Modular Symmetries and the Flavour Problem

João Rafael Freitas Lourenço

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Supervisor: Prof. Ivo de Medeiros Varzielas

Examination Committee

Chairperson: Prof. Mário João Martins Pimenta
Supervisor: Prof. Ivo de Medeiros Varzielas
Member of the Committee: Prof. Maria Margarida Nesbitt Rebelo da Silva

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To my dear parents

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Resumo

Notamos que nenhuma solução é fornecida para o problema do sabor no contexto do Modelo Padrão (SM) mas que este pode ser resolvido pela introdução de múltiplas simetrias modulares. Construímos modelos para os sabores leptónicos baseados em duas simetrias modulares A_4, que são quebradas por um campo bi-tripleto para o subgrupo diagonal A_4^D, resultando em uma simetria modular do sabor efetiva com dois módulos. Utilizámos esses módulos como estabilizadores, que preservam simetrias residuais distintas, permitindo-nos obter a mistura Tri-Maximal 2 (TM_2) com um conteúdo de campos mínimo, sem flavons a baixa energia, abaixo da quebra para um único A_4. Também construímos modelos baseados em duas simetrias modulares A_5, que são quebradas por um bi-quintupleto (se os neutrinos obtêm a sua massa através do operador de Weinberg) ou um campo bi-tripleto (se os neutrinos obtêm a sua massa através do tipo I do mecanismo de *seesaw*), para o subgrupo diagonal A_5^D. Para estes modelos, obtém-se uma mistura que preserva a segunda coluna da mistura do número de ouro (GR), que denominamos GR_2. Os melhores ajustes e gráficos para o decaimento beta sem neutrinos são obtidos para todos estes modelos. Percebeu-se que a ordenação normal (NO) das massas dos neutrinos é a ordenação mais favorecida, sendo os modelos que resultam em GR_2 mais favoráveis do que aqueles que resultam em TM_2. Para todos os ajustes para NO, as massas e ângulos de mistura dos neutrinos, exceto θ_{12}, são compatíveis com os resultados experimentais a 1σ.

Palavras-chave: Problema do Sabor, Múltiplas Simetrias Modulares, Mistura Tti-Maximal 2, Mistura do Número de Ouro, Massas e Ângulos de Mistura dos Neutrinos

Abstract

We note that no solution is provided for the flavour problem in the context of the Standard Model (SM) but that this can be solved by introducing multiple modular symmetries. We construct lepton flavour models based on two A_4 modular symmetries, which are broken by a bi-triplet field to the diagonal subgroup A_4^D, resulting in an effective modular flavour symmetry with two moduli. We employ these moduli as stabilisers, that preserve distinct residual symmetries, enabling us to obtain Tri-Maximal 2 (TM_2) mixing with a minimal field content, flavonless at the effective scale, below the breaking to the single A_4. We also construct models based on two A_5 modular symmetries, which are broken by a bi-quintuplet (if neutrinos get their mass through the Weinberg operator) or a bi-triplet field (if neutrinos get their mass through the type I seesaw mechanism), to the diagonal subgroup A_5^D. For these models, a mixing that preserves the second column of the Golden Ratio (GR) mixing, which we called GR_2, is obtained. Best fit points and plots for the neutrinoless beta decay are obtained for all these models. It was realised that the normal ordering (NO) of neutrino masses is the preferred ordering, being the models that lead to GR_2 more favourable than those that lead to TM_2. For all the best fit values for NO, the neutrino masses and mixing angles except θ_{12} are compatible with experimental results at the 1σ confidence interval.

Keywords: Flavour Problem, Multiple Modular Symmetries, Tri-Maximal 2 Mixing, Golden Ratio Mixing, Neutrino Masses and Mixing Angles

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Nomenclature

Superscripts

† Hermitian conjugate.

* Adjoint.

T Transpose.

Glossary

GR_2	Mixing that preserves the second column of the GR mixing.
GR	Golden Ratio mixing.
IO	Inverted Ordering of neutrino masses.
NO	Normal Ordering of neutrino masses.
SM	Standard Model.
TBM	Tri-bimaximal mixing.
TM_2	Trimaximal 2 mixing, preserving the second column of the TBM mixing.
VEV	Vacuum Expectation Value.
h.c.	Helicity Conjugate.

Chapter 1

Introduction

The current model of particle physics is the Standard Model (SM) [1–3]. Until now, it has been extremely compatible with experimental results. In this model, the fundamental constituents of matter are quarks and leptons. There are three charged electron-like leptons, the electron, the muon, and the tau, and three neutral leptons interacting only weakly, the neutrinos, which come in three flavours as their charged partners. These are organised in triplets of flavour. The SM also describes the interactions between these particles, which are mediated by bosons: the photon, for the electromagnetic interaction, the W and Z bosons for the weak interaction, and the gluons, for the strong interaction.

This model was completed with the discovery of the Higgs boson in 2012 at the LHC [4, 5]. After spontaneous symmetry breaking, when the Higgs field acquires a non zero vacuum expectation value (VEV), the Yukawa terms give mass to the charged leptons. However, neutrinos remain massless in the SM, which is in disagreement with experimental evidence.

Flavour symmetries, both discrete and continuous, have been extensively treated in the literature as a way to solve the puzzling questions associated with flavour. Examples of discrete well-known symmetries applied to flavour are A_4, S_4, A_5 and $\Delta(27)$.

Theories that use modular symmetries, upgrading the Yukawa couplings to modular forms and introducing similar transformations for the particle blocks, were also constructed. Models using multiple S_4 modular symmetries (one for the charged leptons, one or two for the neutrinos, each one with its own modulus field, that treat the symmetry breaking from these multiple symmetry groups to a single symmetry group at low energy) can be found at [6, 7]

Before the mixing angles were observed experimentally with more precision, a commonly used mixing texture for the PMNS matrix was the Tri-BiMaximal mixing (TBM). This ansatz, ruled out since the measurement of non-zero θ_{13} mixing angle, remains an appealing leading order solution with no free parameters. Mixing schemes such as Tri-Maximal 1 (TM$_1$) and Tri-Maximal 2 (TM$_2$) preserve respectively the first and second columns of tri-bimaximal mixing [8], and remain viable. For models that deal with A_4 symmetries I will be particularly interested in the tri-maximal 2 (TM$_2$) mixing, which preserves the second column of the tri-bimaximal mixing matrix:

$$U_{TM_2} = \begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \end{pmatrix}. \tag{1.1}$$

Another mixing I will be particularly interested, in this case with relation to models with A_5, is the Golden Ratio mixing. More specifically, I will be exclusively interested in models that preserve the second column of the golden ratio mixing matrix:

$$U_{GR_2} = \begin{pmatrix} - & \frac{1}{\sqrt{2+\phi}} & - \\ - & \frac{\phi}{\sqrt{4+2\phi}} & - \\ - & \frac{\phi}{\sqrt{4+2\phi}} & - \end{pmatrix}, \tag{1.2}$$

where ϕ is the golden ratio: $\phi = \frac{1+\sqrt{5}}{2}$.

The objective of this dissertation now follows: I will use multiple modular symmetries, either two A_4's or two A_5's, to construct a high energy theory which is then broken to a low energy model with a single modular symmetry, whose moduli fields gain different VEV's, leading to the realisation of different mass textures in the charged lepton and neutrino sectors. It is then possible to obtain a realistic mixing matrix and mass hierarchies, for example TM_2 or GR_2. These modular symmetries are thus able to generate all masses and mixing parameters for the leptons, using a much smaller set of free parameters, almost only using the VEV's of the Higgs and the moduli fields. Additionally, it will be investigated, through the introduction of driving fields, how the VEV's of the fields that are responsible for the breaking from two modular symmetries to a single one are created.

We will now conclude with a brief outline of the present thesis. In Chapter 2, we review the state of the art of the field. We start by reviewing the leptonic sector of the SM model, how neutrino masses can be generated and discuss how the flavour problem arises. We then introduce the concept of modular symmetries, which can be used to solve the flavour problem, and how we can obtain a lagrangian invariant under these symmetries. In Chapter 3, three models using two modular A_4 symmetries are introduced which are then broken to a single A_4, one using the Weinberg operator and two the type I seesaw mechanism. In Chapter 4, the same procedure is introduced for obtaining two models, one using the Weinberg operator, the other the seesaw mechanism, invariant under two A_5 modular symmetries which are similarly broken to a single A_5. In Chapter 5, we review the main conclusions and some aspects of the present work possible to be improved in the future.

Chapter 2

State of the Art

The present state of theoretical particle physics had their main development in the 30's and 40's with the development of Quantum Field Theory (QFT) in the form of Quantum Electrodynamics (QED). In connection with experiment, this framework lead to the establishment of the SM [3, 9, 10]. However, there were still some problems that remained unsolved and lead to the investigation of extensions of this model.

Supersymmetry first appeared in the context of string theory through the introduction of infinitesimal transformations that interchange bosonic and fermionic fields [11, 12] but soon was worked into a form using quantum field theory in four spacetime dimensions [13] (see [14, 15] and their bibliography for the subsequent development). Given the present experimental knowledge, there is no support for this class of models, that are today quite disfavoured in the sense of requiring the superpartners of the known particles to be much heavier. But it was in connection with supersymmetry and string theory that a new type of symmetries started to be applied to extended forms of the SM: modular symmetries. These, similarly to simpler symmetries already used, proved to be a way of generating all the parameters in the leptonic sector of the SM in agreement with experiment.

Flavour symmetries, both discrete and continuous, have been extensively employed in the literature as a way to solve the puzzling questions associated with flavour. Examples of well-known discrete symmetries applied to flavour are S_3, A_4, S_4 and A_5. More recently, these same symmetries are used in flavour models as modular symmetries $\Gamma_2 \simeq S_3$ [16–20], $\Gamma_3 \simeq A_4$ [21–44], $\Gamma_4 \simeq S_4$ [6, 7, 45–51], and $\Gamma_5 \simeq A_5$ [52, 53]. More recently $\Gamma_7 \simeq PSL(2, \mathbb{Z}_7)$ was studied [54] and [55] studied the mass sum rules arising in these models.

As an example, a S_4 flavour model featuring TM$_1$ mixing [56] is constructed in an elegant manner from three S_4 modular symmetries [7]. This work presents a general mechanism of employing multiple modular symmetries to construct a high energy theory which is then broken to a low energy model with a single modular symmetry, which is also broken when these modulus fields gain different VEV's at fixed points of the modular symmetry (stabilisers). The preserved residual symmetries then lead to the realisation of different mass textures in the charged lepton and neutrino sectors. These modular symmetries are thus able to generate all masses and mixing parameters for the leptons, using a much smaller set of free parameters than the present SM. In [6], a similar model that uses only two S_4 modular symmetries is presented.

In the following overview of the topic, I will start by reviewing the leptonic sector of the SM (Section 2.1). After that, some ways of generating neutrino masses will be introduced (Section 2.2), followed by a brief section on lepton mixing (Section 2.3). Some interesting questions that remain unsolved, the so called flavour problem, which is the primary motivation for the present work, are succinctly explained in Section 2.4. It will be introduced afterwards one way of recreating realistic masses and mixing parameters: the addition of modular flavour symmetries (Section 2.5). These are the foundations of the following chapters and their models. For part of this chapter, [57] will be followed.

2.1 The leptonic sector of the SM

In the SM, the strong, weak and electromagnetic interactions are mediated by spin-1 particles that are connected to the local gauge symmetries $SU(3)_C \times SU(2)_L \times U(1)_Y$, where C stands for colour, L for left-handedness, and Y for hypercharge. This symmetry is spontaneously broken to $SU(3)_C \times U(1)_{EM}$ where $U(1)_{EM}$ couples to the electromagnetic charge $Q_{EM} = T_3 + Y$ where T_3 is the third component of the isospin.

In the leptonic sector, one has three generations of charged leptons, that can be both left and right-handed fermions, and three left-handed neutrinos. The left-handed particles are arranged in doublets of $SU(2)_L$:

$$L_{Ll} = \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L \tag{2.1}$$

and the other three charged leptons are singlets of $SU(2)_L$. In the SM model, no right-handed neutrinos are considered because neutrinos do not interact through other force than the weak force and the weak bosons only couple to left-handed particles. Left-handed neutrinos are also known as active neutrinos and right-handed neutrinos are know as sterile neutrinos, since they have no SM interactions.

The only possible interaction terms when imposing $SU(2)_L$ invariance for the charged currents (CC) among neutrinos and their associated charged leptons and the neutral currents (NC) among neutrinos are

$$-\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \sum_l \overline{\nu}_{Ll} \gamma^\mu l_L W^+_\mu + h.c. \tag{2.2}$$

$$-\mathcal{L}_{NC} = \frac{g}{2\cos\theta_W} \sum_l \overline{\nu}_{Ll} \gamma^\mu \nu_{Ll} Z^0_\mu \tag{2.3}$$

where g is the weak coupling constant and θ_W the Weinberg angle.

In the SM the fermions get their mass through a Yukawa term that couples the scalar Higgs field doublet ϕ to a component of a $SU(2)_L$ doublet and a $SU(2)_L$ singlet through a Yukawa coupling Y. For leptons, this term has the following form:

$$- \mathcal{L}_{Yukawa,leptonic} = Y^l_{ij} \overline{L}_{Li} \phi E_{Rj} + h.c.. \quad (2.4)$$

After spontaneous symmetry breaking, when the Higgs acquires the VEV $\langle \phi \rangle = 1/\sqrt{2}(0, v + h(x))$, the charged lepton masses are generated:

$$- \mathcal{L}_{Yukawa,leptonic} = \bar{l}_{Li} m^l_{ij} E_{Rj} + h.c., \quad m^l_{ij} = Y^l_{ij} \frac{v}{\sqrt{2}}. \quad (2.5)$$

The model only contains left-handed neutrinos thus no Yukawa mass terms can be constructed for the neutrinos and these remain massless at the Lagrangian level.

A possible neutrino mass would arise from the bilinear $\overline{L}_L L^c_L$ where $L^c_L = C\overline{L}^T$ is the charged conjugated field, C the charge conjugation matrix representing a charge conjugation operator. However, this term is forbidden in the SM because it violates the total leptonic number by two units thus cannot be induced by loop corrections, and also violates $B - L$ thus cannot be induced by non-perturbative corrections.

But it is a well established result that neutrinos oscillate between flavours. The first clue arose from the discrepancy between theoretical models for the neutrinos produced at the Sun and the experimental results of neutrino rates. This result was explained by the conversion of electron neutrinos into muon and tau neutrinos due to a non-zero probability of measuring muon and tau neutrinos as a initial beam of electron neutrinos propagates through space. This implies that neutrinos have different masses, so at least two of them, although very light, have a mass, which is in disagreement with the SM. Hence the need to go beyond the SM.

2.2 How do neutrinos get their mass?

We consider in this section how terms can be added to the SM to describe the neutrino masses.

2.2.1 Weinberg operator

One possible way of seeing the neutrino masses problem is to consider that new physics only appears above a scale Λ_{NP} and that the SM is simply a effective low energy theory of a high energy theory. In this case, one doesn't have to worry about the renormalisability of the theory and terms with mass dimension larger than 4, although suppressed by $1/\Lambda^{dim-4}_{NP}$, are not forbidden. The least suppressed term is the dimension 5 term:

$$\frac{Z^\nu_{ij}}{\Lambda_{NP}} (\overline{L}_{Li} \tilde{\phi})(\tilde{\phi}^T L^C_{Lj}) + h.c. \quad (2.6)$$

where $\tilde{\phi} = i\tau_2 \phi^*$. It gives rise, after spontaneous symmetry breaking, to the mass terms

$$- \mathcal{L}_{M_\nu} = \frac{Z^\nu_{ij}}{2} \frac{v^2}{\Lambda_{NP}} \overline{\nu}_{Li} \nu^c_{Lj} + h.c. \quad (2.7)$$

which is a Majorana mass term. The suppression points towards the lightness of the known neutrinos. In fact, this model can be interpreted as the low energy limit of the see-saw model discussed in the following section, where m heavy sterile neutrinos are added. In this model, the new physics scale Λ_{NP} is simply the mass scale of the heavy sterile neutrinos.

2.2.2 See-saw mechanism

Other possibility is to consider now the SM with the addition of m sterile neutrinos. Two possible gauge invariant terms can be constructed:

$$-\mathcal{L}_{M_\nu} = M_{Dij}\overline{\nu}_{si}\nu_{Lj} + \frac{1}{2}M_{Nij}\overline{\nu}_{si}\nu^c_{sj} + h.c. \tag{2.8}$$

where M_D is a complex $m \times 3$ matrix, M_N a symmetric $m \times m$ matrix and $\nu^c = C\overline{\nu}^T$ is the charged conjugated neutrino field. The first term arises from the Yukawa terms for the neutrinos after spontaneous symmetry breaking, while the second term is a Majorana term that violates leptonic number. This can be rewritten as

$$-\mathcal{L}_{M_\nu} = \frac{1}{2}(\overline{\nu}^c_L \;\; \overline{\nu}_s)\begin{pmatrix} 0 & M_D^T \\ M_D & M_N \end{pmatrix}\begin{pmatrix} \nu_L \\ \nu^c_s \end{pmatrix} + h.c. \equiv \overline{\vec{\nu}}^c M_\nu \vec{\nu} + h.c.. \tag{2.9}$$

Given that M_ν is a $(3+m)\times(3+m)$ symmetric complex matrix, it is possible to diagonalize it by a unitary V^ν:

$$(V^\nu)^T M_\nu V^\nu = diag(m_1, m_2, \ldots, m_{3+m}). \tag{2.10}$$

This induces a change of basis, from the interaction eigenstates to the mass eigenstates:

$$\nu_{mass} = (V^\nu)^\dagger \vec{\nu}. \tag{2.11}$$

In terms of mass eigenstates, Eq.(2.9) can be rewritten as

$$-\mathcal{L}_{M_\nu} = \frac{1}{2}\sum_{k=1}^{3+m} m_k(\overline{\nu}^c_{mass,k}\nu_{mass,k} + \overline{\nu}_{mass,k}\nu^c_{mass,k}) = \frac{1}{2}\sum_{k=1}^{3+m} m_k\overline{\nu}_{Mk}\nu_{Mk} \tag{2.12}$$

where $\nu_{Mk} = \nu_{mass,k} + \nu^c_{mass,k} = (V^{\nu\dagger}\vec{\nu})_k + (V^{\nu\dagger}\vec{\nu})^c_k$. The ν_M states obey $\nu_M = \nu^c_M$, thus they are Majorana states. This means that one field is enough to describe both neutrino and antineutrino states. While the Dirac fermions have four-component spinor representations where all components are independent, the four-component Majorana spinors can be written in terms of a two-component Weyl spinor. For more details on Dirac, Majorana and Weyl fermions, see e.g. [58]. When working with Dirac neutrinos instead, one has simply to set $M_N = 0$ in Eq.(2.9).

It is possible to get 3 light neutrinos ν_l and m heavy neutrinos N from the previous $3+m$ neutrinos if the mass eigenvalues of M_N are much larger than the electroweak symmetry breaking scale v. This can be written as

$$-\mathcal{L}_{M_\nu} = \frac{1}{2}\overline{\nu}_l M^l \nu_l + \frac{1}{2}\overline{N}M^h N \tag{2.13}$$

where

$$M^l \simeq -V_l^T M_D^T M_N^{-1} M_D V_l \tag{2.14}$$

$$M^h \simeq V_h^T M_N V_h \tag{2.15}$$

$$V^\nu \simeq \begin{pmatrix} \left(1-\frac{1}{2}M_D^\dagger M_N^{*-1} M_N^{-1} M_D\right) V_l & M_D^\dagger M_N^{*-1} V_h \\ -M_N^{-1} M_D V_l & \left(1-\frac{1}{2}M_N^{-1} M_D M_D^\dagger M_N^{*-1}\right) V_h \end{pmatrix} \tag{2.16}$$

where V_l and V_h are respectively 3×3 and $m \times m$ unitary matrices, M^l is the mass matrix for light neutrinos, M^h the mass matrix for heavy neutrinos and V^ν the matrix in Eq.(2.10).

As wanted, the masses of the heavier states are proportional to M_N and the lighter states to $M_D^2 M_N^{-1}$. When the heavy neutrino masses increase, the almost massless neutrinos become lighter, hence the name see-saw mechanism applied to this model.

2.3 Lepton mixing

Previously we proceeded to the diagonalization of the neutrino mass matrix (see Eqs.(2.10)-(2.11)). To work only with mass eigenstates, the mass matrix for the charged leptons needs to be diagonalized too.

In the interaction basis, the mass terms for the charged leptons that arise from the Yukawa terms are

$$-\mathcal{L}_{M_l} = (\overline{e}_L^I \; \overline{\mu}_L^I \; \overline{\tau}_L^I) M_l \begin{pmatrix} \overline{e}_R^I \\ \overline{\mu}_R^I \\ \overline{\tau}_R^I \end{pmatrix} + h.c.. \tag{2.17}$$

It is possible to diagonalize M_l using two 3×3 unitary matrices V_L^l and V_R^l obtaining

$$V_L^l M_l V_R^l = diag(m_e, m_\mu, m_\tau) \tag{2.18}$$

which implies that the mass eigenstates can be written as

$$l_L = V_L^{l\dagger} l_L^I \;\text{ and }\; l_R = V_R^{l\dagger} l_R^I. \tag{2.19}$$

This change from the interaction eigenstates to the mass eigenstates has consequences in the charged current part of the Lagrangian (Eq.(2.2)), that is now

$$-\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \sum_l \bar{l}_L \gamma^\mu U \nu_{mass} W_\mu^- + h.c. \tag{2.20}$$

where the $3 \times (3+m)$ mixing matrix U was introduced. It is defined as

$$U = V_L^{l\dagger} V^\nu, \tag{2.21}$$

where in this product only the first three rows of V^ν are considered.

Consider now the parametrization of this mixing matrix. Since the charged leptons are Dirac particles, three phases can be eliminated by field redefinitions. The same occurs for neutrinos if they are Dirac particles: 3+m phases can be eliminated. However, this is not possible if neutrinos are Majorana particles, in which case only one phase can be eliminated.

If there were only 3 Majorana neutrinos, the situation would be similar to what happens in the quark sector, where the mixing is described by the Cabibbo–Kobayashi–Maskawa (CKM) matrix. In the leptonic sector, the mixing matrix is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. The parametrization is then

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\eta_1} & 0 & 0 \\ 0 & e^{i\eta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.22)$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Without loss of generality, one can take the angles $\theta_{ij} \in [0, \pi/2]$ and the phases $\delta_{CP}, \eta_i \in [0, 2\pi]$. The parametrization for the PMNS matrix above has more two phases than the CKM matrix, due to the Majorana nature of the neutrinos. If neutrinos are Dirac particles, the number of phases is only one, as in the CKM matrix, and η_i can be eliminated.

Although the unitarity condition is not valid for the 3 light neutrinos mixing sub-matrix for a number of neutrinos larger than 3 (we are only interested in this sub-matrix, not the complete mixing matrix), this violation of unitarity is small and the same parametrization can be used in models using the see-saw mechanism.

2.4 The flavour problem

However, even if we are able to modify slightly the SM to account for neutrino masses, we will still have a lot of questions on flavour that remain unanswered. First of all, there is no reason for why there are three families of quarks and leptons.

The mass hierarchies of the quarks and leptons also seem to encode new physics, with the down type quarks and charged leptons having mass values of the same order of magnitude, while the up-type quarks are much more hierarchical and the neutrinos are almost massless. But it is not only when we compare mass hierarchies that flavour for leptons and quarks has a very different behaviour.

The most recent values for the leptonic sector mixing matrix, obtained from the global fit NuFit [59] (other global fits for neutrino oscillation data are also available in the literature, e.g. [60]), is

$$V_{PMNS} = \begin{pmatrix} 0.801 \to 0.845 & 0.513 \to 0.579 & 0.143 \to 0.156 \\ 0.233 \to 0.507 & 0.461 \to 0.694 & 0.631 \to 0.778 \\ 0.261 \to 0.526 & 0.471 \to 0.701 & 0.611 \to 0.761 \end{pmatrix} \quad (2.23)$$

and the quark sector mixing matrix is [57]

$$V_{CKM} = \begin{pmatrix} 0.97401 \pm 0.00011 & 0.22650 \pm 0.00048 & 0.00361^{+0.00011}_{-0.00009} \\ 0.22636 \pm 0.00048 & 0.97320 \pm 0.00011 & 0.04053^{+0.00083}_{-0.00061} \\ 0.00854^{+0.00023}_{-0.00016} & 0.03978^{+0.00082}_{-0.00060} & 0.999172^{+0.000024}_{-0.000035} \end{pmatrix}. \tag{2.24}$$

The differences are clear: the mixing between flavours is much larger in the leptonic sector while the CKM matrix is almost diagonal. In fact, the PMNS mixing angles are much larger than the CKM mixing angles apart from two of them that have the same order of magnitude.

Finally, the SM and slight modifications of it have another conceptual problem: why are there much more parameters in the flavour sector than in the gauge (strong, weak and electromagnetic) sectors?

All these questions, that constitute the so called flavour problem, point towards the need for the introduction of a fundamental flavour symmetry that accounts for this large collection of parameters arising from the Higgs sector. This new symmetry could, from only a few parameters, generate all the fermion masses and mixing parameters.

2.5 Modular symmetries - an introduction

This section provides the general definitions of the modular group and modular forms, and some fundamental aspects of constructing a realistic model with multiple modular symmetries, as in [7]. In the following chapters the modular groups Γ_3 and Γ_5 will be particularly covered and models that obey the general requirements that are here introduced will be constructed.

2.5.1 Modular group and modular forms

The modular group $\overline{\Gamma}$ is the group of linear fractional transformations γ that act on the complex modulus τ, for τ in the upper-half complex plane, i.e. $Im(\tau) > 0$:

$$\gamma : \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{2.25}$$

where a, b, c, d are integers and satisfy $ad - bc = 1$.

It is convenient to use 2×2 matrices to represent the elements of $\overline{\Gamma}$ as

$$\overline{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \{\pm 1\},\ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}. \tag{2.26}$$

Note that, since γ and $-\gamma$ are the same modular transformation, the group $\overline{\Gamma}$ is isomorphic to $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$, where $SL(2, \mathbb{Z})$ is the group of 2×2 matrices with integer entries and determinant one.

The modular group has two generators, S_τ and T_τ, which satisfy $S_\tau^2 = (S_\tau T_\tau)^3 = 1$. One possible choice for these generators is the following:

$$S_\tau : \tau \to -\frac{1}{\tau},\ T_\tau : \tau \to \tau + 1 \tag{2.27}$$

and their corresponding representations are

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.28}$$

It is possible to define subgroups $\overline{\Gamma}(N)$ of $\overline{\Gamma}$ modding out the entries of the representation matrices:

$$\overline{\Gamma}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ (mod\ \text{N}) \right\}. \tag{2.29}$$

Although the groups $\overline{\Gamma}(N)$ are discrete but infinite, the quotient groups $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$ are finite, thus being called finite modular groups. For $N \leq 5$, these groups are isomorphic to well-known groups: $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, $\Gamma_5 \simeq A_5$. These finite modular groups can be obtained by imposing an additional condition, $T_\tau^N = 1$, which implies that $\tau = \tau + N$.

Modular forms of weight $2k$ and level N are holomorphic functions of τ that transform under $\overline{\Gamma}(N)$ in the following way:

$$f(\gamma\tau) = (c\tau + d)^{2k} f(\tau),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}(N), \tag{2.30}$$

where k is a non-negative integer and N is natural (we are only interested in even weights). These modular forms are invariant under $\overline{\Gamma}(N)$, up to the factor $(c\tau+d)^{2k}$, but they transform under the quotient group Γ_N.

Modular forms of weight $2k$ and level N span a linear space of finite dimension $\mathcal{M}_{2k}(\overline{\Gamma}(N))$. It is possible to choose a basis in $\mathcal{M}_{2k}(\overline{\Gamma}(N))$ such that the transformation of the modular forms under Γ_N is described by a unitary representation ρ of Γ_N:

$$f_i(\gamma\tau) = (c\tau + d)^{2k} \rho(\gamma)_{ij} f_j(\tau),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N. \tag{2.31}$$

2.5.2 Models with a single modular symmetry

Considering an N = 1 supersymmetric model invariant under a finite modular symmetry, the action in general takes the form

$$S = \int d^4x d^2\theta d^2\overline{\theta}\ K(\phi_i, \overline{\phi}_i; \tau, \overline{\tau}) + \left(\int d^4x d^2\theta\ W(\phi_i; \tau) + h.c. \right). \tag{2.32}$$

Under Γ_N the Kähler potential K transforms at most by a Kähler transformation and the superpotential W stays invariant:

$$K(\phi_i, \overline{\phi}_i; \tau, \overline{\tau}) \to K(\phi_i, \overline{\phi}_i; \tau, \overline{\tau}) + f(\phi_i; \tau) + \overline{f}(\overline{\phi}_i; \overline{\tau}) \tag{2.33}$$

$$W(\phi_i; \tau) \to W(\phi_i; \tau). \tag{2.34}$$

The superpotential is in general a function of the modulus τ and superfields ϕ_i and can be expanded as:

$$W(\phi_i; \tau) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_{I_Y} (Y_{I_Y} \phi_{i_1} ... \phi_{i_n})_{\mathbf{1}}. \tag{2.35}$$

We want the superpotential to be invariant under Γ_N. This is possible if we assume the couplings Y_{I_Y} to be multiplet modular forms, and the superfields ϕ_i to transform as

$$\phi_i(\tau) \to \phi_i(\gamma\tau) = (c\tau + d)^{-2k_i} \rho_{I_i}(\gamma) \phi_i(\tau) \tag{2.36}$$

$$Y_{I_Y}(\tau) \to Y_{I_Y}(\gamma\tau) = (c\tau + d)^{2k_Y} \rho_{I_Y}(\gamma) Y_{I_Y}(\tau), \tag{2.37}$$

where $-2k_i$ is the modular weight of ϕ_i, I_i is the representation of ϕ_i, $2k_Y$ is the modular weight of Y_{I_Y}, I_Y is the representation of Y_{I_Y} and $\rho_{I_i}(\gamma)$ and $\rho_{I_Y}(\gamma)$ are the unitary representation matrices of $\gamma \in \Gamma_N$. For the superpotential to be invariant as wanted, the sum of the weights needs to equal zero, i.e. $k_Y = k_{i_1} + \ldots + k_{i_n}$, and the multiplication of the representations $I_Y \times I_{i_1} \times \ldots \times I_{i_n}$ has to contain an invariant singlet.

2.5.3 Models with multiple modular symmetries

Consider a theory that has multiple modular symmetries, based on a series of M modular groups $\overline{\Gamma}^1$, $\overline{\Gamma}^2$, ..., $\overline{\Gamma}^M$, where the modulus field for each symmetry $\overline{\Gamma}^J$, $J = 1, \ldots, M$, is denoted as τ_J. The associated modular transformations take the form:

$$\gamma_J : \tau_J \to \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J}. \tag{2.38}$$

A series of finite modular groups $\Gamma^J_{N_J}$ for $J = 1, \ldots, M$ can be obtained by modding out an integer N_J as done for only one modular group in the previous subsection and taking the quotient finite groups. Take into account that N_J does not need to be identical to $N_{J'}$ for $J \neq J'$.

Consider an N = 1 supersymmetric model invariant under multiple modular symmetries; the action in general takes the form:

$$S = \int d^4x d^2\theta d^2\overline{\theta}\; K(\phi_i, \overline{\phi}_i; \tau_1, \ldots, \tau_M, \overline{\tau}_1, \ldots, \overline{\tau}_M) + \left(\int d^4x d^2\theta\; W(\phi_i; \tau_1, \ldots, \tau_M) + h.c. \right). \tag{2.39}$$

Under $\Gamma^J_{N_J}$ for $J = 1, \ldots, M$ the Kähler potential K transforms at most by a Kähler transformation and the superpotential W stays invariant:

$$K(\phi_i, \overline{\phi}_i; \tau_1, \ldots, \tau_M, \overline{\tau}_1, \ldots, \overline{\tau}_M) \to K(\phi_i, \overline{\phi}_i; \tau_1, \ldots, \tau_M, \overline{\tau}_1, \ldots, \overline{\tau}_M) + f(\phi_i; \tau_1, \ldots, \tau_M) + \overline{f}(\overline{\phi}_i; \overline{\tau}_1, \ldots, \overline{\tau}_M) \tag{2.40}$$

$$W(\phi_i;\tau_1,\ldots,\tau_M,\overline{\tau}_1,\ldots,\overline{\tau}_M)\to W(\phi_i;\tau_1,\ldots,\tau_M,\overline{\tau}_1,\ldots,\overline{\tau}_M). \tag{2.41}$$

The superpotential is in general a function of the modulus τ_i and superfields ϕ_i and the expansion in powers of the superfields takes the form

$$W(\phi_i;\tau_1,\ldots,\tau_M)=\sum_n\sum_{\{i_1,\ldots,i_n\}}\sum_{(I_{Y,1},\ldots,I_{Y,M})}(Y_{(I_{Y,1},\ldots,I_{Y,M})}\phi_{i_1}\ldots\phi_{i_n})_{\mathbf{1}}. \tag{2.42}$$

For the superpotential to be invariant under any finite modular transformation $\gamma_1,\ldots,\gamma_M$ in $\Gamma^1_{N_1}\times\Gamma^2_{N_2}\times\ldots\times\Gamma^M_{N_M}$, the couplings $Y_{(I_{Y,1},\ldots,I_{Y,M})}$ must be multiplet modular forms, and the superfields ϕ_i must transform as

$$\begin{aligned}\phi_i(\tau_1,\ldots,\tau_M)&\to\phi_i(\gamma_1\tau_1,\ldots,\gamma_M\tau_M)\\&=\prod_{J=1,\ldots,M}(c_J\tau_J+d_J)^{-2k_{i,J}}\bigotimes_{J=1,\ldots,M}\rho_{I_{i,J}}(\gamma_J)\,\phi_i(\tau_1,\ldots,\tau_M)\end{aligned} \tag{2.43}$$

$$\begin{aligned}Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1,\ldots,\tau_M)&\to Y_{(I_{Y,1},\ldots,I_{Y,M})}(\gamma_1\tau_1,\ldots,\gamma_M\tau_M)\\&=\prod_{J=1,\ldots,M}(c_J\tau_J+d_J)^{2k_{Y,J}}\bigotimes_{J=1,\ldots,M}\rho_{I_{Y,J}}(\gamma_J)\,Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1,\ldots,\tau_M).\end{aligned} \tag{2.44}$$

where $-2k_{i,J}$ is the modular weight of ϕ_i, $I_{i,J}$ is the representation of ϕ_i, $2k_{Y,J}$ is the modular weight of $Y_{I_{Y,J}}$, $I_{Y,J}$ is the representation of $Y_{I_{Y,J}}$ and $\rho_{I_{i,J}}(\gamma)$ and $\rho_{I_{Y,J}}(\gamma)$ are the unitary representation matrices of γ_J with $\gamma_J\in\Gamma^J_{N_J}$. As discussed previously, for the superpotential to be invariant, $k_{Y,J}=k_{i_1,J}+\ldots+k_{i_n,J}$, and $I_{Y,J}\times I_{i_1,J}\times\ldots\times I_{i_n,J}$ must contain an invariant singlet, for $J=1,\ldots,M$.

Chapter 3

Two A_4 Modular Symmetries for Tri-Maximal 2 Mixing

In this chapter, we will use two A_4 modular symmetries to build models that lead to the TM$_2$ mixing, similarly to the use of multiple S_4 modular symmetries in [6, 7], where models that consider the symmetry breaking from multiple modular symmetry groups to a single symmetry group at low energy have been constructed in order to obtain the TM$_1$ mixing. Although the current experimental evidence excludes TBM mixing, TM$_1$ and TM$_2$ remain viable and appealing schemes for lepton mixings. Some of the work here included was already presented at [61].

We note that [25] already employs a single A_4 modular symmetry and two moduli in a model leading to TM$_2$ mixing, where neutrino masses arise through the effective Weinberg operator. In the models constructed here, we will also start by using the Weinberg operator and afterwards we will use the type I see-saw mechanism to generate the neutrino masses. The presence of two distinct moduli is justified by starting with two A_4 symmetries $A_4^l \times A_4^\nu$ which are subsequently broken to the diagonal subgroup A_4^D. But before considering these matters more attentively, we should start by introducing the modular A_4 symmetry group.

3.1 Modular A_4 symmetry and residual symmetries

In the following subsection, some main properties of the modular A_4 symmetry group including the modular forms of level 3 and its stabilisers will be presented. These stabilisers apply for the specific case of A_4 modular symmetries and, as well as the stabilisers for the modular groups from $N = 2$ to 5, can be found in [62] (we note also that the stabilisers or fixed points for $N = 3, 4$ were presented in [63]). The directions at the stabilisers can also be found in [25], although the factors for the modular forms were corrected here.

3.1.1 Modular A_4 symmetry and modular forms of level 3

The group A_4 is the group of even permutations of 4 objects and has 12 elements. It is generated by two operators S_τ and T_τ obeying

$$S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^3 = 1. \tag{3.1}$$

This group has three singlets and one triplet as its irreducible representations and the multiplication rules and other properties can be found in Appendix A.1. In the so-called complex basis (basis where T_τ is diagonal), the triplet representations of the A_4 generators are

$$\rho_{\mathbf{3}}(S_\tau) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \text{ and } \rho_{\mathbf{3}}(T_\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \ \omega = e^{i2\pi/3}. \tag{3.2}$$

The flavour models that are going to be built employ A_4 as a modular symmetry group and the Yukawa couplings are hence going to be modular forms. These are now going to be introduced.

The three linearly independent weight 2 modular forms of level 3, $Y_{\mathbf{3}}^{(2)} = (Y_1, Y_2, Y_3)$, form a triplet of A_4 and can be expressed in terms of the Dedekind eta functions (see Appendix A.2). The modular forms of higher weight can be generated starting from these modular forms of weight 2. For example, the five linearly independent weight 4 modular forms decompose into a triplet $\mathbf{3}$ and two singlets $\mathbf{1}$ and $\mathbf{1}'$. Using the weight 2 modular forms, one obtains the weight 4 modular forms:

$$Y_{\mathbf{3}}^{(4)} = \frac{2}{3}\begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix} \tag{3.3}$$

and

$$Y_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3, \ \ Y_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1Y_2. \tag{3.4}$$

The singlet $\mathbf{1}''$ vanishes because $Y_{\mathbf{3}}^{(2)}(\tau)$ satisfy the constraint

$$Y_{\mathbf{1}''}^{(4)} = Y_2^2 + 2Y_1Y_3 = 0. \tag{3.5}$$

Note that here a factor $2/3$ was included in the definition for $Y_{\mathbf{3}}^{(4)}$ in accordance with [25] although no such factor is present in [21].

Furthermore, the modular forms of weight 6, whose linear space has dimension 7 and decomposes into 2 triplets and 1 singlet, are [21]:

$$Y_{\mathbf{3}_1}^{(6)} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix} \tag{3.6}$$

$$Y^{(6)}_{\mathbf{3}_2} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix} \tag{3.7}$$

and

$$Y^{(6)}_{\mathbf{1}} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3, \tag{3.8}$$

and the other triplet that we are able to construct vanishes:

$$Y^{(6)}_{\mathbf{3}_3} = \begin{pmatrix} Y_2^3 + 2Y_1Y_2Y_3 \\ Y_2^2Y_3 + 2Y_3^2Y_1 \\ Y_2^2Y_1 + 2Y_1^2Y_3 \end{pmatrix} = 0. \tag{3.9}$$

The weight 8 modular forms, which were constructed similarly to the lower weight modular forms, will be useful for the second model that uses the see-saw mechanism. Their linear space has dimension 9 and decompose into three singlets, the first of which is invariant:

$$Y^{(8)}_{\mathbf{1}} = Y_1^4 + 4Y_1^2Y_2Y_3 + 4Y_2^2Y_3^2 \tag{3.10}$$

$$Y^{(8)}_{\mathbf{1'}} = 2Y_1^3Y_2 + 4Y_1Y_2^2Y_3 + Y_1^2Y_3^2 + 2Y_2Y_3^3 \tag{3.11}$$

$$Y^{(8)}_{\mathbf{1''}} = Y_3^4 + 4Y_1Y_2Y_3^2 + 4Y_1^2Y_2^2 \tag{3.12}$$

and two triplets:

$$Y^{(8)}_{\mathbf{3}_1} = \begin{pmatrix} Y_1^4 + Y_1^2Y_2Y_3 - 2Y_2^2Y_3^2 \\ Y_1^2Y_3^2 - Y_1^3Y_2 - 2Y_1Y_2^2Y_3 + 2Y_2Y_3^3 \\ Y_1^2Y_2^2 - Y_1^3Y_3 - 2Y_1Y_2Y_3^2 + 2Y_2^3Y_3 \end{pmatrix} \tag{3.13}$$

$$Y^{(8)}_{\mathbf{3}_2} = \begin{pmatrix} Y_1^4 + Y_1Y_2^3 - 3Y_1^2Y_2Y_3 + Y_1Y_3^3 \\ Y_2^4 + Y_2Y_3^3 - 3Y_1Y_2^2Y_3 + Y_2Y_1^3 \\ Y_3^4 + Y_3Y_2^3 - 3Y_1Y_2Y_3^2 + Y_3Y_1^3 \end{pmatrix}. \tag{3.14}$$

These are all the modular forms that will prove necessary for the models using two modular A_4 symmetries that will be discussed in two posterior sections.

3.1.2 Stabilisers and residual symmetries of modular A_4

But first a really critical property shall be discussed: the stabilisers of the modular symmetry, which play a crucial role in preserving residual symmetries. Given an element γ in the modular group A_4, a stabiliser τ_γ of γ corresponds to a fixed point in the upper half complex plane that transforms as $\gamma\tau_\gamma = \tau_\gamma$. Once the modular field acquires a VEV at this special point, $\langle\tau\rangle = \tau_\gamma$, the modular symmetry is broken but an Abelian residual modular symmetry generated by γ is preserved. Obviously, acting γ on the modular form at its stabiliser leaves the modular form invariant:

$$\gamma : Y_I(\tau_\gamma) \to Y_I(\gamma\tau_\gamma) = Y_I(\tau_\gamma), \tag{3.15}$$

which implies that

$$\rho_I(\gamma)Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k} Y_I(\tau_\gamma). \tag{3.16}$$

This means that, at the stabiliser, the modular form is an eigenvector of the representation matrix $\rho_{\mathbf{3}}(\gamma)$ for the given stabiliser that corresponds to the eigenvalue $(c\tau_\gamma + d)^{-2k}$, and thus the directions of the modular forms at the stabilisers can be easily determined. Furthermore, since the representation matrix is unitary, $|c\tau_\gamma + d| = 1$.

The stabilisers for the A_4 modular group are shown in Table 3.1 [62].

γ	τ_γ
T_τ, T_τ^2	$i\infty,\ \frac{3}{2} + \frac{i}{2\sqrt{3}}$
$S_\tau T_\tau, T_\tau^2 S_\tau$	$1,\ -\frac{1}{2} + \frac{i\sqrt{3}}{2}$
$T_\tau S_\tau T_\tau, S_\tau T_\tau S_\tau$	$0,\ \frac{3}{2} + \frac{i\sqrt{3}}{2}$
$T_\tau S_\tau, S_\tau T_\tau^2$	$-1,\ \frac{1}{2} + \frac{i\sqrt{3}}{2}$
$T_\tau^2 S_\tau T_\tau$	$-1+i,\ \frac{1}{2} + \frac{i}{2}$
S_τ	$i,\ \frac{3}{2} + \frac{i}{2}$
$T_\tau S_\tau T_\tau^2$	$1+i,\ -\frac{1}{2} + \frac{i}{2}$

Table 3.1: Stabilisers for the A_4 elements [62].

For the transformations S_τ, T_τ, $S_\tau T_\tau$ and $T_\tau S_\tau$, the coefficients $(c\tau_\gamma + d)^{-2k}$ are

$$(c\tau_\gamma + d)^{-2k} = \begin{cases} (-1)^k & \gamma = S_\tau, \tau_{S_{\tau 1}} = i \text{ or } \tau_{S_{\tau 2}} = \frac{3}{2} + \frac{i}{2} \\ 1 & \gamma = T_\tau, \tau_{T_{\tau 1}} = i\infty \\ \omega^{2k} & \gamma = T_\tau, \tau_{T_{\tau 2}} = \frac{3}{2} + \frac{i}{2\sqrt{3}} \\ \omega^{2k} & \gamma = S_\tau T_\tau, \tau_{S_\tau T_\tau} = \omega \\ \omega^{2k} & \gamma = T_\tau S_\tau, \tau_{T_\tau S_\tau} = -\omega^2 \end{cases}. \tag{3.17}$$

The directions of the modular forms of weight $2k = 2$, 4, 6 and 8 for the stabilisers of these four elements are shown in Table 3.2. Additionally, we include the factors for each modular form. Although the directions for the modular forms of weight 2 and 4 had been previously introduced in [25], the factors were corrected here for the weight 4 triplets. These factors are written in function of Y, which is defined in general as the first component Y_1 of $Y_{\mathbf{3}}^{(2)}$, except for $\tau_{T_{\tau 2}} = \frac{3}{2} + \frac{i}{2\sqrt{3}}$, when we define it as the third component Y_3 of that triplet since the first component happens to vanish. For Y, the explicit definitions for the weight 2 modular forms in terms of the Dedekind eta function, present in Appendix A.2, were used. The values the modular form singlets of weight 4, 6 and 8 take at the stabilisers are additionally included in Table 3.3.

The other two stabilisers for $S_\tau T_\tau$ and $T_\tau S_\tau$ were not considered in Tables 3.2 and 3.3 since the modular forms approach infinity for these two values of the modulus field. Notice also that the two

τ_γ	weight 2	weight 4	weight 6		weight 8	
	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}_1$	$\mathbf{3}_2$	$\mathbf{3}_1$	$\mathbf{3}_2$
$\tau_{S_\tau T_\tau} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$Y\begin{pmatrix}1\\ \omega\\ -\frac{1}{2}\omega^2\end{pmatrix}$	$Y^2\begin{pmatrix}1\\ -\frac{1}{2}\omega\\ \omega^2\end{pmatrix}$	0	$-\frac{9}{8}Y^3\begin{pmatrix}1\\ -2\omega\\ -2\omega^2\end{pmatrix}$	0	$\frac{27}{8}Y^4\begin{pmatrix}1\\ \omega\\ -\frac{1}{2}\omega^2\end{pmatrix}$
$\tau_{T_\tau S_\tau} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$	$Y\begin{pmatrix}1\\ \omega^2\\ -\frac{1}{2}\omega\end{pmatrix}$	$Y^2\begin{pmatrix}1\\ -\frac{1}{2}\omega^2\\ \omega\end{pmatrix}$	0	$-\frac{9}{8}Y^3\begin{pmatrix}1\\ -2\omega^2\\ -2\omega\end{pmatrix}$	0	$\frac{27}{8}Y^4\begin{pmatrix}1\\ \omega^2\\ -\frac{1}{2}\omega\end{pmatrix}$
$\tau_{S_{\tau 1}} = i$	$Y\begin{pmatrix}1\\ 1-\sqrt{3}\\ -2+\sqrt{3}\end{pmatrix}$	$(4-2\sqrt{3})Y^2\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$	$(6\sqrt{3}-9)Y^3\begin{pmatrix}1\\ 1-\sqrt{3}\\ -2+\sqrt{3}\end{pmatrix}$	$(21\sqrt{3}-36)Y^3\begin{pmatrix}1\\ -2-\sqrt{3}\\ 1+\sqrt{3}\end{pmatrix}$	$(63\sqrt{3}-108)Y^4\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$	0
$\tau_{S_{\tau 2}} = \frac{3}{2} + \frac{i}{2}$	$Y\begin{pmatrix}1\\ 1+\sqrt{3}\\ -2-\sqrt{3}\end{pmatrix}$	$(4+2\sqrt{3})Y^2\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$	$-(6\sqrt{3}+9)Y^3\begin{pmatrix}1\\ 1+\sqrt{3}\\ -2-\sqrt{3}\end{pmatrix}$	$-(21\sqrt{3}+36)Y^3\begin{pmatrix}1\\ -2+\sqrt{3}\\ 1-\sqrt{3}\end{pmatrix}$	$-(63\sqrt{3}+108)Y^4\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$	0
$\tau_{T_{\tau 1}} = i\infty$	$Y\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$	$\frac{2}{3}Y^2\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$	$Y^3\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$	0	$Y^4\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$	$Y^4\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$
$\tau_{T_{\tau 2}} = \frac{3}{2} + \frac{i}{2\sqrt{3}}$	$Y\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$	$\frac{2}{3}Y^2\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$	0	$Y^3\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$	0	$Y^4\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

Table 3.2: Directions for the modular forms of weight 2, 4, 6 and 8 of level 3 for four A_4 elements (Y in Table 3.3).

τ_γ	weight 4		weight 6	weight 8			Y
	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	
$\tau_{S_\tau T_\tau} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$	0	$\frac{9}{4}\omega Y^2$	$\frac{27}{8}Y^3$	0	0	$\frac{81}{16}\omega^2 Y^4$	$0.94867\ldots$
$\tau_{T_\tau S_\tau} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$	0	$\frac{9}{4}\omega^2 Y^2$	$\frac{27}{8}Y^3$	0	0	$\frac{81}{16}\omega Y^4$	$0.94867\ldots$
$\tau_{S_{\tau 1}} = i$	$(6\sqrt{3}-9)Y^2$	$-(6\sqrt{3}-9)Y^2$	0	$(189-108\sqrt{3})Y^4$	$-(189-108\sqrt{3})Y^4$	$(189-108\sqrt{3})Y^4$	$1.02253\ldots$
$\tau_{S_{\tau 2}} = \frac{3}{2} + \frac{i}{2}$	$-(6\sqrt{3}+9)Y^2$	$(6\sqrt{3}+9)Y^2$	0	$(189+108\sqrt{3})Y^4$	$-(189+108\sqrt{3})Y^4$	$(189+108\sqrt{3})Y^4$	$0.54798\ldots$
$\tau_{T_{\tau 1}} = i\infty$	Y^2	0	Y^3	Y^4	0	0	1
$\tau_{T_{\tau 2}} = \frac{3}{2} + \frac{i}{2\sqrt{3}}$	0	Y^2	Y^3	0	0	Y^4	$-4.26903\ldots$

Table 3.3: Singlets for the modular forms of weight 4, 6 and 8 of level 3 for four A_4 elements and factors Y for each stabiliser.

stabilisers of S_τ and T_τ stabilise these two modular transformations but for different, although equivalent, representations in terms of 2×2 matrices, which means then different values for c and d. This explains the different eigenvalues obtained for the two stabilisers of T_τ. However, in spite of the eigenvalues being the same for both stabilisers of S_τ, the directions the modular forms take at these stabilisers of S_τ are indeed different. In this case the difference comes from the existence of two eigenvectors for the same eigenvalue, eigenvectors that are introduced in the example that follows.

For $S_\tau : \tau \to -1/\tau$, and using the stabiliser $\tau_{S_{\tau 1}} = i$, which stabilises the modular transformation represented by the 2×2 matrix in Eq.(2.28), the expression for the modular form at the stabiliser is

$$\rho_{\mathbf{3}}(S_\tau) Y_{\mathbf{3}}^{(2k)}(\tau_{S_{\tau 1}}) = (-\tau_{S_{\tau 1}})^{-2k} Y_{\mathbf{3}}^{(2k)}(\tau_{S_{\tau 1}}) = (-i)^{-2k} Y_{\mathbf{3}}^{(2k)}(\tau_{S_{\tau 1}}) = (-1)^k Y_{\mathbf{3}}^{(2k)}(\tau_{S_{\tau 1}}), \tag{3.18}$$

and thus we obtain its directions from the eigenvectors of the representation matrix for S_τ (Eq.(3.2)) corresponding to the eigenvalue in the previous equation:

$$Y_{\mathbf{3}}^{(2k)}(\tau_{S_{\tau 1}}) = y^{(2k)}_{\tau_{S_{\tau 1}},1} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y^{(2k)}_{\tau_{S_{\tau 1}},2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \; k = 1 \; (mod\; 2) \tag{3.19}$$

$$Y_{\mathbf{3}}^{(2k)}(\tau_{S_{\tau 1}}) = y^{(2k)}_{\tau_{S_{\tau 1}}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; k = 2 \; (mod\; 2). \tag{3.20}$$

For the lowest weights that appear in Table 3.2, $y^{(2)}_{\tau_{S_{\tau 1}},1} = Y$ and $y^{(2)}_{\tau_{S_{\tau 1}},2} = \sqrt{3}Y$, and from the definitions of the modular forms of higher weight in terms of those of weight 2 we have that $y^{(4)}_{\tau_{S_{\tau 1}}} = (4-2\sqrt{3})Y^2$ and the factors for the two triplets with weight 6 are obtained similarly.

3.2 Tri-bimaximal mixing and related mixings

We have already introduced the main aspects of the A_4 modular group that are going to be useful in the models were are going to construct in this chapter. But given that for these three models we want to obtain the same mixing scheme, it seems wiser to start by introducing the TM$_2$ mixing, which is the mixing derived from the tri-bimaximal especially useful for models dealing with A_4 symmetry groups. For the tri-bimaximal matrix we use the definition:

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (3.21)$$

As already mentioned, this matrix is incompatible with the known experimental results due to the non-vanishing value for the angle θ_{13}, which leads to the consideration of mixings that only preserve the first or the second columns of this matrix, the TM$_1$ and TM$_2$ mixings, respectively, which can be written as the TBM matrix times a rotation between the two columns that are not preserved.

For TM$_2$, which is our mixing of interest, the matrix that diagonalizes M_ν is $U = U_{TBM}U_r$, where U_r is a rotation between the first and third columns. Using the parametrization

$$U_r = \begin{pmatrix} \cos\theta e^{i\alpha_1} & 0 & \sin\theta e^{-i\alpha_2} \\ 0 & e^{i\alpha_3} & 0 \\ -\sin\theta e^{i\alpha_2} & 0 & \cos\theta e^{-i\alpha_1} \end{pmatrix}, \quad (3.22)$$

we are then able to diagonalize M_ν. Here, θ is the angle that governs the rotation and the three α_i are introduced such that the neutrino masses m_i take real values.

The angles and phases from the standard parametrization of the PMNS matrix in [57] can be expressed in terms of the model parameters θ, α_1 and α_2 using the expressions between the parameters and the PMNS matrix elements (these expressions are equivalent to the ones in [25])

$$\sin^2\theta_{13} = |U_{e3}|^2 = \frac{2\sin^2\theta}{3} \quad (3.23)$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{1}{3-2\sin^2\theta} \quad (3.24)$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sin 2\theta}{2+\cos 2\theta}\cos(\alpha_1-\alpha_2) \quad (3.25)$$

$$\begin{aligned}\delta &= -\arg\left(\frac{U_{e3}U_{\tau 1}U^*_{e1}U^*_{\tau 3}}{\cos\theta_{12}\sin\theta_{13}\cos^2\theta_{13}\cos\theta_{23}} + \cos\theta_{12}\sin\theta_{13}\cos\theta_{23}\right) \\ &= \arg\left(\left(e^{i(\alpha_1-\alpha_2)}\sin^2\theta - 3e^{-i(\alpha_1-\alpha_2)}\cos^2\theta\right)\sin 2\theta\right). \quad (3.26)\end{aligned}$$

Using the 3σ C.L. range of $\sin^2\theta_{13}$ for NO(IO), $0.02034(0.02053) \to 0.02430(0.02436)$ [59], we obtain the allowed range for $\sin\theta$:

$$0.1747(0.1755) \lesssim |\sin\theta| \lesssim 0.1909(0.1912), \tag{3.27}$$

which implies also ranges for the other mixing angles (using that $-1 \leq \cos(\alpha_1 - \alpha_2) \leq 1$):

$$0.3403(0.3403) \lesssim \sin^2\theta_{12} \lesssim 0.3416(0.3417) \tag{3.28}$$

$$0.3891(0.3890) \lesssim \sin^2\theta_{23} \lesssim 0.6109(0.6110). \tag{3.29}$$

The experimental 1σ region is within the interval found for $\sin^2\theta_{23}$, which overlaps with the 3σ region for this parameter, with our result extending below the lower 3σ limit for this parameter, 0.407(0.411) for NO(IO), and not reaching its upper limit. The range of allowed values for $\sin^2\theta_{12}$ is near the upper allowed limit, which is a characteristic feature of the TM_2 mixing, since the lowest value allowed for $\sin^2\theta_{12}$ is $1/3$ as can be seen from Eq.(3.24).

We conclude that, in spite of the discrepancy found for $\sin^2\theta_{12}$, this is still a mixing that is worth considering.

3.3 Models with two modular A_4 symmetries - using the Weinberg operator

Now that the A_4 modular symmetry and the TBM and related mixings were introduced, the models that use this symmetry in order to get the TM_2 mixing can now be described. We will start by constructing one model where it is assumed that neutrinos get their mass through the Weinberg operator, and afterwards we introduce two models where the see-saw mechanism is used. At high energies, these models are based in two modular symmetries, A_4^l and A_4^ν, with modulus fields denoted by τ_l and τ_ν, respectively. After the modulus fields acquire different VEV's, different mass textures are realised in the charged lepton and neutrino sectors, and thus the PMNS matrix will get the TM_2 mixing form.

In this section we consider that neutrinos get their mass through the Weinberg operator, which is an effective term of the type $\frac{1}{\Lambda} Y L^2 H_u^2$. The transformation properties of fields and Yukawa couplings can be found in Table 3.4.

The Yukawa coefficients are modular forms and their weights were chosen in such a way that we obtain the desired directions when the modular fields gain a VEV at a given stabiliser: Y^l is then a triplet of A_4^l with weight $+6$ and $Y_{\mathbf{3}}$ is a triplet of A_4^ν and trivial singlet of A_4^ν with both weights $+4$ so that they have the directions $(1,0,0)$ and $(1,1,1)$ at their stabilisers, respectively. We also considered the non vanishing weight 4 modular forms that will couple to L^2, $Y_{\mathbf{1}}$ and $Y_{\mathbf{1}'}$, singlets $\mathbf{1}$ and $\mathbf{1}'$ under A_4^l, respectively, and both singlets $\mathbf{1}$ under A_4^ν.

The right-handed lepton fields e^c, μ^c and τ^c are singlets $\mathbf{1}$, $\mathbf{1}''$ and $\mathbf{1}'$ of A_4^l, respectively, and trivial singlets $\mathbf{1}$ of A_4^ν, with weights $2k_l = +4$ and $2k_\nu = -2$. Similarly the lepton doublets L transform as a $\mathbf{3}$

Fields	$SU(2)$	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
L	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{1}$	$+2$	$+2$
e^c	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$+4$	-2
μ^c	$\mathbf{2}$	$\mathbf{1''}$	$\mathbf{1}$	$+4$	-2
τ^c	$\mathbf{2}$	$\mathbf{1'}$	$\mathbf{1}$	$+4$	-2
$H_{u,d}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	0	0
Φ	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0

Yukawas/Masses	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
Y^l	$\mathbf{3}$	$\mathbf{1}$	$+6$	0
$Y_{\mathbf{1}}$	$\mathbf{1}$	$\mathbf{1}$	$+4$	$+4$
$Y_{\mathbf{1'}}$	$\mathbf{1'}$	$\mathbf{1}$	$+4$	$+4$
$Y_{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{3}$	$+4$	$+4$

Table 3.4: Transformation properties of fields and Yukawa couplings for model using the Weinberg operator and two modular A_4.

of A_4^l and a $\mathbf{1}$ of A_4^ν, with weights $2k_l = 2k_\nu = +2$. These are the correct choices for the weights such that the modular forms and fields in each term in the superpotential sum up to zero since the weight for the fields is not $2k$, which are the values that were introduced in this section, but $-2k$ instead (recall the transformation relations for the modular forms and the superfields, Eq.(2.44), and how the signs of the exponents where the weights enter differ). H_d and H_u are the usual Higgs and an additional Higgs doublet as required in supersymmetric models. A bi-triplet Φ, which is a triplet under both A_4^l and A_4^ν, is introduced to describe the breaking from the two modular A_4 groups to a single A_4.

With the fields assigned in this manner, the superpotential for this model, which can be separated into one part containing the mass terms for the charged leptons and the other the neutrino mass terms, has the following form:

$$w = w_e + w_\nu, \tag{3.30}$$

$$w_e = \left(\alpha(LY^l(\tau_l))_{\mathbf{1}}e^c + \beta(LY^l(\tau_l))_{\mathbf{1'}}\mu^c + \gamma(LY^l(\tau_l))_{\mathbf{1''}}\tau^c\right) H_d, \tag{3.31}$$

$$w_\nu = \frac{1}{\Lambda}\left((L^2)_{\mathbf{1}}Y_{\mathbf{1}}(\tau_l,\tau_\nu) + (L^2)_{\mathbf{1''}}Y_{\mathbf{1'}}(\tau_l,\tau_\nu) + \frac{1}{\Lambda}(L^2)_{\mathbf{3}}\Phi Y_{\mathbf{3}}(\tau_l,\tau_\nu)\right) H_u^2, \tag{3.32}$$

where only the symmetric decomposition contributes to $(L^2)_{\mathbf{3}}$.

$A_4^l \times A_4^\nu \to A_4^D$ breaking

We discuss now how the symmetry breaking from two independent $A_4^l \times A_4^\nu$ to a single A_4^D is achieved. We start by discussing the term $1/\Lambda^2(L^2)_{\mathbf{3}}\Phi Y_{\mathbf{3}}(\tau_l,\tau_\nu)H_u^2$. Considering the multiplication rules for two triplets to get a trivial singlet, this term can be explicitly expanded as:

$$\frac{1}{\Lambda^2}(L^2)_{\mathbf{3}}P_{23}\begin{pmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13}\\ \Phi_{21} & \Phi_{22} & \Phi_{23}\\ \Phi_{31} & \Phi_{32} & \Phi_{33}\end{pmatrix} P_{23}Y_{\mathbf{3}}(\tau_l,\tau_\nu)H_u^2, \tag{3.33}$$

where P_{23} is the matrix that permutes the second and third columns/rows:

$$P_{23} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}. \tag{3.34}$$

If Φ acquires the VEV $\langle\Phi\rangle = v_\Phi P_{23}$ (see Appendix A.3 for more details), the symmetry $A_4^l \times A_4^\nu$ is broken but given that the same transformation γ can be performed in A_4^l and A_4^ν simultaneously, there is still a single modular symmetry A_4^D that is conserved (the diagonal subgroup). Under this symmetry, a modular transformation takes the form

$$\gamma : (\tau_l, \tau_\nu) \to (\gamma\tau_l, \gamma\tau_\nu) = \left(\frac{a\tau_l + b}{c\tau_l + d}, \frac{a\tau_\nu + b}{c\tau_\nu + d}\right), \gamma \in A_4. \tag{3.35}$$

Consequently, the term $\frac{1}{\Lambda^2}(L^2)_{\mathbf{3}}\Phi Y_{\mathbf{3}} H_u^2$ gets the form $\frac{v_\Phi}{\Lambda^2}((L^2)_{\mathbf{3}} Y_{\mathbf{3}})_{\mathbf{1}} H_u^2$, which is invariant under the remaining symmetry. This term implies a mass matrix for the neutrinos when the Higgs doublet H_u acquires a VEV.

Consequently, we obtain for w_ν (the w_e terms remain exactly the same):

$$w_\nu = \frac{1}{\Lambda}\left((L^2)_{\mathbf{1}} Y_{\mathbf{1}}(\tau_l, \tau_\nu) + (L^2)_{\mathbf{1}''} Y_{\mathbf{1}'}(\tau_l, \tau_\nu) + \frac{v_\Phi}{\Lambda}(L^2)_{\mathbf{3}} Y_{\mathbf{3}}(\tau_l, \tau_\nu)\right) H_u^2. \tag{3.36}$$

A_4^D breaking

The flavour structure after A_4^D symmetry breaking now follows. We assume that the charged lepton modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T = \frac{3}{2} + \frac{i}{2\sqrt{3}}$, which is a stabiliser of T_τ. At this stabiliser, a residual modular Z_3^T symmetry is preserved in the charged lepton sector. This implies that the modular form Y^l, which has weight $+6$, gets the direction

$$Y^l(\tau_l) \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.37}$$

This direction leads to a diagonal charged lepton mass matrix when the Higgs field H_d acquires a VEV $\langle H_d\rangle = (0, v_d)$:

$$m_e = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{3.38}$$

The masses for the charged leptons can be reproduced by adjusting the parameters α, β and γ. These constants were redefined to include any factor associated with $Y^l(\tau_T)$ and v_d.

For the other modular field τ_ν, we want to find a VEV that leads to a mixing that preserves the second column of the TBM mixing matrix. This occurs for $\langle\tau_\nu\rangle = \tau_S = i$ and $2k_\nu = +4$, and, in this case, a residual modular Z_2^S symmetry is preserved in the neutrino sector. According to Table 3.2, the direction of $Y_{\mathbf{3}}$ at this stabiliser is going to be

$$Y_{\mathbf{3}}(\tau_l, \tau_\nu) \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.39}$$

This implies the following structure for the neutrino mass matrix:

$$M_\nu = g_{\mathbf{1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_{\mathbf{1'}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{g_{\mathbf{3}}}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{3.40}$$

where $g_{\mathbf{1}}$, $g_{\mathbf{1'}}$ and $g_{\mathbf{3}}$ are arbitrary complex constants associated with the respective modular form contribution. Similarly to what was done for α, β and γ, the factors $2v_u^2/\Lambda$ and $2v_u^2 v_\Phi/\Lambda^2$ and any factor coming from the modular forms were also included inside these complex constants.

We want now to diagonalize M_ν, such that $U^T M_\nu U = M_{\nu_d} =$ diag(m_1, m_2, m_3), where m_i are the neutrino masses and U is an unitary matrix. In the present model, when we apply the tri-bimaximal mixing matrix Eq.(3.21) to the neutrino mass matrix we obtain:

$$U_{TBM}^T M_\nu U_{TBM} = \begin{pmatrix} a & 0 & c \\ 0 & \frac{a-b}{2} + \sqrt{3}c & 0 \\ c & 0 & b \end{pmatrix} \tag{3.41}$$

where $a = g_{\mathbf{3}} + g_{\mathbf{1}} - \frac{1}{2} g_{\mathbf{1'}}$, $b = g_{\mathbf{3}} - g_{\mathbf{1}} + \frac{1}{2} g_{\mathbf{1'}}$ and $c = \frac{\sqrt{3}}{2} g_{\mathbf{1'}}$. This matrix has only an element on the second row and second column and four elements on the corners that form a 2×2 symmetric matrix and so it can be fully diagonalized introducing a matrix U_r, which describes a rotation among the first and third columns, and thus preserves the second column. The matrix that diagonalizes M_ν is then $U = U_{TBM} U_r$. This is precisely the TM_2 mixing, and using U_r as defined in Eq.(3.22) we are then able to diagonalize M_ν.

It is also possible to start from the diagonal matrix M_{ν_d} and get $U_{TBM}^T M_\nu U_{TBM}$. We have then:

$$U_r^* M_{\nu_d} U_r^\dagger = \begin{pmatrix} m_1 \cos^2\theta e^{-2i\alpha_1} + m_3 \sin^2\theta e^{2i\alpha_2} & 0 & \frac{1}{2}(-m_1 e^{-i(\alpha_1+\alpha_2)} + m_3 e^{i(\alpha_1+\alpha_2)}) \sin 2\theta \\ 0 & m_2 e^{-2i\alpha_3} & 0 \\ * & 0 & m_1 \sin^2\theta e^{-2i\alpha_2} + m_3 \cos^2\theta e^{2i\alpha_1} \end{pmatrix}, \tag{3.42}$$

where an asterisk was used to omit the non-vanishing off diagonal entry of this symmetric matrix. Comparing this with Eq.(3.41) we obtain that $\alpha_3 = -\frac{1}{2} \arg\left(\frac{a-b}{2} + \sqrt{3}c\right)$ and, more importantly, we get a mass sum rule for m_i:

$$\begin{aligned} m_2 &= \left| \frac{a-b}{2} + \sqrt{3}c \right| \\ &= \left| \frac{m_1}{2} \left(e^{-2i\alpha_1} \cos^2\theta - e^{-2i\alpha_2} \sin^2\theta - \sqrt{3} e^{-i(\alpha_1+\alpha_2)} \sin 2\theta \right) \right. \\ &\qquad \left. - \frac{m_3}{2} \left(e^{2i\alpha_1} \cos^2\theta - e^{2i\alpha_2} \sin^2\theta - \sqrt{3} e^{i(\alpha_1-\alpha_2)} \sin 2\theta \right) \right|. \end{aligned} \tag{3.43}$$

The sum rule Eq.(3.43) and Eqs.(3.23-3.26) are relations between the observables and the parameters of the TM_2 mixing, and hence provide what is needed to do a numerical minimisation using the χ^2

function:

$$\chi^2 = \sum_i \left(\frac{P_i(\{x\}) - BF_i}{\sigma_i} \right)^2 , \tag{3.44}$$

where P_i are the values provided by the considered model, BF the best fit value from NuFit [59] and σ_i is also provided by NuFit, when averaging the upper and lower σ provided. For the fitting, six variables were considered: the three mixing angles, the atmospheric and solar neutrino squared mass differences, and the Dirac neutrino CP violation phase.

The fit parameters obtained for normal ordering (NO) and inverted ordering (IO) of neutrino masses can be found in Table 3.5. The best fit values lie inside the 1σ range for all the observables except θ_{12}, for both orderings near the upper limit of the 3σ range, and θ_{23} for IO. Nonetheless, all the observables are within their 3σ intervals. The best-fit occurs for normal ordering of neutrino masses with a $\chi^2/6 = 1.57$.

It is also possible to obtain the expected $m_{\beta\beta}$ for neutrinoless beta decay using the formula

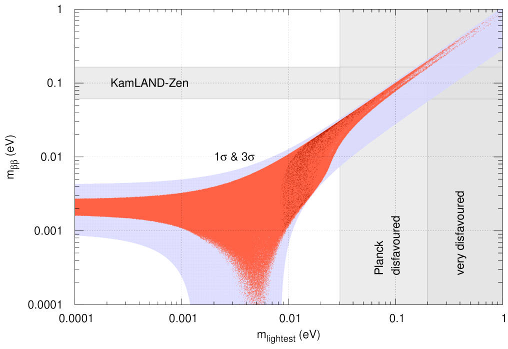
$$\begin{aligned} m_{\beta\beta} &= |(M_\nu)_{(1,1)}| \\ &= \left| \frac{m_1}{6} \left(5e^{-2i\alpha_1} \cos^2\theta - e^{-2i\alpha_2} \sin^2\theta - \sqrt{3} e^{-i(\alpha_1+\alpha_2)} \sin 2\theta \right) - \right. \\ & \qquad \left. - \frac{m_3}{6} \left(e^{2i\alpha_1} \cos^2\theta - 5e^{2i\alpha_2} \sin^2\theta - \sqrt{3} e^{i(\alpha_1-\alpha_2)} \sin 2\theta \right) \right| . \end{aligned} \tag{3.45}$$

Doing a numerical computation, the allowed regions of m_{lightest} vs $m_{\beta\beta}$ of Figure 3.1 (for NO, $m_{\text{lightest}} = m_1$ and, for IO, $m_{\text{lightest}} = m_3$) were obtained, using again as constraints the data from [59]. In both figures it is also shown the current upper limit provided by KamLAND-Zen, $m_{\beta\beta} < 61 - 165$ meV [64]. Results from PLANCK 2018 also constrain the sum of neutrino masses, although different constrains can be obtained depending on the data considered (for more details, see [65]). In the figures are plotted two shadowed regions, a very disfavoured region $\sum m_i > 0.60$ eV (considering the limit 95%C.L.,Planck lensing+BAO+θ_{MC}) and a disfavoured region $\sum m_i > 0.12$ eV (considering the limit 95%C.L.,Planck TT,TE,EE+lowE+lensing+BAO+θ_{MC}). These constraints on $\sum m_i$ can be expressed as constraints on m_{lightest} using the best fit value for the squared mass differences: $m_{\text{lightest}} > 0.198$ eV and $m_{\text{lightest}} > 0.030$ eV for NO and $m_{\text{lightest}} > 0.196$ eV and $m_{\text{lightest}} > 0.016$ eV for IO, for the very disfavoured and the disfavoured regions respectively. We conclude then that only the NO in Table 3.5 is outside the disfavoured regions, although near.

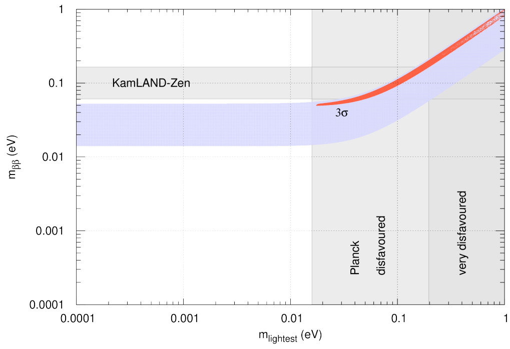
For NO, there are some points compatible with the 1σ ranges of the observables other than θ_{12} (which

NO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			1.57	10.51°	-10.21°	33.17°	0.0227 eV	0.0550 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{31}	$m_{\beta\beta}$
		35.72°	49.4°	8.56°	224°	7.42$\times 10^{-5}$eV^2	2.514$\times 10^{-3}$eV^2	0.0188 eV
IO	Para.		χ^2/6	θ	α_1	α_2	m_3	m_1
			2.74	-10.57°	152.76°	48.71°	0.1095 eV	0.1201 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{32}	$m_{\beta\beta}$
		35.73°	46.5°	8.62°	256°	7.42$\times 10^{-5}$eV^2	-2.497$\times 10^{-3}$eV^2	0.1146 eV

Table 3.5: Parameters (Para.) and observables (Obs.) for the best fit point for normal and inverted orderings for model using the Weinberg operator and two modular A_4.



(a) Normal Ordering



(b) Inverted Ordering

Figure 3.1: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both orderings of neutrino masses compatible with 1σ (dark-red, NO only, except θ_{12}) and 3σ data from [59] for model using the Weinberg operator and two modular A_4. In both figures there were also included the current upper limit from KamLAND-Zen $m_{\beta\beta} < 61 - 165$ meV [64] and cosmological constraints from PLANCK 2018 (disfavoured region 0.12 eV $< \sum m_i <$ 0.60 eV and very disfavoured region $\sum m_i > 0.60$ eV) [65].

is, as already said, always near the upper 3σ limit although below). These points were plotted with a darker red colour. For IO, at least one of the other observables is incompatible with its 1σ region, as happened for the best fit value, hence only the 3σ compatible points are shown for IO.

Only for normal mass orderings do we have points outside the disfavoured region. For IO, the minimum values for the 3σ region are

$$(m_{\text{lightest}})^{\text{IO}}_{\text{min}} \approx 0.018 \text{ eV} \quad (m_{\beta\beta})^{\text{IO}}_{\text{min}} \approx 0.050 \text{ eV}. \tag{3.46}$$

For NO, the compatible 3σ region covers all orders of magnitude, but the 1σ is limited from below:

$$(m_{\text{lightest}})^{\text{NO}}_{\text{min}} \approx 0.008 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.001 \text{ eV}. \tag{3.47}$$

For this model that uses the Weinberg operator to generate the neutrino masses, NO is hence the preferred mass ordering, although this means that smaller orders of magnitude for both m_1 and $m_{\beta\beta}$, which are harder to access experimentally, are still compatible with experimental values for this model.

3.4 Models with two modular A_4 symmetries - using the see-saw mechanism

In this section, we construct two models that consider that neutrinos get their mass through the type I see-saw mechanism, using different weights for the fields and modular forms in each model. Again, both models are based in two modular symmetries, A_4^l and A_4^ν, with modulus fields denoted by τ_l and τ_ν, that will acquire different VEV's, leading to a TM$_2$ mixing.

3.4.1 $A_4^l \times A_4^\nu \to A_4^D$ breaking

First, we start by discussing how the symmetry breaking from two independent $A_4^l \times A_4^\nu$ to a single A_4^D is achieved (this mechanism is shared by both models). The superfields considered for these models are L, which is a doublet of $SU(2)_L$ containing the left-handed leptons and a triplet under A_4^l, ν^c, which is a triplet under A_4^ν containing the conjugate of the right-handed neutrino fields added to the Standard Model, and H_u, an additional Higgs doublet as required in Supersymmetric models. A bi-triplet Φ, which is a triplet under both A_4^l and A_4^ν, is introduced. Y^ν represents the Yukawa couplings that in the case of modular symmetries should be modular forms. One model considers a weight zero modular form (i.e. a modular field independent constant), and the other a singlet and a triplet under A_4^ν.

We consider that neutrinos get their mass through the type I see-saw mechanism and the term from the superpotential that gives rise to a Dirac mass matrix is $\frac{1}{\Lambda} L\Phi Y^\nu \nu^c H_u$. This term is an effective term that can arise from renormalizable interactions of the fields shown with heavy messengers (not shown explicitly - a possibility for the messenger is an electroweak neutral field). Considering the multiplication rules for two triplets to get a trivial singlet, the term $\frac{1}{\Lambda} L\Phi Y^\nu \nu^c H_u$ can be explicitly expanded as:

$$\frac{1}{\Lambda}(L_1, L_2, L_3) P_{23} \begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{pmatrix} P_{23} Y^\nu(\tau_\nu) \otimes \begin{pmatrix} \nu_1^c \\ \nu_2^c \\ \nu_3^c \end{pmatrix} H_u, \tag{3.48}$$

where $Y^\nu \otimes \nu^c$ is the product between Y^ν and ν^c that gives a triplet of A_4^ν, and P_{23} is the matrix that permutes the second and third columns/rows:

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.49}$$

If Φ acquires the VEV $\langle \Phi \rangle = v_\Phi P_{23}$ (see Appendix A.3 for more details), the symmetry $A_4^l \times A_4^\nu$ is broken but given that the same transformation γ can be performed in A_4^l and A_4^ν simultaneously, there is still a single modular symmetry A_4^D, the diagonal subgroup, that is conserved. The term $\frac{1}{\Lambda} L\Phi Y^\nu \nu^c H_u$ gets the form $\frac{v_\Phi}{\Lambda}(L Y^\nu \nu^c)_{\mathbf{1}} H_u$, which implies a Dirac matrix term for the neutrinos when the Higgs doublet H_u acquires a VEV.

3.4.2 Model 1

The first model we consider is a model were the Yukawa coupling Y^ν is simply a constant. The transformation properties of fields, Yukawa couplings and masses for this model are in Table 3.6.

The Yukawa coefficients Y^l for the charged leptons are a modular form which transforms as a triplet of A_4^l with weight $2k_l = +6$, whereas Y^ν is simply a modulus independent constant, a modular form of weight 0. For the right-handed neutrino masses we consider three modular forms transforming under A_4^ν: $M_{\mathbf{1}}$ as a trivial singlet $\mathbf{1}$, $M_{\mathbf{1}'}$ as a singlet $\mathbf{1}'$ and $M_{\mathbf{3}}$ as a triplet $\mathbf{3}$, all with weights $2k_\nu = +4$. Again,

Fields	$SU(2)$	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
L	**2**	$\mathbf{3}$	$\mathbf{1}$	0	-2
e^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$+6$	$+2$
μ^c	$\mathbf{1}$	$\mathbf{1''}$	$\mathbf{1}$	$+6$	$+2$
τ^c	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1}$	$+6$	$+2$
ν^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	$+2$
$H_{u,d}$	**2**	$\mathbf{1}$	$\mathbf{1}$	0	0
Φ	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0

Yukawas/Masses	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
Y^l	$\mathbf{3}$	$\mathbf{1}$	$+6$	0
Y^ν	$\mathbf{1}$	$\mathbf{1}$	0	0
$M_{\mathbf{1}}$	$\mathbf{1}$	$\mathbf{1}$	0	$+4$
$M_{\mathbf{1'}}$	$\mathbf{1}$	$\mathbf{1'}$	0	$+4$
$M_{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{3}$	0	$+4$

Table 3.6: Transformation properties of fields, Yukawa couplings and masses for the right-handed neutrinos for model 1 using the see-saw mechanism and two modular A_4.

the weights were chosen in such a way that the modular forms acquire the desired directions as we show below.

The right-handed electron, muon and tau fields are respectively singlets $\mathbf{1}$, $\mathbf{1''}$ and $\mathbf{1'}$ of A_4^l and trivial singlets $\mathbf{1}$ of A_4^ν, with weights $2k_l = +6$ and $2k_\nu = +2$. The lepton doublets L are arranged as a triplet of A_4^l and a singlet of A_4^ν, with weights $2k_l = 0$ and $2k_\nu = -2$. In this model, the three right-handed neutrinos introduced form a triplet of A_4^ν with weight $2k_\nu = +2$. These are the correct choices for the weights such that the modular forms and fields in each term sum up to zero since the weight for the fields is not $+2k$ but $-2k$ instead (see Eq.(2.44) and how the signs of the exponents where the weights enter differ).

Note that, in spite of the charged leptons only having non-trivial singlet transformations under A_4^l and the right-handed neutrinos only under A_4^ν (which justifies the nomenclature used), the respective weights introduce non-trivial transformations under both modular symmetries for these fields.

With the fields assigned in this manner, the superpotential for this model, which can be separated into one part containing the mass terms for the charged leptons and the other the neutrino mass terms, has the following form:

$$w = w_e + w_\nu, \tag{3.50}$$

$$w_e = \left(\alpha(LY^l(\tau_l))_{\mathbf{1}}e^c + \beta(LY^l(\tau_l))_{\mathbf{1'}}\mu^c + \gamma(LY^l(\tau_l))_{\mathbf{1''}}\tau^c\right) H_d, \tag{3.51}$$

$$w_\nu = \frac{Y^\nu}{\Lambda} L\Phi\nu^c H_u + \frac{1}{2}M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}M_{\mathbf{1'}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1''}} + \frac{1}{2}M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}}. \tag{3.52}$$

The bi-triplet Φ will then acquire a VEV and the two modular symmetries are broken to a single A_4^D, as presented in Section 3.4.1, getting for w_ν (the w_e terms remain exactly the same):

$$w_\nu = y_D(L\nu^c)_{\mathbf{1}}H_u + \frac{1}{2}M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}M_{\mathbf{1'}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1''}} + \frac{1}{2}M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}}, \tag{3.53}$$

where $y_D = Y^\nu v_\Phi/\Lambda$.

A_4^D breaking

We consider now the flavour structure after A_4^D symmetry breaking. As for the model using the Weinberg operator, we assume that the charged lepton modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T =$

$\frac{3}{2} + \frac{i}{2\sqrt{3}}$, which is a stabiliser of T_τ, which implies that a residual modular Z_3^T symmetry is preserved in the charged lepton sector. At this stabiliser, the modular form Y^l, will then acquire the direction

$$Y^l(\tau_l) \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.54}$$

This direction leads to a diagonal charged lepton mass matrix when the Higgs field H_d acquires a VEV $\langle H_d \rangle = (0, v_d)$:

$$m_e = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{3.55}$$

The masses for the charged leptons can be reproduced by adjusting the parameters α, β and γ. These constants were redefined to include the constant associated with $Y^l(\tau_l)$ and v_d.

For the other modular field τ_ν, since we want to obtain the trimaximal mixing TM$_2$, which preserves the second column of the tri-bimaximal mixing matrix, the modular form $M_{\mathbf{3}}$ should acquire the direction

$$M_{\mathbf{3}}(\tau_\nu) \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{3.56}$$

which occurs for the VEV $\langle \tau_\nu \rangle = \tau_S = i$, and thus it should have an even k_ν, as happens for $2k_\nu = +4$. In this case, a residual modular Z_2^S symmetry is preserved in the neutrino sector. This implies the following structure for the right-handed neutrino mass matrix:

$$M_R = c_{\mathbf{1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + c_{\mathbf{1'}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{c_{\mathbf{3}}}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{3.57}$$

where $c_{\mathbf{1}}, c_{\mathbf{1'}}, c_{\mathbf{3}}$ are complex constants associated with the respective modular form.

The Dirac mass matrix that relates the right-handed and active neutrinos after the Higgs field H_u acquires a VEV $\langle H_u \rangle = (0, v_u)$ is simply

$$M_D = y_D v_u P_{23}. \tag{3.58}$$

Consequently, the active neutrino mass matrix for the see-saw mechanism gets the form

$$M_\nu = -M_D M_R^{-1} M_D^T = -y_D^2 v_u^2 P_{23} M_R^{-1} P_{23}. \tag{3.59}$$

We want now to diagonalize M_ν, such that $U^T M_\nu U = M_{\nu_d} =$ diag(m_1, m_2, m_3), where m_i are the neutrino masses and U is an unitary matrix. It is also true that $U^T M_\nu U = -U^T M_D M_R^{-1} M_D^T U = M_{\nu_d}$. So $M_D^T U$ also diagonalizes the matrix M_R^{-1} and thus $V = M_D^\dagger U^*$ diagonalizes M_R such that $V^T M_R V =$

$M_{R_d} = \text{diag}(M_1, M_2, M_3)$ where $M_i = -\frac{y_D^2 v_u^2}{m_i}$. Conversely, $U = M_D^* V^*$ when V diagonalizes M_R.

In the present model, when we apply the tri-bimaximal matrix in Eq.(3.21) to the heavy neutrino mass matrix, we obtain:

$$U_{TBM}^T M_R U_{TBM} = \begin{pmatrix} a & 0 & c \\ 0 & \frac{a-b}{2} + \sqrt{3}c & 0 \\ c & 0 & b \end{pmatrix}, \tag{3.60}$$

where $a = c_{\mathbf{3}} + c_{\mathbf{1}} - \frac{1}{2}c_{\mathbf{1'}}$, $b = c_{\mathbf{3}} - c_{\mathbf{1}} + \frac{1}{2}c_{\mathbf{1'}}$ and $c = \frac{\sqrt{3}}{2}c_{\mathbf{1'}}$. This matrix has only an element on the second row and second column and four elements on the corners that form a 2×2 symmetric matrix and so can be put into block diagonal form by permuting the first and second columns and rows. Thus, the full matrix can be fully diagonalized adding a matrix V_r that introduces a rotation among the first and third columns. This rotation preserves the second column so M_R is diagonalized by a TM_2 mixing matrix, since this mixing matrix can be written as the product of the TBM mixing matrix and a rotation on the first and third columns. For the present model, M_D is only a permutation, so we have that, being $V = U_{TBM}V_r$ the matrix that diagonalizes M_R, the matrix that diagonalizes M_ν is $U = P_{23}U_{TBM}V_r$, which can also be written as $U_{TBM}U_r$, where U_r is a rotation between the first and third columns. Using for U_r the parametrization given by Eq.(3.22), which implies that

$$V_r = \begin{pmatrix} \cos\theta e^{-i\alpha_1} & 0 & \sin\theta e^{i\alpha_2} \\ 0 & e^{-i\alpha_3} & 0 \\ \sin\theta e^{-i\alpha_2} & 0 & -\cos\theta e^{i\alpha_1} \end{pmatrix}, \tag{3.61}$$

we are then able to diagonalize both M_ν and M_R. Here, θ is the angle that governs the rotation and the three α_i are introduced such that M_i are purely real values.

It is also possible to start from the diagonal matrix M_{R_d} and get $U_{TBM}^T M_R U_{TBM}$. We have that

$$V_r^* M_{R_d} V_r^\dagger = \begin{pmatrix} M_1\cos^2\theta e^{2i\alpha_1} + M_3\sin^2\theta e^{-2i\alpha_2} & 0 & \frac{1}{2}(M_1 e^{i(\alpha_1+\alpha_2)} - M_3 e^{-i(\alpha_1+\alpha_2)})\sin 2\theta \\ 0 & M_2 e^{2i\alpha_3} & 0 \\ * & 0 & M_1\sin^2\theta e^{2i\alpha_2} + M_3\cos^2\theta e^{-2i\alpha_1} \end{pmatrix}, \tag{3.62}$$

and comparing with Eq.(3.60) we obtain that $\alpha_3 = \frac{1}{2}\arg\left(\frac{a-b}{2} + \sqrt{3}c\right)$ and, more importantly, we get a mass sum rule for M_i that can also be expressed in terms of the active neutrino masses m_i:

$$\begin{aligned} \frac{1}{m_2} &= -\frac{1}{y_D^2 v_u^2}\left|\frac{a-b}{2} + \sqrt{3}c\right| \\ &= \left|\frac{1}{2m_1}\left(e^{2i\alpha_1}\cos^2\theta - e^{2i\alpha_2}\sin^2\theta + \sqrt{3}e^{i(\alpha_1+\alpha_2)}\sin 2\theta\right) - \right. \\ &\qquad \left. -\frac{1}{2m_3}\left(e^{-2i\alpha_1}\cos^2\theta - e^{-2i\alpha_2}\sin^2\theta + \sqrt{3}e^{-i(\alpha_1+\alpha_2)}\sin 2\theta\right)\right|. \end{aligned} \tag{3.63}$$

This sum rule has obvious similarities with the sum rule for the model using the Weinberg operator, Eq.(3.43), which comes from the fact that M_R in this model, given by Eq.(3.57), and M_ν in the previous model, given by Eq.(3.40), have the same structure. Similarly to what can be found in [55], we can write these sum rules as

$$m_2^\eta = f_1(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\; m_1^\eta + f_3(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\; m_3^\eta \tag{3.64}$$

where

$$f_1(\theta, \alpha_1, \alpha_2, \alpha_3) = \frac{1}{2}\left(e^{-2i\alpha_1}\cos^2\theta - e^{-2i\alpha_2}\sin^2\theta - \sqrt{3}e^{-i(\alpha_1+\alpha_2)}\sin 2\theta\right)e^{2i\alpha_3} \tag{3.65}$$

$$f_3(\theta, \alpha_1, \alpha_2, \alpha_3) = -\frac{1}{2}\left(e^{2i\alpha_1}\cos^2\theta - e^{2i\alpha_2}\sin^2\theta - \sqrt{3}e^{i(\alpha_1+\alpha_2)}\sin 2\theta\right)e^{2i\alpha_3}. \tag{3.66}$$

With these definitions, we can say that for the model where we use the Weinberg operator, we choose for the exponent $\eta = +1$ and thus

$$m_2 = f_1(\theta, \alpha_1, \alpha_2, \alpha_3)\; m_1 + f_3(\theta, \alpha_1, \alpha_2, \alpha_3)\; m_3. \tag{3.67}$$

However, for the model using the see-saw mechanism, since the matrix that diagonalizes the matrix that has the same structure as M_ν in the model using the Weinberg operator is not U_r but V_r instead, apart from having $\eta = -1$ in the exponent, we will also have to exchange all the signs of the angles and complex phases. We will have then for the sum rule:

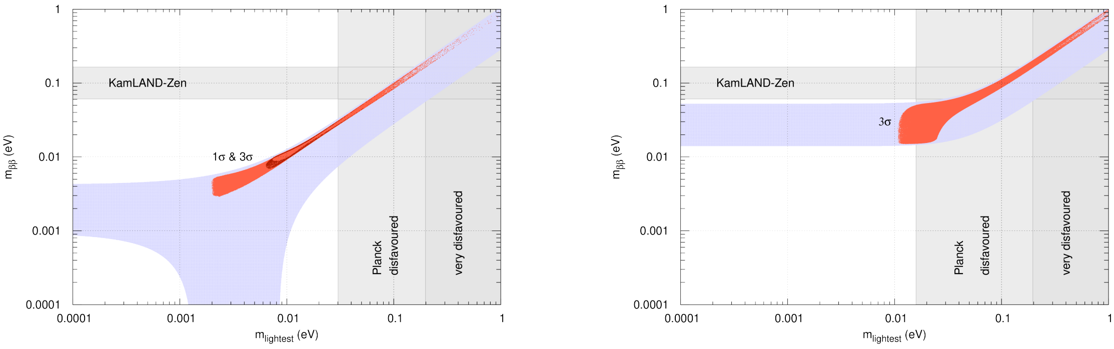
$$\frac{1}{m_2} = f_1(-\theta, -\alpha_1, -\alpha_2, -\alpha_3)\; \frac{1}{m_1} + f_3(-\theta, -\alpha_1, -\alpha_2, -\alpha_3)\; \frac{1}{m_3}. \tag{3.68}$$

The sum rule Eq.(3.63) and Eqs.(3.23-3.26) provide relations between the observables and the parameters of the TM$_2$ mixing, and so we are able to do a numerical minimisation using the χ^2 function Eq.(3.44). For the fitting, the three mixing angles, the atmospheric and solar neutrino squared mass differences and the Dirac neutrino CP violation phase were considered.

The fit parameters obtained for NO and IO of neutrino masses can be found in Table 3.7. The best fit values lie inside the 1σ range for all the observables except θ_{12}, as is characteristic of the TM$_2$ mixing, and δ for IO. Nonetheless, all the observables are within their 3σ intervals. The best-fit occurs for normal ordering of neutrino masses with a $\chi^2/6 = 1.57$.

NO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			1.57	10.51°	-67.60°	-24.26°	0.0141 eV	0.0521 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{31}	$m_{\beta\beta}$
		35.72°	49.4°	8.56°	224°	7.42$\times 10^{-5}$eV^2	2.514$\times 10^{-3}$eV^2	0.0131 eV
IO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			2.04	10.56°	-95.56°	-38.93°	0.0546 eV	0.0236 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{32}	$m_{\beta\beta}$
		35.73°	48.4°	8.61°	237°	7.42$\times 10^{-5}$eV^2	-2.496$\times 10^{-3}$eV^2	0.0174 eV

Table 3.7: Parameters (Para.) and observables (Obs.) for the best fit point for normal and inverted orderings for model 1 using the see-saw mechanism and two modular A_4.



(a) Normal Ordering

(b) Inverted Ordering

Figure 3.2: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both orderings of neutrino masses compatible with 1σ (dark-red, NO only, except θ_{12}) and 3σ data from [59] for model 1 using the see-saw mechanism and two modular A_4. In both figures were also included the current upper limit from KamLAND-Zen and PLANCK 2018 as in Figure 3.1.

It is also possible to obtain the expected $m_{\beta\beta}$ for neutrinoless beta decay using the formula

$$\begin{aligned} m_{\beta\beta} &= \left|(M_\nu)_{(1,1)}\right| = y_D^2 v_u^2 \left|(M_R^{-1})_{(1,1)}\right| \\ &= \frac{1}{3}\left|2m_1 e^{-2i\alpha_1}\cos^2\theta + m_2 e^{-2i\alpha_3} + 2m_3 e^{2i\alpha_2}\sin^2\theta\right| , \end{aligned} \tag{3.69}$$

where m_2 is given by Eq.(3.63). Doing a numerical computation, the allowed regions of m_{lightest} vs $m_{\beta\beta}$ of Figure 3.2 (for NO, $m_{\text{lightest}} = m_1$ and for IO, $m_{\text{lightest}} = m_3$) were obtained, using again as constraints the data from [59]. In both figures it is also shown the current upper limit provided by KamLAND-Zen, $m_{\beta\beta} < 61 - 165$ meV [64]. In both figures are also plotted two shadowed regions that take into account experimental results from PLANCK 2018 [65]. These constrain the sum of neutrino masses and consequently the mass of the lightest neutrino. These regions were previously discussed in Section 3.3: a very disfavoured region $m_{\text{lightest}} > 0.198$ eV for NO and $m_{\text{lightest}} > 0.196$ eV for IO (for which the limit 95%C.L.,Planck lensing+BAO+θ_{MC} was considered) and a disfavoured region $m_{\text{lightest}} > 0.030$ eV for NO and $m_{\text{lightest}} > 0.016$ eV for IO (for which the limit 95%C.L.,Planck TT,TE,EE+lowE+lensing+BAO+θ_{MC} was considered). We conclude then that only the fit for NO in Table 3.7 is outside the disfavoured region.

For NO, the points compatible with the 1σ ranges of the observables other than θ_{12} were plotted with a darker red color. For IO, at least one of the other observables is incompatible with its 1σ region, hence only the 3σ compatible points are shown. Both mass orderings have points outside the disfavoured region, although the non-disfavoured region for IO is smaller. The minimum values considering the 3σ ranges are

$$\begin{aligned} (m_{\text{lightest}})^{\text{NO}}_{\text{min}} &\approx 0.002 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.003 \text{ eV} \\ (m_{\text{lightest}})^{\text{IO}}_{\text{min}} &\approx 0.011 \text{ eV} \quad (m_{\beta\beta})^{\text{IO}}_{\text{min}} \approx 0.015 \text{ eV}, \end{aligned} \tag{3.70}$$

and the the 1σ region for NO is limited by

$$(m_{\text{lightest}})^{\text{NO}}_{\text{min}} \approx 0.006 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.007 \text{ eV} \tag{3.71}$$

Thus, for this first model using the see-saw mechanism, NO is the preferred mass ordering.

3.4.3 Model 2

It should be noted that it is possible to use other weights for the modular forms and still obtain a model with TM_2 mixing. One example is a model where we substitute the fixed constant Y^ν of the previous model by a triplet modular form under A_4^ν. However, in this case, the obtained matrix for the interactions between left and right-handed neutrinos would have a null eigenvalue associated with the eigenvector we want to preserve in TM_2, which would lead to a massless effective neutrino corresponding to the second column of the PMNS. Therefore we need to further introduce an additional term for the model to be viable, or equivalently we should somehow keep the term from the previous model but taking into account that the weights in each term must always sum to zero.

The transformation properties of fields, Yukawa couplings and masses are shown in Table 3.8. The weights for the A_4^l symmetry remain the same as in the previous model given that the content of that symmetry was not changed, only of A_4^ν, and as such just the A_4^ν weights will be introduced below.

Fields	$SU(2)$	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
L	**2**	$\mathbf{3}$	$\mathbf{1}$	0	0
e^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$+6$	0
μ^c	$\mathbf{1}$	$\mathbf{1''}$	$\mathbf{1}$	$+6$	0
τ^c	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1}$	$+6$	0
ν^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	$+4$
$H_{u,d}$	**2**	$\mathbf{1}$	$\mathbf{1}$	0	0
Φ	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0

Yukawas/Masses	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
Y^l	$\mathbf{3}$	$\mathbf{1}$	$+6$	0
$Y_{\mathbf{1}}^\nu$	$\mathbf{1}$	$\mathbf{1}$	0	$+4$
$Y_{\mathbf{3}}^\nu$	$\mathbf{1}$	$\mathbf{3}$	0	$+4$
$M_{\mathbf{1}}$	$\mathbf{1}$	$\mathbf{1}$	0	$+8$
$M_{\mathbf{1'}}$	$\mathbf{1}$	$\mathbf{1'}$	0	$+8$
$M_{\mathbf{1''}}$	$\mathbf{1}$	$\mathbf{1''}$	0	$+8$
$M_{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{3}$	0	$+8$

Table 3.8: Transformation properties of fields, Yukawa couplings and masses for the right-handed neutrinos for model 2 using the see-saw mechanism and two modular A_4.

The Yukawa coefficients $Y^l(\tau_l)$ remain the same. However, $Y_{\mathbf{1}}^\nu(\tau_\nu)$ is now a singlet under both symmetries with weights $2k_l = 0, 2k_\nu = +4$, and $Y_{\mathbf{3}}^\nu(\tau_\nu)$, a triplet under A_4^ν with weight $2k_\nu = +4$, is introduced. It is then possible to assign a null value to $2k_\nu$ for the right-handed charged leptons e^c, μ^c, τ^c and the lepton doublets L, which means that, conversely to what happened for model 1, no factors dependent on τ_ν appear in the transformation relations for these superfields. That is to say that the leptons only transform under A_4^l and the right-handed neutrinos only under A_4^ν, with no addition of $(c\tau_\nu + d)^{-2k_\nu}$ factors for the charged leptons as happened for model 1.

However, it should be pointed out that, even for model 1, it is possible to substitute the modular form of weight 0 by a singlet modular form of weight 4, thus changing the transformation properties under A_4^ν, given that different weights are now attributed. In fact, if we consider model 2 without adding the triplet $Y_{\mathbf{3}}^\nu$, using the same weights, we obtain then the same mass matrices and mixing scheme as in model 1, although now no τ_l dependent factor would appear in the transformation rule for $Y_{\mathbf{1}}^{\nu}$ and the transformation relations for the leptons would look simpler too.

As already said, for the right-handed charged leptons e^c, μ^c, τ^c and the lepton doublets L, the weights are now $2k_\nu = 0$ and consequently, for the three right-handed neutrinos ν^c, the weight is $2k_\nu = +4$ instead. This implies that the right-handed neutrino masses $M_{\mathbf{1}}$, $M_{\mathbf{1}'}$, $M_{\mathbf{1}''}$ and $M_{\mathbf{3}}$ will have now $2k_\nu = +8$. As before, these weights are chosen in such a way that the modular forms acquire the desired directions as we will see.

Again, neutrino masses can be generated through the type I see-saw mechanism, and, with the choices previously described, the superpotential has the following form:

$$w = w_e + w_\nu, \tag{3.72}$$

$$w_e = \left(\alpha(LY^l(\tau_l))_{\mathbf{1}}e^c + \beta(LY^l(\tau_l))_{\mathbf{1}'}\mu^c + \gamma(LY^l(\tau_l))_{\mathbf{1}''}\tau^c\right) H_d, \tag{3.73}$$

$$\begin{aligned} w_\nu = &\frac{1}{\Lambda}L\Phi Y_{\mathbf{1}}^{\nu}(\tau_\nu)\nu^c H_u + \frac{1}{\Lambda}L\Phi Y_{\mathbf{3}}^{\nu}(\tau_\nu)\nu^c H_u + \\ &+ \frac{1}{2}M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}M_{\mathbf{1}'}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}''} + \frac{1}{2}M_{\mathbf{1}''}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}'} + \frac{1}{2}M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}}. \end{aligned} \tag{3.74}$$

The bi-triplet Φ will then acquire a VEV as before and the two modular symmetries are broken to a single one as discussed in Section 3.4.1. As seen previously, w_ν gets the form:

$$\begin{aligned} w_\nu = &\frac{v_\Phi}{\Lambda}(LY_{\mathbf{1}}^{\nu}(\tau_\nu)\nu^c + LY_{\mathbf{3}}^{\nu}(\tau_\nu)\nu^c)_{\mathbf{1}} H_u + \\ &+ \frac{1}{2}M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}M_{\mathbf{1}'}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}''} + \frac{1}{2}M_{\mathbf{1}''}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}'} + \frac{1}{2}M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}}, \end{aligned} \tag{3.75}$$

and w_e does not change.

A_4^D breaking

The flavour structure after A_4^D breaking is now going to be covered. We assume that the modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T = \frac{3}{2} + \frac{i}{2\sqrt{3}}$, stabiliser of T_τ, as in the previous model. A residual modular Z_3^T symmetry is preserved in the charged lepton sector and, when the Higgs field H_d acquires a VEV, the Y^l direction leads to a diagonal charged lepton mass matrix as in Eq.(3.55), and the masses for the charged leptons can be reproduced by adjusting the parameters α, β and γ as before.

For the other modular field τ_ν, a residual Z_2^S symmetry is conserved, given that, as seen previously, the modulus should acquire the VEV $\langle\tau_\nu\rangle = \tau_S = i$ for $M_{\mathbf{3}}$ to have the direction $(1,1,1)$, and thus $M_{\mathbf{3}}$ must have an even k_ν, as happens for $2k_\nu = +8$. Although in general we would have to consider instead of the mass triplet $M_{\mathbf{3}}$ two triplets $M_{\mathbf{3}_1}$ and $M_{\mathbf{3}_2}$ arising from each triplet of weight 8 in Eqs.(3.13-3.14), in this case, given that the second weight 8 triplet vanishes at this stabiliser, we only have to consider the first weight 8 triplet. From Table 3.3 one knows also that none of the three singlets of weight 8 (Eqs.(3.10-3.12)) vanishes at this stabiliser, which accounts for the existence of $M_{\mathbf{1}}$, $M_{\mathbf{1}'}$ and $M_{\mathbf{1}''}$. This implies the following structure for the right-handed neutrino mass matrix:

$$M_R = c_{\mathbf{1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + c_{\mathbf{1}'} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c_{\mathbf{1}''} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{c_{\mathbf{3}}}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{3.76}$$

where $c_{\mathbf{1}}, c_{\mathbf{1}'}, c_{\mathbf{1}''}, c_{\mathbf{3}}$ are complex constants associated with the respective modular forms. The VEV the modulus field τ_ν acquires will also imply a direction $(1,1,1)$ for $Y^\nu_{\mathbf{3}}(\tau_\nu)$.

The term $(LY^\nu_{\mathbf{3}}(\tau_\nu)\nu^c)_{\mathbf{1}}$ will lead to two independent contributions, one when we multiply two of the triplets to get a symmetric triplet and then obtain a singlet by multiplying with the third triplet, and the other when we consider in the first step the antisymmetric contribution instead. In order to obtain the Dirac mass matrix, the $Y^\nu_{\mathbf{1}}$ term is added to these two terms, all with different multiplicative constants, and we thus obtain $g_1 Y^\nu_{\mathbf{1}}(\tau_\nu)(L\nu^c)_{\mathbf{1}} + g_2((LY^\nu_{\mathbf{3}}(\tau_\nu))_{\mathbf{3}_S}\nu^c)_{\mathbf{1}} + g_3((LY^\nu_{\mathbf{3}}(\tau_\nu))_{\mathbf{3}_A}\nu^c)_{\mathbf{1}}$, where the complex constants g_i already account for the constants associated with the modular forms. The Dirac mass matrix that relates the right-handed and active neutrinos after the Higgs field H_u acquires a VEV $\langle H_u \rangle = (0, v_u)$ will then be

$$M_D = y_D v_u g_1 \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{h_2}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \frac{h_3}{2} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \right), \tag{3.77}$$

where $h_i = g_i/g_1$ (g_1 was chosen for the denominator given that, for the model to hold, it can not be taken to zero, as stated in the beginning of this section). Again, the active neutrino mass matrix for the see-saw mechanism is obtained through the following formula:

$$M_\nu = -M_D M_R^{-1} M_D^T. \tag{3.78}$$

The expressions for the entries of the neutrino mass matrix are much more complicated in this case since the Dirac mass matrix is not a permutation matrix as before. Nevertheless, the see-saw mass matrix is diagonalized by the TM$_2$ mixing matrix and again the PMNS matrix corresponds to the TM$_2$ mixing.

Doing a similar derivation to what was done for the first model, the mass sum rule for this model is

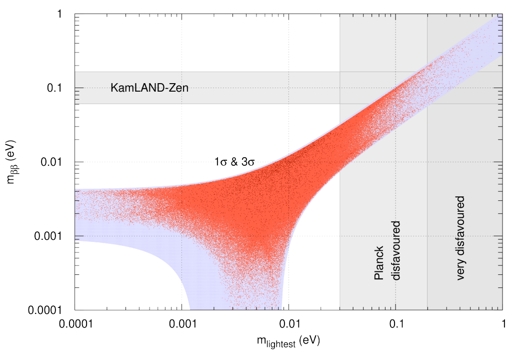
$$\begin{aligned} \frac{1}{m_2} = \Bigg| \frac{1}{8m_1}\Big(& e^{2i\alpha_1}\left(4h_2^2 + 12h_2h_3 - 3h_3^2 + 8h_2 + 12h_3 + 4\right)\cos^2\theta - \\ & - e^{2i\alpha_2}\left(4h_2^2 + 12h_2h_3 - 3h_3^2 - 8h_2 - 12h_3 + 4\right)\sin^2\theta - \\ & - \sqrt{3}e^{i(\alpha_1+\alpha_2)}\left(4h_2^2 - 4h_2h_3 - 3h_3^2 - 4\right)\sin 2\theta\Big) - \\ - \frac{1}{8m_3}\Big(& e^{-2i\alpha_1}\left(4h_2^2 + 12h_2h_3 - 3h_3^2 - 8h_2 - 12h_3 + 4\right)\cos^2\theta - \\ & - e^{-2i\alpha_2}\left(4h_2^2 + 12h_2h_3 - 3h_3^2 + 8h_2 + 12h_3 + 4\right)\sin^2\theta - \\ & - \sqrt{3}e^{-i(\alpha_1+\alpha_2)}\left(4h_2^2 - 4h_2h_3 - 3h_3^2 - 4\right)\sin 2\theta\Big) - 3M_{\mathbf{1}''}\Bigg|, \end{aligned} \tag{3.79}$$

Note that when the extra parameters introduced in model 2 vanish, $h_2 = h_3 = M_{\mathbf{1}''} = 0$, the sum rule for model 1, Eq.(3.63), is recovered.

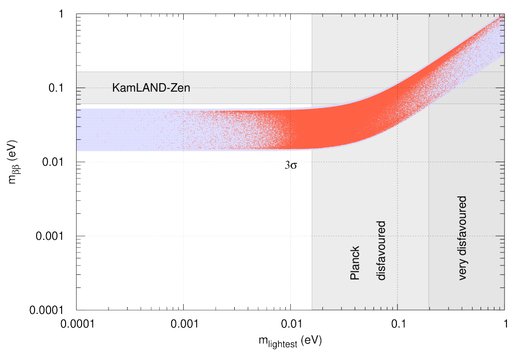
Using this new sum rule with extra parameters, the best fit values shown in Table 3.9 were obtained. For NO, all the observables except θ_{12} are compatible with their 1σ ranges. For IO, δ is also outside its 1σ region, as happened for model 1, even though all of the observables are still within their 3σ ranges. As for model 1, NO provides the best fit, with $\chi^2/6 = 1.57$, which is the same value found for model 1 using the see-saw mechanism and also for the model using the Weinberg operator. This is not surprising given the contribution to the χ^2 is coming not from the masses, but from the mixing angles, and all these models give TM$_2$ mixing.

NO	Para.	$\chi^2/6$	θ	α_1	α_2	h_2	h_3	$M_{\mathbf{1}''}$	m_1	m_3
		1.57	10.51°	102.16°	145.49°	$-1.052-0.375i$	$0.788+0.114i$	$2.099+2.675i$	0.0131 eV	0.0518 eV
	Obs.			θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{31}	$m_{\beta\beta}$
				35.72°	49.4°	8.56°	224°	7.42×10^{-5}eV^2	2.514×10^{-3}eV^2	0.0106 eV
IO	Para.	$\chi^2/6$	θ	α_1	α_2	h_2	h_3	$M_{\mathbf{1}''}$	m_1	m_3
		2.03	-10.56°	-6.17°	-131.13°	$0.767-0.825i$	$-0.419+0.283i$	$1.855-0.727i$	0.0929 eV	0.0788 eV
	Obs.			θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{32}	$m_{\beta\beta}$
				35.73°	48.5°	8.60°	236°	7.42×10^{-5}eV^2	-2.498×10^{-3}eV^2	0.0397 eV

Table 3.9: Parameters (Para.) and observables (Obs.) for the best fit point for normal and inverted orderings for model 2 using the see-saw mechanism and two modular A_4.



(a) Normal Ordering



(b) Inverted Ordering

Figure 3.3: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both orderings of neutrino masses compatible with 1σ (dark-red, NO only, except θ_{12}) and 3σ data from [59] for model 2 using the see-saw mechanism and two modular A_4. In both figures were also included the current upper limit from KamLAND-Zen and PLANCK 2018 as in Figure 3.1.

Furthermore, using Eq.(3.69) with m_2 given by Eq.(3.79), the allowed regions of m_{lightest} vs $m_{\beta\beta}$, shown in Figure 3.3 (as before, for NO, $m_{\text{lightest}} = m_1$ and for IO, $m_{\text{lightest}} = m_3$) were obtained. The experimental constrains that were already discussed in Section 3.3 are also included. As before, there are some points (with a darker red color) in the NO figure compatible with the 1σ ranges for the observables other than θ_{12}. For IO again no 1σ compatible points are shown, given that all points are outside the 1σ range for at least one of the other observables, as occurred for the best fit values.

However, these 1σ points for NO do not form a characteristic structure as happened for model 1 but are dispersed within the other points that have at least an observable other than θ_{12} outside its 1σ region.

For the fits shown in Table 3.9, only NO is outside the disfavoured region. From what has been written, it is inferred that, similarly to what happened for model 1, NO is once more the preferred mass ordering.

This second model is much less restrictive, and thus predictive, than the first one, since m_{lightest} covers all orders of magnitude and almost all the available region for m_{lightest} vs $m_{\beta\beta}$ and, more importantly, the minimum value for $m_{\beta\beta}$ also approaches zero. More specifically, model 1 is simply a special case of model 2 when we neglect all the extra parameters that where introduced in model 2 due to the new terms that appear when we assign a higher weight to the modular forms Y^{ν}.

Chapter 4

Two A_5 Modular Symmetries for Golden Ratio 2 Mixing

In this chapter, we construct two models that use two A_5 modular symmetries in order to obtain the golden ratio mixing plus a rotation among the first and the third columns, one using the Weinberg operator and the other the see-saw mechanism to generate the neutrino masses. At high energies, the model is based in two modular symmetries, A_5^l and A_5^ν, with modulus fields denoted by τ_l and τ_ν, respectively. After the modulus fields acquire different VEV's, different mass textures are realised in the charged lepton and neutrino sectors. We will start by introducing some properties of the A_5 modular symmetry group. Subsequently, the various possibilities of a golden ratio mixing and a rotation among two of its columns are investigated and concluded that only a rotation between the first and third column is compatible with the 3σ confidence interval. Only then will these two models be introduced.

We note once again that [52] already employs a single A_5 modular symmetry and two moduli in models using the Weinberg operator to generate the neutrino masses. The model that uses some fixed points of the modular fields lead to the same mixing we are going to discuss here, although that is not explicit in [52].

4.1 Modular A_5 symmetry and residual symmetries

In the following subsection the A_5 symmetry group is introduced including some of its main properties as the modular forms of level 5 and its stabilisers, which apply for the specific case of A_5 modular symmetries and, as well as the stabilisers for the modular groups from $N = 2$ to 5, can be found in [62].

4.1.1 Modular A_5 symmetry and modular forms of level 5

The group A_5 is the group of even permutations of 5 objects and has 60 elements. It is generated by two operators S_τ and T_τ obeying

$$S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^5 = 1. \tag{4.1}$$

This group has one singlet $\mathbf{1}$, two triplets $\mathbf{3}$ and $\mathbf{3'}$, one quadruplet $\mathbf{4}$ and one quintuplet $\mathbf{5}$ as its irreducible representations. The irreducible representations of the generators and the multiplication rules for the irreducible representations can be found in Appendix B.1.

Similarly to what was done for $\Gamma_3 \sim A_4$, the Yukawa couplings in a theory that is invariant under a $\Gamma_5 \sim A_5$ symmetry are also going to be modular forms, but in this case of level 5. The eleven linearly independent weight 2 modular forms of level 5 form a quintuplet $Y_{\mathbf{5}}^{(2)} = (Y_1, Y_2, Y_3, Y_4, Y_5)$ of A_5, a triplet $\mathbf{3}$ $Y_{\mathbf{3}}^{(2)} = (Y_6, Y_7, Y_8)$ and a triplet $\mathbf{3'}$ $Y_{\mathbf{3'}}^{(2)} = (Y_9, Y_{10}, Y_{11})$. These modular functions can be expressed in terms of the third theta function (see Appendix B.2 for more details). The modular forms of higher weight are generated starting from these eleven modular forms of weight 2.

The space of the weight 4 modular forms of level 5 has dimension 21 and decomposes into a singlet $\mathbf{1}$, one triplet $\mathbf{3}$, one triplet $\mathbf{3'}$, a quadruplet $\mathbf{4}$ and two quintuplets $\mathbf{5}$. Using the weight 2 modular forms, one obtains the following expressions for the weight 4 modular forms [52]:

$$Y_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5, \tag{4.2}$$

$$Y_{\mathbf{3}}^{(4)} = \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}Y_5Y_7 + \sqrt{3}Y_2Y_8 \\ \sqrt{3}Y_2Y_6 + Y_1Y_7 - \sqrt{6}Y_3Y_8 \\ \sqrt{3}Y_5Y_6 - \sqrt{6}Y_4Y_7 + Y_1Y_8 \end{pmatrix}, \tag{4.3}$$

$$Y_{\mathbf{3'}}^{(4)} = \begin{pmatrix} \sqrt{3}Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}Y_2Y_7 - \sqrt{2}Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}Y_3Y_7 - \sqrt{2}Y_5Y_8 \end{pmatrix}, \tag{4.4}$$

$$Y_{\mathbf{4}}^{(4)} = \begin{pmatrix} 2Y_4^2 + \sqrt{6}Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}Y_1Y_5 \end{pmatrix}, \tag{4.5}$$

$$Y_{\mathbf{5}_1}^{(4)} = \begin{pmatrix} \sqrt{2}Y_1^2 + \sqrt{2}Y_3Y_4 - 2\sqrt{2}Y_2Y_5 \\ \sqrt{3}Y_4^2 - 2\sqrt{2}Y_1Y_2 \\ \sqrt{2}Y_1Y_3 + 2\sqrt{3}Y_4Y_5 \\ 2\sqrt{3}Y_2Y_3 + \sqrt{2}Y_1Y_4 \\ \sqrt{3}Y_3^2 - 2\sqrt{2}Y_1Y_5 \end{pmatrix}, \tag{4.6}$$

$$Y_{\mathbf{5}_2}^{(4)} = \begin{pmatrix} \sqrt{3}Y_5Y_7 - \sqrt{3}Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}Y_1Y_7 - \sqrt{2}Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}Y_4Y_7 + \sqrt{3}Y_1Y_8 \end{pmatrix}. \tag{4.7}$$

Furthermore, the modular forms of weight 6, whose linear space has dimension 31 and decomposes into one singlet $\mathbf{1}$, two triplets $\mathbf{3}$, two triplets $\mathbf{3'}$, two quadruplet $\mathbf{4}$ and two quintuplets $\mathbf{5}$, are the following according to [52]:

$$Y_{\mathbf{1}}^{(6)} = 3\sqrt{3}\left(Y_2Y_3^2 + Y_4^2Y_5\right) + \sqrt{2}Y_1\left(Y_1^2 + 3Y_3Y_4 - 6Y_2Y_5\right), \tag{4.8}$$

$$Y_{\mathbf{3}_1}^{(6)} = \left(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5\right)\begin{pmatrix} Y_6 \\ Y_7 \\ Y_8 \end{pmatrix}, \tag{4.9}$$

$$Y_{\mathbf{3}_2}^{(6)} = \begin{pmatrix} \left(Y_5Y_6 - \sqrt{2}Y_4Y_7\right)Y_7 + \left(\sqrt{2}Y_3Y_8 - Y_2Y_6\right)Y_8 \\ \left(\sqrt{3}Y_1Y_6 - Y_5Y_7\right)Y_7 - \sqrt{2}Y_3Y_6Y_8 + \left(Y_6^2 - Y_7Y_8\right)Y_2 \\ \left(Y_2Y_8 - \sqrt{3}Y_1Y_6\right)Y_8 + \sqrt{2}Y_4Y_6Y_7 - \left(Y_6^2 - Y_7Y_8\right)Y_5 \end{pmatrix}, \tag{4.10}$$

$$Y_{\mathbf{3'}_1}^{(6)} = \left(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5\right)\begin{pmatrix} Y_9 \\ Y_{10} \\ Y_{11} \end{pmatrix}, \tag{4.11}$$

$$Y_{\mathbf{3'}_2}^{(6)} = \begin{pmatrix} \left(Y_4Y_6 - \sqrt{2}Y_3Y_7 - \sqrt{2}Y_5Y_8\right)Y_{10} - \left(Y_3Y_6 - \sqrt{2}Y_2Y_7 - \sqrt{2}Y_4Y_8\right)Y_{11} \\ \left(Y_3Y_6 - \sqrt{2}Y_2Y_7 - \sqrt{2}Y_4Y_8\right)Y_9 - \left(\sqrt{3}Y_1Y_6 + Y_5Y_7 + Y_2Y_8\right)Y_{10} \\ \left(\sqrt{3}Y_1Y_6 + Y_5Y_7 + Y_2Y_8\right)Y_{11} - \left(Y_4Y_6 - \sqrt{2}Y_3Y_7 - \sqrt{2}Y_5Y_8\right)Y_9 \end{pmatrix}, \tag{4.12}$$

$$Y_{\mathbf{4}_1}^{(6)} = \begin{pmatrix} \sqrt{2}\left(\sqrt{6}Y_3Y_8 - \sqrt{3}Y_2Y_6 - Y_1Y_7\right)Y_9 - \left(\sqrt{3}Y_5Y_6 - \sqrt{6}Y_4Y_7 + Y_1Y_8\right)Y_{10} \\ \left(\sqrt{3}Y_5Y_6 - \sqrt{6}Y_4Y_7 + Y_1Y_8\right)Y_{11} + \sqrt{2}\left(\sqrt{3}Y_5Y_7 - 2Y_1Y_6 + \sqrt{3}Y_2Y_8\right)Y_{10} \\ \left(\sqrt{3}Y_2Y_6 + Y_1Y_7 - \sqrt{6}Y_3Y_8\right)Y_{10} + \sqrt{2}\left(\sqrt{3}Y_5Y_7 - 2Y_1Y_6 + \sqrt{3}Y_2Y_8\right)Y_{11} \\ \sqrt{2}\left(\sqrt{6}Y_4Y_7 - \sqrt{3}Y_5Y_6 - Y_1Y_8\right)Y_9 - \left(\sqrt{3}Y_2Y_6 + Y_1Y_7 - \sqrt{6}Y_3Y_8\right)Y_{11} \end{pmatrix}, \tag{4.13}$$

$$Y_{\mathbf{4}_2}^{(6)} = \begin{pmatrix} \sqrt{2}\left(\sqrt{3}Y_1Y_6 + Y_5Y_7\right)Y_7 + \left(Y_3Y_6 - \sqrt{2}Y_4Y_8\right)Y_8 \\ \sqrt{2}\left(\sqrt{2}Y_2Y_7 - Y_3Y_6\right)Y_6 + \left(Y_4Y_6 + \sqrt{2}Y_3Y_7 + \sqrt{2}Y_5Y_8\right)Y_8 \\ \sqrt{2}\left(\sqrt{2}Y_5Y_8 - Y_4Y_6\right)Y_6 + \left(Y_3Y_6 + \sqrt{2}Y_2Y_7 + \sqrt{2}Y_4Y_8\right)Y_7 \\ \sqrt{2}\left(\sqrt{3}Y_1Y_6 + Y_2Y_8\right)Y_8 + \left(Y_4Y_6 - \sqrt{2}Y_3Y_7\right)Y_7 \end{pmatrix}, \tag{4.14}$$

$$Y_{\mathbf{5}_1}^{(6)} = \left(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5\right)\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}, \tag{4.15}$$

$$Y_{\mathbf{5}_2}^{(6)} = \begin{pmatrix} \sqrt{3}\left(\sqrt{3}Y_1Y_6 + Y_5Y_7 + Y_2Y_8\right)Y_6 \\ \left(Y_5Y_7 + \sqrt{3}Y_1Y_6\right)Y_7 + \left(3Y_2Y_7 + 2Y_4Y_8 - \sqrt{2}Y_3Y_6\right)Y_8 \\ \left(Y_3Y_6 - \sqrt{2}Y_2Y_7\right)Y_6 + 2\left(Y_5Y_8 + Y_3Y_7 - \sqrt{2}Y_4Y_6\right)Y_8 \\ \left(Y_4Y_6 - \sqrt{2}Y_5Y_8\right)Y_6 + 2\left(Y_2Y_7 + Y_4Y_8 - \sqrt{2}Y_3Y_6\right)Y_7 \\ \left(Y_2Y_8 + \sqrt{3}Y_1Y_6\right)Y_8 + \left(3Y_5Y_8 + 2Y_3Y_7 - \sqrt{2}Y_4Y_6\right)Y_7 \end{pmatrix}. \tag{4.16}$$

4.1.2 Stabilisers and residual symmetries of modular A_5

As explained in Section 3.1.2, stabilisers of the symmetry play a crucial role in residual symmetries. Given an element γ in the modular group A_5, a stabiliser of γ corresponds to a fixed point in the upper half complex plane that transforms as $\gamma\tau_\gamma = \tau_\gamma$. Once the modular field acquires a VEV at this special point, $\langle\tau\rangle = \tau_\gamma$, the modular symmetry is broken but an Abelian residual modular symmetry generated by γ is preserved. Obviously, acting γ on the modular form at its stabiliser leaves the modular form invariant, which implies that, at the stabiliser, the modular form is an eigenvector of the representation matrix $\rho_I(\gamma)$ for the given stabiliser that corresponds to the eigenvalue $(c\tau_\gamma + d)^{-2k}$, and thus the directions of the modular forms at the stabilisers can be easily determined (see Eq.(3.16)).

The stabilisers for the A_5 modular group are shown in Table 4.1 and can be found in [62].

γ	τ_γ
$T_\tau\,,\,T_\tau^2\,,\,T_\tau^3\,,\,T_\tau^4$	$i\infty\,,\,\frac{8}{5}$
S_τ	$i\,,\,-\frac{70}{29}+\frac{i}{29}$
$T_\tau S_\tau\,,\,T_\tau S_\tau T_\tau S_\tau$	$\frac{1}{2}+\frac{i\sqrt{3}}{2}\,,\,-\frac{37}{26}+\frac{i}{26\sqrt{3}}$
$S_\tau T_\tau\,,\,S_\tau T_\tau S_\tau T_\tau$	$-\frac{1}{2}+\frac{i\sqrt{3}}{2}\,,\,\frac{91}{38}+\frac{i\sqrt{3}}{38}$

Table 4.1: Stabilisers for some of the A_5 elements [62].

For the transformations S_τ, T_τ, $S_\tau T_\tau$ and $T_\tau S_\tau$, the coefficients $(c\tau_\gamma + d)^{-2k}$ are

$$(c\tau_\gamma + d)^{-2k} = \begin{cases} (-1)^k & \tau_{S_{\tau 1}} = i \\ 1 & \tau_{T_{\tau 1}} = i\infty \end{cases} . \tag{4.17}$$

The directions of the modular forms of weight $2k = 2$ and 4 for the stabilisers of the generators S and T are shown in Table 4.2. Additionally, we include the factors for each modular form. These factors are written in function of Y, which is defined in general as the first component Y_1 of $Y_{\mathbf{5}}^{(2)}$. For Y, the definitions for the weight 2 modular forms present in Appendix B.2 were used. The value the modular form singlet of weight 4 takes at the stabilisers is also included.

4.2 Golden ratio mixing and related mixings

The golden ratio (GR) mixing is a mixing associated in previous works with models based in the A_5 symmetry, and this is not different for models using multiple modular A_5. The mixing matrix that we will use is

$$U_{GR} = \begin{pmatrix} \frac{\phi}{\sqrt{2+\phi}} & \frac{1}{\sqrt{2+\phi}} & 0 \\ -\frac{1}{\sqrt{4+2\phi}} & \frac{\phi}{\sqrt{4+2\phi}} & 1/\sqrt{2} \\ -\frac{1}{\sqrt{4+2\phi}} & \frac{\phi}{\sqrt{4+2\phi}} & -1/\sqrt{2} \end{pmatrix}, \tag{4.18}$$

where $\phi = \frac{1+\sqrt{5}}{2}$. This mixing has the same problem as the TBM mixing: it is incompatible with the experimental results for θ_{13}, and thus we want to work with models that preserve only the first or the second columns of the GR mixing matrix, that can be written as the GR matrix times a rotation between the other two columns.

For a model where the second column is preserved, the matrix that diagonalizes M_ν is $U = U_{GR}U_r$,

τ_γ		$\tau_{S_{\tau 1}} = i$	$\tau_{T_{\tau 1}} = i\infty$
weight 2	$\mathbf{5}$	$Y \begin{pmatrix} 1 \\ \frac{-1-\sqrt{7-4\phi}}{\sqrt{6}} \\ \frac{-1-\sqrt{18-11\phi}}{\sqrt{6}} \\ \frac{-1+\sqrt{18-11\phi}}{\sqrt{6}} \\ \frac{-1+\sqrt{7-4\phi}}{\sqrt{6}} \end{pmatrix}$	$Y \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
	$\mathbf{3}$	$Y \begin{pmatrix} \sqrt{\frac{58-31\phi}{15}} \\ \frac{-9+8\phi+\sqrt{27-4\phi}}{\sqrt{30}} \\ \frac{9-8\phi+\sqrt{27-4\phi}}{\sqrt{30}} \end{pmatrix}$	$\sqrt{\frac{3}{5}} Y \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\mathbf{3'}$	$Y \begin{pmatrix} -\sqrt{\frac{3+4\phi}{15}} \\ \frac{7-4\phi+\sqrt{2+\phi}}{\sqrt{30}} \\ \frac{-7+4\phi+\sqrt{2+\phi}}{\sqrt{30}} \end{pmatrix}$	$-\sqrt{\frac{3}{5}} Y \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
weight 4	$\mathbf{1}$	$\frac{15\sqrt{5}-25}{6} Y^2$	Y^2
	$\mathbf{3}$	$-\sqrt{\frac{100-40\sqrt{5}}{3}} Y^2 \begin{pmatrix} 1 \\ -\frac{\sqrt{3-\sqrt{5}}}{2} \\ -\frac{\sqrt{3-\sqrt{5}}}{2} \end{pmatrix}$	$-2\sqrt{\frac{3}{5}} Y^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\mathbf{3'}$	$\sqrt{\frac{125-55\sqrt{5}}{2}} Y^2 \begin{pmatrix} 1 \\ \frac{\sqrt{3+\sqrt{5}}}{2} \\ \frac{\sqrt{3+\sqrt{5}}}{2} \end{pmatrix}$	$\frac{3}{\sqrt{5}} Y^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\mathbf{4}$	$\frac{Y^2}{12} \begin{pmatrix} 25 - 15\sqrt{5} - 5\sqrt{10-2\sqrt{5}} \\ 25 - 15\sqrt{5} + 5\sqrt{130-58\sqrt{5}} \\ 25 - 15\sqrt{5} - 5\sqrt{130-58\sqrt{5}} \\ 25 - 15\sqrt{5} + 5\sqrt{10-2\sqrt{5}} \end{pmatrix}$	0
	$\mathbf{5}_1$	$Y^2 \begin{pmatrix} \frac{1}{6}\sqrt{15\sqrt{5}+35} \\ \frac{-11\sqrt{5}+2\sqrt{250-110\sqrt{5}}+35}{4\sqrt{3}} \\ -\frac{-7\sqrt{5}+2\sqrt{5\left(5-2\sqrt{5}\right)}+15}{2\sqrt{3}} \\ \sqrt{\frac{5}{3}\left(5-2\sqrt{5}\right)} + \sqrt{\frac{5}{6}\left(47-21\sqrt{5}\right)} \\ \frac{-11\sqrt{5}-2\sqrt{250-110\sqrt{5}}+35}{4\sqrt{3}} \end{pmatrix}$	$\sqrt{2} Y^2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
	$\mathbf{5}_2$	$Y^2 \begin{pmatrix} -\frac{7-\sqrt{45}}{\sqrt{3}} \\ -\frac{1}{3}\sqrt{-173\sqrt{5}+8\sqrt{10-2\sqrt{5}}+407} \\ \frac{2}{3}\sqrt{-61\sqrt{5}+4\sqrt{1930-862\sqrt{5}}+143} \\ -\frac{2}{3}\sqrt{-61\sqrt{5}-4\sqrt{1930-862\sqrt{5}}+143} \\ -\frac{1}{3}\sqrt{-173\sqrt{5}-8\sqrt{10-2\sqrt{5}}+407} \end{pmatrix}$	0
Y		$2.594\ldots i$	$\sqrt{\frac{2}{3}}\pi i$

Table 4.2: Directions for the modular forms of weight 2 and 4 of level 5 for the A_5 generators.

where U_r is a rotation between the first and third columns. Using the parametrisation

$$U_r = \begin{pmatrix} \cos\theta e^{i\alpha_1} & 0 & \sin\theta e^{-i\alpha_2} \\ 0 & e^{i\alpha_3} & 0 \\ -\sin\theta e^{i\alpha_2} & 0 & \cos\theta e^{-i\alpha_1} \end{pmatrix}, \tag{4.19}$$

we are then able to diagonalize M_ν. Here, θ is the angle that governs the rotation and the three α_i are introduced such that m_i are purely real values.

The angles and phases from the standard parametrisation of the PMNS matrix in [57] can be expressed in terms of the model parameters θ, α_1 and α_2 using the expressions between the parameters and the PMNS matrix elements:

$$\sin^2\theta_{13} = |U_{e3}|^2 = \frac{5+\sqrt{5}}{10}\sin^2\theta \tag{4.20}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{3-\sqrt{5}}{4-\sqrt{5}+\cos 2\theta} \tag{4.21}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{4-\sqrt{5}+\cos 2\theta - 2\sqrt{5-2\sqrt{5}}\sin 2\theta\cos(\alpha_1-\alpha_2)}{8-2\sqrt{5}+2\cos 2\theta} \tag{4.22}$$

$$\begin{aligned}\delta &= -\arg\left(\frac{U_{e3}U_{\tau 1}U^*_{e1}U^*_{\tau 3}}{\cos\theta_{12}\sin\theta_{13}\cos^2\theta_{13}\cos\theta_{23}} + \cos\theta_{12}\sin\theta_{13}\cos\theta_{23}\right)\\ &= \arg\left(\sin 2\theta\left(\frac{5+\sqrt{5}}{2}e^{-i(\alpha_1-\alpha_2)}\cos^2\theta - e^{i(\alpha_1-\alpha_2)}\sin^2\theta\right)\right).\end{aligned} \tag{4.23}$$

Using the 3σ C.L. range of $\sin^2\theta_{13}$ for NO(IO), $0.02034(0.02053) \to 0.02430(0.02436)$ [59], we obtain the allowed range for $\sin\theta$:

$$0.1677(0.1684) \lesssim |\sin\theta| \lesssim 0.1833(0.1835), \tag{4.24}$$

which implies also ranges for the other mixing angles (using that $-1 \leq \cos(\alpha_1-\alpha_2) \leq 1$):

$$0.2821(0.2822) \lesssim \sin^2\theta_{12} \lesssim 0.2833(0.2833) \tag{4.25}$$

$$0.4029(0.4028) \lesssim \sin^2\theta_{23} \lesssim 0.5971(0.5972). \tag{4.26}$$

The 1σ NuFit region is within the interval found for $\sin^2\theta_{23}$, which overlaps with the 3σ region for this parameter, with our result extending below 0.407(0.411) for NO(IO) and not reaching its upper limit. The range of allowed values for $\sin^2\theta_{12}$ is near the lowest limit of the 1σ region although outside.

For a model where the first column is preserved instead, the rotation matrix U_r between the second and third columns can be parametrised as:

$$U_r = \begin{pmatrix} e^{i\alpha_3} & 0 & 0 \\ 0 & \cos\theta e^{i\alpha_1} & \sin\theta e^{-i\alpha_2} \\ 0 & -\sin\theta e^{i\alpha_2} & \cos\theta e^{-i\alpha_1}\end{pmatrix}, \tag{4.27}$$

Again, θ is the angle that governs the rotation and the three α_i are introduced such that the three neutrino masses m_i have purely real values.

For this model, the expressions for the angles and phases from the standard parametrisation of the PMNS matrix in [57] in terms of the model parameters θ, α_1 and α_2 are

$$\sin^2\theta_{13} = |U_{e3}|^2 = \frac{5-\sqrt{5}}{10}\sin^2\theta \tag{4.28}$$

$$\sin^2\theta_{12} = \frac{|U_{e2}|^2}{1-|U_{e3}|^2} = \frac{2\cos^2\theta}{4+\sqrt{5}+\cos 2\theta} \tag{4.29}$$

$$\sin^2\theta_{23} = \frac{|U_{\mu 3}|^2}{1-|U_{e3}|^2} = \frac{4+\sqrt{5}+\cos 2\theta + 2\sqrt{5+2\sqrt{5}}\sin 2\theta\cos(\alpha_1-\alpha_2)}{8+2\sqrt{5}+2\cos 2\theta} \tag{4.30}$$

$$\begin{aligned}\delta &= -\arg\left(\frac{U_{e3}U_{\tau 1}U_{e1}^*U_{\tau 3}^*}{\cos\theta_{12}\sin\theta_{13}\cos^2\theta_{13}\cos\theta_{23}} + \cos\theta_{12}\sin\theta_{13}\cos\theta_{23}\right)\\ &= \arg\left(\sin 2\theta\left(\frac{5-\sqrt{5}}{2}e^{-i(\alpha_1-\alpha_2)}\cos^2\theta - e^{i(\alpha_1-\alpha_2)}\sin^2\theta\right)\right).\end{aligned} \tag{4.31}$$

Using the 3σ C.L. range of $\sin^2\theta_{13}$ for NO(IO), $0.02034(0.02053) \to 0.02430(0.02436)$ [59], we obtain the allowed range for $\sin\theta$:

$$0.2713(0.2725) \lesssim |\sin\theta| \lesssim 0.2965(0.2969), \tag{4.32}$$

which implies also ranges for the other mixing angles (using that $-1 \leq \cos(\alpha_1-\alpha_2) \leq 1$):

$$0.2584(0.2583) \lesssim \sin^2\theta_{12} \lesssim 0.2614(0.2612) \tag{4.33}$$

$$0.2531(0.2528) \lesssim \sin^2\theta_{23} \lesssim 0.7469(0.7472). \tag{4.34}$$

We conclude that the range of allowed values for $\sin^2\theta_{12}$ is outside the 3σ region and thus the class of models that preserve the first column of the golden ratio mixing matrix, which we call GR$_1$ mixing, are disfavoured by experiment.

Consequently, in the following we are only interested in models that preserve the second column of the golden ratio mixing, which we call GR$_2$, although, as pointed out previously, even for these models $\sin^2\theta_{12}$ is outside the experimental 1σ interval.

4.3 Models with two modular A_5 symmetries - using the Weinberg operator

Now that the A_5 modular symmetry and the mixing derived from the GR mixing were introduced, the models that use this symmetry in order to get what we called the GR$_2$ mixing can now be described. We will start by constructing one model where it is assumed that neutrinos get their mass through the Weinberg operator, and afterwards another model where the see-saw mechanism is used is introduced. At high energies, these models are based in two modular symmetries, A_5^l and A_5^ν, with modulus fields denoted by τ_l and τ_ν, respectively. After the modulus fields acquire different VEV's, different mass textures are realised in the charged lepton and neutrino sectors, in such a way that the GR$_2$ mixing is recovered for the PMNS.

In this section we consider that neutrinos get their mass through an effective term of the type $\frac{1}{\Lambda}YL^2H_u^2$. The transformation properties of fields and Yukawa couplings can be found in Table 4.3.

All the Yukawa coefficients Y^l and Y^ν are modular forms of weight 4. The right-handed lepton fields E^c are arranged as a triplet $\mathbf{3}$ or $\mathbf{3}^{(\prime)}$ of A_5^l and singlets $\mathbf{1}$ of A_5^ν, with weights $2k_l = +4$ and $2k_\nu = -2$.

Fields	$SU(2)$	A_5^l	A_5^ν	$2k_l$	$2k_\nu$
L	$\mathbf{2}$	$\mathbf{3}^{(\prime)}$	$\mathbf{1}$	0	$+2$
E^c	$\mathbf{2}$	$\mathbf{3}^{(\prime)}$	$\mathbf{1}$	$+4$	-2
$H_{u,d}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	0	0
Φ	$\mathbf{1}$	$\mathbf{5}$	$\mathbf{5}$	0	0

Yukawas/Masses	A_5^l	A_5^ν	$2k_l$	$2k_\nu$
$Y_{\mathbf{1}}^l$	$\mathbf{1}$	$\mathbf{1}$	$+4$	0
$Y_{\mathbf{3}^{(\prime)}}^l$	$\mathbf{3}^{(\prime)}$	$\mathbf{1}$	$+4$	0
$Y_{\mathbf{5}}^l$	$\mathbf{5}$	$\mathbf{1}$	$+4$	0
$Y_{\mathbf{1}}^\nu$	$\mathbf{1}$	$\mathbf{1}$	0	$+4$
$Y_{\mathbf{5}_1}^\nu$	$\mathbf{1}$	$\mathbf{5}$	0	$+4$
$Y_{\mathbf{5}_2}^\nu$	$\mathbf{1}$	$\mathbf{5}$	0	$+4$

Table 4.3: Transformation properties of fields and Yukawa couplings for model using the Weinberg operator and two modular A_5.

Similarly the lepton doublets L transform as a $\mathbf{3}^{(\prime)}$ of A_5^l and a $\mathbf{1}$ of A_5^ν, with weights $2k_l = 0$ and $2k_\nu = +2$. These are the correct choices for the weights such that the modular forms and fields in each term sum up to zero since the weight for the fields is not $2k$, which are the values that were introduced in this section, but $-2k$ instead. H_d and H_u are the usual Higgs and an additional Higgs doublet as required in supersymmetric models. A bi-quintuplet Φ, which is a quintuplet under both A_5^l and A_5^ν, is introduced.

The multiplication of two triplets has the decomposition $\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} = \mathbf{1} \oplus \mathbf{3}^{(\prime)} \oplus \mathbf{5}$, where the $\mathbf{3}^{(\prime)}$ component is antisymmetric. This means that L^2 only decomposes as $\mathbf{1} \otimes \mathbf{5}$, and so it must combine with a singlet or quintuplet. This implies that we have only to consider the contributions from $Y_{\mathbf{1}}^\nu$, $Y_{\mathbf{5}_1}^\nu$ and $Y_{\mathbf{5}_2}^\nu$, each associated with a different complex constant g_i. For Y^ν, we only consider the contribution from $\mathbf{5}_1$ since the other weight 4 $\mathbf{5}_2$ will vanish at the chosen stabiliser for τ_ν as is shown below.

With the fields assigned in this manner, the superpotential for this model, which can be separated into one part containing the mass terms for the charged leptons and the other the neutrino mass terms, has the following form:

$$w = w_e + w_\nu, \tag{4.35}$$

$$w_e = \left(\alpha_1 Y_{\mathbf{1}}^l(\tau_l)(LE^c)_{\mathbf{1}} + \alpha_2 Y_{\mathbf{3}^{(\prime)}}^l(\tau_l)(LE^c)_{\mathbf{3}^{(\prime)}} + \alpha_3 Y_{\mathbf{5}}^l(\tau_l)(LE^c)_{\mathbf{5}}\right) H_d, \tag{4.36}$$

$$w_\nu = \frac{1}{\Lambda} L^2 \left[Y_{\mathbf{1}}^\nu(\tau_\nu) + \frac{1}{\Lambda}\Phi\left(Y_{\mathbf{5}_1}^\nu(\tau_\nu) + Y_{\mathbf{5}_2}^\nu(\tau_\nu)\right)\right] H_u^2. \tag{4.37}$$

$A_5^l \times A_5^\nu \to A_5^D$ breaking

Considering the multiplication rules for two quintuplets to get a trivial singlet, the term $\frac{1}{\Lambda^2}L^2\Phi Y^\nu H_u^2$ can be explicitly expanded as:

$$\frac{1}{\Lambda^2}(L^2)_{\mathbf{5}}^T P_\pi \Phi P_\pi Y_{\mathbf{5}}^\nu(\tau_\nu) H_u^2, \tag{4.38}$$

where P_π is the matrix that describes the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, \tag{4.39}$$

which is explicitly

$$P_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.40}$$

If Φ acquires the VEV $\langle \Phi \rangle = v_\Phi P_\pi$ (see Appendix B.3 for more details), the term in Eq.(4.38)

$$\frac{v_\Phi}{\Lambda^2}(L^2)^T_{\mathbf{5}} P_\pi Y^\nu_{\mathbf{5}}(\tau_\nu) H_u^2, \tag{4.41}$$

which implies that w_ν gets the form (the w_e terms remain exactly the same):

$$w_\nu = \frac{1}{\Lambda}\left[(L^2)_{\mathbf{1}} Y^\nu_{\mathbf{1}}(\tau_\nu) + \frac{v_\Phi}{\Lambda}\left((L^2)_{\mathbf{5}} Y^\nu_{\mathbf{5}_1}(\tau_\nu) + (L^2)_{\mathbf{5}} Y^\nu_{\mathbf{5}_2}(\tau_\nu)\right)_{\mathbf{1}}\right] H_u^2. \tag{4.42}$$

This means that the symmetry $A_5^l \times A_5^\nu$ is broken but given that the same transformation γ can be performed in A_5^l and A_5^ν simultaneously and being the terms in the superpotential above all left invariant by such a transformation, there is still a single modular symmetry A_5^D, the diagonal subgroup, that is conserved.

The superpotential above implies a neutrino mass matrix. Expanding $Y^\nu_{\mathbf{5}_1}$ and $Y^\nu_{\mathbf{5}_2}$ in terms of the weight 2 modular functions gives the results already derived in [52]. If the triplets L, E^c and ν^c are triplets $\mathbf{3}$, which we will simply write as $\rho_L \sim \mathbf{3}$, the neutrino mass matrix after the Higgs field acquires the VEV $\langle H_u \rangle = (0, v_u)$ gets the form:

$$\begin{aligned} M_\nu^{\mathbf{3}} = {} & g_{\mathbf{1}}(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ & + g_{\mathbf{5}_1}\begin{pmatrix} Y_5Y_7 - Y_2Y_8 & -\frac{1}{2}Y_5Y_6 - \frac{1}{\sqrt{2}}Y_4Y_7 - \frac{\sqrt{3}}{2}Y_1Y_8 & \frac{1}{2}Y_2Y_6 + \frac{1}{\sqrt{2}}Y_3Y_8 + \frac{\sqrt{3}}{2}Y_1Y_7 \\ * & Y_5Y_8 + \sqrt{2}Y_4Y_6 & -\frac{1}{2}Y_5Y_7 + \frac{1}{2}Y_2Y_8 \\ * & * & -Y_2Y_7 - \sqrt{2}Y_3Y_6 \end{pmatrix} \\ & + g_{\mathbf{5}_2}\begin{pmatrix} Y_1^2 + Y_3Y_4 - 2Y_2Y_5 & -\frac{3}{2\sqrt{2}}Y_3^2 + \sqrt{3}Y_1Y_5 & -\frac{3}{2\sqrt{2}}Y_4^2 + \sqrt{3}Y_1Y_2 \\ * & 3Y_2Y_3 + \sqrt{\frac{3}{2}}Y_1Y_4 & -\frac{1}{2}Y_1^2 - \frac{1}{2}Y_3Y_4 + Y_2Y_5 \\ * & * & 3Y_4Y_5 + \sqrt{\frac{3}{2}}Y_1Y_3 \end{pmatrix}, \end{aligned} \tag{4.43}$$

where asterisks were used to omit the off diagonal entries of symmetric matrices and $g_{\mathbf{1}}$, $g_{\mathbf{5}_1}$ and $g_{\mathbf{5}_2}$ are arbitrary complex constants associated with the respective modular form contribution. The factors $2v_u^2/\Lambda$ and $2v_u^2 v_\Phi/\Lambda^2$ are included inside these constants.

If the triplets L, E^c and ν^c are triplets $\mathbf{3}'$ instead, which can be equivalently expressed as $\rho_L \sim \mathbf{3}'$, one obtains:

$$
\begin{aligned}
M_\nu^{\mathbf{3}'} &= g_{\mathbf{1}}(Y_1^2+2Y_3Y_4+2Y_2Y_5)\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\\
&+g_{\mathbf{5}_1}\begin{pmatrix}Y_5Y_7-Y_2Y_8 & -Y_4Y_6-\frac{1}{\sqrt{2}}Y_5Y_8 & Y_3Y_6+\frac{1}{\sqrt{2}}Y_2Y_7\\ * & -Y_3Y_8-\frac{1}{\sqrt{2}}Y_2Y_6-\sqrt{\frac{3}{2}}Y_1Y_7 & -\frac{1}{2}Y_5Y_7+\frac{1}{2}Y_2Y_8\\ * & * & Y_4Y_7+\frac{1}{\sqrt{2}}Y_5Y_6+\sqrt{\frac{3}{2}}Y_1Y_8\end{pmatrix}\\
&+g_{\mathbf{5}_2}\begin{pmatrix}Y_1^2+Y_3Y_4-2Y_2Y_5 & -\frac{3}{\sqrt{2}}Y_2Y_3-\frac{\sqrt{3}}{2}Y_1Y_4 & -\frac{3}{\sqrt{2}}Y_4Y_5-\frac{\sqrt{3}}{2}Y_1Y_3\\ * & \frac{3}{2}Y_4^2-\sqrt{6}Y_1Y_2 & -\frac{1}{2}Y_1^2-\frac{1}{2}Y_3Y_4+Y_2Y_5\\ * & * & \frac{3}{2}Y_3^2-\sqrt{6}Y_1Y_5\end{pmatrix},
\end{aligned}
\tag{4.44}
$$

where again $g_{\mathbf{1}}$, $g_{\mathbf{5}_1}$ and $g_{\mathbf{5}_2}$ are arbitrary complex constants associated with the respective modular form contribution that absorbed the factors $2v_u^2/\Lambda$ and $2v_u^2 v_\Phi/\Lambda^2$.

A_5^D breaking

The flavour structure after A_5^D symmetry breaking will now be covered. We assume that the charged lepton modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T = i\infty$. This is a stabiliser of T_τ which means that a residual modular Z_5^T symmetry is preserved in the charged lepton sector. The directions the modular forms take at this stabiliser are in Table 4.2. These directions lead to an almost diagonal charged lepton mass matrix when the Higgs field H_d acquires a VEV $\langle H_d\rangle = (0, v_d)$:

$$
m_e = v_d\alpha_1\begin{pmatrix}1+2\frac{\alpha_3}{\alpha_1} & 0 & 0\\ 0 & 0 & 1-\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1}\\ 0 & 1+\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1} & 0\end{pmatrix}. \tag{4.45}
$$

The masses for the charged leptons can be reproduced by adjusting the parameters α_i. These constants were redefined to include the constant associated with $Y^l(\tau_l)$. This matrix can be diagonalized by multiplying on the left and right by P_L and P_R ($P_L^T m_e P_R = m_{e_d}$) by taking P_L as the identity matrix and $P_R = P_{23}$. Consequently, the PMNS matrix is simply the matrix that diagonalizes the mass matrix for the neutrinos. These considerations are valid whether we choose the triplets in the model to be $\mathbf{3}$ or $\mathbf{3}'$.

For the other modular field τ_ν, we want to find a VEV that leads to a mixing that preserves the second column of the GR mixing matrix. This occurs for $\langle\tau_\nu\rangle = \tau_S = i$ and for Y^ν with weight 4 (see Table 4.2 for the directions the modular forms get at this stabiliser). In this case, a residual modular Z_2^S symmetry is preserved in the neutrino sector.

If $\rho_L \sim \mathbf{3}$, this implies the following structure for the neutrino mass matrix:

$$
M_\nu^{\mathbf{3}} = g_{\mathbf{1}}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}
$$

$$+ g_{\mathbf{5}_1} \begin{pmatrix} 1 & -\sqrt{\frac{241}{8}+13\sqrt{5}-\sqrt{1525+682\sqrt{5}}} & -\sqrt{\frac{241}{8}+13\sqrt{5}+\sqrt{1525+682\sqrt{5}}} \\ * & -3-2\sqrt{5}+\sqrt{50+22\sqrt{5}} & -\frac{1}{2} \\ * & * & -3-2\sqrt{5}-\sqrt{50+22\sqrt{5}} \end{pmatrix}$$

$$+ g_{\mathbf{5}_2} \begin{pmatrix} 1 & -\frac{3}{2}\sqrt{\frac{949}{2}-212\sqrt{5}-2\sqrt{103445-46262\sqrt{5}}} & -\frac{3}{2}\sqrt{\frac{949}{2}-212\sqrt{5}+2\sqrt{103445-46262\sqrt{5}}} \\ * & \frac{3}{2}\left(18-8\sqrt{5}+\sqrt{130-58\sqrt{5}}\right) & -\frac{1}{2} \\ * & * & \frac{3}{2}\left(18-8\sqrt{5}-\sqrt{130-58\sqrt{5}}\right) \end{pmatrix}$$

$$\approx \begin{pmatrix} g_{\mathbf{1}}+g_{\mathbf{5}_1}+g_{\mathbf{5}_2} & -1.99176 g_{\mathbf{5}_1}-0.578608 g_{\mathbf{5}_2} & -10.6968 g_{\mathbf{5}_1}-1.30628 g_{\mathbf{5}_2} \\ * & 2.48746 g_{\mathbf{5}_1}+0.999728 g_{\mathbf{5}_2} & g_{\mathbf{1}}-\frac{1}{2}g_{\mathbf{5}_1}-\frac{1}{2}g_{\mathbf{5}_2} \\ * & * & -17.4317 g_{\mathbf{5}_1}-0.665359 g_{\mathbf{5}_2} \end{pmatrix} \quad (4.46)$$

where $g_{\mathbf{1}}$, $g_{\mathbf{5}_1}$ and $g_{\mathbf{5}_2}$ were redefined to include factors coming from the modular forms $Y^{\nu}_{\mathbf{1}}$, $Y^{\nu}_{\mathbf{5}_1}$ and $Y^{\nu}_{\mathbf{5}_2}$.

We want now to diagonalize M_ν, such that $U^T M_\nu U = M_{\nu_d} = \text{diag}(m_1, m_2, m_3)$, where m_i are the neutrino masses and U is an unitary matrix. When we apply the golden ratio mixing matrix Eq.(4.18) to the neutrino mass matrix for triplets $\mathbf{3}$ one obtains:

$$U^T_{GR} M^{\mathbf{3}}_\nu U_{GR} = \begin{pmatrix} \frac{1}{10}\left(\left(7\sqrt{5}+5\right)a+\left(7\sqrt{5}-5\right)b+16\sqrt{5}c\right) & 0 & 0 \\ 0 & a & c \\ 0 & c & b \end{pmatrix} \quad (4.47)$$

where $a = g_{\mathbf{1}} - \frac{13\sqrt{5}+25}{4} g_{\mathbf{5}_1} + \frac{27\sqrt{5}-65}{4} g_{\mathbf{5}_2}$, $b = -2g_{\mathbf{1}} - \frac{4\sqrt{5}+5}{2} g_{\mathbf{5}_1} + \frac{55-24\sqrt{5}}{2} g_{\mathbf{5}_2}$ and $c = \left(3\sqrt{5}+5\right) g_{\mathbf{5}_1} + \frac{3\left(7\sqrt{5}-15\right)}{2} g_{\mathbf{5}_2}$.

This implies that the PMNS is simply the Golden Ratio matrix times a rotation among the second and third columns, conserving only its first column. We have already discussed the compatibility of the GR_1 mixing and experimental values in Section 4.2, where it has already been seen that this mixing is incompatible with the 3σ confidence interval for θ_{12}. For this reason, we will not further develop the case $\rho_L \sim \mathbf{3}$.

We now turn our attention to $M^{\mathbf{3}'}_\nu$. For $\rho_L \sim \mathbf{3}'$, we have the following structure for the neutrino mass matrix:

$$M^{\mathbf{3}'}_\nu = g_{\mathbf{1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$+ g_{\mathbf{5}_1} \begin{pmatrix} 1 & -\sqrt{\frac{79}{2}+17\sqrt{5}-\sqrt{2770+1238\sqrt{5}}} & \sqrt{\frac{79}{2}+17\sqrt{5}+\sqrt{2770+1238\sqrt{5}}} \\ * & \frac{9}{2}+2\sqrt{5}+2\sqrt{5+2\sqrt{5}} & -\frac{1}{2} \\ * & * & \frac{9}{2}+2\sqrt{5}-2\sqrt{5+2\sqrt{5}} \end{pmatrix}$$

$$+ g_{\mathbf{5}_2} \begin{pmatrix} 1 & -\frac{3}{2}\sqrt{387-173\sqrt{5}+2\sqrt{41810-18698\sqrt{5}}} & \frac{3}{2}\sqrt{387-173\sqrt{5}-2\sqrt{41810-18698\sqrt{5}}} \\ * & -\frac{51}{2}+12\sqrt{5}+3\sqrt{85-38\sqrt{5}} & -\frac{1}{2} \\ * & * & -\frac{51}{2}+12\sqrt{5}-3\sqrt{85-38\sqrt{5}} \end{pmatrix}$$

$$\approx \begin{pmatrix} g_{\mathbf{1}}+g_{\mathbf{5}_1}+g_{\mathbf{5}_2} & -1.75890 g_{\mathbf{5}_1}-0.706914 g_{\mathbf{5}_2} & 12.3261 g_{\mathbf{5}_1}+0.47048 g_{\mathbf{5}_2} \\ * & 15.1275 g_{\mathbf{5}_1}+1.84736 g_{\mathbf{5}_2} & g_{\mathbf{1}}+0.5 g_{\mathbf{5}_1}+0.5 g_{\mathbf{5}_2} \\ * & * & 2.81677 g_{\mathbf{5}_1}+0.818275 g_{\mathbf{5}_2} \end{pmatrix} \quad (4.48)$$

where once again $g_{\mathbf{1}}$, $g_{\mathbf{5}_1}$ and $g_{\mathbf{5}_2}$ were redefined to include the factors coming from the modular forms $Y_{\mathbf{1}}^{\nu}$, $Y_{\mathbf{5}_1}^{\nu}$ and $Y_{\mathbf{5}_2}^{\nu}$.

When we apply the golden ratio mixing matrix Eq.(4.18) to the neutrino mass matrix for triplets $\mathbf{3}'$ we obtain:

$$U_{GR}^T M_{\nu}^{\mathbf{3}'} U_{GR} = \begin{pmatrix} a & 0 & c \\ 0 & \frac{1}{10}\left(\left(5-\sqrt{5}\right)a - \left(5+\sqrt{5}\right)b - 8\sqrt{5}c\right) & 0 \\ c & 0 & b \end{pmatrix}, \tag{4.49}$$

where $a = g_{\mathbf{1}} - \frac{5+\sqrt{5}}{2} g_{\mathbf{5}_1} + \frac{39\sqrt{5}-85}{2} g_{\mathbf{5}_2}$, $b = -g_{\mathbf{1}} + \left(5+2\sqrt{5}\right) g_{\mathbf{5}_1} + \left(12\sqrt{5}-25\right) g_{\mathbf{5}_2}$ and $c = -\left(5+3\sqrt{5}\right) g_{\mathbf{5}_1}$ $-\frac{3}{2}\left(7\sqrt{5}-15\right) g_{\mathbf{5}_2}$. This matrix has only an element on the second row and second column and four elements on the corners that form a 2×2 symmetric matrix and so it can be fully diagonalized by introducing a matrix U_r that describes a rotation among the first and third columns. The matrix that diagonalizes M_ν is then $U = U_{GR}U_r$, where U_r is given by Eq.(4.19). We are then able to diagonalize M_ν and the lepton mixing obeys a GR_2 mixing.

It is also possible to start from the diagonal matrix M_{ν_d} and get $U_{GR}^T M_\nu U_{GR}$. We have that:

$$U_r^* M_{\nu_d} U_r^\dagger = \begin{pmatrix} m_1 e^{-2i\alpha_1}\cos^2\theta + m_3 e^{2i\alpha_2}\sin^2\theta & 0 & \frac{1}{2}(-m_1 e^{-i(\alpha_1+\alpha_2)} + m_3 e^{i(\alpha_1+\alpha_2)})\sin 2\theta \\ 0 & m_2 e^{-2i\alpha_3} & 0 \\ * & 0 & m_1 e^{-2i\alpha_2}\sin^2\theta + m_3 e^{2i\alpha_1}\cos^2\theta \end{pmatrix}, \tag{4.50}$$

and comparing with (4.49) we obtain that $\alpha_3 = -\frac{1}{2}\arg\left(\left(5-\sqrt{5}\right)a - \left(5+\sqrt{5}\right)b - 8\sqrt{5}c\right)$ and, more importantly, we get a mass sum rule for m_i:

$$\begin{aligned} m_2 &= \Big|\frac{1}{10}\left(\left(5-\sqrt{5}\right)a - \left(5+\sqrt{5}\right)b - 8\sqrt{5}c\right)\Big| \\ &= \frac{1}{10}\Big|m_1\left(\left(5-\sqrt{5}\right)e^{-2i\alpha_1}\cos^2\theta - \left(5+\sqrt{5}\right)e^{-2i\alpha_2}\sin^2\theta + 4\sqrt{5}e^{-i(\alpha_1+\alpha_2)}\sin 2\theta\right) - \\ &\qquad - m_3\left(\left(5+\sqrt{5}\right)e^{2i\alpha_1}\cos^2\theta - \left(5-\sqrt{5}\right)e^{2i\alpha_2}\sin^2\theta + 4\sqrt{5}e^{i(\alpha_1+\alpha_2)}\sin 2\theta\right)\Big|. \end{aligned} \tag{4.51}$$

The sum rule (4.51) and (4.20-4.23) give us relations between the observables and the parameters of the GR_2 mixing, and hence we can do a numerical minimisation using the χ^2 function:

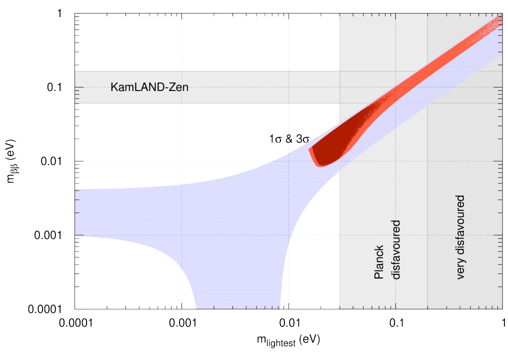
$$\chi^2 = \sum_i \left(\frac{P_i(\{x\}) - BF_i}{\sigma_i}\right)^2, \tag{4.52}$$

where P_i are the values provided by the considered model, BF the best fit value from NuFit [59] and σ_i is also provided by NuFit, when averaging the upper and lower σ provided. For the fitting, the three mixing angles, the atmospheric and solar neutrino squared mass differences and the Dirac neutrino CP violation phase were considered.

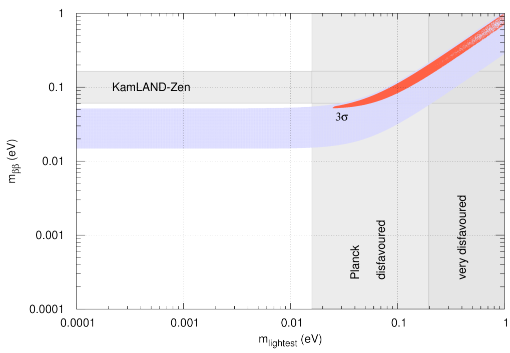
The fit parameters obtained for normal ordering (NO) and inverted ordering (IO) of neutrino masses can be found in Table 4.4. The best fit values lie inside the 1σ range for all the observables except θ_{12}, for both orderings near the lower limit of the 1σ range, and θ_{23} for IO. Nonetheless, all the observables are within their 3σ intervals. The best-fit occurs for NO with a $\chi^2/6 = 0.55$.

NO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			0.55	-10.09°	-12.97°	24.16°	0.0372 eV	0.0624 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{31}	$m_{\beta\beta}$
		32.12°	49.3°	8.57°	218°	7.42$\times 10^{-5}$eV^2	2.514$\times 10^{-3}$eV^2	0.0276 eV
IO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			1.80	10.16°	-24.53°	-130.52°	0.1209 eV	0.1104 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{32}	$m_{\beta\beta}$
		32.12°	46.5°	8.63°	254°	7.42$\times 10^{-5}$eV^2	-2.497$\times 10^{-3}$eV^2	0.1091 eV

Table 4.4: Parameters (Para.) and observables (Obs.) for the best fit point for normal and inverted orderings for model using the Weinberg operator and two modular A_5.



(a) Normal Ordering



(b) Inverted Ordering

Figure 4.1: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both orderings of neutrino masses compatible with 3σ data from [59] for model using the Weinberg operator and two modular A_5. For NO, the points having $\chi^2/6 < 1$ were plotted in dark-red. In both figures were also included the current upper limit from KamLAND-Zen and PLANCK 2018 as in Figure 3.1.

It is also possible to obtain the expected $m_{\beta\beta}$ for neutrinoless beta decay using the formula

$$\begin{aligned} m_{\beta\beta} &= |(M_\nu)_{(1,1)}| \\ &= \left| \frac{5+\sqrt{5}}{10} m_1 e^{-2i\alpha_1} \cos^2\theta + \frac{2m_2 e^{-2i\alpha_3}}{5+\sqrt{5}} + \frac{5+\sqrt{5}}{10} m_3 e^{2i\alpha_2} \sin^2\theta \right|, \end{aligned} \tag{4.53}$$

where m_2 is given by Eq.(4.51). Doing a numerical computation, the allowed regions of m_{lightest} vs $m_{\beta\beta}$ of Figure 4.1 (for NO, $m_{\text{lightest}} = m_1$ and for IO, $m_{\text{lightest}} = m_3$) were obtained, using again as constraints the data from [59]. The experimental constrains that were already discussed in Section 3.3, and arise from experimental results provided by KamLAND-Zen [64] and PLANCK 2018 [65]), are also included. We conclude then that both fits in Table 4.4 are in the disfavoured region.

For NO, the points that have $\chi^2/6 < 1$ were plotted with a darker red colour. Only for normal mass orderings do we have points outside the disfavoured region. The minimum values considering the 3σ ranges are

$$(m_{\text{lightest}})^{\text{NO}}_{\text{min}} \approx 0.015 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.008 \text{ eV}$$

$$(m_{\text{lightest}})^{\text{IO}}_{\text{min}} \approx 0.025 \text{ eV} \quad (m_{\beta\beta})^{\text{IO}}_{\text{min}} \approx 0.052 \text{ eV}, \tag{4.54}$$

and the minimum values for the points that have $\chi^2/6 < 1$ are

$$(m_{\text{lightest}})^{\text{NO}}_{\text{min}} \approx 0.017 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.009 \text{ eV}. \tag{4.55}$$

Taking these considerations into account, we conclude that NO is once again the preferred mass ordering.

4.4 Models with two modular A_5 symmetries - using the see-saw mechanism

In this section it is assumed that neutrinos get their mass through the type I see-saw mechanism, the effective term from the superpotential that gives rise to a Dirac mass matrix being of the form $\frac{1}{\Lambda}LY^{\nu}\nu^c H_u$. Again, at high energies this model is based in two modular symmetries, A_5^l and A_5^{ν}, with modulus fields denoted by τ_l and τ_ν, that will acquire different VEV's, leading to a GR_2 mixing.

We will assume that the Yukawa coupling Y^ν is simply a constant. The transformation properties of fields, Yukawa couplings and masses for this model are shown in Table 4.5.

Fields	$SU(2)$	A_5^l	A_5^ν	$2k_l$	$2k_\nu$
L	$\mathbf{2}$	$\mathbf{3}^{(\prime)}$	$\mathbf{1}$	0	-2
E^c	$\mathbf{2}$	$\mathbf{3}^{(\prime)}$	$\mathbf{1}$	$+4$	$+2$
ν^c	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{3}^{(\prime)}$	0	$+2$
$H_{u,d}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	0	0
Φ	$\mathbf{1}$	$\mathbf{3}^{(\prime)}$	$\mathbf{3}^{(\prime)}$	0	0

Yukawas/Masses	A_5^l	A_5^ν	$2k_l$	$2k_\nu$
$Y^l_{\mathbf{1}}$	$\mathbf{1}$	$\mathbf{1}$	$+4$	0
$Y^l_{\mathbf{3}^{(\prime)}}$	$\mathbf{3}^{(\prime)}$	$\mathbf{1}$	$+4$	0
$Y^l_{\mathbf{5}}$	$\mathbf{5}$	$\mathbf{1}$	$+4$	0
Y^ν	$\mathbf{1}$	$\mathbf{1}$	0	0
$M_{\mathbf{1}}$	$\mathbf{1}$	$\mathbf{1}$	0	$+4$
$M_{\mathbf{5}_1}$	$\mathbf{1}$	$\mathbf{5}$	0	$+4$
$M_{\mathbf{5}_2}$	$\mathbf{1}$	$\mathbf{5}$	0	$+4$

Table 4.5: Transformation properties of fields, Yukawa couplings and masses for the right-handed neutrinos for model using the see-saw mechanism and two modular A_5.

The Yukawa coefficients for the charged leptons are a modular form which transforms as a triplet $\mathbf{3}^{(\prime)}$ of A_5^l with weight $2k_l = +4$, whereas Y^ν is simply a modulus independent constant, a modular form of weight 0. For the right-handed neutrino masses we consider three modular forms transforming under A_5^ν: $M_{\mathbf{1}}$ as a singlet, and $M_{\mathbf{5}_1}$ and $M_{\mathbf{5}_2}$ as two quintuplets, all with weights $2k_\nu = +4$. The weights were chosen in such a way that the modular forms acquire the desired directions as we show below.

The right-handed charged leptons are arranged in a triplet $\mathbf{3}^{(\prime)}$ of A_5^l and trivial singlet $\mathbf{1}$ of A_5^ν, with weights $2k_l = +4$ and $2k_\nu = +2$. The lepton doublets L are arranged as a triplet $\mathbf{3}^{(\prime)}$ of A_5^l and a singlet of A_5^ν, with weights $2k_l = 0$ and $2k_\nu = -2$. In this model, the three right-handed neutrinos that were introduced also form a triplet $\mathbf{3}^{(\prime)}$ of A_5^ν with weight $2k_\nu = +2$. These are the correct choices for the weights such that the modular forms and fields in each term sum up to zero since the weight for the fields is not $+2k$, which are the values that were introduced in this section, but $-2k$ instead.

Note once again that, in spite of the charged leptons only having non-trivial singlet transformations under A_5^l and the right-handed neutrinos only under A_5^ν (which justifies the nomenclature used), the respective weights introduce non-trivial factors in the transformations under both modular symmetries for these fields.

With the fields assigned in this manner, the superpotential for this model, which can be separated into one part containing the mass terms for the charged leptons and the other the neutrino mass terms, has the following form:

$$w = w_e + w_\nu, \tag{4.56}$$

$$w_e = \left(\alpha_1 Y^l_{\mathbf{1}}(\tau_l)(LE^c)_{\mathbf{1}} + \alpha_2 Y^l_{\mathbf{3}^{(\prime)}}(\tau_l)(LE^c)_{\mathbf{3}^{(\prime)}} + \alpha_3 Y^l_{\mathbf{5}}(\tau_l)(LE^c)_{\mathbf{5}}\right) H_d, \tag{4.57}$$

$$w_\nu = \frac{Y^\nu}{\Lambda} L\Phi\nu^c H_u + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}\left(M_{\mathbf{5}_1}(\tau_\nu) + M_{\mathbf{5}_2}(\tau_\nu)\right)(\nu^c\nu^c)_{\mathbf{5}}. \tag{4.58}$$

From this superpotential, we can obtain the mass matrix for the right-handed neutrinos. As for the models discussed in the previous section, using he Weinberg operator, we expand the weight 4 triplets $M_{\mathbf{5}_1}$ and $M_{\mathbf{5}_2}$ in terms of the weight 2 modular functions Y_1, Y_8. If $\rho_L \sim \mathbf{3}'$, the mass matrix for the right-handed neutrinos after the Higgs field acquires the VEV $\langle H_u \rangle = (0, v_u)$ gets the form:

$$\begin{aligned} M_R^{\mathbf{3}'} &= c_{\mathbf{1}}(Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &+ c_{\mathbf{5}_1}\begin{pmatrix} Y_5Y_7 - Y_2Y_8 & -Y_4Y_6 - \frac{1}{\sqrt{2}}Y_5Y_8 & Y_3Y_6 + \frac{1}{\sqrt{2}}Y_2Y_7 \\ * & -Y_3Y_8 - \frac{1}{\sqrt{2}}Y_2Y_6 - \sqrt{\frac{3}{2}}Y_1Y_7 & -\frac{1}{2}Y_5Y_7 + \frac{1}{2}Y_2Y_8 \\ * & * & Y_4Y_7 + \frac{1}{\sqrt{2}}Y_5Y_6 + \sqrt{\frac{3}{2}}Y_1Y_8 \end{pmatrix} \\ &+ c_{\mathbf{5}_2}\begin{pmatrix} Y_1^2 + Y_3Y_4 - 2Y_2Y_5 & -\frac{3}{\sqrt{2}}Y_2Y_3 - \frac{\sqrt{3}}{2}Y_1Y_4 & -\frac{3}{\sqrt{2}}Y_4Y_5 - \frac{\sqrt{3}}{2}Y_1Y_3 \\ * & \frac{3}{2}Y_4^2 - \sqrt{6}Y_1Y_2 & -\frac{1}{2}Y_1^2 - \frac{1}{2}Y_3Y_4 + Y_2Y_5 \\ * & * & \frac{3}{2}Y_3^2 - \sqrt{6}Y_1Y_5 \end{pmatrix}. \end{aligned} \tag{4.59}$$

where asterisks were used to omit the off diagonal entries of symmetric matrices and where $c_{\mathbf{1}}, c_{\mathbf{5}_1}$ and $c_{\mathbf{5}_2}$ are arbitrary complex constants associated with the respective modular form contribution. We have redefined the constants associated with the quintuplets in order to have simpler factors for the first column first row entry in the matrix above. This matrix $M_R^{\mathbf{3}'}$ has the same structure as $M_\nu^{\mathbf{3}'}$ for the models using the Weinberg operator instead, and the same is also valid for $M_R^{\mathbf{3}}$ and $M_\nu^{\mathbf{3}}$.

$A_5^l \times A_5^\nu \to A_5^D$ breaking

Considering the multiplication rules for two triplets to get a trivial singlet, the term $\frac{Y^\nu}{\Lambda} L\Phi\nu^c H_u$ can be explicitly expanded as:

$$\frac{Y^\nu}{\Lambda} L^T P_{23}\Phi P_{23}\nu^c H_u^2, \tag{4.60}$$

where the superscript T as usual stands for the transpose of a vector and P_{23} is the matrix that describes the permutation of the second and third columns or rows:

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{4.61}$$

If Φ acquires the VEV $\langle\Phi\rangle = v_\Phi P_{23}$ (see Appendix B.4 for more details), the term in Eq.(4.60)

$$\frac{Y^\nu v_\Phi}{\Lambda} L^T P_{23} \nu^c H_u, \tag{4.62}$$

which is precisely the contraction rule from two triplets $\mathbf{3}$ or two triplets $\mathbf{3}'$ to a singlet. This implies that w_ν gets the form (the w_e terms remain exactly the same):

$$w_\nu = y_D \left(L\nu^c\right)_{\mathbf{1}} H_u + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2}\left(M_{\mathbf{5}_1}(\tau_\nu) + M_{\mathbf{5}_2}(\tau_\nu)\right)(\nu^c\nu^c)_{\mathbf{5}}, \tag{4.63}$$

where $y_D = Y^\nu v_\Phi/\Lambda$. This means that the symmetry $A_5^l \times A_5^\nu$ is broken but given that the same transformation γ can be performed in A_5^l and A_5^ν simultaneously and being the term in the superpotential above left invariant by such a transformation, the diagonal subgroup A_5^D is conserved.

A_5^D breaking

The flavour structure after A_5^D symmetry breaking now follows. As for the models using the Weinberg operator, we assume that the charged lepton modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T = i\infty$, which is a stabiliser of T_τ and thus a residual modular Z_5^T symmetry is preserved in the charged lepton sector. The directions the modular forms take at this stabiliser are in Table 4.2 and lead to an almost diagonal charged lepton mass matrix as in Eq.(4.45). The masses for the charged leptons can be reproduced by adjusting the parameters α_i and the mass matrix for the charged leptons can be diagonalized by multiplying on the left by the identity matrix and on the right by P_{23} and thus the PMNS matrix is simply the matrix that diagonalizes the mass matrix for the neutrinos.

For the other modular field τ_ν, we want to find a VEV that leads to a mixing that preserves the second column of the golden ratio GR mixing matrix. Again, this occurs for $\rho_L \sim \mathbf{3}'$, when the modular field acquires the VEV $\langle\tau_\nu\rangle = \tau_S = i$ and k_ν is even (the simplest case is $2k_\nu = +4$). In this case, a residual modular Z_2^S symmetry is preserved in the neutrino sector (see Table 4.2 for the directions at this stabiliser). This implies the following structure for the neutrino mass matrix for the right-handed neutrinos:

$$M_R^{\mathbf{3}'} = c_{\mathbf{1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$+\,c_{\mathbf{5}_1}\begin{pmatrix} 1 & -\sqrt{\frac{79}{2}+17\sqrt{5}-\sqrt{2770+1238\sqrt{5}}} & \sqrt{\frac{79}{2}+17\sqrt{5}+\sqrt{2770+1238\sqrt{5}}} \\ * & \frac{9}{2}+2\sqrt{5}+2\sqrt{5+2\sqrt{5}} & -\frac{1}{2} \\ * & * & \frac{9}{2}+2\sqrt{5}-2\sqrt{5+2\sqrt{5}} \end{pmatrix}$$
$$+\,c_{\mathbf{5}_2}\begin{pmatrix} 1 & -\frac{3}{2}\sqrt{387-173\sqrt{5}+2\sqrt{41810-18698\sqrt{5}}} & \frac{3}{2}\sqrt{387-173\sqrt{5}-2\sqrt{41810-18698\sqrt{5}}} \\ * & -\frac{51}{2}+12\sqrt{5}+3\sqrt{85-38\sqrt{5}} & -\frac{1}{2} \\ * & * & -\frac{51}{2}+12\sqrt{5}-3\sqrt{85-38\sqrt{5}} \end{pmatrix}$$
$$\approx\begin{pmatrix} c_{\mathbf{1}}+c_{\mathbf{5}_1}+g_{\mathbf{5}_2} & -1.75890c_{\mathbf{5}_1}-0.706914c_{\mathbf{5}_2} & 12.3261c_{\mathbf{5}_1}+0.47048c_{\mathbf{5}_2} \\ * & 15.1275c_{\mathbf{5}_1}+1.84736c_{\mathbf{5}_2} & c_{\mathbf{1}}+0.5c_{\mathbf{5}_1}+0.5c_{\mathbf{5}_2} \\ * & * & 2.81677c_{\mathbf{5}_1}+0.818275c_{\mathbf{5}_2} \end{pmatrix} \tag{4.64}$$

where $c_{\mathbf{1}}$, $c_{\mathbf{5}_1}$ and $c_{\mathbf{5}_2}$ were redefined to include the factors coming from the modular forms $M_{\mathbf{1}}$, $M_{\mathbf{5}_1}$ and $M_{\mathbf{5}_2}$.

For this model the VEV the field τ_ν acquires has no implication on the term that generates the Dirac mass matrix that relates the right-handed and active neutrinos. This matrix after the Higgs field H_u acquires a VEV $\langle H_u\rangle = (0, v_u)$ is simply

$$M_D = y_D v_u P_{23}. \tag{4.65}$$

Consequently, the active neutrino mass matrix for the see-saw mechanism gets the form

$$M_\nu = -M_D M_R^{-1} M_D^T = -y_D^2 v_u^2 P_{23} M_R^{-1} P_{23}. \tag{4.66}$$

We want now to diagonalize M_ν, such that $U^T M_\nu U = M_{\nu_d} = \text{diag}(m_1, m_2, m_3)$, where m_i are the neutrino masses and U is an unitary matrix. As derived in Section 3.4.2, it is also true that $U^T M_\nu U = -U^T M_D M_R^{-1} M_D^T U = M_{\nu_d}$. So $M_D^T U$ also diagonalizes the matrix M_R^{-1} and thus $V = M_D^\dagger U^*$ diagonalizes M_R such that $V^T M_R V = M_{R_d} = \text{diag}(M_1, M_2, M_3)$ where $M_i = -\frac{y_D^2 v_u^2}{m_i}$. Conversely, $U = M_D^* V^*$ when V diagonalizes M_R.

In the present model, when we apply the golden ratio matrix in Eq.(4.18) to the heavy neutrino mass matrix, we obtain:

$$U_{GR}^T M_R U_{GR} = \begin{pmatrix} a & 0 & c \\ 0 & \frac{1}{10}\left(\left(5-\sqrt{5}\right)a-\left(5+\sqrt{5}\right)b-8\sqrt{5}c\right) & 0 \\ c & 0 & b \end{pmatrix}, \tag{4.67}$$

where where $a = g_{\mathbf{1}} - \frac{5+\sqrt{5}}{2} g_{\mathbf{5}_1} + \frac{39\sqrt{5}-85}{2} g_{\mathbf{5}_2}$, $b = -g_{\mathbf{1}} + \left(5+2\sqrt{5}\right) g_{\mathbf{5}_1} + \left(12\sqrt{5}-25\right) g_{\mathbf{5}_2}$ and $c = -\left(5+3\sqrt{5}\right) g_{\mathbf{5}_1} - \frac{3}{2}\left(7\sqrt{5}-15\right) g_{\mathbf{5}_2}$. This matrix can be fully diagonalized adding a matrix V_r that introduces a rotation among the first and third columns. This rotation preserves the second column so M_R is diagonalized by a matrix that has the second column of the GR mixing matrix. For the present model, M_D is only a permutation, so we have that, being $V = U_{GR} V_r$ the matrix that diagonalizes M_R, the matrix that diagonalizes M_ν is $U = P_{23} U_{GR} V_r$, which can also be written as $U_{GR} U_r$, where U_r is a rotation between the first and third columns. If we define U_r as in Eq.(4.19), the definition for the matrix

V_r is going to be

$$V_r = \begin{pmatrix} \cos\theta e^{-i\alpha_1} & 0 & \sin\theta e^{i\alpha_2} \\ 0 & e^{-i\alpha_3} & 0 \\ \sin\theta e^{-i\alpha_2} & 0 & -\cos\theta e^{i\alpha_1} \end{pmatrix}, \tag{4.68}$$

where θ is the angle that governs the rotation and the three α_i are introduced such that M_i, and m_i too, are purely real values. We are then able to diagonalize both M_ν and M_R.

It is also possible to start from the diagonal matrix M_{R_d} and get $U_{GR}^T M_R U_{GR}$. We have that

$$V_r^* M_{R_d} V_r^\dagger = \begin{pmatrix} M_1\cos^2\theta e^{2i\alpha_1} + M_3\sin^2\theta e^{-2i\alpha_2} & 0 & \frac{1}{2}(M_1 e^{i(\alpha_1+\alpha_2)} - M_3 e^{-i(\alpha_1+\alpha_2)})\sin 2\theta \\ 0 & M_2 e^{2i\alpha_3} & 0 \\ * & 0 & M_1\sin^2\theta e^{2i\alpha_2} + M_3\cos^2\theta e^{-2i\alpha_1} \end{pmatrix}, \tag{4.69}$$

and comparing with Eq.(4.67) we obtain that $\alpha_3 = \frac{1}{2}\arg\left(\frac{1}{10}\left(\left(5-\sqrt{5}\right)a - \left(5+\sqrt{5}\right)b - 8\sqrt{5}c\right)\right)$ and, more importantly, we get a mass sum rule for M_i that can also be expressed in terms of the active neutrino masses m_i:

$$\begin{aligned} \frac{1}{m_2} &= -\frac{1}{y_D^2 v_u^2}\Big|\frac{1}{10}\left(\left(5-\sqrt{5}\right)a - \left(5+\sqrt{5}\right)b - 8\sqrt{5}c\right)\Big| \\ &= \frac{1}{10}\Big|\frac{1}{m_1}\left(\left(5-\sqrt{5}\right)e^{2i\alpha_1}\cos^2\theta - \left(5+\sqrt{5}\right)e^{2i\alpha_2}\sin^2\theta - 4\sqrt{5}e^{i(\alpha_1+\alpha_2)}\sin 2\theta\right) - \\ &\quad - \frac{1}{m_3}\left(\left(5+\sqrt{5}\right)e^{-2i\alpha_1}\cos^2\theta - \left(5-\sqrt{5}\right)e^{-2i\alpha_2}\sin^2\theta - 4\sqrt{5}e^{-i(\alpha_1+\alpha_2)}\sin 2\theta\right)\Big|. \end{aligned} \tag{4.70}$$

For the models constructed in the previous chapter with two A_4 modular symmetries, we found that the model using the Weinberg operator and the first model using the see-saw mechanism could be expressed in a simpler sum rule. This occurred because the matrices M_ν using the Weinberg operator and M_R using the see-saw mechanism had the same structure. Since the same is valid for the models constructed in this chapter with two A_5 modular symmetries, we can easily see that the sum rule can be expressed similarly as in [55] as

$$m_2^\eta = f_1(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\ m_1^\eta + f_3(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\ m_3^\eta \tag{4.71}$$

where

$$f_1(\theta,\alpha_1,\alpha_2,\alpha_3) = \frac{1}{10}\left(\left(5-\sqrt{5}\right)e^{-2i\alpha_1}\cos^2\theta - \left(5+\sqrt{5}\right)e^{-2i\alpha_2}\sin^2\theta + 4\sqrt{5}e^{-i(\alpha_1+\alpha_2)}\sin 2\theta\right)e^{2i\alpha_3} \tag{4.72}$$

$$f_3(\theta,\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{10}\left(\left(5+\sqrt{5}\right)e^{2i\alpha_1}\cos^2\theta - \left(5-\sqrt{5}\right)e^{2i\alpha_2}\sin^2\theta + 4\sqrt{5}e^{i(\alpha_1+\alpha_2)}\sin 2\theta\right)e^{2i\alpha_3}. \tag{4.73}$$

With these definitions, for the model using the Weinberg operator to generate the neutrino masses, we choose for the exponent $\eta = +1$ and thus:

$$m_2 = f_1(\theta,\alpha_1,\alpha_2,\alpha_3)\ m_1 + f_3(\theta,\alpha_1,\alpha_2,\alpha_3)\ m_3. \tag{4.74}$$

and we recover Eq.(4.51). For the model using the see-saw mechanism, we chose $\eta = -1$ for the exponent, and since there is that difference between in U_r and V_r already discussed in Section 3.4.2, in this case we will also have to exchange all the signs of the angles and complex phases. We will have then for the model using the see-saw mechanism:

$$\frac{1}{m_2} = f_1(-\theta, -\alpha_1, -\alpha_2, -\alpha_3)\,\frac{1}{m_1} + f_3(-\theta, -\alpha_1, -\alpha_2, -\alpha_3)\,\frac{1}{m_3}, \tag{4.75}$$

which recovers Eq.(4.70).

Before considering how well experimental results agree with these models, we stop here to consider briefly the case $\rho_L \sim \mathbf{3}$ for the present model using the see-saw mechanism. We would obtain for M_R the same structure as M_ν in the previous model, which was diagonalized by the GR mixing matrix times a rotation among the second and third columns. This implies that, for the simple model using the see-saw mechanism we are now considering, where M_D is simply a permutation, the mixing obtained for $\rho_L \sim \mathbf{3}$ using the see-saw mechanism will also be GR$_1$, which, as stated in Section 4.2, is incompatible with the experimental 3σ confidence interval for θ_{12}. For this reason, we will not develop more the case $\rho_L \sim \mathbf{3}$.

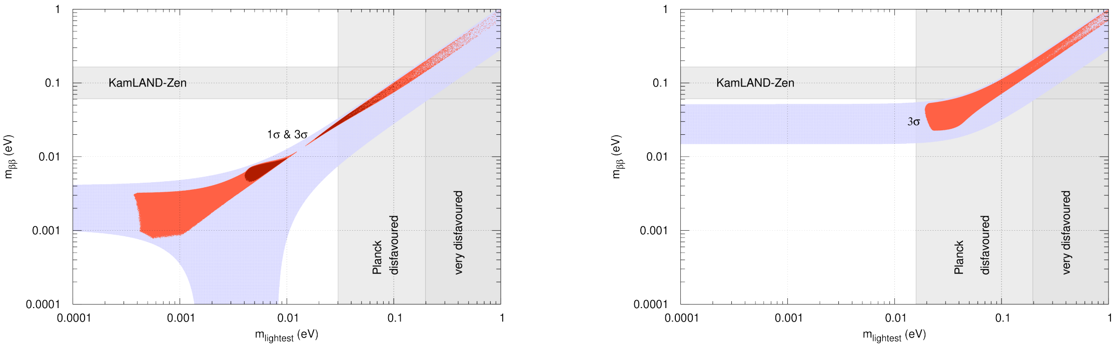
We turn now to the agreement between the model using $\rho_L \sim \mathbf{3}'$ and experiment. We can use the sum rule Eq.(4.70) and Eqs.(4.20-4.23), which are relations between the observables and the parameters of the GR$_2$ mixing, to do a numerical minimisation using the χ^2 function, Eq.(4.52). For the fitting, the three mixing angles, the atmospheric and solar neutrino squared mass differences and the Dirac neutrino CP violation phase were considered.

The fit parameters obtained for NO and IO of neutrino masses can be found in Table 4.6. For NO, the best fit values lie inside the 1σ range for all the observables except θ_{12}, for both orderings near the lower limit of the 1σ range. For IO, θ_{23} and δ also lie outside the 1σ confidence intervals. Nonetheless, all the observables are within their 3σ intervals. The best-fit occurs for normal ordering of neutrino masses with a $\chi^2/6 = 0.55$.

It is also possible to obtain the expected $m_{\beta\beta}$ for neutrinoless beta decay using Eq.(4.53), but now using Eq.(4.70) for m_2. Doing a numerical computation, the allowed regions of m_{lightest} vs $m_{\beta\beta}$ of Figure 4.2 (for NO, $m_{\text{lightest}} = m_1$ and for IO, $m_{\text{lightest}} = m_3$) were obtained, using again as constraints the data from [59]. The experimental constrains that were already discussed in Section 3.3 are also included.

NO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			0.55	-10.09°	-102.67°	-68.40°	0.0045 eV	0.0503 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{31}	$m_{\beta\beta}$
		32.12°	49.4°	8.57°	215°	7.42$\times10^{-5}$eV^2	2.514$\times10^{-3}$eV^2	0.0068 eV
IO	Para.		χ^2/6	θ	α_1	α_2	m_1	m_3
			1.58	10.14°	-181.33°	68.58°	0.0687 eV	0.0480 eV
	Obs.	θ_{12}	θ_{23}	θ_{13}	δ	Δm^2_{21}	Δm^2_{32}	$m_{\beta\beta}$
		32.12°	46.8°	8.61°	250°	7.42$\times10^{-5}$eV^2	-2.497$\times10^{-3}$eV^2	0.0335 eV

Table 4.6: Parameters (Para.) and observables (Obs.) for the best fit point for normal and inverted orderings for model using the see-saw mechanism and two modular A_5.



(a) Normal Ordering

(b) Inverted Ordering

Figure 4.2: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both orderings of neutrino masses compatible with 3σ data from [59] for model using the see-saw mechanism and two modular A_5. For NO, the points with $\chi^2/6 < 1$ were plotted in dark-red. In both figures were also included the current upper limit from KamLAND-Zen and PLANCK 2018 as in Figure 3.1.

We conclude then that only the fit for NO in Table 4.6 is outside the disfavoured region.

For NO, the points that are compatible with the 1σ ranges of the observables other than θ_{12} (which is, as already said, always near the lower 1σ limit although outside), which are inside a larger group containing the points that have $\chi^2/6 < 1$. These points were plotted with a darker red colour. For IO, at least one of the other observables is outside its 1σ region, hence only the 3σ compatible region is plotted for IO. As happened for the model using the Weinberg operator, only for normal ordering do we have points outside the disfavoured region. The minimum values considering the 3σ ranges are

$$(m_{\text{lightest}})^{\text{NO}}_{\text{min}} \approx 0.0004 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.0008 \text{ eV}$$
$$(m_{\text{lightest}})^{\text{IO}}_{\text{min}} \approx 0.019 \text{ eV} \quad (m_{\beta\beta})^{\text{IO}}_{\text{min}} \approx 0.023 \text{ eV}, \tag{4.76}$$

and the minimum values for the points that have $\chi^2/6 < 1$ are

$$(m_{\text{lightest}})^{\text{NO}}_{\text{min}} \approx 0.004 \text{ eV} \quad (m_{\beta\beta})^{\text{NO}}_{\text{min}} \approx 0.005 \text{ eV}. \tag{4.77}$$

We conclude that, when using the see-saw mechanism, NO is once again the preferred mass ordering, although, when comparing the present model with the model using the Weinberg operator discussed in Section 4.3 the $m_{\beta\beta}$ vs $m_{lightest}$ region extends to lower orders of magnitude.

Chapter 5

Conclusions

In this work, we employed the framework of multiple modular symmetries to build models with minimal field content that are able to reproduce viable mixings. For the models using two A_4 modular symmetries, the tri-maximal 2 mixing was obtained, and, for the models using two A_5 modular symmetries, a variation of the golden ratio mixing where only the second column is preserved, which was called GR_2, was obtained instead.

We described how the multiple A_4 and A_5 modular symmetries can be broken to a single symmetry group and showed possible assignments of fields and weights under these two modular symmetries leading to the desired mixing scheme. Three explicit models for A_4 and two for A_5 were built (with different weights and using the Weinberg operator or the seesaw mechanism to generate the neutrino masses) and shown to be predictive and to reproduce the observed mixing angles and mass differences with good fits.

Neutrinoless double beta decay is expected, with the inverted ordering possibility almost entirely disfavoured by cosmological observations and less compatible with the 1σ best fit intervals for the experimental observables than the normal ordering of neutrino masses. This occurs for all the models, independent of the mechanism that generates the masses. Furthermore, the χ^2 values obtained for all the models, which depended mainly on the $\sin^2\theta_{12}$ deviation from the best fit point, favour the GR_2 mixing scheme more than the TM_2 mixing.

It should be noted that this work is possible to be continued and will be continued. First of all, in October 2021, new results from NuFit were published at `http://www.nu-fit.org/` which seems to mean that the connection between our results and the results from this global fit needs to be updated. The results differ more significantly from the July 2020 data in the best fit points for $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$, and also on their 3σ range, but these are still not much significant differences. Thus, we expect that no noticeable changes seem to apply. Nevertheless, it would be a good idea to update the analysis considering these more recent confidence intervals, which can be easily done.

Secondly, for the bi-quintuplet Φ for the models using A_5, the vacuum alignments are still being studied and should be improved in the near future. All the solutions were not obtained fully for the alignment of the bi-quintuplet, and for the bi-triplet, no equations that can be fully solved were obtained so far. We conclude that more driving fields of different nature need to be added to the present model to account for the Φ VEV when using A_5.

In conclusion, the models shown in this dissertation maintain their valid results and prove to be in agreement with experiment, and so, despite the present incompleteness of the A_5 alignments in its present version, this thesis is a useful addendum to the field of modular field symmetries.

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Appendix A

Modular A_4 Symmetry Group

A.1 A_4 multiplication rules

The group A_4 is the group of even permutations of four objects and is the symmetry group of the tetrahedron, see e.g. [25]. It has 12 elements and two generators, S and T:

$$S^2 = (ST)^3 = T^3 = 1. \tag{A.1}$$

A_4 has four conjugacy classes: $C_1 = \{e\}$, $C_2 = \{T, ST, TS, STS\}$, $C_3 = \{T^2, ST^2, T^2S, TST\}$, $C_4 = \{S, T^2ST, TST^2\}$ [21].

This group has four irreducible representations: an invariant singlet **1**, two non-invariant singlets **1′** and **1″**, and a triplet **3**. The representations for the generators are in Table A.1. The three dimensional representation is not determined uniquely but up to an unitary transformation, representing a change of basis. Two possible basis are the complex basis, in which T is diagonal, and the real basis, in which S is diagonal.

	1	**1′**	**1″**	**3** - complex basis - ρ	**3** - real basis - $\tilde{\rho}$
S	1	1	1	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
T	1	ω	ω^2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Table A.1: Representation for the two generators of A_4, where $\omega = e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$.

To transform from one basis to the other, we use

$$\tilde{\rho}_{\mathbf{3}}(\gamma) = U_\omega \rho_{\mathbf{3}}(\gamma) U_\omega^\dagger, \tag{A.2}$$

where the change of basis matrix is

$$U_\omega = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \tag{A.3}$$

and obeys $U_\omega^\dagger = U_\omega P_{23}$.

The product of two triplets decomposes as $\mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3}_S + \mathbf{3}_A$ where $\mathbf{3}_{S(A)}$ denotes the symmetric (antisymmetric) combination. In the complex basis, this decomposition is [25]

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1}'} \oplus (a_2b_2 + a_3b_1 + a_1b_3)_{\mathbf{1}''}$$
$$\oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}_S} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}_A}, \tag{A.4}$$

and in the real basis it is [21]

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_2 + a_3b_3)_{\mathbf{1}} \oplus (a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3)_{\mathbf{1}'} \oplus (a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3)_{\mathbf{1}''}$$
$$\oplus \begin{pmatrix} a_2b_3 + a_3b_2 \\ a_3b_1 + a_1b_3 \\ a_1b_2 + a_2b_1 \end{pmatrix}_{\mathbf{3}_S} \oplus \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}_{\mathbf{3}_A}. \tag{A.5}$$

Finally, the multiplication rules for the singlets are

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}, \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'', \quad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}', \quad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}. \tag{A.6}$$

A.2 Modular forms of weight 2 for A_4

The three linearly independent weight 2 modular forms of level 3 $Y_{1,2,3}^{(2)}$ form a triplet of A_4. In [21], these modular forms were expressed in terms of the Dedekind eta functions

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n), \quad q = e^{i2\pi\tau}. \tag{A.7}$$

The triplet modular forms $Y_{1,2,3}^{(2)}$ can then be expressed as

$$Y_1^{(2)}(\tau) = \frac{i}{2\pi}\left[\frac{\eta'(\frac{\tau}{3})}{\eta(\frac{\tau}{3})} + \frac{\eta'(\frac{\tau+1}{3})}{\eta(\frac{\tau+1}{3})} + \frac{\eta'(\frac{\tau+2}{3})}{\eta(\frac{\tau+2}{3})} - 27\frac{\eta'(3\tau)}{\eta(3\tau)}\right] \tag{A.8}$$

$$Y_2^{(2)}(\tau) = -\frac{i}{\pi}\left[\frac{\eta'(\frac{\tau}{3})}{\eta(\frac{\tau}{3})} + \omega^2\frac{\eta'(\frac{\tau+1}{3})}{\eta(\frac{\tau+1}{3})} + \omega\frac{\eta'(\frac{\tau+2}{3})}{\eta(\frac{\tau+2}{3})}\right] \tag{A.9}$$

$$Y_3^{(2)}(\tau) = -\frac{i}{\pi}\left[\frac{\eta'(\frac{\tau}{3})}{\eta(\frac{\tau}{3})} + \omega\frac{\eta'(\frac{\tau+1}{3})}{\eta(\frac{\tau+1}{3})} + \omega^2\frac{\eta'(\frac{\tau+2}{3})}{\eta(\frac{\tau+2}{3})}\right]. \tag{A.10}$$

A.3 Vacuum alignments for bi-triplet Φ in A_4

In this Appendix we consider how to align the VEV of the bi-triplet Φ. Following from [7] where such an alignment was obtained in the context of S_4, we add two driving fields, with the properties present in Table A.2.

Fields	A_4^l	A_4^ν	$2k_l$	$2k_\nu$
$\chi_{l\nu}$	$\mathbf{3}$	$\mathbf{3}$	0	0
χ_l	$\mathbf{3}$	$\mathbf{1}$	0	0

Table A.2: Transformation properties of the fields responsible for the vacuum alignment of the bi-triplet Φ for the models with two modular A_4.

The superpotential responsible for the vacuum alignment that will be minimized with relation to the driving fields is

$$w = \Phi\Phi\chi_{l\nu} + M\Phi\chi_{l\nu} + \Phi\Phi\chi_l. \tag{A.11}$$

Care should be taken given that we are here dealing with A_4 groups, rather than S_4. The main differences are the presence of the anti-symmetric triplet $\mathbf{3}_A$ in the contraction of $\mathbf{3} \times \mathbf{3}$ (in S_4 it is a different inequivalent $\mathbf{3}'$), and that S_4 has a doublet (which decomposes into the two non-trivial singlets of A_4).

As the alignment superpotential above features only contractions into the trivial singlet of A_4 and $\Phi\Phi$ contractions (where Φ appears twice), the equations are analogous to those in the S_4 case and in general the solutions of these equations are the same as for the S_4 case, presented in [7]. Still, the new contraction in A_4 that gives a antisymmetric triplet introduces a small difference. When considering the term $\Phi\Phi\chi_l$, we contract $\Phi\Phi$ into a singlet of A_4^ν and a triplet of A_4^l and thus the only non-vanishing contribution is the symmetric triplet of A_4^ν that is finally combined with χ_l into a singlet of A_4^ν. Here, no difference appears with relation to S_4. However, for the term $\Phi\Phi\chi_{\nu l}$, we are now contracting $\Phi\Phi$ into triplets of both symmetries, which means that we will have to consider separately when we contract $\Phi\Phi$ into both symmetric triplets of A_4^l and A_4^ν, and antisymmetric triplets of A_4^l and A_4^ν. The other possibility, i.e. considering simultaneously a symmetric triplet under one symmetry and a antisymmetric triplet under the other, always vanishes.

It is simpler to solve the relations that arise from the minimisation of this superpotential working in the real basis. In fact, the multiplication of two triplets in the real basis can be simply expressed by a Levi-Civita tensor. From Eq.(A.11), we have that

$$(a \otimes b)_{\mathbf{3}_S i} = |\epsilon_{ijk}| a_j b_k \tag{A.12}$$

$$(a \otimes b)_{\mathbf{3}_A i} = \epsilon_{ijk} a_j b_k \tag{A.13}$$

We get the constraints:

$$\sum_{j,k=1,2,3}\sum_{\beta,\gamma=1,2,3}(g_S|\epsilon_{ijk}||\epsilon_{\alpha\beta\gamma}|+g_A\epsilon_{ijk}\epsilon_{\alpha\beta\gamma})(\tilde{\Phi})_{j\beta}(\tilde{\Phi})_{k\gamma}+M(\tilde{\Phi})_{i\alpha}=0 \text{ for } i=1,2,3, \alpha=1,2,3 \quad \text{(A.14)}$$

$$\sum_{j,k=1,2,3}\sum_{\alpha=1,2,3}|\epsilon_{ijk}|(\tilde{\Phi})_{j\alpha}(\tilde{\Phi})_{k\alpha}=0 \text{ for } i=1,2,3. \quad \text{(A.15)}$$

where g_A and g_S are constants that account for the combination of both indices of $\Phi\Phi$ symmetrically and anti-symmetrically. The solutions for general values of g_S and g_A, with $g_A \neq g_S$ can be written as 3×3 unitary matrices.

$$\langle\tilde{\Phi}\rangle = v_\Phi \left\{ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}, \right.$$
$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\-1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&-1\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&-1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&1&0\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&-1&0\\-1&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix},$$
$$\left. \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&1\\0&-1&0\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&-1\\0&1&0\end{pmatrix} \right\}.$$

where v_Φ is a constant that depends on g_A, g_S and M. These are precisely the representations of the elements of S_4 in the real basis, $\tilde{\rho}_{\mathbf{3}}(\gamma), \gamma \in S_4$, half of which correspond also to representations of A_4 in the real basis. Returning to the complex basis used in the main text, we find simply that

$$\langle\Phi\rangle = v_\Phi \rho_{\mathbf{3}}(\gamma) P_{23},\ \gamma \in S_4. \quad \text{(A.16)}$$

However, in the specific case that $g_A = g_S$, only half of these 24 solutions are valid solutions, more precisely the first twelve solutions in Eq.(A.16), which are the A_4 elements in the real basis, and thus,

$$g_A = g_S\ :\ \langle\Phi\rangle = v_\Phi \rho_{\mathbf{3}}(\gamma) P_{23},\ \gamma \in A_4. \quad \text{(A.17)}$$

In the main text we have used as VEV the identity in the real basis, $\delta_{i\alpha}$, first solution in Eq.(A.16), which in the complex basis becomes $\langle\Phi\rangle = v_\Phi P_{23}$. This specific VEV leads to the recovering of the usual multiplication of two triplets to give a singlet. In the following we will show that it is still possible to construct an invariant term under the single A_4 symmetry that remains after the symmetry breaking of the two independent symmetries when choosing one of the other eleven VEV's.

We choose then one of the twelve VEV's $\langle\Phi\rangle = v_\Phi \rho_{\mathbf{3}}(\gamma_1) P_{23},\ \gamma_1 \in A_4$. We consider that the fields transform under the single A_4 as

$$E^c \to (c_2\tau_l + d_2)^{-2k^l_{E^c}}(c_2\tau_\nu + d_2)^{-2k^\nu_{E^c}}\rho(\gamma_2)E^c \tag{A.18}$$

$$L \to (c_2\tau_\nu + d_2)^{-2k^\nu_L}\rho(\gamma_2)L \tag{A.19}$$

$$\nu^c \to (c_2\tau_\nu + d_2)^{-2k^\nu_{\nu^c}}\rho(\gamma_1^{-1}\gamma_2\gamma_1)\nu^c \tag{A.20}$$

$$Y^l \to (c_2\tau_l + d_2)^{2k^l_{Y^l}}\rho(\gamma_2)Y^l \tag{A.21}$$

$$Y^\nu_{\mathbf{1}} \to (c_2\tau_\nu + d_2)^{2k^\nu_{Y^\nu_{\mathbf{1}}}}Y^\nu_{\mathbf{1}} \tag{A.22}$$

$$Y^\nu_{\mathbf{3}} \to (c_2\tau_\nu + d_2)^{2k^\nu_{Y^\nu_{\mathbf{3}}}}\rho(\gamma_1^{-1}\gamma_2\gamma_1)Y^\nu_{\mathbf{3}} \tag{A.23}$$

$$M_{\mathbf{1}} \to (c_2\tau_\nu + d_2)^{2k^\nu_{M_{\mathbf{1}}}}M_{\mathbf{1}} \tag{A.24}$$

$$M_{\mathbf{1}'} \to (c_2\tau_\nu + d_2)^{2k^\nu_{M_{\mathbf{1}'}}}\rho(\gamma_2)M_{\mathbf{1}'} \tag{A.25}$$

$$M_{\mathbf{1}''} \to (c_2\tau_\nu + d_2)^{2k^\nu_{M_{\mathbf{1}''}}}\rho(\gamma_2)M_{\mathbf{1}''} \tag{A.26}$$

$$M_{\mathbf{3}} \to (c_2\tau_\nu + d_2)^{2k^\nu_{M_{\mathbf{3}}}}\rho(\gamma_1^{-1}\gamma_2\gamma_1)M_{\mathbf{3}} \tag{A.27}$$

where E^c stands for e^c, ν^c and τ^c. For the singlets, it was taken into account that $\rho(\gamma_1^{-1}\gamma_2\gamma_1) = \rho(\gamma_2)$. We require here that the triplets ν^c, $Y^\nu_{\mathbf{3}}$ and $M_{\mathbf{3}}$, instead of transforming under $\gamma_2 \in A_4$, transform under the conjugate element of γ_2, which belongs to A_4 if γ_1 also belongs to A_4. Obviously for the other twelve solutions that belong to S_4 but not to A_4 this is not verified.

The transformation rules for ν^c, $Y^\nu_{\mathbf{3}}$ and $M_{\mathbf{3}}$ are equivalent to the following ones:

$$[\rho(\gamma_1)\nu^c] \to (c_2\tau_\nu + d_2)^{-2k^\nu_{\nu^c}}\rho(\gamma_2)\,[\rho(\gamma_1)\nu^c] \tag{A.28}$$

$$[\rho(\gamma_1)Y^\nu_{\mathbf{3}}] \to (c_2\tau_\nu + d_2)^{2k^\nu_{Y^\nu_{\mathbf{3}}}}\rho(\gamma_2)\,[\rho(\gamma_1)Y^\nu_{\mathbf{3}}] \tag{A.29}$$

$$[\rho(\gamma_1)M_{\mathbf{3}}] \to (c_2\tau_\nu + d_2)^{2k^\nu_{M_{\mathbf{3}}}}\rho(\gamma_2)\,[\rho(\gamma_1)M_{\mathbf{3}}] \tag{A.30}$$

which implies that, with a suitable redefinition of ν^c, $Y^\nu_{\mathbf{3}}$ and $M_{\mathbf{3}}$, we recover the single A_4 subgroup under which all the terms after Φ gains a VEV are invariant. In conclusion, we found that, in general, half of the values the VEV of Φ can have (12 in 24) lead to the same results discussed in the main text and nothing new is left to be said about these other 11 solutions, and interestingly these twelve equivalent solutions are the only possible values for the VEV when $g_A = g_S$.

Here we dealt with the specific case of the seesaw mechanism used for the two models in Section 3.4. For the Weinberg operator in Section 3.3 the same conclusions are valid: there is no difference in using the other eleven VEV's for Φ. In fact, the reasoning is even simpler in this case since fewer fields are used.

Appendix B

Modular A_5 Symmetry Group

B.1 A_5 multiplication rules

The group A_5 is the group of even permutations of five objects and is the symmetry group of the icosahedron and its dual solid the dodecahedron. It has 60 elements and two generators, S and T:

$$S^2 = (ST)^3 = T^5 = 1. \tag{B.1}$$

A_5 has five conjugacy classes:

$$C_1 = \{e\} \tag{B.2}$$

$$C_2 = \{T^3ST^2ST, ST^2ST^3, ST^2ST^2ST, ST^3ST, T^3ST^3, T^2ST^2, TS, TSTS, ST^3STS, T^2ST^2ST, STST^3, T^3ST, ST^3ST^2, T^3ST^2S, T^3STS, TST^3, ST, STST, TST^3ST, ST^2ST^2S\} \tag{B.3}$$

$$C_3 = \{STST^2, T^2ST^3STS, ST^3ST^2S, T^2ST^3, S, ST^3ST^2ST, ST^2ST^3ST, T^2ST^3ST^2, STST^3ST^2, TST^2S, ST^2ST^3ST^2, ST^2ST, T^3ST^2, T^2STS, TST^3ST^2S\} \tag{B.4}$$

$$C_4 = \{T, T^4, TST, STS, STST^2S, TST^2, T^3S, ST^2, T^2S, ST^3, ST^2STS, T^2ST\} \tag{B.5}$$

$$C_5 = \{T^2, T^3, ST^2S, TST^2ST, STST^3ST^2S, TST^3ST^2, STST^3ST, ST^2ST^2, T^2ST^2S, TST^3STS, T^2ST^3ST, ST^2ST^3STS\} \tag{B.6}$$

This group has five irreducible representations: an invariant singlet $\mathbf{1}$, two triplets $\mathbf{3}$ and $\mathbf{3'}$, a quadruplet $\mathbf{4}$ and a quintuplet $\mathbf{5}$. The representations for the generators are in Table B.1.

The product of two irreps decomposes in the following way:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5} \tag{B.7}$$

$$\mathbf{3} \otimes \mathbf{3'} = \mathbf{4} \oplus \mathbf{5} \tag{B.8}$$

$$\mathbf{3} \otimes \mathbf{4} = \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5} \tag{B.9}$$

$$\mathbf{3} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5} \tag{B.10}$$

	S	T
$\mathbf{1}$	1	1
$\mathbf{3}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & 1/\phi \\ -\sqrt{2} & 1/\phi & -\phi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$
$\mathbf{3'}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1/\phi & \phi \\ \sqrt{2} & \phi & -1/\phi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}$
$\mathbf{4}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1/\phi & \phi & -1 \\ 1/\phi & -1 & 1 & \phi \\ \phi & 1 & -1 & 1/\phi \\ -1 & \phi & 1/\phi & 1 \end{pmatrix}$	$\begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & \zeta^4 \end{pmatrix}$
$\mathbf{5}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 1/\phi^2 & -2\phi & 2/\phi & \phi^2 \\ \sqrt{6} & -2\phi & \phi^2 & 1/\phi^2 & 2/\phi \\ \sqrt{6} & 2/\phi & 1/\phi^2 & \phi^2 & 2\phi \\ \sqrt{6} & \phi^2 & 2/\phi & -2\phi & 1/\phi^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^4 \end{pmatrix}$

Table B.1: Representation for the two generators of A_5, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\zeta = e^{2\pi i/5}$.

$$\mathbf{3'} \otimes \mathbf{3'} = \mathbf{1} \oplus \mathbf{3'} \oplus \mathbf{5} \tag{B.11}$$

$$\mathbf{3'} \otimes \mathbf{4} = \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5} \tag{B.12}$$

$$\mathbf{3'} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5} \tag{B.13}$$

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5} \tag{B.14}$$

$$\mathbf{4} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}_1 \oplus \mathbf{5}_2 \tag{B.15}$$

$$\mathbf{5} \otimes \mathbf{5} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4}_1 \oplus \mathbf{4}_2 \oplus \mathbf{5}_1 \oplus \mathbf{5}_2 \tag{B.16}$$

The factors considered for the representation in Table B.1 lead to the following decomposition, with the Clebsch-Gordan coefficients in [52]:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ -\sqrt{3}a_1b_2 - \sqrt{3}a_2b_1 \\ \sqrt{6}a_2b_2 \\ \sqrt{6}a_3b_3 \\ -\sqrt{3}a_1b_3 - \sqrt{3}a_3b_1 \end{pmatrix}_{\mathbf{5}} \tag{B.17}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3'}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3'}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3'}} \oplus \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ \sqrt{6}a_3b_3 \\ -\sqrt{3}a_1b_2 - \sqrt{3}a_2b_1 \\ -\sqrt{3}a_1b_3 - \sqrt{3}a_3b_1 \\ \sqrt{6}a_2b_2 \end{pmatrix}_{\mathbf{5}} \tag{B.18}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3'}} = \begin{pmatrix} \sqrt{2}a_2b_1 + a_3b_2 \\ -\sqrt{2}a_1b_2 - a_3b_3 \\ -\sqrt{2}a_1b_3 - a_2b_2 \\ \sqrt{2}a_3b_1 + a_2b_3 \end{pmatrix}_{\mathbf{4}} \oplus \begin{pmatrix} \sqrt{3}a_1b_1 \\ a_2b_1 + \sqrt{2}a_3b_2 \\ a_1b_2 - \sqrt{2}a_3b_3 \\ a_1b_3 - \sqrt{2}a_2b_2 \\ a_3b_1 + \sqrt{2}a_2b_3 \end{pmatrix}_{\mathbf{5}} \tag{B.19}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_{\mathbf{4}} = \begin{pmatrix} -\sqrt{2}a_2b_4 - \sqrt{2}a_3b_1 \\ \sqrt{2}a_1b_2 - a_2b_1 + a_3b_3 \\ \sqrt{2}a_1b_3 - a_3b_4 + a_2b_2 \end{pmatrix}_{\mathbf{3'}} \oplus \begin{pmatrix} a_1b_1 - \sqrt{2}a_3b_2 \\ -a_1b_2 - \sqrt{2}a_2b_1 \\ a_1b_3 + \sqrt{2}a_3b_4 \\ -a_1b_4 + \sqrt{2}a_2b_3 \end{pmatrix}_{\mathbf{4}} \oplus$$
$$\oplus \begin{pmatrix} \sqrt{6}a_2b_4 - \sqrt{6}a_3b_1 \\ 2\sqrt{2}a_1b_1 + 2a_3b_2 \\ -\sqrt{2}a_1b_2 + a_2b_1 + 3a_3b_3 \\ \sqrt{2}a_1b_3 - a_3b_4 - 3a_2b_2 \\ -2\sqrt{2}a_1b_4 - 2a_2b_3 \end{pmatrix}_{\mathbf{5}} \tag{B.20}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3'}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_{\mathbf{4}} = \begin{pmatrix} -\sqrt{2}a_2b_3 - \sqrt{2}a_3b_2 \\ \sqrt{2}a_1b_1 + a_2b_4 - a_3b_3 \\ \sqrt{2}a_1b_4 + a_3b_1 - a_2b_2 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} a_1b_1 + \sqrt{2}a_3b_3 \\ a_1b_2 - \sqrt{2}a_3b_4 \\ -a_1b_3 + \sqrt{2}a_2b_1 \\ -a_1b_4 - \sqrt{2}a_2b_2 \end{pmatrix}_{\mathbf{4}} \oplus$$
$$\oplus \begin{pmatrix} \sqrt{6}a_2b_3 - \sqrt{6}a_3b_2 \\ \sqrt{2}a_1b_1 - 3a_2b_4 - a_3b_3 \\ 2\sqrt{2}a_1b_2 + 2a_3b_4 \\ -2\sqrt{2}a_1b_3 - 2a_2b_1 \\ -\sqrt{2}a_1b_4 + 3a_3b_1 + a_2b_2 \end{pmatrix}_{\mathbf{5}} \tag{B.21}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}_{\mathbf{5}} = \begin{pmatrix} -2a_1b_1 + \sqrt{3}a_2b_5 + \sqrt{3}a_3b_2 \\ \sqrt{3}a_1b_2 + a_2b_1 - \sqrt{6}a_3b_3 \\ \sqrt{3}a_1b_5 + a_3b_1 - \sqrt{6}a_2b_4 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} \sqrt{3}a_1b_1 + a_2b_5 + a_3b_2 \\ a_1b_3 - \sqrt{2}a_2b_2 - \sqrt{2}a_3b_4 \\ a_1b_4 - \sqrt{2}a_2b_3 - \sqrt{2}a_3b_5 \end{pmatrix}_{\mathbf{3'}} \oplus$$
$$\oplus \begin{pmatrix} 2\sqrt{2}a_1b_2 - \sqrt{6}a_2b_1 + a_3b_3 \\ -\sqrt{2}a_1b_3 + 2a_2b_2 - 3a_3b_4 \\ \sqrt{2}a_1b_4 - 2a_2b_5 + 3a_2b_3 \\ -2\sqrt{2}a_1b_5 + \sqrt{6}a_3b_1 - a_2b_4 \end{pmatrix}_{\mathbf{4}} \oplus \begin{pmatrix} \sqrt{3}a_2b_5 - \sqrt{3}a_3b_2 \\ -a_1b_2 - \sqrt{3}a_2b_1 - \sqrt{2}a_3b_3 \\ -2a_1b_3 - \sqrt{2}a_2b_2 \\ 2a_1b_4 + \sqrt{2}a_3b_5 \\ a_1b_5 + \sqrt{3}a_3b_1 + \sqrt{2}a_2b_4 \end{pmatrix}_{\mathbf{5}} \tag{B.22}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3'}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}_{\mathbf{5}} = \begin{pmatrix} \sqrt{3}a_1b_1 + a_2b_4 + a_3b_3 \\ a_1b_2 - \sqrt{2}a_2b_5 - \sqrt{2}a_3b_4 \\ a_1b_5 - \sqrt{2}a_2b_3 - \sqrt{2}a_3b_2 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} -2a_1b_1 + \sqrt{3}a_2b_4 + \sqrt{3}a_3b_3 \\ \sqrt{3}a_1b_3 + a_2b_1 - \sqrt{6}a_3b_5 \\ \sqrt{3}a_1b_4 + a_3b_1 - \sqrt{6}a_2b_2 \end{pmatrix}_{\mathbf{3'}} \oplus$$

$$\oplus \begin{pmatrix} \sqrt{2}a_1b_2 - 2a_3b_4 + 3a_2b_5 \\ 2\sqrt{2}a_1b_3 - \sqrt{6}a_2b_1 + a_3b_5 \\ -2\sqrt{2}a_1b_4 + \sqrt{6}a_3b_1 - a_2b_2 \\ -\sqrt{2}a_1b_5 + 2a_2b_3 - 3a_3b_2 \end{pmatrix}_{\mathbf{4}} \oplus \begin{pmatrix} \sqrt{3}a_2b_4 - \sqrt{3}a_3b_3 \\ 2a_1b_2 + \sqrt{2}a_3b_4 \\ -a_1b_3 - \sqrt{3}a_2b_1 - \sqrt{2}a_3b_5 \\ a_1b_4 + \sqrt{3}a_3b_1 + \sqrt{2}a_2b_2 \\ -2a_1b_5 - \sqrt{2}a_2b_3 \end{pmatrix}_{\mathbf{5}} \qquad \text{(B.23)}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}_{\mathbf{4}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}_{\mathbf{4}} = (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1)_{\mathbf{1}} \oplus \begin{pmatrix} a_2b_3 - a_3b_2 + a_4b_1 - a_1b_4 \\ \sqrt{2}a_2b_4 - \sqrt{2}a_4b_2 \\ \sqrt{2}a_1b_3 - \sqrt{2}a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus$$

$$\oplus \begin{pmatrix} a_2b_3 - a_3b_2 + a_1b_4 - a_4b_1 \\ \sqrt{2}a_3b_4 - \sqrt{2}a_4b_3 \\ \sqrt{2}a_1b_2 - \sqrt{2}a_2b_1 \end{pmatrix}_{\mathbf{3}} \oplus \begin{pmatrix} a_3b_3 + a_2b_4 + a_4b_2 \\ a_1b_1 + a_3b_4 + a_4b_3 \\ a_4b_4 + a_1b_2 + a_2b_1 \\ a_2b_2 + a_1b_3 + a_3b_1 \end{pmatrix}_{\mathbf{4}} \oplus$$

$$\oplus \begin{pmatrix} \sqrt{3}a_1b_4 + \sqrt{3}a_4b_1 - \sqrt{3}a_2b_3 - \sqrt{3}a_3b_2 \\ 2\sqrt{2}a_3b_3 - \sqrt{2}a_2b_4 - \sqrt{2}a_4b_2 \\ -2\sqrt{2}a_1b_1 + \sqrt{2}a_3b_4 + \sqrt{2}a_4b_3 \\ -2\sqrt{2}a_4b_4 + \sqrt{2}a_1b_2 - \sqrt{2}a_2b_1 \\ 2\sqrt{2}a_2b_2 - \sqrt{2}a_1b_3 - \sqrt{2}a_3b_1 \end{pmatrix}_{\mathbf{5}} \qquad \text{(B.24)}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}_{\mathbf{4}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}_{\mathbf{5}} = \begin{pmatrix} 2\sqrt{2}a_1b_5 - 2\sqrt{2}a_4b_2 + \sqrt{2}a_3b_3 - \sqrt{2}a_2b_4 \\ 3a_3b_4 + 2a_2b_5 - a_4b_3 - \sqrt{6}a_1b_1 \\ -3a_2b_3 - 2a_3b_2 + a_1b_4 + \sqrt{6}a_4b_1 \end{pmatrix}_{\mathbf{3}} \oplus$$

$$\oplus \begin{pmatrix} 2\sqrt{2}a_2b_4 - 2\sqrt{2}a_3b_3 + \sqrt{2}a_1b_5 - \sqrt{2}a_4b_2 \\ 3a_1b_2 + 2a_4b_4 - a_3b_5 - \sqrt{6}a_2b_1 \\ -3a_4b_5 - 2a_1b_3 + a_2b_2 + \sqrt{6}a_3b_1 \end{pmatrix}_{\mathbf{3'}} \oplus$$

$$\oplus \begin{pmatrix} \sqrt{3}a_1b_1 - \sqrt{2}a_2b_5 + \sqrt{2}a_3b_4 - 2\sqrt{2}a_4b_3 \\ -\sqrt{2}a_1b_2 - \sqrt{3}a_2b_1 + 2\sqrt{2}a_3b_5 + \sqrt{2}a_4b_4 \\ \sqrt{2}a_1b_3 + 2\sqrt{2}a_2b_2 - \sqrt{3}a_3b_1 - \sqrt{2}a_4b_5 \\ -2\sqrt{2}a_1b_4 + \sqrt{2}a_2b_3 - \sqrt{2}a_3b_2 + \sqrt{3}a_4b_1 \end{pmatrix}_{\mathbf{4}} \oplus$$

$$
\oplus \begin{pmatrix} \sqrt{2}a_1b_5 - \sqrt{2}a_2b_4 - \sqrt{2}a_3b_3 + \sqrt{2}a_4b_2 \\ -\sqrt{2}a_1b_1 - \sqrt{3}a_3b_4 - \sqrt{3}a_4b_3 \\ \sqrt{3}a_1b_2 + \sqrt{2}a_2b_1 + \sqrt{3}a_3b_5 \\ \sqrt{3}a_2b_2 + \sqrt{2}a_3b_1 + \sqrt{3}a_4b_5 \\ -\sqrt{3}a_1b_4 - \sqrt{3}a_2b_3 - \sqrt{2}a_4b_1 \end{pmatrix}_{\mathbf{5}_1} \oplus
$$

$$
\oplus \begin{pmatrix} 2a_1b_5 + 4a_2b_4 + 4a_3b_3 + 2a_4b_2 \\ 4a_1b_1 + 2\sqrt{6}a_2b_5 \\ -\sqrt{6}a_1b_2 + 2a_2b_1 - \sqrt{6}a_3b_5 + 2\sqrt{6}a_4b_4 \\ 2\sqrt{6}a_1b_3 - \sqrt{6}a_2b_2 + 2a_3b_1 - \sqrt{6}a_4b_5 \\ 2\sqrt{6}a_3b_2 + 4a_4b_1 \end{pmatrix}_{\mathbf{5}_2} \tag{B.25}
$$

$$
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}_{\mathbf{5}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}_{\mathbf{5}} = (a_1b_1 + a_2b_5 + a_5b_2 + a_3b_4 + a_4b_3)_{\mathbf{1}} \oplus
$$

$$
\oplus \begin{pmatrix} a_2b_5 - a_5b_2 + 2a_3b_4 - 2a_4b_3 \\ \sqrt{3}a_2b_1 - \sqrt{3}a_1b_2 + \sqrt{2}a_3b_5 - \sqrt{2}a_5b_3 \\ \sqrt{3}a_1b_5 - \sqrt{3}a_5b_1 + \sqrt{2}a_2b_4 - \sqrt{2}a_2b_4 \end{pmatrix}_{\mathbf{3}} \oplus
$$

$$
\oplus \begin{pmatrix} a_4b_3 - a_3b_4 + 2a_2b_5 - 2a_5b_2 \\ \sqrt{3}a_1b_3 - \sqrt{3}a_3b_1 + \sqrt{2}a_4b_5 - \sqrt{2}a_5b_4 \\ \sqrt{3}a_4b_1 - \sqrt{3}a_1b_4 + \sqrt{2}a_2b_3 - \sqrt{2}a_3b_4 \end{pmatrix}_{\mathbf{3'}} \oplus
$$

$$
\oplus \begin{pmatrix} 4\sqrt{3}a_4b_4 + 3\sqrt{2}a_1b_2 + 3\sqrt{2}a_2b_1 - \sqrt{3}a_3b_5 - \sqrt{3}a_5b_3 \\ 4\sqrt{3}a_2b_2 + 3\sqrt{2}a_1b_3 + 3\sqrt{2}a_3b_1 - \sqrt{3}a_4b_5 - \sqrt{3}a_5b_4 \\ 4\sqrt{3}a_5b_5 + 3\sqrt{2}a_1b_4 + 3\sqrt{2}a_4b_1 - \sqrt{3}a_3b_2 - \sqrt{3}a_2b_3 \\ 4\sqrt{3}a_3b_3 + 3\sqrt{2}a_1b_5 + 3\sqrt{2}a_5b_1 - \sqrt{3}a_2b_4 - \sqrt{3}a_4b_2 \end{pmatrix}_{\mathbf{4}_1} \oplus
$$

$$
\oplus \begin{pmatrix} \sqrt{2}a_1b_2 - \sqrt{2}a_2b_1 + \sqrt{3}a_3b_5 - \sqrt{3}a_5b_3 \\ \sqrt{2}a_3b_1 - \sqrt{2}a_1b_3 + \sqrt{3}a_4b_5 - \sqrt{3}a_5b_4 \\ \sqrt{2}a_4b_1 - \sqrt{2}a_1b_4 + \sqrt{3}a_3b_2 - \sqrt{3}a_2b_3 \\ \sqrt{2}a_1b_5 - \sqrt{2}a_5b_1 + \sqrt{3}a_4b_2 - \sqrt{3}a_2b_4 \end{pmatrix}_{\mathbf{4}_2} \oplus
$$

$$
\oplus \begin{pmatrix} 2a_1b_1 + a_2b_5 + a_5b_2 - 2a_3b_4 - 2a_4b_3 \\ a_1b_2 + a_2b_1 + \sqrt{6}a_3b_5 + \sqrt{6}a_5b_3 \\ \sqrt{6}a_2b_2 - 2a_1b_3 - 2a_3b_1 \\ \sqrt{6}a_5b_5 - 2a_1b_4 - 2a_4b_1 \\ a_1b_5 + a_5b_1 + \sqrt{6}a_2b_4 + \sqrt{6}a_4b_2 \end{pmatrix}_{\mathbf{5}_1} \oplus
$$

$$\oplus \begin{pmatrix} 2a_1b_1 + a_3b_4 + a_4b_3 - 2a_2b_5 - 2a_5b_2 \\ \sqrt{6}a_4b_4 - 2a_1b_2 - 2a_2b_1 \\ a_1b_3 + a_3b_1 + \sqrt{6}a_4b_5 + \sqrt{6}a_5b_4 \\ a_1b_4 + a_4b_1 + \sqrt{6}a_2b_3 + \sqrt{6}a_3b_2 \\ \sqrt{6}a_3b_3 - 2a_1b_5 - 2a_5b_1 \end{pmatrix}_{\mathbf{5}_2} \tag{B.26}$$

B.2 Modular forms of weight 2 for A_5

The linear space of modular forms of level $N = 5$ and weight 2 has dimension 11. These modular functions are arranged into two triplets $\mathbf{3}$ and $\mathbf{3}'$ and a quintuplet $\mathbf{5}$ of Γ_5. Modular forms of higher weight can be constructed from polynomials of these eleven modular functions.

The weight 2 modular functions can be expressed as linear combinations of logarithmic derivatives of some functions $\alpha_{i,j}(\tau)$, closed under the action of A_5, and these can be in terms of the theta function $\theta_3(z(\tau), t(\tau))$:

$$\theta_3(z,t) = \sum_{k\in\mathbb{Z}} q^{k^2} e^{2\pi i k z} = 1 + 2\sum_{k\in\mathbb{N}} q^{k^2} \cos(2\pi k z)\,,\ q = e^{\pi i t} \tag{B.27}$$

The seed functions $\alpha_{i,j}(\tau)$ are explicitly:

$$\begin{aligned}
\alpha_{1,-1}(\tau) &\equiv \theta_3\left(\frac{\tau+1}{2}, 5\tau\right), & \alpha_{2,-1}(\tau) &\equiv e^{2\pi i\tau/5}\theta_3\left(\frac{3\tau+1}{2}, 5\tau\right), \\
\alpha_{1,0}(\tau) &\equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau}{5}\right), & \alpha_{2,0}(\tau) &\equiv \theta_3\left(\frac{\tau+7}{10}, \frac{\tau}{5}\right), \\
\alpha_{1,1}(\tau) &\equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+1}{5}\right), & \alpha_{2,1}(\tau) &\equiv \theta_3\left(\frac{\tau+8}{10}, \frac{\tau+1}{5}\right), \\
\alpha_{1,2}(\tau) &\equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+2}{5}\right), & \alpha_{2,2}(\tau) &\equiv \theta_3\left(\frac{\tau+9}{10}, \frac{\tau+2}{5}\right), \\
\alpha_{1,3}(\tau) &\equiv \theta_3\left(\frac{\tau+2}{10}, \frac{\tau+3}{5}\right), & \alpha_{2,3}(\tau) &\equiv \theta_3\left(\frac{\tau}{10}, \frac{\tau+3}{5}\right), \\
\alpha_{1,4}(\tau) &\equiv \theta_3\left(\frac{\tau+3}{10}, \frac{\tau+4}{5}\right), & \alpha_{2,4}(\tau) &\equiv \theta_3\left(\frac{\tau+1}{10}, \frac{\tau+4}{5}\right).
\end{aligned} \tag{B.28}$$

The linear combination of the logarithmic derivatives of these functions,

$$Y(c_{1,-1}, \ldots, c_{1,4}; c_{2,-1}, \ldots, c_{2,4}|\tau) \equiv \sum_{i,j} c_{i,j} \frac{\mathrm{d}}{\mathrm{d}\tau} \log \alpha_{i,j}(\tau), \quad \text{with } \sum_{i,j} c_{i,j} = 0, \tag{B.29}$$

span the linear space of the modular forms of level $N = 5$ and weight 2. These are then divided into the multiplets:

$$Y_{\mathbf{5}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{\sqrt{6}} Y(-5,1,1,1,1,1;-5,1,1,1,1,1|\tau) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta;0,1,\zeta^4,\zeta^3,\zeta^2,\zeta|\tau) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2;0,1,\zeta^3,\zeta,\zeta^4,\zeta^2|\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3;0,1,\zeta^2,\zeta^4,\zeta,\zeta^3|\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4;0,1,\zeta,\zeta^2,\zeta^3,\zeta^4|\tau) \end{pmatrix}, \tag{B.30}$$

$$Y_{\mathbf{3}}(\tau) = \begin{pmatrix} Y_6(\tau) \\ Y_7(\tau) \\ Y_8(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} Y\left(-\sqrt{5}, -1, -1, -1, -1, -1; \sqrt{5}, 1, 1, 1, 1, 1 \middle| \tau\right) \\ Y(0, 1, \zeta^4, \zeta^3, \zeta^2, \zeta; 0, -1, -\zeta^4, -\zeta^3, -\zeta^2, -\zeta|\tau) \\ Y(0, 1, \zeta, \zeta^2, \zeta^3, \zeta^4; 0, -1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4|\tau) \end{pmatrix}, \tag{B.31}$$

$$Y_{\mathbf{3'}}(\tau) = \begin{pmatrix} Y_9(\tau) \\ Y_{10}(\tau) \\ Y_{11}(\tau) \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} Y\left(\sqrt{5}, -1, -1, -1, -1, -1; -\sqrt{5}, 1, 1, 1, 1, 1 \middle| \tau\right) \\ Y(0, 1, \zeta^3, \zeta, \zeta^4, \zeta^2; 0, -1, -\zeta^3, -\zeta, -\zeta^4, -\zeta^2|\tau) \\ Y(0, 1, \zeta^2, \zeta^4, \zeta, \zeta^3; 0, -1, -\zeta^2, -\zeta^4, -\zeta, -\zeta^3|\tau) \end{pmatrix}, \tag{B.32}$$

where $\zeta = e^{2\pi i/5}$.

B.3 Vacuum alignments for bi-quintuplet Φ in A_5

In this Appendix we consider how to align the VEV of the bi-quintuplet Φ. Following from [7] and A.3 where an alignment was obtained in the context of S_4 and A_4, we add two driving fields, with the properties present in Table B.2.

Fields	A_5^l	A_5^ν	$2k_l$	$2k_\nu$
$\chi_{l\nu}$	$\mathbf{5}$	$\mathbf{5}$	0	0
χ_l	$\mathbf{5}$	$\mathbf{1}$	0	0

Table B.2: Transformation properties of the fields responsible for the vacuum alignment of the bi-quintuplet Φ for the models with two modular A_5.

The superpotential responsible for the vacuum alignment that will be minimised with relation to the driving fields is

$$w = \Phi\Phi\chi_{l\nu} + M\Phi\chi_{l\nu} + \Phi\Phi\chi_l. \tag{B.33}$$

With this field content, we are only interested in contractions of quintuplets to give quintuplets or singlets. Minimising this superpotential in order to the driving fields leads us to the constraints:

$$\sum_{j,k=1,\ldots,5} \sum_{\beta,\gamma=1,\ldots,5} \left(P_{ijk}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_1}} + cP_{ijk}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_2}}\right)\left(P_{\alpha\beta\gamma}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_1}} + cP_{\alpha\beta\gamma}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_2}}\right)(\Phi)_{j\beta}(\Phi)_{k\gamma} + M(\Phi)_{i\alpha} = 0, \tag{B.34}$$

for $i = 1, \ldots, 5, \alpha = 1, \ldots, 5$

$$\sum_{j,k=1,\ldots,5} \sum_{\alpha,\beta=1,\ldots,5} P_{\alpha\beta}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{1}}} \left(P_{ijk}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_1}} + cP_{ijk}^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_2}}\right)(\Phi)_{j\alpha}(\Phi)_{k\beta} = 0 \text{ for } i = 1, \ldots, 5. \tag{B.35}$$

where 5×5 matrices that describe the multiplication rules in Section B.1 were introduced:

$$P^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{1}}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{B.36}$$

$$P^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_1}} = \left(\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \tag{B.37}$$

$$P^{(\mathbf{5}\otimes\mathbf{5})_{\mathbf{5}_2}} = \left(\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & \sqrt{6} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix} \right) \tag{B.38}$$

It can be easily verified that

$$\langle \Phi \rangle = v_\Phi \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{B.39}$$

which is the VEV used in the main text for Φ, obeys the constraints Eq.(B.35). This is one of the elements of the A_5 in the five dimensional representation. In fact, all the representations of the elements of A_5 are solutions of these constraints. Note that we only verified that these solve the equations, we did not solve fully these equations. This is the matter of a still ongoing study.

B.4 Vacuum alignments for bi-triplet Φ in A_5

In this Appendix we consider how to align the VEV of the bi-triplet Φ for the model using the seesaw mechanism to generate the neutrino masses. In this symmetry group, conversely to what happened in A_4, there is no symmetry contraction of two triplets to another triplet, the contraction of two triplets to a triplet is by definition antisymmetric. This reasoning is valid either for $\mathbf{3}$ and $\mathbf{3}'$, since their multiplication rules only differ in the quintuplet decomposition. We conclude then that in A_4 we had both triplets (as we saw in Section A.3), in S_4 only the symmetric contribution appeared and for A_5 only the antisymmetric one appears.

This means that adding a driving field like χ_l in Section A.3 does not provide additional constraints since Φ will not couple to χ_l in a term like $\Phi\Phi\chi_l$. Thus, we will try to add only one driving field, with the properties present in Table B.3. We state again that is not important if the L, E^c and ν^c are triplets $\mathbf{3}$ or $\mathbf{3}'$ and thus if Φ is a bi-triplet $\mathbf{3}$ or $\mathbf{3}'$ given that the contraction rules $\mathbf{3} \times \mathbf{3} \to \mathbf{3}$ and $\mathbf{3}' \times \mathbf{3}' \to \mathbf{3}'$ are the same, and the same happens for $\mathbf{3} \times \mathbf{3} \to \mathbf{1}$ and $\mathbf{3}' \times \mathbf{3}' \to \mathbf{1}$.

Fields	A_5^l	A_5^ν	$2k_l$	$2k_\nu$
$\chi_{l\nu}$	$\mathbf{3}^{(\prime)}$	$\mathbf{3}^{(\prime)}$	0	0

Table B.3: Transformation properties of the fields responsible for the vacuum alignment of the bi-triplet Φ for the models with two modular A_5.

The superpotential responsible for the vacuum alignment that will be minimized with relation to the driving field is

$$w = \Phi\Phi\chi_{l\nu} + M\Phi\chi_{l\nu}. \tag{B.40}$$

From Eq.(B.40) and working in the complex basis, in which T is diagonal, we are then able to derive the constraints:

$$\sum_{j,k=1,2,3} \sum_{\beta,\gamma=1,2,3} \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}(\Phi)_{j\beta}(\Phi)_{k\gamma} + M(\Phi)_{i\alpha} = 0 \text{ for } i = 1,2,3, \alpha = 1,2,3. \tag{B.41}$$

This system of equations is not fully determined but substituting we conclude that P_{23}, the vacuum used in the main text for $\langle\Phi\rangle$ is indeed one possible solution. This specific VEV leads to the recovering of the usual multiplication of two triplets to give a singlet. As for the alignment in Section B.3, and in this case even more so, a more complete discussion of this alignment will be considered in the future and is beyond the scope of this thesis.