Q$ onto $V' \times Q'$.

Now we recall a well know result which uses the inf-sup condition concept. Before the result let us introduce the imbbeding operator $E_0 : V' \to V_0'$ given by,

$$\langle E_0\phi, v\rangle_{V_0',V_0} = \langle \phi, v\rangle_{V',V} \text{ for every } \phi \in V' \text{ and } v \in V_0$$

Notice that this operator is bounded. In fact for every $\phi \in V'$,

$$|\langle E_0\phi, v\rangle_{V_0',V_0}| = |\langle \phi, v\rangle_{V',V}| \leq ||\phi||_{V'}||v||_V$$

and thus $||E_0\phi||_{V_0'} \leq ||\phi||_{V'}$.

Proposition 5.1. *Let V, Q be two real Hilbert spaces with inner products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_Q$ which induce the complete norms $||\cdot||_V$ and $||\cdot||_Q$ respectively. Then the following properties are equivalent:*

(i) There exists a constant $\beta_{is} > 0$ such that

$$\inf_{q\in Q:q\neq 0} \sup_{v\in V:v\neq 0} \frac{b(v, q)}{||v||_V ||q||_Q} \geq \beta_{is} \tag{120}$$

(ii) The operator B' is an isomorphism from Q onto $\hat{V}'$ and

$$||B'q||_{V'} \geq ||\beta_{is}||q||_Q \textit{ for all } q \in Q \tag{121}$$

(iii) The operator B is an isomorphism from $V_0^\perp$ onto Q' and

$$||Bv|| \geq \beta_{is}||v||_V \textit{ for all } v \in V_0^\perp \tag{122}$$

The result of the proposition 5.1 is fundamental to obtain the well-posedness of the saddle point problem. The following result makes that connection, and we show the proof to demonstrate how the inf-sup condition is fundamental for the resolution of a general saddle point problem.

Theorem 5.1. *The problem (118), (with $r = 0$ in our case) is well-posed if and only if the following two conditions are satisfied:*

(i) The operator $E_0 \circ A$ is an isomorphism from V onto V_0'

(ii) The bi-linear form $b(\cdot, \cdot)$ satisfies the inf-sup condition.

Proof: The proof is guided by [31]. First we see that the conditions (i) and (ii) are sufficient. Thus, suppose that (i) and (ii) are satisfied. Now we show that the reduced problem,

$$a(u,v) = \langle f, v\rangle_{V',V} \text{ , for all } v \in V_0 = Ker(B) \tag{123}$$

has a unique solution. The equation (123) can be written in an operatorial form

$$E_0 \circ A(w) = E_0(f) \text{ in } V_0' \tag{124}$$

and since $E_0 \circ A$ is, by (i), an isomorphism from V_0 to V_0', we conclude that existence and uniqueness of the solution w for the equation (124), ant therefore there's a unique solution to (123).
Now we see that with the w obtained above, we can construct in an unique way, a pressure p such that (w,p) are the unique solution to (118) with $r = 0$ in Q'. Since $E_\circ A$ is an isomorphism, is by definition, a bijection from V_0 onto V_0' and also continuous. Therefore by the Banach's Open Map theorem we have too that $(E_0 \circ A)^{-1}$ is bounded.
Thus,

$$||w||_V = ||(E_0 \circ A)^{-1} \circ E_0 f||_V \leq C||E_0 \circ f|| \leq C||f||_{V'} \tag{125}$$

and so, we have a stability condition for the solution w.
Now we construct the unique pressure, by using the inf-sup condition. Notice that since, by (123) we have that

$$a(w,v) - \langle f, v\rangle_{V',V} \Leftrightarrow \langle Aw - f, v\rangle_{V',V} = 0 \text{ for all } v \in V_0$$

that is,$Aw - f \in \hat{V}'$, and by the proposition (5.1), we have a unique $p \in Q$ that satisfies

$$B'p = Aw - f \text{ in } V' \tag{126}$$

Moreover we have the estimate, which again cames from the proposition (5.1)

$$||p||_Q \leq \frac{1}{\beta_{is}}||f - Aw||_{V'} \leq C'||f||_{V'} \tag{127}$$

Therefore the application of the definition is a isomorphism.

Now we show that the condition (i) and (ii) are also necessary. For that we assume that the application $\Psi : V \times Q \to V' \times Q'$ is an isomorphism.
We start by seeing that in this conditions the inf-sup condition (ii) is satisfied. For that, consider the operator $B^\perp$ which is defined as the restriction of B to the subset $V_0^\perp \subset V$.
Thus since V is a Hilbert space and V_0 is a closed subset of V, we have the algebraic and topological decomposition

$$V = V_0 \oplus V_0^\perp \tag{128}$$

and so, for every $u \in V$, u assumes a unique decomposition of the form $u = u_0 + \tilde{u}$, where $u_0 \in V_0$ and $\tilde{u} \in V_0^\perp$. With this decomposition we can observe that

$$B(u) = B(u_0 + \tilde{u}) = B(\tilde{u}) = B^\perp(\tilde{u}) \tag{129}$$

Now since the application Ψ is, by hypotheses, an isomorphism, we must have that $Range(B) = Q'$, but from (129), $Range(B) = Range(B^\perp) = Q'$, that is $B^\perp$ is surjective.
To see the injectivity, suppose that exists $u_1, u_2 \in V_0^\perp$, with $B^\perp u_1 = B^\perp u_2 = \phi$. Then $B^\perp(u_1 - u_2) = 0$ and therefore $u_1 - u_2 \in V_0$. But since $V_0^\perp$ is a vector space, and u_1, u_2 both belong to $V_0^\perp$, then $u_1 - u_2$ is also in $V_0^\perp$. But the only element that is in $V_0^\perp \cap V_0$ is the zero element, thus $u_1 = u_2$ and $B^\perp$ is injective, and then is a bijection from $V_0^\perp$ onto Q'.
By the Banach's Open Map Theorem we also can conclude that, since $B^\perp$ is linear and bounded, that $(B^\perp)^{-1}$ is also linear and bounded, and therefore exists a strictly positive constant C such that

$$||(B^\perp)^{-1}\phi||_V \leq C||\phi||_{Q'} \tag{130}$$

and by choosing in (130) $\phi = B^\perp v$ with $v \in V_0$ we get

$$\frac{1}{C}||v||_V \leq ||Bv||_{Q'} \tag{131}$$

and by the proposition (5.1) the inf-sup condition is satisfied.

Now let us see that the condition (i) is also verified. Let $\phi \in V_0'$ an arbitrary element. By the Hanh-Banach's theorem exists at least a function $f \in V'$ such that,

$$E_0 f = \phi \text{ in } V_0'$$

Defining $(u, p) = \Psi(f, 0)$, then the element u is in V_0, since it satisfies $Bu = 0$ (second equation of the definition of Ψ), and we have

$$Au + B'p = f \text{ in } V' \tag{132}$$

On the other hand

$$\langle E_0 \circ B'p, v\rangle_{V_0', V_0} = \langle B'p, v\rangle_{V', V} = \langle Bv, p\rangle_{Q', Q} = 0 \text{ for all } v \in V_0 \tag{133}$$

Therefore $E_0 \circ B'p = 0$ in V_0', and using the equation (132) we get

$$E_0 \circ Au = E_0 f = \phi$$

and since the element $\phi \in V_0'$ was chosen in an arbitrary way, we conclude that $E_\circ A$ is surjective. To see the injectivity we argue by contradiction. Suppose that exists $v_1 \neq v_2$ in V_0 with $E_0 \circ Av_1 = E_0 \circ v_2 = \phi$, then $E_0 \circ A(v1 - v2) = 0$. Let $q \in Q$ arbitrary and we define

$$Av_1 + B'q = f1 \qquad\qquad Av_2 + B'q2$$

where we must have, by the hypotheses that Ψ is an isomorphism and $Bv1 = Bv2 = 0$, that $f_1 \neq f_2$.
Therefore we have $A(v_1 - v_2) = f_1 - f_2 \Leftrightarrow \Psi(v_1 - v_2, 0) = (f_1 - f_2, 0)$.
Now since, $E_0 \circ A(v_1 - v_2) = 0$, we have,

$$0 = \langle E_0 \circ A(v_1 - v_2), v\rangle_{V_0', V_0} = \rangle A(v_1 - v_2), v\rangle_{V', V} = \rangle f_1 - f_2, v\rangle_{v', V}$$

and therefore $f_1 - f_2$ belongs to $\hat{V}'$, and by the proposition[22] (5.1) we get that $\exists! p \in Q$ such that $B'q = f_1 - f_2$.
But then we would have,

$$\Psi(v_1 - v_2, 0) = (f_1 - f_2, 0) = \Psi(0, q)$$

which is a contradiction since the application Ψ is an isomorphism. □

The following result, gives a sufficient condition on the bi-linear form $a(\cdot, \cdot)$, for the well-posedness of the problem (118).

Lemma 5.3. *If the bi-linear form $a : V \times V \to \mathbb{R}$ is V_0-elliptic, i.e. , exists $\alpha > 0$ such that*

$$a(v, v) \geq \alpha ||v||_V^2 \text{ for all } v \in V_0$$

then we only need $b(\cdot, \cdot)$ to verify the inf-sup condition, in order to the problem (118) to be well-posed.

As we said at the beginning of this chapter, we will need, in the fully discrete-method, at each time iteration, to solve a saddle point problem. Therefore we give an analysis of this type of problems in a discrete form, which are very similar to what we have done in the continuous case. We begin to introduce the discretized form of the problem (118), which is given by,

Find $(u^h, p^h) \in V^h \times Q^h$ such that,

$$\begin{cases} a_h(u^h, v^h) + b_h(v^h, p^h) = \langle f, v^h\rangle_{V', V} \text{ for all } v^h \in V^h \\ b_h(u^h, q^h) = \langle r, q^h\rangle_{Q', Q} \text{ for all } q^h \in Q^h \end{cases} \tag{134}$$

where V^h and Q^h are finite dimension spaces, which are said to be conforming if they satisfy $V^h \subset V$ and $Q^h \subset Q$. Also the applications $a_h : V^h \times V^h \to \mathbb{R}$, $b_h : V^h \times Q^h \to \mathbb{R}$ are the discrete

[22] Notice that the inf-sup condition was already verified.

forms of the applications a, b. In the case of V^h, Q^h are conforming, the discrete maps, are simply the restriction of the continuous maps, to the finite dimensional subsets V^h and Q^h.
With analogy to the continuous case we also introduce the discrete inf-sup condition. We say that the application $b_h : V^h \times Q^h \to \mathbb{R}$ satisfies a discrete inf-sup condition, if exists a constant $\beta_{dis} > 0$ such that

$$\inf_{q^h \in Q_h, q^h \neq 0} \sup_{v^h \in V^h, v^h \neq 0} \frac{b_h(v^h, q^h)}{||v_h||_{V^h} ||q^h||_{Q^h}} \geq \beta_{dis} \tag{135}$$

As we said,in the case that V^h, Q^h are conforming b_h is equal to b. Also in that cases the norms in (135) are the norms of the bigger spaces V, Q.

Continuing with the analogy with the continuous case, we define the operator

$$B_h : V^h \to \left(Q^h\right)' \qquad \langle B_h v^h, q^h \rangle_{Q^h{}', Q^h} = b_h(v^h, q^h)$$

and the adjoint

$$B_h' : Q^h \to \left(V^h\right)' \qquad \langle v^h, B_h' q^h \rangle_{V^h, V^h} = b_h(v^h, q^h)$$

In our case the application $b : V \times Q \to \mathbb{R}$ is the negative divergent, i.e. ,

$$b(u, q) = -\int_\Omega (\nabla \cdot u) q dx \text{ for } u \in V = H^1(\Omega), q \in Q = L^2(\Omega) \tag{136}$$

and therefore in this case $B_h = -div_h$ and the adjoint is the discrete gradient $(B_h)' = grad_h$.

We saw that the kernel of B plays an important role in the theory, and therefore seems natural to introduce the discrete kernel, which we will call the set of discretely divergence-free functions

$$V_{div}^h := \{v^h \in V^h : b_h(v^h, q^h) = 0 \text{ for all } q^h \in Q^h\} \tag{137}$$

It is known that the Taylor-Hood spaces P_k/P_{k-1} with $k \geq 2$, are finite element spaces which make the discrete form b_h of the application b of (136) satisfy the discrete inf-sup condition (135). We will in particular use the case P_2/P_1, and with this choice we can see that if v^h is in V_{div}^h then, since[23] $\nabla \cdot P_2 \subset P_1$ we have

$$0 = b(v^h, \nabla \cdot v^h) = \int_\Omega \nabla \cdot v^h(x)^2 dx$$

and therefore v^h has in fact a null divergence in the $L^2(\Omega)$ sense.

Since we are dealing with finite dimensional subspaces of real Hilbert spaces, V^h and Q^h are also[24] real Hilbert spaces. Thus, by using the spaces $P2/P1$ for the space discretization, the inf-sup condition is valid for b, and by the fact that a is also coercive, the same construction as in the continuous case can be done, in order to establish the well-posedness of the discrete problem (134).

Now we introduce the discrete spaces that we will use in the numerical simulations.

First we define a polygonal domain $\widetilde{\Omega} \approx \Omega$ see the images (3),(We do not do a analysis of this approximation error). This approximate domain in our case was obtained by using the Gmesh software, where using an image of the real domain Ω we define, by a sufficient number of boundary points, a contour which is done by connecting the boundary dots with splines. Then using a Gmesh command we can define a mesh for that contour. The final product, which can be seen in

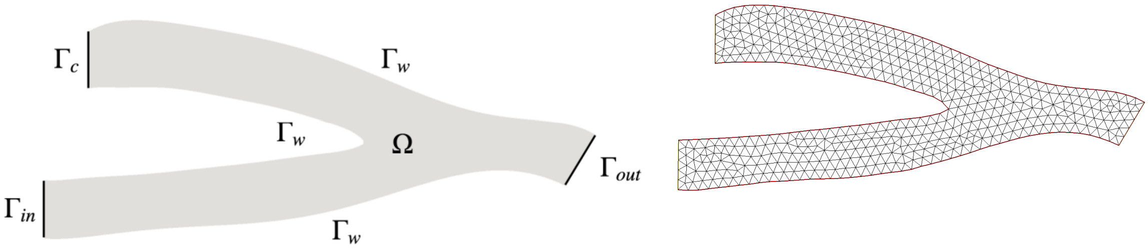
[23] Let us see that this is true.
Let u be a function of P_2, then it has the form $u(x, y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2$ with $a_i, i \in \{1, ...6\} \in \mathbb{R}$. If we calculate the divergence of u we get

$$\nabla \cdot u(x, y) = a_2 + a_3 + a_4(y + x) + 2a_5 x + 2a_6 y \in P_1$$

Therefore we $\nabla \cdot P_2 \subset P_1$.

[24] Recall that every finite dimensional subspace is closed. Therefore every finite dimensional subspace of a Hilbert spaces is also a Hilbert space for the same inner product.

image (3-right) is an approximation $\widetilde{\Omega}$ of Ω, where the borders are define as the arests of frontier triangles, and so the domain is polygonal. Is important to notice that the boundary part Γ_c has no error of approximation when we use the $\widetilde{\Omega}$.



(a) Original Domain (b) Our Ω

Figure 3: Our $\widetilde{\Omega}$ shown by using the Freefem++ software. This is our coarser mesh.

For the mesh refinement we used a Freefem++ command (the splitmesh command) which divides every triangle on the previews mesh by a given number, which we call the order of refinement. In image (4) we applied a splitmesh of order two to the original mesh, a process that divides every original triangle in a regular form. If we continue to apply this type of refinement of can define a regular family of triangulation $\mathscr{T}_h$, h being the a mesh parameter.

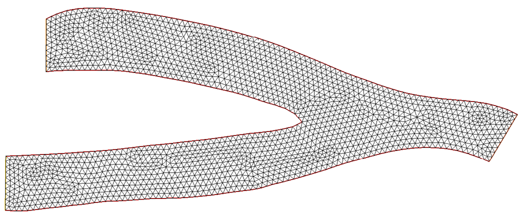


Figure 4: Example of a refinement of order two.

For this type of triangulation we will use the classical $P_2 - P_1$ Taylor-Hood finite elements spaces to define the discrete functional space for the velocity and pressure, V_h, P_h respectively, given by

$$P_h := \left\{ q_h : q_h \in C^0(\overline{\Omega}) \text{ with } q_h|_K \in P^1(K), \forall K \in \mathscr{T}_h \right\}$$
$$V_h := \left\{ v_h : v_h \in C^0(\overline{\Omega}) \times C^0(\overline{\Omega}) \text{ with } v_h|_K \in P^2(K) \times P^2(K), \forall K \in \mathscr{T}_h \right\}$$

It is well known that this choice of functional discrete spaces for the velocity and pressure, satisfy the discrete inf-sup condition[25].

Now, let N_v,N_p be the number of velocity and pressure nodes respectively. The Lagrangian base functions are in this case given by continuous functions $\{\phi_i\}_{i=1}^{N_v}, \{\psi_i\}_{i=1}^{N_p}$ in $\widetilde{\Omega}$[26] such that

$$\phi_i(v_j) = \delta_{ij} \qquad\qquad \psi_i(p_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta and $\{v_j\}_{j=1}^{N_n}, \{p_j\}_{j=1}^{N_p}$ are the velocity and pressure nodes respectively.

Since the velocity vector field has dimension two, we need $2 \times N_v$ base functions to fully describe the discrete velocity. This base functions are given by

$$\{\boldsymbol{\phi_i}\}_{i=1}^{2N_v} = \begin{cases} (\phi_i, 0) \text{ if } i \in \{1, ..., N_v\} \\ (0, \phi_{i \bmod N_v}) \text{ if } i \in \{N_v + 1, ..., 2N_v\} \end{cases}$$

[25]In the 2-dimensional case, a sufficient condition for $P_2 - P_1$ discretization to satisfy the inf-sup condition, is that the triangulation at hand , has at least 3 triangles. This result can be seen in [Volker lemma 3.128]

[26]We will also write sometimes Ω.

Every $u^h \in V^h$ and $p^h \in P^h$ assumes the form

$$u^h(x,y) = \sum_{i=1}^{2N_v} u_i \boldsymbol{\phi}_i(x,y) \qquad\qquad p^h(x,y) = \sum_{i=1}^{N_p} p_i \psi_i(x,y) \tag{138}$$

This representation is fundamental for the sake of obtaining the variational formulations in the discrete case. We take here the opportunity of introducing the following matrices which are used in the numerical process.

$$\widetilde{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \qquad \text{with } M_{i=1,j=1}^{N_x,N_x} = \int_\Omega \phi_i \phi_j \tag{139}$$

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \qquad \text{with } A_{i=1,j=1}^{N_x,N_x} = \mu \int_\Omega \nabla\phi_i : \nabla\phi_j \tag{140}$$

$$B = \begin{bmatrix} B_x & B_y \end{bmatrix} \qquad \text{with } {B_x}_{i=1,j=1}^{N_x,N_p} = \int_\Omega -\frac{\phi_i}{dx}\phi_j^p \qquad {B_y}_{i=1,j=1}^{N_x,N_p} = \int_\Omega -\frac{\phi_i}{dy}\phi_j^p \tag{141}$$

In practice the values on the matrices are not exactly the integrals, which are calculated by using quadrature rules, and here is made another error of approximation for the Stokes solution. In the case of a vorticity minimizing functional cost is also useful to introduce the vorticity matrix, since for the computation of the cost it is necessary to calculate

$$\int_{\widetilde{\Omega}} (\nabla \times \psi)(\nabla \times \varphi)$$

for two given functions $\psi, \varphi \in V^h$. Thus
In 2-D the vorticity of a vector field $\mathbf{v}$ is a scalar function given by

$$\nabla \times \mathbf{v} = \frac{\partial \mathbf{v}_2}{\partial x} - \frac{\partial \mathbf{v}_1}{\partial y}$$

so if we use two base function $\boldsymbol{\phi}_i, \boldsymbol{\phi}_j$ we get the following 4 cases

$$(Vor)_{i,j} = \int_{\widetilde{\Omega}} (\nabla \times \boldsymbol{\phi}_i)(\nabla \times \boldsymbol{\phi}_j) = \begin{cases} \int_{\widetilde{\Omega}} \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \text{ if } i,j \in \{1,...,N_v\} \\ \int_{\widetilde{\Omega}} -\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial x} \text{ if } i \in \{1,...,N_v\} j \in \{N_v+1,...,2N_v\} \\ \int_{\widetilde{\Omega}} -\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial y} \text{ if } j \in \{1,...,N_v\} i \in \{N_v+1,...,2N_v\} \\ \int_{\widetilde{\Omega}} \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_i}{\partial x} \text{ if } i,j \in \{N_v+1,...,2N_v\} \end{cases}$$

We also defined introduce the following discrete spaces [27]:

$$\begin{aligned}
&V_{0h} := \{v_h \in V_h : v_h|_{\Gamma_w} = 0\} \\
&V_{00h} := \{v_h \in V_h : v_h|_{\Gamma_w \cup \Gamma_{in}} = 0\} \\
&V_{000h} := \{v_h \in V_h : v_h|_{\Gamma_c \cup \Gamma_w \cup \Gamma_{in}} = 0\} \\
&T^h = \gamma_D V_h := \{\mu_h : \mu_h = \gamma_D v_h \text{ with } v_h \in V_h\} \\
&T_1^h = \gamma_1 V_h := \{\mu_h : \mu_h = \gamma_1 v_{0h} \text{ with } v_h \in V_h\}
\end{aligned}$$

Now suppose that the conditions H_2 are valid and. We set $V = H^1(\widetilde{\Omega} =$ and $P = L^2(\widetilde{\Omega})$, which are approximated by the finite element spaces P_2/P_1.
We denote by $\widetilde{\Gamma_D}$ the discretization of the Dirichlet boundary of $\widetilde{\Omega}$, and in the same way are defined $\Gamma_c, \Gamma_{in}, \Gamma_{out}$.

[27] Here we used the same notation of the article [4]

Now we focus on the time discretization. For that, let $[0,T]$ be the temporal interval of analysis and we define the time step Δt (also denoted by dt sometimes). Therefore we have a list of time instants $\{t_n\}_{n=0}^{n=N_t}$ where $t_n = \Delta t \times n$ and $N_t = \dfrac{T}{dt}$.
This time instants will be the nodes for the linear Lagrangian functions with respect with time

$$\phi_i^t(t_j) = \delta_{i,j}$$

With this we may define the following time dependent discrete spaces (notation from [16]):

$$V_{dt}^h := \{y : [0,T] \times \widetilde{\Omega} \to \mathbb{R} : y(t,\cdot)|_{\widetilde{\Omega}} \in V^h, y(\cdot,(x,y))|_{I_n} \in \mathbb{P}^1, n = 1,...,N_t\}$$
$$P_{dt}^h := \{\pi : [0,T] \times \widetilde{\Omega} \to \mathbb{R} : \pi(t,\cdot)|_{\widetilde{\Omega}} \in P^h, \pi(\cdot,(x,y))|_{I_n} \in \mathbb{P}^1, n = 1,...,N_t\}$$

where $I_n = [t_{n-1}, t_n]$. The spaces $V_{dt}^{0h}, V_{dt}^{00h}$ are define on a similar way. In this framework every function of V_{dt}^h can be written in the form

$$V_{dt}^h \ni y(t,(x,y)) = \sum_{n=0}^{N_t} \phi_n^t(t) \sum_{k=1}^{2N_v} y_k^n \boldsymbol{\phi}_k(x,y)$$

and a function of V_{dt}^h is uniquely defined if we have the collection of values $\{y_k^n\}_{n=0,k=1}^{N_t,2N_v}$. Therefore in order to define solutions for the Stokes non-stationary problem we need a way of determining the coefficients $\{y_k^n\}_{n=0,k=1}^{N_t,2N_v}$. This can be achieved by using the Euler implicit scheme that we will shortly define.

We close this part by noticing that, (we used the same notation of [4]) if we define

$$\mathscr{M}_h := \{z \in V_{00h} : \text{ for every } K \in \mathscr{T}_h \text{ if } K \cap \Gamma_c = \emptyset \implies z|_K = 0\}$$

The functions of $\mathscr{M}_h$ are the discrete liftings for the boundary data on Γ_c. In particular, given a function $y \in V_{00h}$, the can be written in the form[28]

$$y = \sum_{i \in I \setminus I_c} \alpha_i \boldsymbol{\phi}_i + \sum_{i \in I_c} \alpha_i \boldsymbol{\phi}_i$$

and notice that

$$\sum_{i \in I \setminus I_c} \alpha_i \boldsymbol{\phi}_i \in V_{000h} \qquad\qquad \sum_{i \in I_c} \alpha_i \boldsymbol{\phi}_i \in \mathscr{M}_h$$

Since this decomposition is unique for every element in V_{00h} we have

$$V_{00h} = \mathscr{M}_h \oplus V_{000h} \tag{142}$$

This fact will important for the gradient construction.

5.2 Euler Implicit Scheme

In the following we will denote $y(t_n,(x,y))$ by y_n^h for every function $y \in V_{dt}^h$ or the analogous spaces. Also $u_0^h = \Pi^h(u_0)$ is a discretization of the initial condition, where $\Pi^h : V \to V^h$ is an interpolation operator, and lastly, we denote by f^n the value of $f^n|_{\widetilde{\Omega}}$, and analogously g^n .

With this, the Implicit Euler scheme is given by,

For $n = 1,...,N_t$ *find* $(y_{n+1}^h, \pi_{n+1}^h) \in V^h \times P^h$ *such that*

$$\begin{cases} \displaystyle\int_{\widetilde{\Omega}} \frac{1}{\Delta t}(y_n^h - y_{n-1}^h)\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla y_n^h : \nabla \varphi^h - \int_{\widetilde{\Omega}} \pi_n^h (\nabla \cdot \varphi^h) = \int_{\widetilde{\Omega}} f^n \varphi^h \text{ for all } \varphi^h \in V_{0h} \\ \displaystyle\int_{\widetilde{\Omega}} (\nabla \cdot y_n^h) q^h = 0 \text{ for all } q^h \in P^h \\ y_n^h = g^n \text{ in } \widetilde{\Gamma_D^h} \end{cases}$$

[28]The set indexes $\widetilde{I}$ is given by indexes of the nodes which are not in $\Gamma_{in} \cup \Gamma_w$ and therefore $\widetilde{I} \subset \{1,...,N_v\}$. Then, in order to duplicate for the y component of the velocity, we define $I = \widetilde{I} + (\widetilde{I} + N_v) \subset \{1...,2N_v\}$.

This scheme defines a collection of N_t systems of equations on for each fixed $n \in \{1, ..., N_t\}$ we have a system to determine the coefficients $\{y_k^n\}_k^{2N_v}$ and $\{p_k^n\}_k^{N_p}$.
Let us now see that this scheme gives for an appropriate y_0, f, g a unique discrete solution (y^h, π^h), that is stable. In the following we will use an equivalent system for the Implicit Euler scheme, where we introduce the boundary conditions in a weak form: *For $n = 1, ..., N_t$ find $(y_{n+1}^h, \pi_{n+1}^h) \in V^h \times P^h$ such that*

$$\begin{cases} \int_{\widetilde{\Omega}} \frac{1}{\Delta t}(y_n^h - y_{n-1}^h)\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla y_n^h : \nabla \varphi^h - \int_{\widetilde{\Omega}} \pi_n^h (\nabla \cdot \varphi^h) + \langle s_n^h, \lambda \rangle_{T_1^h, (T_1^h)^*} = \\ = \int_{\widetilde{\Omega}} f^n \varphi^h \text{ for all } \varphi^h \in V_{0h} \\ \int_{\widetilde{\Omega}} (\nabla \cdot y_n^h) q^h = 0 \text{ for all } q^h \in P^h \\ \langle \gamma_1(y_n^h), \lambda \rangle_{T_1^h, (T_1^h)^*} = \langle g^n, \lambda \rangle_{T_1^h, (T_1^h)^*} \text{ for all } \lambda \in (T_1^h)^* \end{cases} \tag{143}$$

The equivalence between (in the sense that if (y_n^h, π_n^h, s_n^h) is a solution to the weak form then (y_n^h, π_n^h) is also a solution to the original formulation) the above scheme and the original one is given by the fact that, since for every $\lambda \in (T_1^h)^*$ we have $\langle \gamma_1(y_n^h)\lambda \rangle_{T_1^h, (T_1^h)^*} = \langle g^n, \lambda \rangle T_1^h, (T_1^h)^*$ we must also have that $\gamma_1(y_n^h) = g^n$ in $\widetilde{\Gamma_1}$.

We have the following property of this numerical scheme.

Lemma 5.4. *The implicit Euler scheme produces a unique solution which is stable.*

Proof: Uniqueness: To analyze the uniqueness we will focus on analysing the uniqueness for each time step, since this is sufficient. Suppose that $n \in \{1, ..., N_t\}$ is fixed and that the functions on the previews times steps had already been calculated (in the case $n = 1$, notice that the $y_0^h = \Pi^h(y_0)$ which is a give data), so to determine $(y_n^h, \pi_n^h, s_n^h) \in V^h \times P^h \times (T_1^h)^*$

$$\begin{cases} \int_{\widetilde{\Omega}} \frac{y_n^h}{\Delta t}\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla y_n^h : \nabla \varphi^h + \widetilde{b}^h(\varphi, (\pi_n^h, s_n^h)) = \int_{\widetilde{\Omega}} \frac{y_{n-1}^h}{\Delta t}\varphi + \int_{\widetilde{\Omega}} f^n \varphi^h \text{ for all } \varphi^h \in V_{0h} \\ \widetilde{b}(y^h, (q, \lambda)) = \langle G, (q, \lambda) \rangle_{Q^*, Q} \text{ for all } (q, \lambda) \in Q \end{cases} \tag{144}$$

where we have $Q = P^h \times (T_1^h)^*$, and

$$\widetilde{b}^h(\varphi, (q, \lambda)) = -\int_{\widetilde{\Omega}} (\nabla \cdot \varphi) q + \langle \lambda, \varphi \rangle_{\Gamma_1}, \text{ for } (\varphi, q, \lambda) \in V_{0h} \times Q \quad \langle G^n, (q, \lambda) \rangle_{Q^*, Q} = \langle g^n, \lambda \rangle_{\Gamma_1}$$

The continuous analogous $\widetilde{b}$ to the operator $\widetilde{b}^h$ is given by

$$\widetilde{b}(\varphi, (q, \lambda)) = -\int_{\widetilde{\Omega}} (\nabla \cdot \varphi) q + \langle \lambda, \varphi \rangle_{\Gamma_1}, \text{ for } (\varphi, q, \lambda) \in H^1_{\Gamma_1}(\widetilde{\Omega}) \times L^2(\widetilde{\Omega}) \times (\mathscr{T}_1)^* \quad \langle G^n, (q, \lambda) \rangle_{Q^*, Q} = \langle g^n, \lambda \rangle_{\Gamma_1}$$

where $\mathscr{T}_1 = H_{00}^{1/2}(\Gamma_1)$.
It is possible to proof that the map $\widetilde{b}$ satisfies the inf-sup condition (see [24]). Now, usually the b application for the saddle point problem in the Stokes Equations framework in a bounded and polygonal domain Ω, is only given by

$$b : H^1(\Omega) \times L^2(\Omega) \to \mathbb{R} \qquad b(v, p) = -\int_{\Omega} (\nabla \cdot v) p$$

and in this case, when the space discretization is done by using conforming finite element spaces (for example as in our case the P_2/P_1 Taylor-Hood), the discrete analogous b^h of b is given by

$$b^h : H^1(\Omega)^h \times L^2(\Omega)^h \to \mathbb{R} \qquad b^h(v^h, p^h) = b(v^h, p^h) \text{ for all } (v^h, p^h) \in H^1(\omega)^h \times L^2(\Omega)^h$$

and therefore since $H^1(\Omega)^h \times L^2(\Omega)^h \subset H^1(\Omega) \times L^2(\Omega)$ we get that

$$b^h(v^h, p^h) = b(v^h, p^h) \geq \beta ||v^h||_{H^1(\Omega)^h} ||p^h||_{L^2(\Omega)^h}$$

that is, if b has the inf-sup condition, then b^h has it too.
However, in our case, the above argument is not so linear since a linear functional $\lambda \in (T_1^h)^*$ is not in general contained in $(\mathscr{T}_1)^*$. Notice that $T_1^h \subset \mathscr{T}_1$. Therefore every linear functional $\lambda \in (T_1^h)^*$ has, by the Hahn-Banach theorem, an extension (which may not be unique) $\widetilde{\lambda} \in \mathscr{T}_1$, which has also $||\lambda||_{(T_1^h)^*} = ||\widetilde{\lambda}||_{\mathscr{T}_1^*}$.
Thus, for every $(v^h, p^h, s^h) \in V_{0h}^h \times P^h \times (T_1^h)^*$

$$
\begin{aligned}
\widetilde{b}^h(v^h,(p^h,s^h)) &= -\int_{\widetilde{\Omega}} (\nabla \cdot v^h)p^h + \langle s^h, v^h\rangle_{(T_1^h)^*, T_1^h} \\
&= -\int_{\widetilde{\Omega}} (\nabla \cdot v^h)p^h + \langle \widetilde{s}^h, v^h\rangle_{\mathscr{T}_1^*, \mathscr{T}_1} \\
&\geq \beta ||v^h||_{V_{0h}} \sqrt{||p^h||^2_{P^h} + ||\widetilde{s}^h||^2_{\mathscr{T}_1^*}} \\
&= \beta ||v^h||_{V_{0h}} \sqrt{||p^h||^2_{P^h} + ||s^h||^2_{(T_1^h)^*}} \\
&= \beta ||v^h||_{V_{0h}} ||(p^h, s^h)||_Q
\end{aligned}
$$

and in this case we also have that $\widetilde{b}^h$. It is simple to see that the right-hand-side of the system (144) is a linear functional in $V_{0h} \times Q$. Also if we define the map $a_h : v_{0h} \times V_{0h} \to \mathbb{R}$ by

$$
a_h(u^h, v^h) = \frac{1}{\Delta t}\int_{\widetilde{\Omega}} u^h v^h + \mu \int_{\widetilde{\Omega}} \nabla u^h : \nabla v^h \tag{145}
$$

The map a is $V_{0h} - coercive$ since

$$
a(v^h, v^h) = \frac{||v^h||^2_{L^2(\widetilde{\Omega})}}{\Delta t} + \mu ||\nabla v^h||^2_{L^2(\widetilde{\Omega})} \geq \mu ||\nabla v^h||^2_{L^2(\widetilde{\Omega})}
$$

by consequence, we get that the saddle point problem (144) has a unique solution on each step.

Stability: This is important to see that in finite time the approximate solution does not blow up in finite time.
For each $n \in \{1, ..., N_t\}$ if we choose as a test function in (144) $\varphi = \widetilde{y_n^h}$, of the decomposition $y_n^h = \widehat{y_n^h} + \widetilde{y_n^h}$ with $\widehat{y_n^h}$ being the lifting and $\widetilde{y_n^h} \in V^h \cap Ker(\widetilde{B})$.

Then we get

$$
\begin{aligned}
&\int_{\widetilde{\Omega}} \frac{y_n^h - y_{n-1}^h}{\Delta t}\widetilde{y_n^h} + \mu \int_{\widetilde{\Omega}} \nabla y_n^h : \nabla \widetilde{y_n^h} = \int_{\widetilde{\Omega}} f^n \widetilde{y_n^h} \Leftrightarrow \\
&\int_{\widetilde{\Omega}} \frac{\widetilde{y_n^h} - \widetilde{y_{n-1}^h}}{\Delta t}\widetilde{y_n^h} + \mu \int_{\widetilde{\Omega}} \nabla \widetilde{y_n^h} : \nabla \widetilde{y_n^h} = \int_{\widetilde{\Omega}} \frac{\widehat{y_{n-1}^h} - \widehat{y_n^h}}{\Delta t}\widetilde{y_n^h} + \int_{\widetilde{\Omega}} f^n \widetilde{y_n^h} - \mu \int_{\widetilde{\Omega}} \nabla \widehat{y_n^h} : \nabla \widetilde{y_n^h}
\end{aligned}
$$

Now we have the following identity, and in what follows we denote the $L^2(\widetilde{\Omega})$ inner product by the symbol $(\cdot, \cdot)$,

$$
\left(\frac{\widetilde{y_n^h} - \widetilde{y_{n-1}^h}}{\Delta t}, \widetilde{y_n^h}\right) \geq \frac{1}{2\Delta t}||\widetilde{y_n^h}||^2 - \frac{1}{2\Delta t}||\widetilde{y_{n-1}^h}||^2 \tag{146}
$$

Also from the inf-sup condition,

$$
||\nabla \widehat{g}_n^h|| \leq \frac{1}{\beta}||g^n||_{L^2(\Gamma_1)} \tag{147}
$$

By using the classical Cauchy-Schwarz inequality, the Poincaré inequality and (146),(147),

$$
||\widetilde{y_n^h}||^2 + 2\mu\Delta t||\nabla\widetilde{y_n^h}||^2 \leq 2\Delta t C_p ||f^n|| \, ||\nabla \widetilde{y_n^h}|| + 2\Delta t\mu ||\nabla \widehat{y_n^h}|| \, ||\nabla \widetilde{y_n^h}|| + \frac{2C_p}{\beta}||g^n - g^{n-1}||_{L^2(\Gamma_c)}||\nabla \widetilde{y_n^h}|| + ||\widetilde{y_{n-1}^h}||^2 \tag{148}
$$

Now by Young's inequality exists $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that

$$2\Delta t C_p ||f^n|| ||\nabla \widetilde{y_n^h}|| \leq 2\Delta t C_p \Big(\frac{\epsilon_1 ||f^n||^2}{2} + \frac{||\nabla \widetilde{y_n^h}||^2}{2\epsilon_1} \Big) \tag{149}$$

$$2\Delta t \mu ||\nabla \widehat{y_n^h}|| ||\nabla \widetilde{y_n^h}|| \leq 2\Delta t C_p \Big(\frac{\epsilon_2 ||g_n^h||^2_{L^2(\Gamma_1)}}{2\beta^2} + \frac{||\nabla \widetilde{y_n^h}||^2}{2\epsilon_2} \Big) \tag{150}$$

$$\frac{2C_p}{\beta} ||g^n - g^{n-1}||_{L^2(\Gamma_1)} ||\nabla \widetilde{y_n^h}|| \leq \frac{2C_p}{\beta} \Big(\frac{\epsilon_3 ||g^n - g^{n-1}||^2_{L^2(\Gamma_1)}}{2\beta^2} + \frac{||\nabla \widetilde{y_n^h}||^2}{2\epsilon_3} \Big) \tag{151}$$

Choosing $\epsilon_1 = \dfrac{3C_p}{\mu}, \epsilon_2 = 3, \epsilon_3 = \dfrac{3C_p}{\Delta t \mu}$ yields,

$$||\widetilde{y_n^h}||^2 + \mu \Delta t ||\nabla \widetilde{y_n^h}||^2 \leq \Delta t \Big(K_1 ||f^n||^2 + K_2 ||g^n||^2 + K_3 \Big|\Big| \frac{g^n - g^{n-1}}{\Delta t} \Big|\Big|^2 \Big) + ||\widetilde{y_{n-1}^h}||^2 \tag{152}$$

and by adding to (152) the term $\mu \Delta t ||\nabla \widehat{g_n^h}||^2$, and using the inequality (147), we get ($K_4 = K_2 + \frac{1}{\beta^2}$)

$$||\widetilde{y_n^h}||^2 + \frac{\mu \Delta t}{2} ||\nabla y_n^h||^2 \leq \Delta t \Big(K_1 ||f^n||^2 + K_4 ||g^n||^2 + K_3 \Big|\Big| \frac{g^n - g^{n-1}}{\Delta t} \Big|\Big|^2 \Big)$$

Summing up from $n = 0$ to $n = N_t$, and using the telescopic sum we have

$$||\widetilde{y_{N_t}^h}||^2 + \sum_{n=1}^{N_t} \frac{\mu \Delta t}{2} ||\nabla y_n^h||^2 \leq \sum_{n=1}^{N_t} \Delta t \Big(K_1 ||f^n||^2 + K_4 ||g^n||^2 + K_3 \Big|\Big| \frac{g^n - g^{n-1}}{\Delta t} \Big|\Big|^2 \Big) + ||\widetilde{y_0^h}||^2 \tag{153}$$

Now since $g \in L^2(L^2(\Gamma_1)), g' \in L^2(L^2(\Gamma_1))$ and $f \in L^2(L^2(\widetilde{\Omega}))$ the Riemann sums on the right hand side is bounded independently of Δt and arrive at

$$\sum_{n=1}^{N_t} \Delta t ||\nabla y_n^h||^2 \leq K \tag{154}$$

and by the Poincaré inequality also

$$\sum_{n=1}^{N_t} \Delta t ||y_n^h||^2 \leq C_p K \tag{155}$$

□

5.3 First Discretize then Optimize

In this section we will study the method discretize then optimize for the minimization problem at hand.
This method consists in first discretizing all the terms in the PDE and the cost functional, determining a gradient for the discrete minimizing problem, and then apply a descent method to find a possible minimum for the cost. There are other processes to find a possible minimum for the discrete functional cost, but here we will only focus on the descent method.

Let $\Gamma_c \subset \Gamma = \partial\Omega$ be the boundary segment where we do the control, and let $\Sigma_c = \Gamma_c \times [0, T]$ be the lateral boundary. We denote by $\Sigma_c^{h_t,h_x}$ (also denoted by Σ_c^h for short) the discretization of the lateral control boundary Σ_c.

$\Sigma_c^{h_t,h_x}$ is composed by a collection of $N_t + 1$ sets of N_c nodes, being N_t the number of time intervals, and N_c the number of boundary nodes in Γ_c. For example, in figure (5) we considered 3 time intervals, or equivalently 4 times states, given by the number of lines in the figure, (which gives $N_t = 3$), and for the space discretization we considered $N_c = 4$ nodes for each time state, given by the number of red nodes. This gives a total of $N_t \times N_c = 12$ parameters which we desire to control.

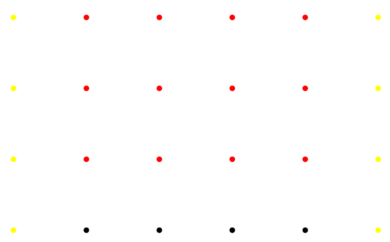


Figure 5: Example time boundary discretization

Observation: Notice that we do not consider the nodes in the lateral boundary, given by the orange nodes, since these, already make part of the Γ_w^h (boundary with the no-slip condition), and therefore are fixed with the value zero. Also the nodes in the initial time ($t = 0$) given by the color black, are also not considered in the control since they are fixed by the initial condition imposed on the problem.

Now the controls will be given by tensors in $\mathbb{R}^{(N_t)\times N_c \times 2}$, the number 2 comes from the number of directions for the vector velocities at each node. The state solution is composed by the velocity $y^h \in \mathbb{R}^{(N_t+1)\times N_x \times 2}$ and the pressure $\pi^h \in \mathbb{R}^{(N_t+1)\times N_p}$, where N_x is the number of velocities nodes and N_p is the number of pressure nodes.

Given an initial condition u_0^h, we define the Stokes solver $\mathscr{S}_h^{u_0^h} : U^h \to V_{dt}^h$, by the operator that for every control $u^h \in U^h$ gives the velocity solution y^h which is the solution for the Implicit Euler scheme.

We use the cost functional (in this case the velocity tracking cost)

$$J(u^h) = \frac{\tau}{2}\int_0^T \int_{\Gamma_c} ||\nabla u^h||^2_{\Gamma_c} dsdt + \frac{\alpha}{2}\int_0^T \int_{\widetilde{\Omega}} ||S_h^{u_0^h}(u^h) - z_d||^2 dxdt$$

which can be discretized to

$$J^h(u^h) = \frac{\tau}{2}\sum_{n=1}^{N_t} \int_{\Gamma_c} ||\nabla u^h||^2_{\Gamma_c} ds + \frac{\alpha}{2}\sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} ||S_h^{u_0^h}(u^h) - z_d||^2 dxdt$$

where the time integrals were approximated by a trapezoidal rule.

We recall that the stokes solver is determined by the following collection of systems of equations

$$S_h^{y_0^h}(u^h) = y^h \textit{ such that } \begin{cases} y_0^h = \Pi^h(y_0) \\ \int_\Omega (\frac{y_n^h - y_{n-1}^h}{dt}) \cdot \varphi + \mu \int_\Omega \nabla y_n^h : \nabla \varphi - \int_\Omega \pi_n^h (\nabla \cdot \varphi) = 0, \forall \varphi \in V_{000}^h \\ \int_\Omega (\nabla \cdot y_n^h) q = 0 \text{ , } \forall q \in P_h \\ y_n^h = g^n \text{ in } \widetilde{\Gamma^{in}} \\ y_n^h = u_n^h \text{ in } \widetilde{\Gamma^{c}} \\ y_n^h = 0 \text{ in } \widetilde{\Gamma^{w}} \end{cases}$$

which is equivalent to determine in each time step the solution (y_n^h, π_n^h) for the linear system

$$\begin{bmatrix} \frac{\widetilde{M}}{dt} + \widetilde{A} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} y_n^h \\ \pi_n^h \end{bmatrix} = \begin{bmatrix} \frac{\widetilde{M}}{dt} y_{n-1}^h \\ 0 \end{bmatrix} \tag{156}$$

where the block matrices were defined in previews subsection. To solve this system we only used a pressure stabilizer, by transforming the Stokes system matrix to

$$\begin{bmatrix} \frac{\widetilde{M}}{dt} + \widetilde{A} & B^T \\ B & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{\widetilde{M}}{dt} + \widetilde{A} & B^T \\ B & C \end{bmatrix}$$

where

$$C_{i,j}^{N_p,N_p} = \int_\Omega \epsilon \phi_i^p(x,y) \phi_j^p(x,y)$$

Let us now differentiate the cost functional with respect to the control. Since in the derivative will appear the Fréchet derivative of the Stokes solver we start by calculate that.
Notice that $\mathscr{S}_h^{u_0^h}(u^h) = \mathscr{S}_h^0(u^h) + \mathscr{S}_h^{u_0^h}(0)$. In fact, if $\mathscr{S}_h^0(u^h) = z^h$ and $\mathscr{S}^{u_0^h}(0) = v^h$ then,

$$\begin{cases} z_0^h = 0 \\ \displaystyle\int_{\widetilde{\Omega}} \frac{1}{\Delta t}(z_n^h - z_{n-1}^h)\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla z_n^h : \nabla \varphi^h - \int_{\widetilde{\Omega}} p_n^h(\nabla \cdot \varphi^h) = 0 \text{ for all } \varphi^h \in V_{0h} \\ \displaystyle\int_{\widetilde{\Omega}} (\nabla \cdot z_n^h) q^h = 0 \text{ for all } q^h \in P^h \\ z_n^h = u_n^h \text{ in } \widetilde{\Gamma_c^h} \\ z_n^h = 0 \text{ in } \widetilde{\Gamma_{in}^h} \cup \widetilde{\Gamma_D^h} \end{cases}$$

$$\begin{cases} v_0^h = u_0^h \\ \displaystyle\int_{\widetilde{\Omega}} \frac{1}{\Delta t}(v_n^h - v_{n-1}^h)\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla v_n^h : \nabla \varphi^h - \int_{\widetilde{\Omega}} \xi_n^h(\nabla \cdot \varphi^h) = \int_{\widetilde{\Omega}} f^n \varphi^h \text{ for all } \varphi^h \in V_{0h} \\ \displaystyle\int_{\widetilde{\Omega}} (\nabla \cdot v_n^h) q^h = 0 \text{ for all } q^h \in P^h \\ v_n^h = 0 \text{ in } \widetilde{\Gamma_D^h} \cup \widetilde{\Gamma_c^h} \\ v_n^h = g_{in}^g \text{ in } \widetilde{\Gamma_{in}^h} \end{cases}$$

then the sum satisfies the scheme of the Stokes solver.
The operator $\mathscr{S}_h^0$ is the linear part of $\mathscr{S}_h^{u_0^h}$ and therefore, since is also bounded, is the Fréchet derivative of the Stokes solver.
Now, let $v^h \in U^h$ be a control direction. then the derivative of the cost functional is given by[29]:

$$(J_1'^h(u^h), v^h) = \tau \Delta t \sum_{n=1}^{N_t} \int_{\Gamma_c} \nabla u_n^h : \nabla v_n^h + \alpha \Delta t \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \big(\mathscr{S}_h^{u_0^h}(u^h)_n - z_n^d\big)\mathscr{S}_h^0(v^h)_n$$

$$(J_2'^h(u^h), v^h) = \tau \Delta t \sum_{n=1}^{N_t} \int_{\Gamma_c} \nabla u_n^h : \nabla v_n^h + \alpha \Delta t \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \big(\nabla \times \mathscr{S}_h^{u_0^h}(u^h)_n\big)\big(\nabla \times \mathscr{S}_h^0(v^h)_n\big)$$

$$(J_3'^h(u^h), v^h) = \tau \Delta t \sum_{n=1}^{N_t} \int_{\Gamma_c} \nabla u_n^h : \nabla v_n^h + \alpha \int_{\widetilde{\Omega}} \big(\mathscr{S}_h^{u_0^h}(u^h)_{N_t} - z^d\big)\mathscr{S}_h^0(v^h)_{N_t}$$

Now we do the analysis for the type J_1^h but the other are completely analogous. Our goal is to calculate the gradient of this discrete cost.

To simplify the notation, given a direction δv^h for the derivative, we will denote the solution to the linearized Stokes solver $\mathscr{S}_h^0(\delta v^h)$ by δy^h, which is given by

$$\begin{cases} \delta y_0^h = 0 \\ \displaystyle\int_{\widetilde{\Omega}} \frac{1}{\Delta t}(\delta y^h n - \delta y_{n-1}^h)\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla \delta y_n^h : \nabla \varphi^h - \int_{\widetilde{\Omega}} \delta p_n^h(\nabla \cdot \varphi^h) = 0 \text{ for all } \varphi^h \in V_{00h} \\ \displaystyle\int_{\widetilde{\Omega}} (\nabla \cdot \delta y_n^h) q^h = 0 \text{ for all } q^h \in P^h \\ \delta y_n^h = \delta v_n^h \text{ in } \widetilde{\Gamma_c^h} \\ \delta y_n^h = 0 \text{ in } \widetilde{\Gamma_w^h} \cup \widetilde{\Gamma_{in}^h} \end{cases} \tag{157}$$

Now suppose given a function $z^h \in V_{dt}^{00h}$ we will use for every $n \in \{1, ..., N_t\}$ the function z_n^h as a test function for the equation (157-2), to obtain for each $n \in \{1, ..., N_t\}$

$$\int_{\widetilde{\Omega}} \frac{1}{\Delta t}(\delta y^h n - \delta y_{n-1}^h) z^h + \mu \int_{\widetilde{\Omega}} \nabla \delta y_n^h : \nabla z^h - \int_{\widetilde{\Omega}} \delta p_n^h(\nabla \cdot z^h) = 0$$

[29]the gradient appearing in the boundary integral is the tangential gradient

By doing partial summation and using the fact that $\delta y_0^h = 0$ we get

$$\sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} z_n^h(\delta y_n^h - \delta_{n-1}^h) = \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \delta y_n^h(z_n^h - z_{n+1}^h) + \int_{\widetilde{\Omega}} \delta y_{N_t}^h z_{N_t+1}^h$$

where we also needed to introduce the element $z_{N_t+1}^h \in V_{00h}$.

Now by using the decomposition (142) we have that, for each $n \in \{1, ...N_t\}$, $\delta y_n^h = \widetilde{\delta y_n^h} + \widehat{\delta y_n^h}$ with $\widetilde{\delta y_n^h} \in \mathscr{M}_h$ and $\widehat{\delta y_n^h} \in V_{000h}$.
Therefore,

$$\sum_{n=1}^{N_t} \frac{1}{\Delta t} \int_{\widetilde{\Omega}} \widehat{\delta y_n^h}(z_n^h - z_{n+1}^h) + \int_{\widetilde{\Omega}} \frac{1}{\Delta t} \widehat{\delta y_{N_t}^h} z_{N_t+1}^h + \sum_{n=1}^{N_t} \mu \int_{\widetilde{\Omega}} \nabla \widehat{\delta y_n^h} : \nabla z_n^h - \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \delta p_n^h (\nabla \cdot z_n^h)$$
$$= -\sum_{n=1}^{N_t} \frac{1}{\Delta t} \int_{\widetilde{\Omega}} \widetilde{\delta y_n^h}(z_n^h - z_{n+1}^h) - \int_{\widetilde{\Omega}} \frac{1}{\Delta t} \widetilde{\delta y_{N_t}^h} z_{N_t+1}^h + \sum_{n=1}^{N_t} \mu \int_{\widetilde{\Omega}} \nabla \widetilde{\delta y_n^h} : \nabla z_n^h$$

Let $z^h \in V_{dt}^{000h}$ but with time interval analysis $[dt, T+dt]$, satisfy the following system of equations,

$$\begin{cases} z_{N_t+1}^h = 0 \\ \displaystyle\int_{\widetilde{\Omega}} \frac{1}{\Delta t}(z^h n - z_{n+1}^h)\varphi^h + \mu \int_{\widetilde{\Omega}} \nabla z_n^h : \nabla \varphi^h - \int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \varphi^h) = \alpha \int_{\widetilde{\Omega}} (y_n^h - z_d^n)\varphi \text{ for all } \varphi \in V_{000h} \\ \displaystyle\int_{\widetilde{\Omega}} (\nabla \cdot z_n^h) q = 0 \text{ for all } q \in P^h \end{cases} \tag{158}$$

where $(z_n^h, \sigma_n^h) \in V_{000h} \times P^h$ for each $n \in \{1, ..., N_t\}$. Th right hand side of (158-2) changes with the cost functional. For example, we also may have

$$\alpha \int_{\widetilde{\Omega}} (\nabla \times y_n^h)(\nabla \times \varphi) \text{ , } n \in \{1, .., N_t\} \text{ if it is the vorticity cost functional}$$

$$\begin{cases} 0 \\ \alpha \displaystyle\int_{\widetilde{\Omega}} (y_{N_t}^h - z_d)\varphi \end{cases} \quad \text{if it is the final state functional}$$

The solution z^h of (158) is called the discrete adjoint state. By using the adjoint z^h in the linearized equation we get , (notice that $\int_{\widetilde{\Omega}} (\nabla \cdot z_n^h) p^h = 0$ for all $p^h \in P^h$)

$$\sum_{n=1}^{N_t} \frac{1}{\Delta t} \int_{\widetilde{\Omega}} \widehat{\delta y_n^h}(z_n^h - z_{n+1}^h) + \sum_{n=1}^{N_t} \mu \int_{\widetilde{\Omega}} \nabla \widehat{\delta y_n^h} : \nabla z_n^h$$
$$= -\sum_{n=1}^{N_t} \frac{1}{\Delta t} \int_{\widetilde{\Omega}} \widetilde{\delta y_n^h}(z_n^h - z_{n+1}^h) - \int_{\widetilde{\Omega}} \frac{1}{\Delta t} \widetilde{\delta y_{N_t}^h} z_{N_t+1}^h + \sum_{n=1}^{N_t} \mu \int_{\widetilde{\Omega}} \nabla \widetilde{\delta y_n^h} : \nabla z_n^h$$

but since $\widehat{\delta y_n^h} \in V_{000h}$ the adjoint equation is satisfied by using $\widehat{\delta y_n^h}$ as a test function and we arrive at,

$$\sum_{n=1}^{N_t} \alpha \int_{\widetilde{\Omega}} (y_n^h - z_d^n) \widehat{\delta y_n^h} + \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \widehat{\delta y_n^h}) = -\sum_{n=1}^{N_t} \frac{1}{\Delta t} \int_{\widetilde{\Omega}} \widetilde{\delta y_n^h}(z_n^h - z_{n+1}^h) - \int_{\widetilde{\Omega}} \frac{1}{\Delta t} \widetilde{\delta y_{N_t}^h} z_{N_t+1}^h + \sum_{n=1}^{N_t} \mu \int_{\widetilde{\Omega}} \nabla \widetilde{\delta y_n^h} : \nabla z_n^h \tag{159}$$

Also notice that

$$0 = \int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \delta y_n^h) \implies \int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \widehat{\delta y_n^h}) = -\int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \widetilde{\delta y_n^h}) \tag{160}$$

and that

$$\int_{\widetilde{\Omega}} (y_n^h - z_d^n) \delta y_n^h = \int_{\widetilde{\Omega}} (y_n^h - z_d^n) \widehat{\delta y_n^h} + \int_{\widetilde{\Omega}} (y_n^h - z_d^n) \widetilde{\delta y_n^h} \tag{161}$$

By using (159),(160) and (161) we get,

$$\langle \partial J^h(u^h), \delta v^h \rangle = \tau \Delta t \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \nabla u^h : \nabla \delta v^h + \sum_{n=1}^{N_t} \Delta t \Big(\int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \widetilde{\delta_n^h}) - \mu \int_{\widetilde{\Omega}} \nabla z_n^h :_\nabla \widetilde{\delta y_n^h} + \alpha \int_{\widetilde{\Omega}} (y_n^h - z_n^d) \widetilde{\delta y_n^h} \Big) - \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} (z_n^h - z_{n-1}^h) \widetilde{\delta y_n^h}$$

the above expression leads us to an expression for the gradient of the discrete cost J^h at a point u^h. However this representation is not easy when we see the above calculations in the functional sense, but they become much more easy in the matrices sense. To transform to the matricial form, notice that the space U^h is of finite dimension[30] and therefore is isomorphic to a subspace of $\mathbb{R}^{N_t \times 2N_c}$, where the components of this vectors have a particular order, in fact the, given a function $u^h \in u^h$ (see footnote 31) the coefficients are mapped to a tensor

$$u^h \mapsto [YY]_{n=1,i=1}^{N_t,2N_c} \qquad YY_{n=\tilde{n},i=\tilde{i}} = \alpha_{\tilde{i}}^{\tilde{n}}$$

and the vector $u \in \mathbb{R}^{N_t \times 2N_t}$ is given by the vertical concatenation of the lines of YY in such a way that, to simplify, we will denote $u_k = YY_{n=k, i \in I_c}$ the k-th block of the concatenation, which is related to the solution at the time $t_k = k \times \Delta t$. In $\mathbb{R}^{N_t \times 2N_c}$ we introduce the inner product[31]

$$\langle v, u \rangle = \Delta t \sum_{n=1}^{N_t} v_n^T M_{\Gamma_c} u_n \text{ for all } u, v \in \mathbb{R}^{N_t \times N_c \times 2}$$

where $(\cdot,\cdot)_2$ denotes the usual $\mathbb{R}^N$ inner product. Now we transform the gradient expression to obtain a Riezs representative for the gradient, with respect to inner product in $\mathbb{R}^{N_t \times 2N_c}$, which is the scalar product in the control space.

$$\begin{aligned}\langle \partial J^h(u^h), \delta v^h \rangle =& \tau \Delta t \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} \nabla u^h : \nabla \delta v^h + \sum_{n=1}^{N_t} \Delta t \Big(\int_{\widetilde{\Omega}} \sigma_n^h (\nabla \cdot \widetilde{\delta_n^h}) - \mu \int_{\widetilde{\Omega}} \nabla z_n^h :_\nabla \widetilde{\delta y_n^h} + \alpha \int_{\widetilde{\Omega}} (y_n^h - z_n^d) \widetilde{\delta y_n^h} \Big) - \\ & \sum_{n=1}^{N_t} \int_{\widetilde{\Omega}} (z_n^h - z_{n-1}^h) \widetilde{\delta y_n^h} \\ &= \tau \Delta t \sum_{n=1}^{N_t} (u_n^h)^T M_{\Gamma_c} (\delta v_n^h) \\ &+ \Delta t \sum_{n=1}^{N_t} \Big(B^T \sigma_n^h - \mu A z_n^h - M(z_n^h - z_{n+1}^h) + \alpha M (y_n^h - z_d^n) \Big)_{i \in I_c} (\delta v_n^h)\end{aligned}$$

where for a vector $L \in \mathbb{R}^{N_t \times 2N_v}$ we denoted by $(L)_{i \in I_c}$ the vector composed by the components of the original L which have indexes $i \in I_c$.
Now it is easy to see that a representation for the gradient is given by the vector

$$R(\nabla J^h(u^h))_n = \tau u_n^h + M_{\Gamma_c}^{-1} \Big(B^T \sigma_n^h - \mu A z_n^h - M(z_n^h - z_{n+1}^h) + \alpha M (y_n^h - z_d^n) \Big)_{i \in I_c} \text{ for } n \in 1, ..., N_t$$

This leads us to the following descent method for the purpose of finding the solutions to the discrete minimum problem,

[30] Every function in $u^h \in U^h$ assumes a representation

$$u^h(t,x,y) = \sum_{n=1}^{N_t} \pi^t(t) \sum_{i \in I_c} \alpha_i^n \boldsymbol{\phi_i}(x,y)$$

[31] it is easy to see that the application is in fact an inner product, since the matrix M_{Γ_c} is positive definite.

Algorithm:

1. Set $m = 0$

(i) Define an initial control $(u^h)_0$

(ii) Define the solution to the state equation$(y^h)_0$

(iii) calculate the initial cost $J^h((u^h)_0)$

2. Minimization Loop, set $m = m + 1$

(iv) Solve the adjoint equation to get $(z^h, \sigma^h)_m$ by using the state equation $(y^h)_{m-1}$

(v) Determine the gradient $J^h((u^h)_m)$ by using the expression (...)

(vi) Step size Loop (Armijo Rule) $k \in \{1, ..., 10\}$

$(vi).(1)$ Define the candidate to new control $\widetilde{(u^h)}_{m+1} = (u^h)_m - (2.0)^{(-k)} \nabla J^h((u^h)_m)$

$(vi).(2)$ Determine the new state solution $(y^h)_{m+1}$

$(vi).(3)$ if $J^h(\widetilde{(u^h)}_{m+1}) \leq J^h(u^h_m)$ set $(u^h)_{m+1} = \widetilde{(u^h)}_{m+1}$ otherwise go to (vi).(1) and set $k = k + 1$

(vii) Stop criterion test if $J^h((u^h)_{m+1}) - J^h((u^h)_m) < \epsilon^{tol}$ stop; otherwise go to (iv)

Lastly, we restricted our simulation to the observation on a small sub domain $\omega \subset \Omega$. In this case the discrete functional will be given by

$$J^h_\omega(u^h) = \frac{\tau}{2} \sum_{n=1}^{N_t} \Delta t \int_{\Gamma_c} |\nabla u^h|^2 ds + \frac{\alpha}{2} \sum_{n=1}^{N_t} \int_\omega |y^h - z^h|^2 dx \tag{162}$$

The purpose will be to see the influence of the ω choice, in the rate of convergence in the minimizing process (in the case it converges). In practice we are interested in evaluating the influence when ω is in the following positions

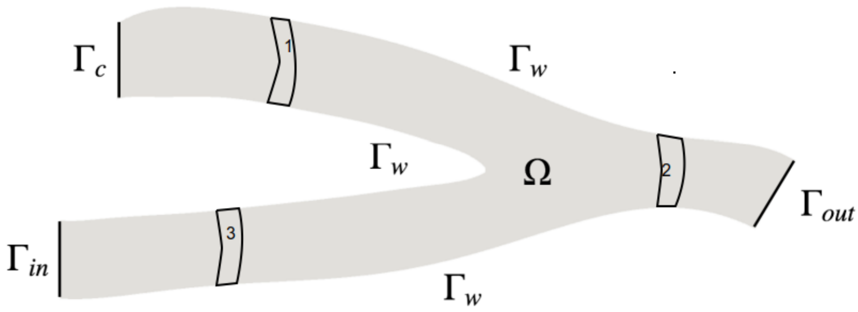


Figure 6: positions.

Our intuition says that when ω assumes the position 1, the minimization process still must work, in position 2 may work and that lastly in the case 3 does not work.

6 Numerical Results

In this section we give examples of application of the above numerical method. Our main goal is to obtain a minimization process, for the cost functional J, which can be a velocity tracking functional or a vorticity functional, in the bifurcation domain (see image 1) and analyze the influence of the observations domains and the cost parameters τ, α in the efficiency of the minimization process.

Before doing our main purpose, we show the application for a collection of simple problems, with an increase on complexity, where we know what to expect the solution to be, by using the technique of manufactured solutions. This technique consists on defining analytic solutions, in our case set the target velocity and the respective pressure, and then define the right hand side such that they all combined satisfy the non-stationary Stokes equations. This first step it is useful to test the behavior of the numerical method implemented.

In the following, the PDE is on a rectangular domain Ω, $\Gamma = \partial\Omega$ will be divided by Γ_1, Γ_3 being the bottom and top of the rectangular, respectively and Γ_2, Γ_4 the right and left sides respectively.

In our first test, and also the simpler, the target flow is given by the stationary velocity field, which is a type of a Poiseuille flow, $Z_d(x, y) = \big(y(1-y), 0\big)$ and the corresponding pressure is set to be $p(x, y) = -2x + 4$. Notice that, both the velocity and pressure are independent of time, and therefore in particular $\frac{\partial Z_(x,y)}{\partial t} = 0$. This velocity field and pressure combined, satisfy the following Stokes system (notice that $\nu = 1$ by simplification)

$$(Test-1)\begin{cases} -\Delta Z_d + \nabla p = 0 \text{ in } \Omega \times (0,1) \\ \nabla \cdot Z_d = 0 \text{ in } \Omega \times (0,1) \\ Z_d = 0 \text{ in } \Gamma_{1,3} \times (0,1) \\ Z_d = \big(y(1-y), 0\big) \text{ in } \Gamma_4 \times (0,1) \\ \dfrac{\partial Z_d}{\partial \boldsymbol{n}} - p\boldsymbol{n} = 0 \text{ in } \Gamma_2 \times (0,1) \\ u_0 = \big(y(1-y), 0\big) \text{ in } \Omega \end{cases} \tag{163}$$

The second test consists in introducing to the above Poiseuille flow, a time dependency. This can be done by, for instance, defining the target flow as $Z_d(x, y, t) = \big(\sin(\pi t)y(1-y), 0\big)$ and the corresponding pressure to be $p(x, y, t) = \sin(\pi t)(-2x + 4)$. In this case we have to solve the problem

$$(Test-2)\begin{cases} \dfrac{\partial Z_d}{\partial t} - \Delta Z_d + \nabla p = \pi\cos(\pi t)\big(y(1-y), 0\big) \text{ in } \Omega \times (0,1) \\ \nabla \cdot Z_d = 0 \text{ in } \Omega \times (0,1) \\ Z_d = 0 \text{ in } \Gamma_{1,3} \times (0,1) \\ Z_d = \sin(\pi t)\big(y(1-y), 0\big) \text{ in } \Gamma_4 \times (0,1) \\ \dfrac{\partial Z_d}{\partial \boldsymbol{n}} - p\boldsymbol{n} = 0 \text{ in } \Gamma_2 \times (0,1) \\ u_0 = \big(y(1-y), 0\big) \text{ in } \Omega \end{cases} \tag{164}$$

In the last example we considered,

$$(Test-3)\begin{cases} \dfrac{\partial Z_d}{\partial t} - \Delta Z_d + \nabla p = F \text{ in } \Omega \times (0,T) \\ \nabla \cdot Z_d = 0 \text{ in } \Omega \times (0,T) \\ Z_d = 0 \text{ in } \Gamma_{1,3} \times (0,T) \\ Z_d = \sin(\pi t)\big(y(1-y), 0\big) \text{ in } \Gamma_4 \times (0,T) \\ \dfrac{\partial Z_d}{\partial \boldsymbol{n}} - p\boldsymbol{n} = 0 \text{ in } \Gamma_2 \times (0,T) \\ u_0 = \big(y(1-y), 0\big) \text{ in } \Omega \end{cases} \tag{165}$$

where in this case $\Omega = [0.1, 0.5] \times [0, 2]$, $T = 2$ and the field F is given by

$$\text{F(x,y,t)=}\begin{bmatrix} \sin(\pi t)\pi^2(\cos(\pi x)\sin(\pi y) - \sin(\pi y)) + \pi\cos(\pi t) * (\cos(\pi x)\sin(\pi y) - \sin(\pi y)) \\ \sin(\pi t)\pi^2(-3 * \cos(\pi y)\sin(\pi x) + \sin(\pi x)) + \pi\cos(\pi t)(-\cos(\pi y)\sin(\pi x) + \sin(\pi x)) \end{bmatrix}$$

In the third problem the target flow is $Z_d(x, y, t) = \sin(\pi t)\Big(\cos(\pi x)\sin(\pi y) - \sin(\pi y), -\cos(\pi y)\sin(\pi x) + \sin(\pi x)\Big)$ and the corresponding pressure $p(x, y, t) = \sin(\pi t)\big(-\sin(\pi x)\sin(\pi y)\pi\big)$. In this work we were not concerned in analyzing the numerical method (which is the implicit Euler scheme) to solve the Stokes system in the non-stationary case. However, is good practice to see that in fact this first step of the minimizing process is well constructed, and so, in order to see that the Euler scheme is sufficiently good for the problem at hand, we in the test 3 (the more complex one) show the errors committed in the numerical approximation of the target velocity. In table (7), we show the errors $||Z_d^h - Z_d||_{L^2(\Omega)}$ at the final time $t = T$, and in table (8) we present the approximation errors in the and maximum norm of the collection $||(Z_d^h)_n - (Z_d)_n||_{L^2(\Omega)}$. The errors committed in the numerical approximations are sufficient low for our purpose. It is also possible to see that in this case seems that a refinement in the time step has more impact than a refinement on the space mesh.

h	Error
0.2	0.00990016
0.1	0.00667197
0.05	0.00650449
0.025	0.00649772
0.0125	0.00649747

Figure 7: Space discretization error.

Let us now turn our attention to the minimization process results. In what follows we will always consider that the maximum number of iteration is 50 and that the time discretization is $dt = 0.05$, if is not said nothing more. Also the minimization algorithm may stop in 4 cases: by reaching the maximum number of iterations, by the relative error criterion, by the step-size criterion, or finally by the small decreasing criterion. The latter 3 stopping criterion consist in stopping if

(i)relative error criterion: the following relative error is smaller then 0.01

$$\sum_{n=1}^{N_t} \frac{dt}{2} ||u_n^h - (z_d^h)_n||^2 < 0.01$$

(ii) The step-size routine reaches the maximum value of iterations and even in that case we do not see any decrease in the cost functional value.

(iii) If the difference $J^h\big((u^h)^{n+1}\big) - J\big((u^h)^n\big) < \epsilon_{tol}$

As said before, we start by analyzing the numerical method in the square domain with the Test-1. For the cost functional to minimize we choose the velocity tracking

$$J^h(u^h, v^h) = \frac{\tau}{2} \int_{\Gamma_4} |\nabla v^h|^2 + \alpha \int_{\Omega} |u^h - Z_d^h|^2$$

where in this and the following cases we will denote by u^h the state equation with respect to the boundary value (control) v^h. In figure (9) we can observe the evolution of the absolute error $||u_n^h - (Z_d^h)_n||_{L^2}$ along time. In (9) with fixed $\alpha = 3$ we observe that the optimal solution strongly depends on the parameter τ. This is expected since we increase the relative importance of the term $||u_n^h - (Z_d^h)_n||_{L^2}$ in the cost functional, when we decrease τ, leading the optimal solution to the target flow. From (9) we may extrapolate that α needs to be $1000\times$ higher than τ to one get a optimal solution obtained by the stopping criterion of the relative error, and not from the criterion of the maximum number of iterations. It is common to, in this type of minimization problems, consider values of τ much lower then the ones choosed for α.

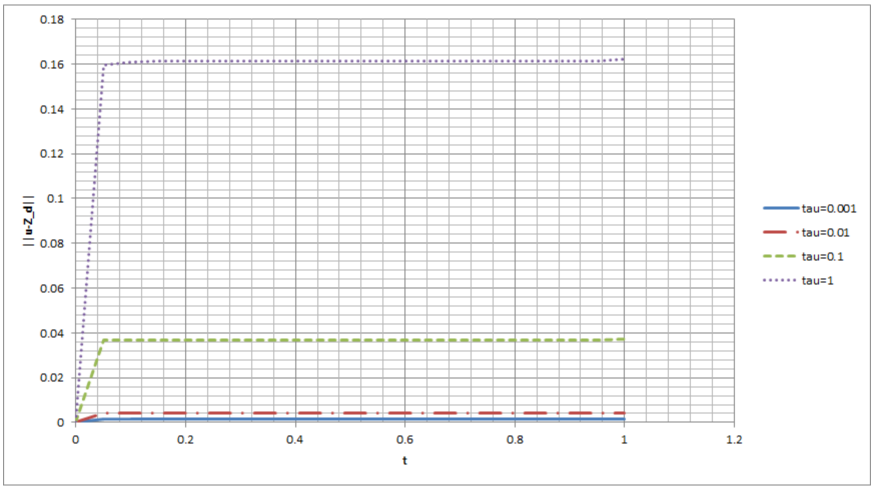
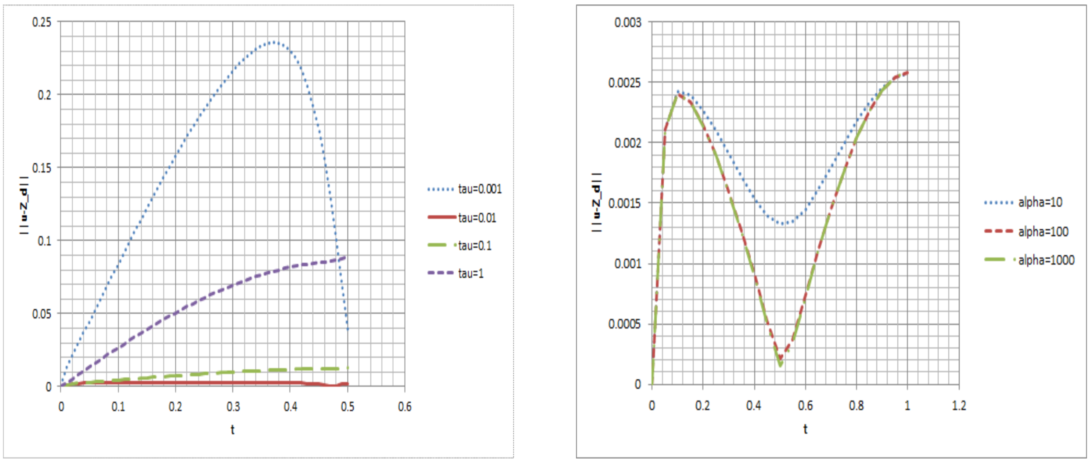


Figure 9: With fixed $\alpha = 3$, varying τ.

On the case of the Test-2, when we introduce time we increase the difficulty and even in this simple case we start to obtain some results which are contrary to some obtain in the stationary case. For example in image (10a) we see that for the same α, in general when we decrease τ we obtain optimal solutions which are more closer to what we expect, which is the known target flow. However, for the case $\tau = 0.001$ the algorithm stops by the maximum iteration number criterion, not having time to converge to the expected solution.
Also we observed that, in the case a bigger time interval, in order to have a change of growth of the target flow, we see that this change makes the optimal solution to have some difficulties to follow this change.

dt	Error
0.5	0.0251171
0.1	0.00577978
0.05	0.00137282
0.01	0.000177682
0.005	0.000134537

Figure 8: Time discretization error.



(a) With fixed $\alpha = 10$, varying τ.

(b) With fixed $\tau = 0.01$, varying α.

Figure 10: Comparison of the numerical solution with the target flow.

Now we turn to our main goal, doing the minimization on the bifurcation. In this work, our domain has the form of the set in image (6), with $x = 0.09$ and $y \in [0.5143, 0.6086]$ for Γ_c, $x = 0.0021$ and $y \in [0.2682, 0.3649]$ for Γ_{in} and lastly $x \in [0.87.92], y \in [0.0.35, 0.43]$ for Γ_{out}. Also this domain is discretized in our experiments in such a way that we have 3616 triangles, 7541 velocity nodes (counting the ones on the Dirichlet boundary) and 1963 pressure nodes.
Similarly to [24] here we consider two different scenarios being, (i) a total occlusion of the host artery which is done by imposing $v_{in} = 0$ in Γ_{in}, and (ii) the presence of a residual blood flow in

which case we have $v_{in} \neq 0$ in Γ_{in}. In the second case we will consider the following type of input functions

$$v_{in}(y) = c\exp(-(y-\overline{y})^2/(2b^2))$$

where c is a positive constant, $\overline{y}$ is the mid point of Γ_{in} and $b = 0.01$. This type of functions can modulate a strong obstruction but not total. Also notice that this type of functions do not have zero trace, but in numerical terms their value on $\partial\Gamma_{in}$ is so small that the computer makes it zero.

Starting with the case $v_{in} = 0$, we defined the target flow in Ω to be the Stokes solution with a parabolic inflow, given by[32] $-10000(y-0.6086)(y-0.5143)$ which leads to the numerical solution in (12a).

	α=10		α=100		α=1000	
Initial cost	32.4038		324.038		3240.38	
iteration	cost	Rel.error	Cost	Rel.error	Cost	Rel. error
1	32.0107	0.993929	285.922	0.939285	626.925	0.392895
2	31.6968	0.988501	259.29	0.888334	450.765	0.215586
3	31.4461	0.983649	240.682	0.845575	438.648	0.163479
4	31.2459	0.979312	227.681	0.809692	437.795	0.148082
5	31.0861	0.975436	218.596	0.779577	437.733	0.143508
6	30.9585	0.97197	212.248	0.754305	437.729	0.142144
7	30.8566	0.968872	207.812	0.733095	437.728	0.141735
8	30.7752	0.966102	204.713	0.715295	--	--
9	30.7102	0.963627	202.547	0.700356	--	--
10	30.6583	0.961414	201.033	0.687819	--	--

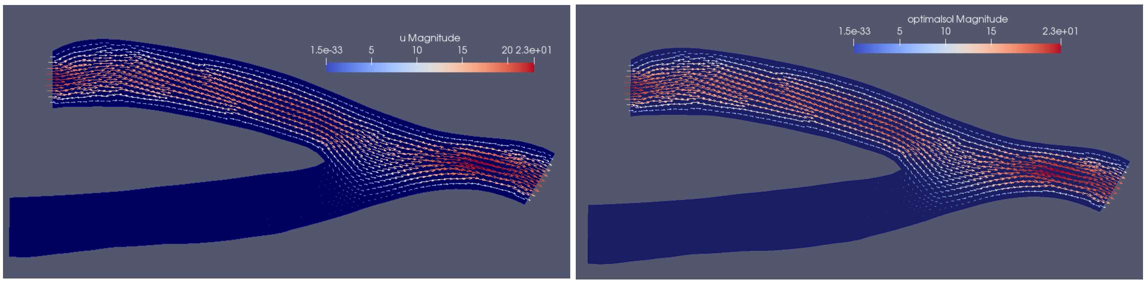
Figure 11: Costs and relative errors, with $\tau = 0.1$ and α varying.

In the table (11) we show the value of the cost functional and the relative error for the first 10 iterations of the minimization process. In this process we consider $dt = 0.05$, $\tau = 0.1$ and the initial guess as the null control $(v^h)_0 = 0$, which yields the initial state equation to be also null.
It is possible to observe that in fact the minimization algorithm works, since in each iteration we obtain a cost smaller then the previews one.
In the case of $\alpha = 1000$, we see that the algorithm stops at the 7-th iteration, but not because the target flow was reached, but because the difference between two consecutive iterations was smaller than 0.0001. In particular we can see that in the case of this alpha the optimal solution is not near the target flow, being however another solution which is near a stationary point of the cost functional.

In the cases of $\alpha = 10, 100$, the algorithm stops at iterations 37, 36 respectively. By the observation of table (11) we could argue that for this $\alpha's$ the optimization method is converging to the desired velocity, since the relative error is always decreasing. However when the process stops we saw that for $\alpha = 10$ the relative error was near 0.94 and for $\alpha = 100$ was near 0.6 being the descent method, in fact, converging to a minimizer which does not correspond to the target flow. It is also possible to observe, that in the first iterations we have a strong decrease in the cost which starts to fade. This is a typical characteristic of the steepest descent method. We also observed that the optimal solutions are given for each of this α cases, by a parabolic flow but with different maximum values for the parabola, being the highest value obtained in the case of the highest α, what makes sense, since in that case we are giving a minor importance to the boundary cost functional parameter.

In image (12) we represent the target flow (left) and the optimal solution (right) for the case of $\alpha = 1000$, $\tau = 0.01$, and stopping in 9 iterations. This choice of parameters appears to be sufficient to one obtain a very precise approximation of the target flow, with a relative error under 0.01.

[32]We multiply the boundary function by the term 10000 only to have bigger values for the cost functional.



(a) Target flow.

(b) Optimal solution.

Figure 12: Comparison of the numerical solution with the target flow, with $\alpha = 1000$ and $\tau = 0.01$.

For the second case, with $v_{in} \neq 0$ we considered the function

$$v_{in}(y) = 40 \exp\big(\frac{-(y-0.3165)^2}{2(0.01)^2}\big)$$

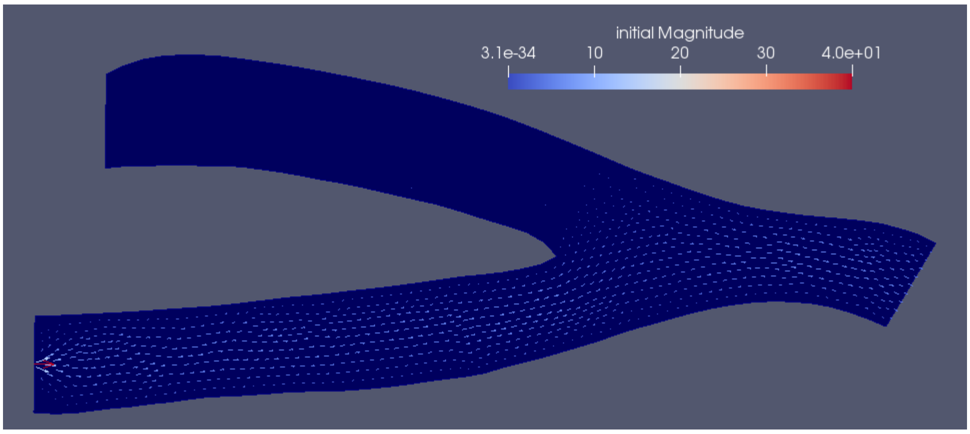


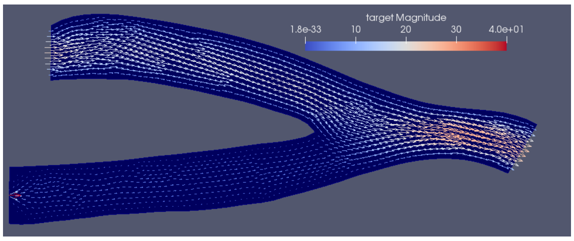
Figure 13: Final state of the initial guess.

For this case, we also started with a null initial guess for the control $(v^h)^0 = 0$, which defines the numerical solution represented in (13). The results of the first 10 iterations of the minimizing process are presented in the table (14). In image (15) we show the comparison between the target flow and the numerical solution. It is interesting to see that in the case of $\alpha = 10, 100$ the costs obtained in the first iterations are equal to the ones obtained in the case $v_{in} = 0$. This would suggest that for the minimization process the change in the boundary condition on Γ_{in}, from a null to a non-null boundary function, does not influnciate the minimizing process, since we are obtaining the some control values on each iteration, and therefore the same cost values. However, for the case $\alpha = 1000$ the costs are different.

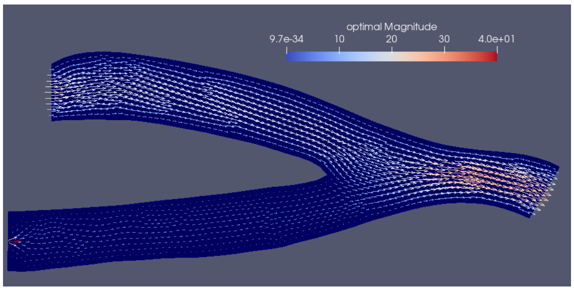
	α=10		α=100		α=1000	
Initial cost	32.4038		324.038		3240.38	
iteration	cost	Rel.error	Cost	Rel.error	Cost	Rel. error
1	32.0107	0.904874	285.922	0.855127	440.068	0.357692
2	31.6968	0.899933	259.29	0.808741	98.1015	0.146545
3	31.4461	0.895516	240.682	0.769813	55.8823	0.065579
4	31.2459	0.891568	227.681	0.737144	50.6053	0.0344236
5	31.0861	0.888038	218.596	0.709728	49.9366	0.0223984
6	30.9585	0.884883	212.248	0.68672	49.8506	0.0177446
7	30.8566	0.882062	207.812	0.667411	49.8394	0.0159387
8	30.7752	0.879541	204.713	0.651206	49.8378	0.0152359
9	30.7102	0.877287	202.547	0.637605	49.8376	0.0149617
10	30.6583	0.875273	201.033	0.626192	--	--

Figure 14: Caption

Notice that the optimal solution is very similar to the target flow, and was obtained in 9 iterations with a relative error under 0.01.



(a) Target flow.



(b) Optimal solution.

Figure 15: Comparison of the numerical solution with the target flow, with $\alpha = 100$ and $\tau = 0.01$.

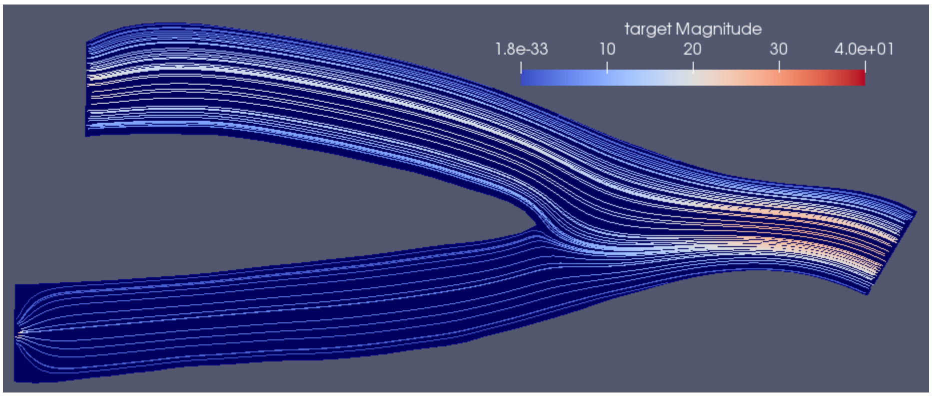


Figure 16: Stream lines of the target flow.

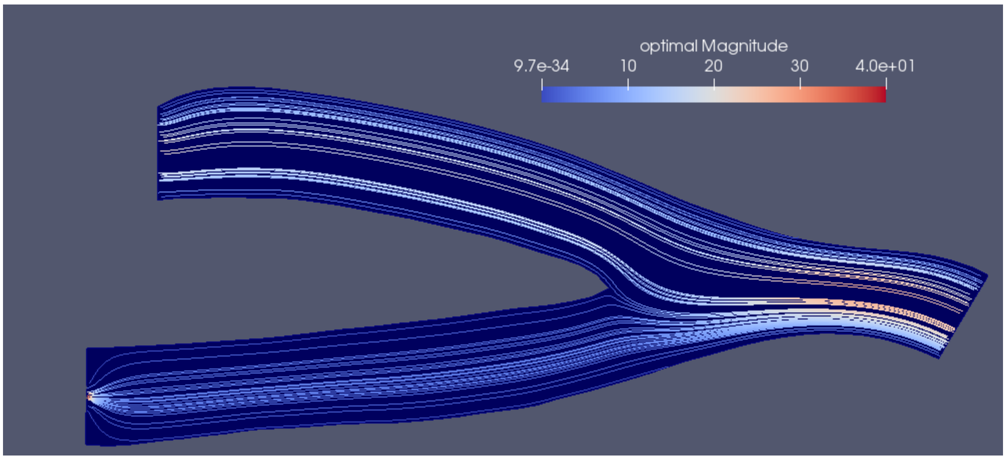


Figure 17: Stream lines of the optimal flow.

Now we analyze the influence of the choice of the observations set ω (see image (6)). We will on a first study, consider the functional

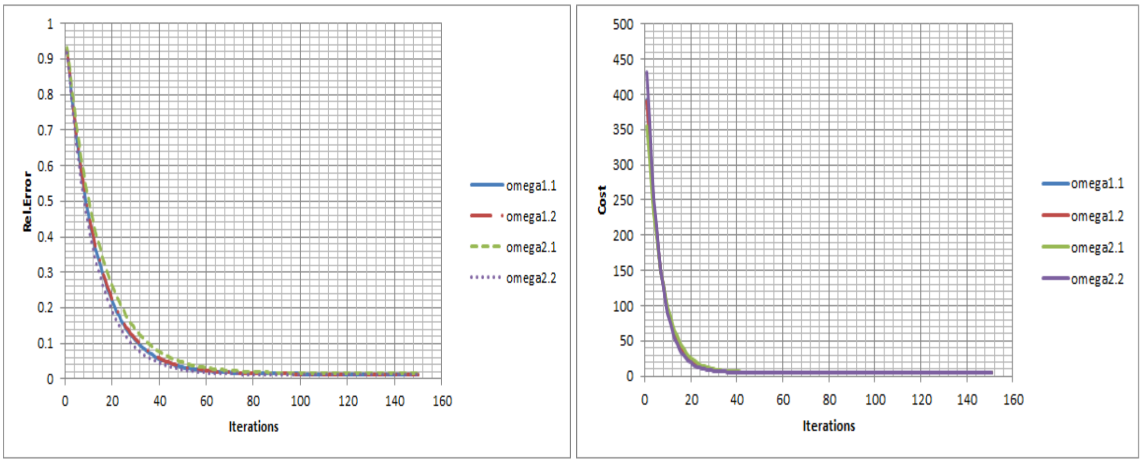
$$J_\omega^h(v^h, u^h) = \frac{\tau}{2}\sum_{n=1}^{N_t} \Delta t \int_{\Gamma_c} |\nabla v_n^h|^2 ds + \frac{\alpha}{2}\sum_{n=1}^{N_t} \int_\omega |u_n^h - (Z_d^h)_n|^2 dx$$

where the target flow is represented in the image (12a), and again we start with the initial guess $(v^h)^0 = 0$.

In this case we are more interested in the relative error reduction than the cost functional reduction, since our analysis is motivated by seeing where to choose the subset ω in such a way that we can easily obtain the target flow, without the need of using all the domain. In terms of applications this would correspond to choose the best subset ω (possibly with some restrictions on this choice, but here we are not concerned about that) on which we focus our observations in order to better control the blood behavior on the bifurcation.

Notice that in the case of a partial observation set for the velocity tracking functional, a reduction on the functional cost does not necessarily mean a reduction on the relative error of approximation to the target flow. Because of this fact, the optimization method when applied on a subset ω of zone-3, for the velocity tracking functional, will not reduce the Rel.error. In this case this cames from the fact that, starting with the null control, $((v^h)_0) = 0$, the initial cost $J^h((v^h)_0) = 0$ which is a minimum and therefore $\nabla J^h((v^h)_0) = 0$, and the process gives in the next iteration always the same solution. For the case where we choose $v_{in} \neq 0$ happened the same, and we observed that the relative error did not change.
Now turning our attention to the zones 1 and 2 we will define the following subsets of Ω: $\omega_{1.1}, \omega_{1.2}$ by being the intersection of $[0.1, 0.2] \times \mathbb{R}, [0.3, 0.4] \times \mathbb{R}$, with the upper channel, and $\omega_{2.1}, \omega_{2.2}$ by being the intersection of $[0.55, 0.65] \times \mathbb{R}, [0.7, 0.8] \times \mathbb{R}$ with Ω. Therefore, $\Omega_{1.i}$, $i \in \{1, 2, 3\}$ are subsets in the zone-1 and, $\Omega_{2.i}$, $i \in \{1, 2\}$ are subsets in the zone-2. For the partial tests we fixed the parameters in $\alpha = 10000$ and $\tau = 0.001$ in order to guarantee that the parameter has enough importance.

In the case $v_{in} = 0$ and the target flow is given by the one represented in (12a), and we set the initial guess to be the null control. In images (18) we have the evolution of the functional costs and the relative errors committed along the iterations.



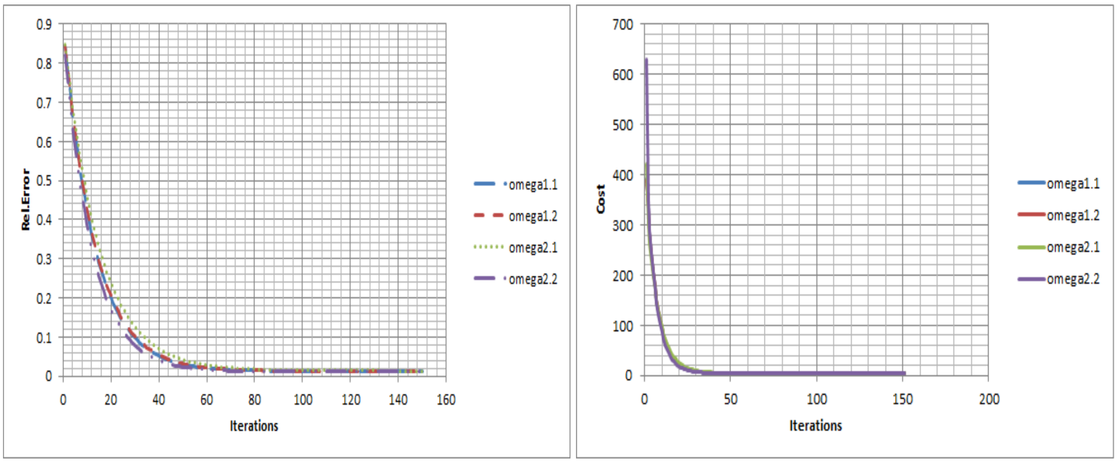
(a) Relative errors evolution.

(b) Costs evolution.

Figure 18: Partial observations in the case $v_{in} = 0$.

It is possible to observe that the subset $\omega_{2.2}$ is the one who shows a stronger decrease, in the relative error. However this is not enough to suggest that the zone $\omega_{2.2}$ is better that the other ones, when the objective is to choose a subset that permits a faster (and a convergence) approach to the target flow.

In the case $v_{in} \neq 0$, we chose the target flow to be given by the one represented in (27a) and we set again, the initial guess to be the null control.

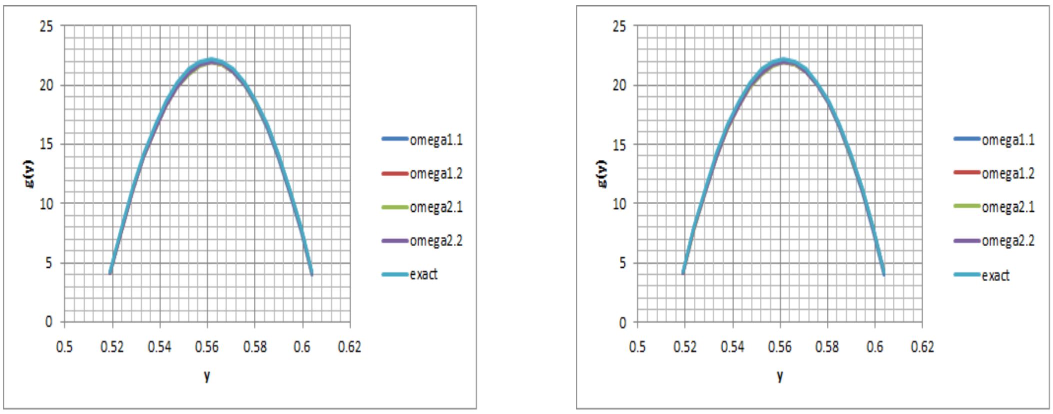


(a) Relative errors evolution. (b) Costs evolution.

Figure 19: Partial observations in the case $v_{in} \neq 0$.

In image (19a) we can observe the evolution of the relative errors along the iterations, and again it seems that the zone $\omega 2.2$ shows a faster decrease in the rel.error. However in comparison with the other zones, the change in the rate of decrease is so low, and they all attained the same finite relative error, that we can not argue that there is a better zone. Here we also considered 150 iterations, since we observed that after this number of iterations the relative error shows a very low decrease.

We also show in image (20), the inlet function in Γ_c for the various simulations, depending on the zone of observation in image, and for the two cases of $v_{in} = 0$ or $v_{in \neq 0}$. It is possible to see that they are very close the exact inlet, for the various cases. This is contrary to what we were expecting, since we thought that observations closer to the control boundary would impose a better approximation to the exact inlet then the observations made far way from the control zone. Also, since we are not giving to much importance to the parameter τ, which controls the gradient of the control, we were expecting inlet functions (at least in the case of far way observations, such as in the case of $\omega 2.1, \omega 2.2$), to have a less accentuated curvature, but permitting the some amount of flux.



(a) Inlet for the case of $v_{in} = 0$. (b) Costs evolution.

Figure 20: Inlet for the case of $v_{in} \neq 0$.

We now consider the case of the vorticity functional cost given by

$$J^h(u^h, v^h) = \sum_{n=1}^{N_t} \frac{dt}{2}\Big(\frac{\tau}{2}\int_{\Gamma_c} |\nabla v_n^h|^2 + \alpha \int_{\Omega} |\nabla \times u_n^h|^2\Big)$$

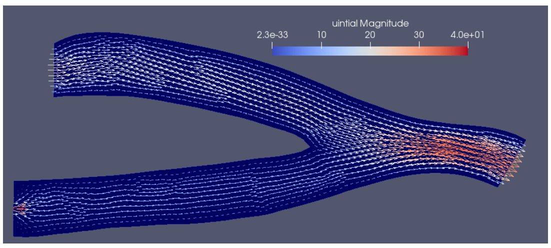
For this case we obtained the values in the table (21) where we considered $\alpha = 10$ and a varying τ. In there we considered the initial guess as the parabolic flow of (12a).

	τ=10	τ=1	τ=0.1	τ=0.01
Initial cost	83437.3	37879	33323.1	32867.5
1	94.5306	289667.7	14100.5	12967.8
2	0.883766	22862	6385.19	5511.29
3	0.059648	19745.2	3674.26	3062.11
4	0.00629	532.964	193.96	171.078
5	0.000771	126.213	27.2943	23.3615
6	0.000102	43.3271	6.48122	5.29643

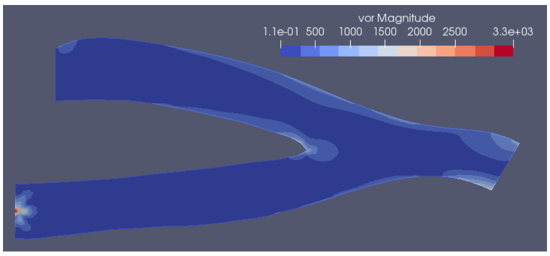
Figure 21: Evolution of the cost functional values for different values of τ, $\alpha = 10$.

In the case of $\tau = 10$, we gave too much force to this parameter, and in this case, the numerical method will tend do the null solution, and we observed that this convergence is fast (in 5 iterations). If we start to decrease τ we are permitting the minimization process to tend to non-null solutions, but however this rate of convergence is slower (notice that in 6-th iteration with $\tau = 1$ we still have a cost value of 43.3271).

Now we consider the case with $v_{in} \neq 0$ and starting with a function which is illustrated in image (final state) (22a), which has the vorticity magnitude illustrated in (22b). After we applied the minimization process, with the vorticity cost functional, and obtained the velocity field (final state) showed in (24a) with a vorticity magnitude in (24b). Notice that the numerical solutions has an exit of fluid in the control boundary Γ_c. Also this process seems to make the vorticity almost uniform along the domain.



(a) Final state for the initial guess.



(b) Initial Vorticity

Figure 22: Initial vorticity.

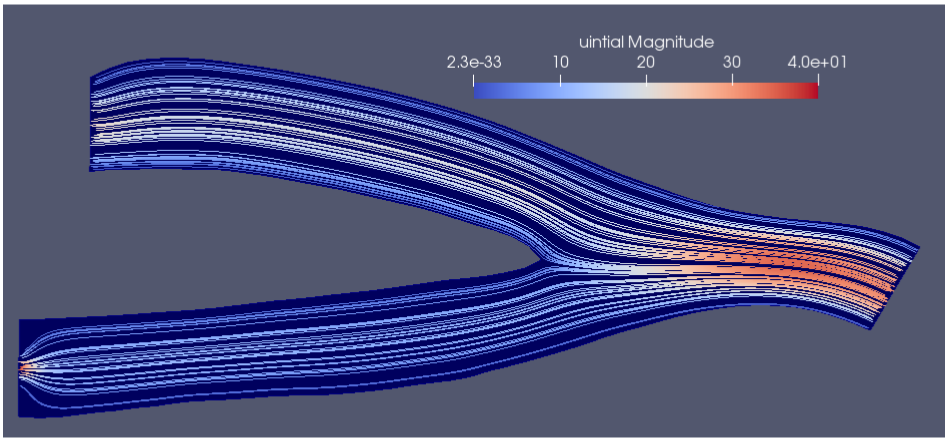
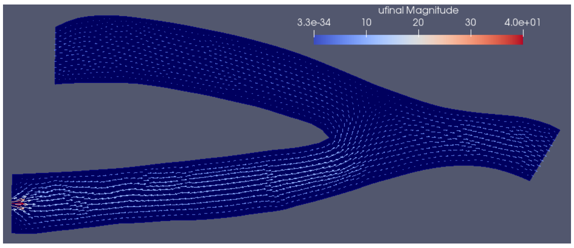
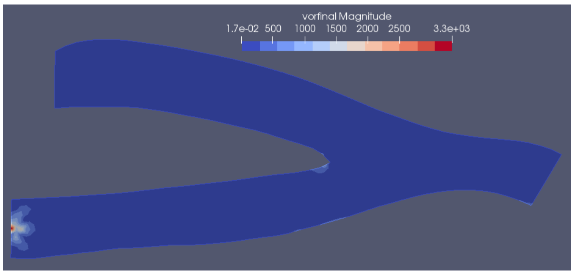


Figure 23: Stream lines of the initial guess, final state.



(a) Final state



(b) Final vorticity.

Figure 24: Final vorticity.

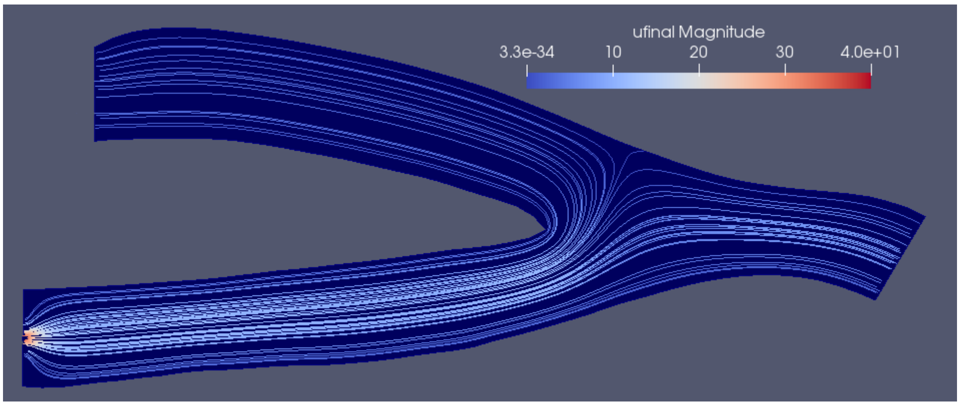
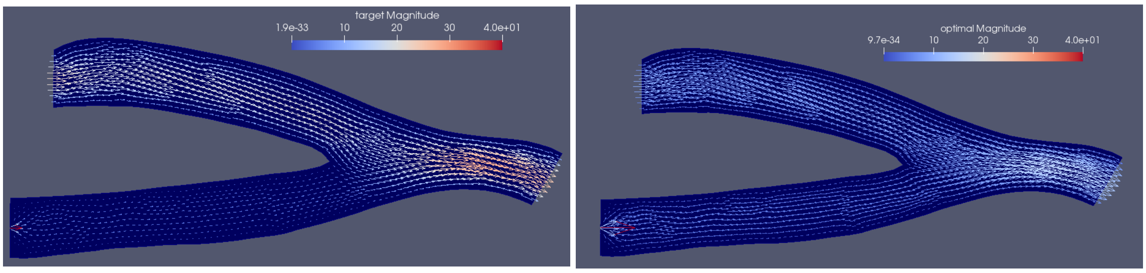


Figure 25: Stream lines of the optimal solution, final state.

We close the numerical tests by giving an example with the functional cost

$$J^h(u^h, v^h) = \sum_{n=1}^{N_t} \frac{dt}{2}\Big(\frac{\tau}{2}\int_{\Gamma_c} |\nabla v_n^h|^2 + \alpha\int_\Omega |(u^h - Z_d^h)_n|^2 + \alpha_1 \int_\Omega |\nabla \times u_n^h|^2\Big)$$

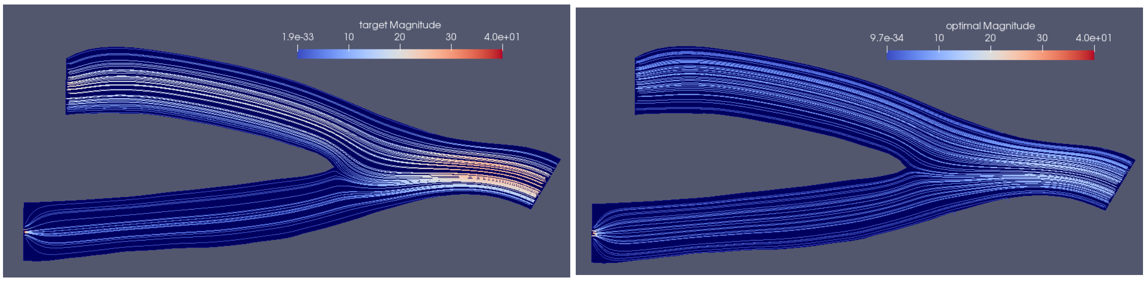
where the target flow is the one in image (18a) and the parameters are given by $\tau = 0.1, \alpha = 10000, \alpha_1 = 10$. As is expected the optimal solution should look closer to the target flow since we are given much more importance to the least square part of the functional cost than the control and vorticity term. We star with the initial guess represented in (27a), and after (x) iteration we obtain the function represented in (27b). Notice that from the initial guess we the method starts to converge towards a numerical solution which starts to have a parabolic flow on the upper channel, and remains unchangeable the bottom channel. Also we do not exactly obtain the target flow due to the vorticity minimization. The initial cost was 32408 and the minimized cost has the value 24797 what corresponds to a decrease of 23 per cent. We also obtain for the optimal solution a rel.error of 0.51 in with respect to target flow.



(a) Target flow.

(b) Optimal solution.

Figure 26: Comparison of the numerical solution with the target flow, with $\alpha = 10000$, $\alpha_1 = 10$ and $\tau = 0.001$.



(a) Target flow.

(b) Optimal solution.

Figure 27: Comparison of the stream lines of numerical solution with the target flow, with $\alpha = 10000$, $\alpha_1 = 10$ and $\tau = 0.001$.

7 Appendix

7.1 Functional Analysis

In this subsection we recall some important results from functional analysis, that we took from the books [20] and [5].

(R1) If $f_n \xrightarrow{w} f$ in $\sigma(E^*, E^{**})$ (that is a weak convergent sequence of functionals in E^*), then $f_n \xrightarrow{w^*} f$ in $\sigma(E, E^*)$ that is $\langle f_n, x\rangle_{E^*,E} \to \langle f, x\rangle_{E^*,E}$ for all $x \in E$.

(R2) This is a simple result but is just for completeness. Let $A : X \to Y$ be a continuous application, where X, Y are Banach spaces. If $x_n \xrightarrow{w} x$ in X then $Ax_n \xrightarrow{w} Ax$ in Y.

(R3) Every continuous and convex functional $f : E \to \mathbb{R}$ is weakly lower semicontinuous, that is, for every weakly convergent sequence $x_n \xrightarrow{w} x$ in E we have

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

(R4) Every convex and closed subset of a Banach space is weakly sequential closed.

(R5) Every bounded subset of a reflexive Banach space is weakly sequentially relatively compact. In particular we have that in every Hilbert space, the bounded sequences have at least one subsequence that converges weakly to some element of the space.

The above results are crucial for the proof of an optimal solution for the control problem. In our work, we do not, in general treat with linear and continuous operators, but instead, with continuous and affine linear operators.
Notice that if an operator is only affine linear and continuous, the result (R2) is still valid.
In fact, let $T : X \to Y$ be an affine linear and continuous operator. Then T assumes the form $T = L+b$ where $L : X \to Y$ is a liner and continuous, and b is a fixed element of Y. Now if $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x in X, we have, by (R2), that $Lx_n \xrightarrow{w} Lx$. Also, since $Tx_n = Lx_n + b$, and the term b of the right hand side can be seen as a constant sequence in Y, and therefore strongly converges to b, we have that $Tx_n = Lx_n + b \xrightarrow{w} Lx + b = Tx$ and therefore $Tx_n \xrightarrow{w} Tx$.

Now, we mention some auxiliary simple results, that were used during the constructions of some of the results in this thesis

In this work we use for many estimates the following result (from [31]),

Lemma 7.1 (Young's inequality). *Suppose that $a, b \geq 0$ then for all $\epsilon > 0$ we have*

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \tag{166}$$

This result allows us to obtain some upper estimates for $||x + y||_X^2$, where $x, y \in X$ and $|| \cdot ||_X$ is a norm. In fact due to the triangular inequality, and the fact that the square function $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ is an increasing function in $[0, \infty[$, we have that

$$||x + y||_X^2 \leq \big(||x||_X + ||y||_X\big)^2$$

Now since $|| \cdot ||_X$ is a norm, is therefore non-negative, and we can apply the lemma 7.1, where we chose $\epsilon = 1$

$$\big(||x||_X+||y||_X\big)^2 = ||x||_X^2+||y||_Y^2+2||x||_X||y||_X \leq ||x||_X^2+||y||_X^2+2\big(\frac{1}{2}||x||_X^2+\frac{1}{2}||y||_X^2\big) = 2(||x||_X^2+||y||_X^2)$$

Thus we get that for every $x, y \in X$

$$||x + y||_X^2 \leq 2\big(||x||_X^2 + ||y||_X^2\big) \tag{167}$$

In the proof of the existence of an optimal solution for the minimization problem, we need a convexity result which is given by the next simple lemma.

Lemma 7.2. *Let $g : \mathbb{R} \to \mathbb{R}$ and $f : X \to \mathbb{R}$ be two convex functions, where g is also assumed to be an increasing function. Then the composition $g \circ f : X \to \mathbb{R}$ is a convex function.*

Proof: Let $\lambda \in [0, 1]$ and u, v be in X. Then,

$$g\Big(f\big(\lambda u + (1 - \lambda)v\big)\Big) \le g\Big(\lambda f(u) + (1 - \lambda)f(v)\Big) \le \lambda g(f(u)) + (1 - \lambda)g(f(v))$$

□

As a simple example of application of the lemma 7.2, one can see that the function $X \ni x \mapsto ||x||_X^2$ is a convex function. In fact by defining $g : \mathbb{R} \to \mathbb{R}$, the function $g(x) = x^2$, and $f : X \to \mathbb{R}$ by $f(x) = ||x||_X$ we get the result.

To finish this first subsection of the appendix, we recall a result from [1] with respect to the equivalence of Cartesian norms.
Given two Banach spaces X, Y the product vector space $X \times Y$ with the norm $||(x, y)||_{X\times Y} = \big(||x||_X^2 + ||y||_Y^2\big)^{1/2}$ is also again a Banach space. Moreover, since the product vector space $X \times Y$ has finite dimension, the following norms are all equivalent to the norm considered

$$||(x, y)||_{X\times Y p} = \big(||x||_X^p + ||y||_Y^p\big)^{1/p} \ , \ p \in [1, \infty[$$

$$||(x, y)||_{X\times Y \infty} = \max\{||x||_X, ||y||_Y\} \ , \ p = \infty$$

Thus, without loss of generality we always consider the norm of the vector product space to be the norm with $p = 2$.

7.2 Sobolev Spaces

In this subsection we recall the definition of the Sobolev spaces, and give some classical results regarding them. We were mainly guided by [1], [33], [8], [25] or [22]. The Sobolev spaces are defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ , \text{ for } |\alpha| \le k\}$$

where $p \in [1, \infty]$ and $k \in \mathbb{N}$, with the norm

$$||u||_{k,p} := \Big(\sum_{|\alpha|\le k} ||D^\alpha u||_{L^p(\Omega)}^p\Big)^{1/p}$$

with which this spaces became complete. In the case of $p = 2$, we denote $W^{k,2}(\Omega) := H^k(\Omega)$ to highlight, that in this case we have a Hilbert space.
Still for the case $p = 2$, we define the Sobolev Spaces of fractional order $H^k(\Omega)$ $0 \le k$, which are defined by using interpolation techniques, (see [22]).

The following result is from [25].

Theorem 7.1 (Trace Theorem)**.** *Suppose that Ω is bounded and has a Lipschitz boundary. Then there exists a linear and continuous operator $\widehat{\tau} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, called trace operator, such that*

$$\widehat{\tau}(u) = u|_{\partial\Omega} \text{ for all } u \in c^1(\overline{\Omega})$$

$$Ker(\widehat{\tau}) = W_0^{1,p}(\Omega)$$

$$Range(\widehat{\tau}) = W_0^{1-\frac{1}{p},p}(\partial\Omega) \subset L^p(\partial\Omega)$$

and exists a constant $c > 0$ such that for all $u \in W^{1,p}(\Omega)$

$$||\widehat{\tau}(u)||_{L^p(\partial\Omega)} \le ||\widehat{\tau}(u)||_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le C||u||_{W^{1,p}(\Omega)}$$

From theorem 7.1 we see that, for instance, the space $H^{1/2}(\partial\Omega)$ may be characterize by being the set of functions which are traces of functions in $H^1(\Omega)$. This functions, are in particular in a subset (strict subset) of $L^2(\partial\Omega)$, and the theorem 7.1, also says that the trace operator is continuous if we consider in the image space the L^2 or the Sobolev $H^{1/2}$ norm.

In this work we use traces operator only on a subset of the boundary, $\Gamma_D \subset \Gamma = \partial\Omega$, and therefore it is crucial to define the traces operators for this boundary parts, and analyze their properties. We will only consider the case of the domain $H^1(\Omega)$ for these operators.
A trace operator of this kind can be constructed in the following way. Let χ_D be the characteristic function

$$\chi_D(x) = \begin{cases} 1 \text{ if } x \in \Gamma_D \\ 0 \text{ otherwise} \end{cases}$$

The we define $\widehat{\tau_D} : H^1(\Omega) \to L^2(\Gamma_D)$ as the composition

$$\widehat{\tau_D} = \chi_D \circ \widehat{\tau} \tag{168}$$

Therefore for a given $u \in H^1(\Omega)$, the image $\widehat{\tau_D}(u)$ is the restriction of the trace of u, $\widehat{\tau}(u)$, to the subset Γ_D of Γ. It is easy to see that this function is in $L^2(\Gamma_D)$ since,

$$\int_{\Gamma_D} |\widehat{\tau_D}(u)|^2 ds = \int_{\Gamma} \chi_D |\widehat{\tau}(u)|^2 ds \leq \int_{\Gamma} |\widehat{\tau}(u)|^2 ds < \infty \tag{169}$$

Also, (169) tells us that the operator $\widehat{\tau_D}$ is bounded, since by theorem 7.1, for every $u \in H^1(\Omega)$

$$||\widehat{\tau_D}(u)||_{L^2(\Gamma_D)} \leq ||\widehat{\tau}(u)||_{L^2(\Gamma)} \leq C||u||_{H^1(\Omega)}$$

By the fact that $\widehat{\tau_D}$ is also linear, we have the continuity of this operators.
For the case of the traces for the lateral boundary, we can use the above operators to define them. In fact, by using the same reasoning as done in [30] we have that, since the operator $\widehat{\tau}$ is linear and continuous from $H^1(\Omega)$ to $L^2(\Gamma)$, we also have that the operator (we denote by the same letter) $\widehat{\tau} : L^2(0,T; H^1(\Omega)) \to L^2(0,T; L^2(\Gamma))$ is linear and continuous. Therefore we can construct the trace operators on subsets $\Sigma_D = \Gamma_D \times (0,T) \subset \Sigma$ is the same way as we constructed $\widehat{\tau_D}$, and get linear and continuous trace operators, for boundary sub parts.

The following two results are from [31].

Theorem 7.2 (Sobolev Imbeddings)**.** *Suppose that $\Omega \subset \mathbb{R}^d$ is bounded and has a Lipschitz boundary. Let j and m be non-negative integers and let p satisfy $1 \leq p < \infty$ Then we have the following cases:*

(i) If $mp < d$, then the imbedding

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \ \textit{for } 1 \leq q \leq \frac{dp}{d-mp}$$

In particular we have that

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \ \textit{for } 1 \leq q \leq \frac{dp}{d-mp}$$

(ii) If $mp = d$. Then

$$W^{m,p}(\Omega) \hookrightarrow L^q(\omega), \textit{ for } 1 \leq q \leq \infty$$

(iii) Suppose that $mp > d$ then

$$W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$$

where

$$C_B(\Omega) := \{f \in C(\Omega) : f \textit{ is bounded } \}$$

(iv) Rellich–Kondrachov theorem: the imbeddings (i)-(ii) are compact for this same conditions on Ω

As a simple application of the theorem 7.2, we can see that, if Ω is a Lipschitz bounded set, then $H^1(\Omega)$ is compactly imbbeded in $L^2(\Omega)$, that is, the application $I : H^1(\Omega) \to L^2(\Omega)$ is compact. As a consequence, every bounded sequence in $H^1(\Omega)$ is mapped via I to a sequence in $L^2(\Omega)$ which has at least a subsequence that converges strongly to an element of $L^2(\Omega)$.

Theorem 7.3 (General Poincaré inequality)**.** *Let $f \in W_0^{1,p}(\Omega)$, then*

$$||f||_{L^p(\Omega)} \leq C_P ||\nabla f||_{L^p(\Omega)} \text{ for } p \in [1, \infty[$$

In particular if $f \in H^1_{\Gamma_0}$, where $\Gamma_0 \subset \partial\Omega$ with positive Lebesgue measure $|\Gamma_0| > 0$, then the above inequality is again valid.

7.3 $L^p(0,T;X)$ Spaces

In this section we introduce the spaces $L^p(0,T;X)$ where X is a separable Banach space.
This spaces are fundamental when dealing with evolution PDE's such as for example the Heat equation. Basically, they introduce the notion of measurability, integrability and weak differentiability for functions of the type

$$t \in [0,T] \mapsto y(t) \in X$$

For detail about the next definitions and results see for example [8], [32], [33] or, on a more simple way [16].

To define this spaces we start by the introduction of the Bochner integral, which is a generalization of the Lebesgue integral to vector valued functions [33].

Definition 7.1. *(i) A function $s : [0,T] \to X$ is called a simple function, if it has the form*

$$s(t) = \sum_{i=1}^{m} 1_{E_i}(t) y_i$$

where the sets $E_i \subset [0,T]$ are Lebesgue mensurable, and $y_i \in X$.

(ii) A function $f : [0,T] \to X$ is called strongly mensurable, if there exists a sequence $\{s_n\}_{n\in\mathbb{N}}, s_n : [0,T] \to X$ of simple functions, such that

$$s_n(t) \to f(t) \text{ , for almost every } t \in (0,T)$$

With the definition 7.1, it is possible to introduce the notion of integrability of vector valued functions.

Definition 7.2. *(i) For a simple function $s(t) = \sum_{i=1}^{m} 1_{E_i}(t) y_i$ we define the integral in $[0,T]$ by*

$$\int_0^T s(t)dt = \sum_{i=1}^{m} |E_i| y_i$$

where $|E_i|$ is the Lebesgue measure of the set $E_i \subset [0,T]$.

(ii) We say that the function $f : [0,T] \to X$ is Bochner-integrable if there exists a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$, $s_n : [0,T] \to X$ such that

$$\int_0^T ||s_n(t) - f(t)||_X dt \to 0 \text{ , when } n \to \infty$$

[33] We call to a function $y : [0,T] \to X$, where X is a Banach space, a vector valued function.

(iii) In the case that f is Bochner integrable, the integral of f in the interval $[0,T]$ is defined by

$$\int_0^T f(t)dt = \lim_{n\to\infty} \int_0^T s_n(t)dt$$

To see that the concepts, introduced in (7.2) are well defined[34], see [32], where it is used the Pettis's theorem.
Also notice that with this definition the integral is not a value in $\mathbb{C}$, as usual, but a element of the Banach space X.
We have the following property for these integrals, which were took from [8].

Theorem 7.4. *A strongly mensurable function $f : [0,T] \to X$ is Bochner-integrable if and only if the function $t \mapsto ||f(t)||_X$ is Lebesgue integrable, and in this case*

$$\Big|\Big| \int_0^T f(t)dt \Big|\Big|_X \leq \int_0^T ||f(t)||_X dt$$

Also, for every F in X^* we have

$$\Big\langle F, \int_0^T f(t)dt \Big\rangle_{X^*,X} = \int_0^T \langle F, f(t)\rangle_{X^*,X} dt$$

This notion of integrability, motivated the introduction of a class function spaces, that are analogous to Lebesgue spaces, which are used for evolutionary equations.

Definition 7.3 ($L^p(0,T;X)$ spaces)**.** *Let X be a separable Banach space. We define for $1 \leq p < \infty$ the space ,*

$$L^p(0,T;X) := \Big\{ y : [0,T] \to X \text{ strongly mensurable such that } ||y||_{L^p(0,T;X)} := \Big(\int_0^T ||y(t)||_X^p dt \Big)^{1/p} < \infty \Big\}$$

Moreover, for the case $p = \infty$

$$L^\infty(0,T;X) := \Big\{ y : [0,T] \to X \text{ strongly mensurable such that } ||y||_{L^\infty(0,T;X)} := ess\ sup_{t\in[0,T]} ||y(t)||_X \Big\}$$

When there is no confusion, we simplify the notation of $L^p(0,T;X)$ by $L^p(X)$. This spaces are Banach spaces when equipped with this norms (see theorem (7.5)), and when we see the elements as equivalence classes, that is, $y \in L^p(0,T;X)$ is a equivalent class, composed by all the functions $\tilde{y} \in \mathscr{L}^p(0,T;X)$ which are equal almost everywhere to y.

We also enunciate the two following technical results, from [16].

Lemma 7.3. *For any $y \in L^p(0,T;X)$, $1 \leq p < \infty$ there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of simple functions with $s_n \to y$ a.e in $(0,T)$, and $s_n \to y$ in $L^p(0,T;X)$. Moreover functions of the form*

$$\sum_{i=1}^m \varphi_i(t) y_i, \ , \varphi \in C_c^\infty((0,T)), y_i \in X$$

are dense in $L^p(0,T;X)$ for $1 \leq p < \infty$.

Theorem 7.5. *Let X a Banach separable space. Then for $1 \leq p < \infty$ the spaces $L^p(0,T;X)$ are also Banach spaces.*
For $1 \leq p < \infty$ the dual space of $L^p(0,T;X)$ can be isometrically $L^q(0,T;X^*)$, where q is the conjugate of p, by means of the pairings

$$\langle v, y\rangle_{L^q(0,T;X^*),L^p(0,T;X)} = \int_0^T \langle v(t), y(t)\rangle_{X^*,X} dt$$

[34] In particular that the integral is independent of the sequence $\{s_n\}_{n\in\mathbb{N}}$, of simple functions.

If X is a separable Hilbert space, the $L^2(0,T;X)$ is a Hilbert space, for the inner product,

$$(v,y)_{L^2(0,T;X)} := \int_0^T (v(t),y(t))_X dt$$

In this work we use multiple times the following lemma.

Lemma 7.4. *If X, Y are Banach spaces which $X \hookrightarrow Y$ then $L^2(X) \hookrightarrow L^2(Y)$.*

Proof: Since $X \hookrightarrow Y$ exists a constant $c > 0$ such that for every $x \in X$ $||x||_Y \leq c||x||_X$.

Now suppose that x is a function of $L^2(X)$. Let us see that we have $||x(t)||_Y \leq c||x(t)||_X$ a.e. in $t \in (0,T)$. We need to see that this a class property, what means, that if $\tilde{x}$ and $\hat{x}$ are two different representatives of $x(t)$ then the inequality holds.
Thus let $\hat{x}(t)$ be a representative of the class $x(t) \in L^2(X)$. Therefore, the pontual evaluation makes sense, and since the function $\hat{x}(t)$ is defined and is equal almost everywhere, to $x(t)$, we get that

$$||\hat{x}(t)||_Y \leq c||\hat{x}(t)||_X = c||x(t)||_X \text{ for a.e. } t \in (0,T)$$

by the embedding $X \hookrightarrow Y$.
Let now, $\tilde{x}(t)$ be another representative of the class $x(t) \in L^2(X)$. Since $\tilde{x}(t) = \hat{x}(t)$ a.e. in $(0,T)$ we also have

$$||\tilde{x}(t)||_Y \leq c||\hat{x}(t)||_X \text{ a.e. } t \in (0,T) \tag{170}$$

Therefore, the inequality (170) is independent of the representatives $\tilde{x}, \hat{x}$ of the class $x \in L^2(X)$ and then is a class property. Thus we can write $||x(t)||_Y \leq c||x(t)||_X$ and since $x \in L^2(X)$ we conclude that

$$\Big(\int_0^T ||x(t)||_Y^2 dt\Big)^{1/2} \leq c\Big(\int_0^T ||x(t)||_X^2 dt\Big)^{1/2}$$

which is equivalent to say that for ever $x \in L^2(X)$, x is also in $L^2(Y)$ and satisfies, $||x||_{L^2(Y)} \leq c||x||_{L^2(X)}$, or in other terms $L^2(X) \hookrightarrow L^2(Y)$. □

7.4 Gelfand Triples

In this section we introduce the concept of Gelfand triples which play an important role in the theory of abstract parabolic equations. Here we will mainly use the results given in [33]. We start by recalling a definition and a classical result from functional analysis.

Definition 7.4. *Let A be a non-empty set of X. We denote by $A^{\perp}$ the set*

$$A^{\perp} := \{f \in X^* : f(x) = 0 \textit{ for all } x \in A\}$$

Let $B \subset X^*$, we define the set

$$\perp B := \{x \in X : f(x) = 0 \textit{ for all } f \in B\}$$

Lemma 7.5. *Let X, Y be two Banach spaces, and let $T \in L(X,Y)$ with the adjoint $T \in L(Y^*, X^*)$. Then the following are equivalent,*

(i) $Im(T)$ is closed in Y.

(ii) $Im(T^*)$ is closed in X^*.

(iii) $Im(T^*) = \big(Ker(T)\big)^{\perp}$.

(iv) $Im(T) = \perp\big(Ker(T^*)\big)$

As a consequence,

Lemma 7.6. *Let X, Y be two Banach spaces and $A : X \to Y$ be a continuous and linear operator. Then,*

$$Im(A) \text{ is dense in } Y \Leftrightarrow A^* \text{ is injective}$$

In general the canonical application $J : X \to X^{**}$ is not an isomorphism, that is in general X is not reflexive. But in the case it is, we may do the identification $X = X^{**}$ which yields that in this case

$$A \text{ is injective} \Leftrightarrow Im(A^*) \text{ is dense in } X^* \tag{171}$$

Also recall that form the Riesz's representation theorem, every Hilbert space is reflexive. In the case of H to be a Hilbert space, is usual to make the identification $H = H^*$, this corresponds to identify every $u \in H$ with the map $(u, \cdot)_H$ in H^*, or vice-versa. Now we introduce the concept of a Gelfand triple.

Definition 7.5 (Gelfand Triple)**.** *Let V be a reflexive Banach space and H a Hilbert space, and therefore we make the identification $H = H^*$. Assume that the embedding $V \underset{i}{\to} H$ is continuous, injective and that the image of i, $Im(i)$, is dense in H. Thus, we have that the injection map $i^* : H \to V^*$ is, by lemma 7.6, injective. Also by (171), $Im(i^*)$ is dense in V^*. Therefore both i, i^* are injective, continuous and have dense images. In this case we write*

$$V \hookrightarrow_i H \hookrightarrow_{i^*} V^*$$

A scheme of this kind is called a Gelfand triple.

Since the map i is continuous

$$||iv||_H \leq c||v||_V \ , \forall v \in V$$

If we re-norm the norm of V, we obtain an equivalent norm and we can achieve,

$$||iv||_H \leq ||v||_V \ , \forall v \in V \implies ||i|| \leq 1$$

and since $||i^*|| = ||i|| \leq 1$, we have that

$$||i^* iv||_{V^*} \leq ||iv||_H \leq ||v||_V \ , \forall v \in V$$

or, by using an abuse of notation,

$$||v||_{V^*} \leq ||v||_H \leq ||v||_V \ , \forall v \in V$$

(This chain of inequalities must be seen carefully since the same v denotes different objects).
Now denoting by $(\cdot, \cdot)_H$ the inner product in H, by definition of the operator i^*, we have that for every $h \in H$, and for every $v \in V$,

$$\langle i^* h, v\rangle_{V^*,V} = \langle h, iv\rangle_H = (h, iv)_H$$

(where we identified h with the functional $(h, \cdot)_H$). Also we have that

$$|(h, iv)_H| = |\langle i^* h, v\rangle_{V^*,V}| \leq ||i^* h||_{V^*}||v||_V \leq ||h||_H||v||_V$$

Thus, every $h \in H$ can be seen as a functional in V. Since $Im(i^*)$ is dense in V^*, for every $f \in V^*$ we have a sequence $h_n \in H$ such that $i'(h_n) \to f$ in V^*, or in other terms,

$$\langle f, v\rangle_{V^*,V} = \lim_{n\to\infty} (h_n, iv)_H$$

Thus for every v in the unit ball on V, we have that the map $(\cdot, iv)_H$ can be uniformly continuously extended to V^*. We denote by the same symbol the continuous extension of the map $(\cdot, i\cdot)_H$ to $V^* \times V$. Therefore sometimes can appear the expression $(f, v)_H$ with $f \in V^*$ and $v \in V$, and we must interpret this operation in the new extended operator sense.

As an instructive example (see [33]) we consider the Gelfand triple

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) = [H_0^1(\Omega)]^*$$

Since the space $\mathscr{D}(\Omega)$ is by definition dense in $H_0^1(\Omega)$, and it is possible to see that is also dense in $L^2(\Omega)$, thus $H_0^1(\Omega)$ is densely and continuously embedded in $L^2(\Omega)$ and we get a Gelfand triple. Also by the Riesz representation theorem, exists an isomorphism $R : H^{-1}(\Omega) \to H_0^1(\Omega)$, which coincides with the identity when H_0^1 is equipped with it's inner product.

Let $v = Rf$ for some $f \in H^{-1}(\Omega)$, then for every $\varphi \in H_0^1(\Omega)$

$$\begin{aligned}\langle f, \varphi \rangle = (v, \varphi)_1 &= \int_\Omega (\sum_i^2 \frac{\partial v}{\partial_i} \frac{\partial \varphi}{\partial_i} + v \cdot \varphi) \\ &= - \int_\Omega (-\Delta v + v)\varphi = (-\Delta v + v, \varphi)_{L^2}\end{aligned}$$

where the integrals must be understood in the distributional sense $\mathscr{D}^*(\Omega)$.
Now using the continuous extension map we obtain

$$\langle f, \varphi \rangle_{H^{-1}, H_0^1} = (f, \varphi)_{L^2}$$

and notice that in general f does not belong to $L^2(\Omega)$. From this we take that

$$(f, \varphi) = (-\Delta v + v, \varphi) \text{ for every } \varphi \in \mathscr{D}(\Omega)$$

what implies that $f = R^{-1}v = -\Delta v + v$

Remark: It may happen that V, H are both Hilbert spaces, but in this case, since we are using the inner product of H, the Riesz isomorphism from $V \to V^*$ does not coincide with the identity.

7.5 Space $W(0, T)$

Definition 7.6 (Distributions with values on a Hilbert space)**.** *Let Ω be an open set in $\mathbb{R}^n$ and H be a Hilbert space. We say that a linear map $T : \mathscr{D}(\Omega) \to H$ is a distribution of $\mathscr{D}^*(\Omega, H)$, if for every compact subset $K \subset\subset \Omega$ there exists constants $p, c \geq 0$ with*

$$||\langle T, \varphi \rangle|| \leq c \cdot \sup_{x \in K} \sum_{|s| \leq p} |D^s(\varphi(x))| \ , \ \varphi \in \mathscr{D}(\Omega) \textit{ with supp}\varphi \subset K$$

Now we introduce the concept of differentiation for this type of distributions.

Definition 7.7 (Differentiation)**.** *Let $s = (s_1, ..., s_n)$ the derivative D^sT is defined by*

$$\langle D^sT, \varphi \rangle := (-1)^{|s|} \langle T, D^s\varphi \rangle \textit{ for all } \varphi \in \mathscr{D}(\Omega)$$

It is possible to see that the derivative of a distribution, of any order, is again a distribution, that is,

$$\text{If } T \in \mathscr{D}^*(\Omega, H) \implies D^sT \in \mathscr{D}^*(\Omega, H)$$

Let $f : \Omega \to H$ be locally integrable, that is, for all $K \subset\subset \Omega \subset \mathbb{R}^n$, K compact, $f \in L^1(K, H)$. We associate to f the distribution from $\mathscr{D}^*(\Omega, H)$

$$f \to T_f \text{ by } \langle T_f, \varphi \rangle := \int_\Omega f(x)\varphi(x)dx \ , \ \varphi \in \mathscr{D}(\Omega)$$

where the integral is in the Bochner sense. We have the following result.

Lemma 7.7. *The map $L^2(\Omega, H) \to \mathscr{D}^*(\Omega, H)$ is injective and continuous.*

Lemma 7.8. *Let H_1, H_2 be two Hilbert spaces with $H_1 \hookrightarrow H_2$ where the inclusion is continuous. The we also have $\mathscr{D}^*(\Omega, H_1) \hookrightarrow \mathscr{D}^*(\Omega, H_2)$ continuously.*

Now suppose that we have the Gelfand triple, $V \hookrightarrow H \hookrightarrow V^*$.
Then, if $f \in L^2((0,T);V)$, we have that $\dfrac{df}{dt} \in \mathscr{D}^*((0,T);V)$ and since $V \hookrightarrow V^*$, we also have from lemma 7.8, that $\dfrac{df}{dt} \in \mathscr{D}^*((0,T);V^*)$.
This motivates the definition of the space $W(0,T)$ where we ask more regularity for the derivative, being a functions and not only a distribution,

$$W(0,T) := \left\{ f \in L^2((0,T);V) : \frac{df}{df} \in L^2((0,T);V^*) \right\}$$

equipped with the inner product

$$(f,g)_W := \int_0^T \big((f(t), g(t))\big)dt + \int_0^T \Big(\frac{df(t)}{dt}, \frac{dg(t)}{dt}\Big)_{V^*} dt$$

We have the following and important result.

Theorem 7.6. *The space $W(0,T)$ is a Hilbert space.*

Lastly is also possible to see that the functions in $W(0,T)$ are continuous.

Theorem 7.7. *We have the continuous imbedding*

$$W(0,T) \hookrightarrow C([0,T], H)$$

The next result can be seen in [29] or in [28], and makes the connection between the derivative in the sense of the vector valued distributions and the derivative in the sense of the scalar distributions.

Lemma 7.9. *Let X be a given Banach space with dual X^* and let u and g be two functions belonging to $L^1(0,T;X)$. The following three conditions are equivalent,*

$$(i)\ ,\ u(t) = \xi + \int_0^t g(s)ds,\ \ ,\ \xi \in X\ \ a.e\ t \in (0,T)$$

$$(ii)\ ,\ \int_0^T u(t)\phi'(t)dt = -\int_0^T g(t)\phi(t)dt\ ,\ \forall \phi \in \mathscr{D}((0,T))$$

$$(iii)\ ,\ \frac{d}{dt}\langle u(t), \eta\rangle = \langle g(t), \eta\rangle\ ,\ \forall \eta \in X^*\ ,\ \text{and in the sense of the scalar distributions}$$

(iii) is equivalent to say, that for all $\phi \in \mathscr{D}((0,T))$ we have

$$\int_0^T \frac{d}{dt}\langle u(t), \eta\rangle \phi(t)dt = -\int_0^T \langle u(t), \eta\rangle \phi(t)dt$$

Also, notice that the item (ii) is the derivative of $u(t)$ in the sense of vector-valued distributions. This connection is important to the conclusion of theorem 3.1.

7.6 Weak Derivative

In this section we give the proof that if $v(t) = V\varphi(t)$ where $\varphi(t)$ is a regular function (at least $C^1[0,T]$) and $V \in X$ (X being a Banach space), then the weak derivative in time (derivative in

the vector distribution sense) coincides with the classical derivative $v'(t) = V\varphi'(t)$.
The classical derivative is indeed given by $v'(t) = V\varphi'(t)$ since

$$\lim_{t\to t_0} \frac{||\varphi(t)V - \varphi(t_0)V - \varphi'(t_0)V||_X}{|t-t_0|} = \lim_{t\to t_0} \frac{||V||_X|\varphi(t) - \varphi(t_0) - \varphi'(t_0)}{|t-t_0|} = ||V||_X o(|t-t_0|)$$

Let $f \in X^*$ then, for every $\phi(t) \in \mathscr{D}((0,T))$. Notice that for every $t \in [0,T]$, $\langle f, \varphi(t)V\rangle = \langle f, V\rangle\varphi(t)$, from the linearity of $f \in X^*$.
Thus

$$\begin{aligned}\int_0^T \langle f, V\varphi(t)\rangle_{X^*,X}\phi'(t)dt = \int_0^T \langle f, V\rangle_{X^*,X}\varphi(t)\phi'(t)dt \\ \langle f, V\rangle_{X^*,X}\int_0^T \varphi(t)\phi'(t)dt \\ = \langle f, V\rangle_{X^*,X}\int_0^T (\varphi(t)\phi(t))'dt - \langle f, V\rangle_{X^*,X}\int_0^T \varphi'(t)\phi(t)dt \\ = -\int_0^T \langle f, V\varphi'(t)\rangle_{X^*,X}\phi(t)dt\end{aligned}$$

and thus we have proofed that $\forall f \in X^*$, $\frac{d}{dt}\langle f, v(t)\rangle = \frac{d}{dt}\langle f, \varphi(t)V\rangle = \frac{d}{dt}\langle f, \varphi'(t)V\rangle$, which by lemma 7.9, is equivalent to say that the derivative, in the vector distribution sense, of $v(t)$ is the function $\varphi'(t)V$, as we wanted.

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