Modular Symmetries and the Flavour Problem

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Abstract

We note that no solution is provided for the flavour problem in the context of the Standard Model (SM) but that this can be solved by introducing multiple modular symmetries. We construct lepton flavour models based on two A_4 modular symmetries, which are broken to the diagonal subgroup A_4^D, resulting in an effective modular flavour symmetry with two moduli. We employ these moduli as stabilisers, that preserve distinct residual symmetries, enabling us to obtain Tri-Maximal 2 (TM_2) mixing with a minimal field content, flavonless at the effective scale, below the breaking to the single A_4. We also construct models based on two A_5 modular symmetries, where a mixing that preserves the second column of the Golden Ratio (GR) mixing, which we called GR_2, is obtained. Best fit points and plots for the neutrinoless beta decay are obtained for all these models. It was realised that the normal ordering (NO) of neutrino masses is the preferred ordering, being the models that lead to GR_2 more favourable than those that lead to TM_2. For all the best fit values for NO, the neutrino masses and mixing angles except θ_{12} are compatible with experimental results at the 1σ confidence interval.
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1. Introduction

The current model of particle physics is the Standard Model (SM) [1, 2]. Until now, it has been extremely compatible with experimental results. In this model, the fundamental constituents of matter are quarks and leptons while the strong, weak and electromagnetic interactions are mediated by spin-1 particles that are connected to the local gauge symmetries $SU(3)_C \times SU(2)_L \times U(1)_Y$, where C stands for colour, L for left-handedness, and Y for hypercharge. This symmetry is spontaneously broken to $SU(3)_C \times U(1)_{EM}$ where $U(1)_{EM}$ couples to the electromagnetic charge $Q_{EM} = T_3 + Y$ where T_3 is the third component of the isospin.

In the leptonic sector, one has three generations of charged leptons, that can be both left and right-handed fermions, and three left-handed neutrinos. The left-handed particles are arranged in doublets of $SU(2)_L$ and the other three charged leptons are singlets of $SU(2)_L$. In the SM model, no right-handed neutrinos are considered because neutrinos do not interact through other forces than the weak force and the weak bosons only couple to left-handed particles. Left-handed neutrinos are also known as active neutrinos and right-handed neutrinos are know as sterile neutrinos, since they have no SM interactions.

In the SM the fermions get their mass through a Yukawa term that couples the scalar Higgs field doublet ϕ to a component of a $SU(2)_L$ doublet and a $SU(2)_L$ singlet through a Yukawa coupling. After spontaneous symmetry breaking, when the Higgs acquires the VEV $\langle\phi\rangle = 1/\sqrt{2}(0, v + h(x))$, the charged lepton masses are generated. The model only contains left-handed neutrinos thus no Yukawa mass terms can be constructed for the neutrinos and these remain massless at the Lagrangian level.

But it is a well established result that neutrinos oscillate between flavours. The first clue arose from the discrepancy between theoretical models for the neutrinos produced at the Sun and the experimental results of neutrino rates. This result was explained by the conversion of electron neutrinos into muon and tau neutrinos due to a non-zero probability of measuring muon and tau neutrinos as a initial beam of electron neutrinos propagates through space. This implies that neutrinos have different masses, so at least two of them, although very light, have a mass, which is in disagreement with the SM. Hence the need to go beyond the SM.

The objective of the present work was to use multiple modular symmetries, either two A_4's or two A_5's, to construct a high energy theory which is then broken to a low energy model with a single modular symmetry, whose moduli fields gain different VEV's, leading to the realisation of different mass textures in the charged lepton and neutrino sectors. It is then possible to obtain a realistic mixing matrix and mass hierarchies. These modular symmetries are thus able to generate all masses and mixing parameters for the leptons, using a much smaller set of free parameters, almost only using the VEV's of the Higgs and the moduli fields. Additionally, it will be investigated, through the introduction of driving fields, how the VEV's of the fields that are responsible for the breaking from two modular symmetries to a single one are created.

We will now conclude with a brief outline of the present work. In Section 2 we start by reviewing how can neutrino masses by generated and discuss what is the flavour problem. In Section 3, we introduce the concept of modular symmetries, which can be used to solve the flavour problem, and how we can obtain a lagrangian invariant under these symmetries. In Section 4 we review the main results for the models obtained when two A_4 modular symmetries are introduced and in Section 5 for the models invariant under two A_5 modular symmetries instead. In Section 6 we review the main conclusions and some aspects of the present work possible to be improved in the future.

2. State of the Art

First we should consider how we can introduce terms in the SM to describe the neutrino masses. One possible way of seeing the neutrino masses problem is to consider that new physics only appears above a scale Λ_{NP} and that the SM is simply a effective low energy theory of a high energy theory. In this case, one doesn't have to worry about the renormalisability of the theory and terms with mass dimension larger than 4, although suppressed, are not forbidden. The least suppressed term that can generate neutrino masses, known as the Weinberg operator, is the dimension 5 term:

$$\frac{Z_{ij}^{\nu}}{\Lambda_{NP}}(\overline{L}_{Li}\tilde{\phi})(\tilde{\phi}^T L_{Lj}^C) + h.c. \qquad (1)$$

where $\tilde{\phi} = i\tau_2\phi^*$.

Other possibility is to consider now the SM with the addition of m sterile neutrinos. Two possible gauge invariant terms can be constructed:

$$-\mathcal{L}_{M_\nu} = M_{Dij}\overline{\nu}_{si}\nu_{Lj} + \frac{1}{2}M_{Nij}\overline{\nu}_{si}\nu_{sj}^c + h.c. \qquad (2)$$

where M_D is a complex $m \times 3$ matrix, M_N a symmetric $m \times m$ matrix and $\nu^c = C\overline{\nu}^T$ is the charged conjugated neutrino field. The first term arises from the Yukawa terms for the neutrinos after spontaneous symmetry breaking, while the second term is a Majorana term that violates leptonic number.

It is possible to get 3 light neutrinos ν_l and m heavy neutrinos N from the previous $3+m$ neutrinos if the mass eigenvalues of M_N are much larger than the electroweak symmetry breaking scale v. The masses of the heavier states will be proportional to M_N and the lighter states to $M_D^2 M_N^{-1}$. When the heavy neutrino masses increase, the almost massless neutrinos become lighter, hence the name see-saw mechanism applied to this model. Other interesting property is that the lighter neutrinos are almost left-handed and the heavy ones almost right-handed.

To work only with mass eigenstates, the mass matrix for the charged leptons and the neutrino mass matrix need both to be diagonalized. This change from the interaction eigenstates to the mass eigenstates has consequences in the charged current part of the Lagrangian, introducing a mixing matrix between the mass states of the leptons, similar to the mixing matrix between quarks. In the leptonic sector, the mixing matrix is known as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix.

However, even if we are able to modify slightly the SM to account for neutrino masses, we will still have a lot of questions on flavour that remain unanswered. First of all, there is no reason for why there are three families of quarks and leptons.

The mass hierarchies of the quarks and leptons also seem to encode new physics, with the down type quarks and charged leptons having mass values of the same order of magnitude, while the up-type quarks are much more hierarchical and the neutrinos are almost massless. But it is not only when we compare mass hierarchies that flavour for leptons and quarks has a very different behaviour. The mixing between flavours is much larger in the leptonic sector while the CKM matrix is almost diagonal. In fact, the PMNS mixing angles are much larger or have the same order of magnitude than the CKM mixing angles.

Finally, the SM and slight modifications of it have another conceptual problem: why are there much more parameters in the flavour sector than in the gauge (strong, weak and electromagnetic) sectors?

All these questions, that constitute the so called flavour problem, point towards the need for the introduction of a fundamental flavour symmetry that accounts for this large collection of parameters arising from the Higgs sector. This new symmetry could, from only a few parameters, generate all the fermion masses and mixing parameters.

Flavour symmetries, both discrete and continuous, have been extensively employed in the literature as a way to solve the puzzling questions associated with flavour. Examples of well-known discrete symmetries applied to flavour are S_3, A_4, S_4 and A_5. More recently, these same symmetries are used in flavour models as modular symmetries $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ and $\Gamma_5 \simeq A_5$. More recently [3] studied the mass sum rules arising in these models.

Models using multiple S_4 modular symmetries can be found at [4, 5]. In these, a general mechanism of employing multiple modular symmetries to construct a high energy theory which is then broken to a low energy model with a single modular symmetry when their modulus fields gain different VEV's at fixed points of the modular symmetry (stabilisers). The preserved residual symmetries then lead to the realisation of different mass textures in the charged lepton and neutrino sectors in modular flavour models without flavons. These modular symmetries are thus able to generate all masses and mixing parameters for the leptons, using a much smaller set of free parameters. The same procedure might be applied to other modular symmetries, which is precisely what is behind the present work.

3. Modular Symmetries - an Introduction

This section provides the general definitions of the modular group and modular forms, and some fundamental aspects of constructing a realistic model with multiple modular symmetries.

3.1. Modular group and modular forms

The modular group $\overline{\Gamma}$ is the group of linear fractional transformations γ that act on the complex modulus τ, for τ in the upper-half complex plane, i.e. $Im(\tau) > 0$:

$$\gamma : \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{3}$$

where a, b, c, d are integers and satisfy $ad - bc = 1$.

It is convenient to use 2×2 matrices to represent the elements of $\overline{\Gamma}$ as

$$\overline{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \{\pm 1\},\ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}. \tag{4}$$

Note that, since γ and $-\gamma$ are the same modular transformation, the group $\overline{\Gamma}$ is isomorphic to $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$, where $SL(2, \mathbb{Z})$ is the group of 2×2 matrices with integer entries and determinant one.

The modular group has two generators, S_τ and T_τ, which satisfy $S_\tau^2 = (S_\tau T_\tau)^3 = 1$. One possible choice for these generators is the following:

$$S_\tau : \tau \to -\frac{1}{\tau},\ T_\tau : \tau \to \tau + 1 \tag{5}$$

and their corresponding representations are

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

It is possible to define subgroups $\overline{\Gamma}(N)$ of $\overline{\Gamma}$ modding out the entries of the representation matrices. Although the groups $\overline{\Gamma}(N)$ are discrete but infinite, the quotient groups $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$ are finite, thus being called finite modular groups. For $N \leq 5$, these groups are isomorphic to well-known groups: $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$, $\Gamma_5 \simeq A_5$. These finite modular groups can be obtained by imposing an additional condition, $T_\tau^N = 1$, which implies that $\tau = \tau + N$.

Modular forms of weight $2k$ and level N are holomorphic functions of τ that transform under $\overline{\Gamma}(N)$ in the following way:

$$f(\gamma\tau) = (c\tau + d)^{2k} f(\tau),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}(N), \tag{7}$$

where k is a non-negative integer and N is natural (we are only interested in even weights). These modular forms are invariant under $\overline{\Gamma}(N)$, up to the factor $(c\tau + d)^{2k}$, but they transform under the quotient group Γ_N.

Modular forms of weight $2k$ and level N span a linear space of finite dimension $\mathcal{M}_{2k}(\overline{\Gamma}(N))$. It is possible to choose a basis in $\mathcal{M}_{2k}(\overline{\Gamma}(N))$ such that the transformation of the modular forms under Γ_N is described by a unitary representation ρ of Γ_N:

$$f_i(\gamma\tau) = (c\tau + d)^{2k} \rho(\gamma)_{ij} f_j(\tau),\ \gamma \in \Gamma_N. \tag{8}$$

3.2. Models with multiple modular symmetries

Consider a theory that has multiple modular symmetries, based on a series of M modular groups $\overline{\Gamma}^1$, $\overline{\Gamma}^2$, ..., $\overline{\Gamma}^M$, where the modulus field for each symmetry $\overline{\Gamma}^J$, $J = 1, \ldots, M$, is denoted as τ_J. The associated modular transformations take the form:

$$\gamma_J : \tau_J \to \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J}. \tag{9}$$

A series of finite modular groups $\Gamma^J_{N_J}$ for $J = 1, \ldots, M$ can be obtained by modding out an integer N_J and taking the quotient finite groups. Take into account that N_J does not need to be identical to $N_{J'}$ for $J \neq J'$.

Consider an N = 1 supersymmetric model invariant under multiple modular symmetries; the action in general takes the form:

$$S = \int d^4x d^2\theta d^2\overline{\theta}\ K(\phi_i, \overline{\phi}_i; \tau_1, \ldots, \tau_M, \overline{\tau}_1, \ldots, \overline{\tau}_M) + \\ + \left(\int d^4x d^2\theta\ W(\phi_i; \tau_1, \ldots, \tau_M) + h.c. \right). \tag{10}$$

Under $\Gamma^J_{N_J}$ for $J = 1, \ldots, M$ the Kähler potential K transforms at most by a Kähler transformation and the superpotential W stays invariant. The superpotential can be expanded in general in powers of the superfields ϕ_i. For the superpotential to be invariant under any finite modular transformation $\gamma_1, \ldots, \gamma_M$ in $\Gamma^1_{N_1} \times \Gamma^2_{N_2} \times \ldots \times \Gamma^M_{N_M}$, the couplings $Y_{(I_{Y,1}, \ldots, I_{Y,M})}$ must be multiplet modular forms, and the superfields ϕ_i must transform as in Eqs.(11)-(12) where $-2k_{i,J}$ is the modular weight of ϕ_i, $I_{i,J}$ is the representation of ϕ_i, $2k_{Y,J}$ is the modular weight of $Y_{I_{Y,J}}$, $I_{Y,J}$ is the representation of $Y_{I_{Y,J}}$ and $\rho_{I_{i,J}}(\gamma)$ and $\rho_{I_{Y,J}}(\gamma)$ are the unitary representation matrices of γ_J with $\gamma_J \in \Gamma^J_{N_J}$. As discussed previously, for the superpotential to be invariant, the sum of the weights for each modular symmetry needs to equal zero, i.e. $k_{Y,J} = k_{i_{1,J}} + \ldots + k_{i_{n,J}}$, and and the multiplication of the representations $\rho_{I_{Y,J}} \times \rho_{i_{1,J}} \times \ldots \times \rho_{i_{n,J}}$ must contain an invariant singlet, for $J = 1, \ldots, M$.

$$\begin{aligned} \phi_i(\tau_1, \ldots, \tau_M) &\to \phi_i(\gamma_1\tau_1, \ldots, \gamma_M\tau_M) \\ &= \prod_{J=1,\ldots,M} (c_J\tau_J + d_J)^{-2k_{i,J}} \bigotimes_{J=1,\ldots,M} \rho_{I_{i,J}}(\gamma_J)\ \phi_i(\tau_1, \ldots, \tau_M) \end{aligned} \tag{11}$$

$$\begin{aligned} Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1, \ldots, \tau_M) &\to Y_{(I_{Y,1},\ldots,I_{Y,M})}(\gamma_1\tau_1, \ldots, \gamma_M\tau_M) \\ &= \prod_{J=1,\ldots,M} (c_J\tau_J + d_J)^{2k_{Y,J}} \bigotimes_{J=1,\ldots,M} \rho_{I_{Y,J}}(\gamma_J)\ Y_{(I_{Y,1},\ldots,I_{Y,M})}(\tau_1, \ldots, \tau_M). \end{aligned} \tag{12}$$

3.3. Stabilisers of the Modular Symmetry

The stabilisers of the modular symmetry play a crucial role in preserving residual symmetries. Given an element γ in the modular group Γ_N, a stabiliser τ_γ of γ corresponds to a fixed point in the upper half complex plane that transforms as $\gamma\tau_\gamma = \tau_\gamma$. Once the modular field acquires a VEV at this special point, $\langle\tau\rangle = \tau_\gamma$, the modular symmetry is broken but an Abelian residual modular symmetry generated by γ is preserved. Obviously, acting γ on the modular form at its stabiliser leaves the modular form invariant, which implies that

$$\rho_I(\gamma)Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k}Y_I(\tau_\gamma). \tag{13}$$

This means that, at the stabiliser, the modular form is an eigenvector of the representation matrix $\rho_I(\gamma)$ for the given stabiliser that corresponds to the eigenvalue $(c\tau_\gamma + d)^{-2k}$, and thus the directions of the modular forms at the stabilisers can be easily determined. Furthermore, since the representation matrix is unitary, $|c\tau_\gamma + d| = 1$.

4. Two A_4 modular symmetries for Tri-Maximal 2 mixing

Two A_4 modular symmetries were used to build models that lead to the TM_2 mixing, similarly to the use of multiple S_4 modular symmetries in [4, 5], where models that consider the symmetry breaking from multiple modular symmetry groups to a single symmetry group at low energy have been constructed in order to obtain the TM_1 mixing. Although the TBM mixing form is incompatible with experimental results due to the non-vanishing value for the angle θ_{13}, mixings that only preserve the first or the second columns of the matrix for the TBM mixing, the TM_1 and TM_2 mixings respectively [6], remain viable and appealing schemes for lepton mixings. Their matrices can be described as the TBM matrix times a rotation among the columns that are nor preserved. I will be particularly interested in the tri-maximal 2 (TM_2) mixing, which preserves the second column of the tribimaximal mixing matrix: $(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$.

We note that [7] already employs a single A_4 modular symmetry and two moduli in a model leading to TM_2 mixing, where neutrino masses arise through the effective Weinberg operator (WO). In the models constructed here, we also start by using the WO and afterwards we used the type I seesaw (SS) mechanism to generate the neutrino masses (part of the work here included was already presented at [8]). Here, the presence of two distinct moduli is justified by starting with two A_4 symmetries $A_4^l \times A_4^\nu$ which are subsequently broken to the diagonal subgroup A_4^D.

For TM_2, our mixing of interest for A_4, using the 3σ C.L. range of $\sin\theta_{13}$ [9], we obtain the allowed ranges on the other neutrino mixings angles:

$$0.3403(0.3403) \lesssim \sin^2\theta_{12} \lesssim 0.3416(0.3417) \tag{14}$$

$$0.3891(0.3890) \lesssim \sin^2\theta_{23} \lesssim 0.6109(0.6110). \tag{15}$$

The experimental 1σ region is within the interval found here for $\sin^2\theta_{23}$, which overlaps with the 3σ region for this parameter, with our result extending below the lower 3σ limit for this parameter, 0.407(0.411) for NO(IO), and not reaching its upper limit. The range of allowed values for $\sin^2\theta_{12}$ is near the upper allowed limit, which is a characteristic feature of the TM_2 mixing, since the lowest value allowed for $\sin^2\theta_{12}$ is 1/3. We conclude that, in spite of the discrepancy found for $\sin^2\theta_{12}$, this is still a mixing that is worth considering.

4.1. Modular A_4 symmetry and residual symmetries

The group A_4 is the group of even permutations of 4 objects and has 12 elements. It is generated by two operators S_τ and T_τ obeying

$$S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^3 = 1. \tag{16}$$

This group has three singlets and one triplet as its irreducible representations. The flavour models that are going to be built employ A_4 as a modular symmetry group and the Yukawa couplings are hence going to be modular forms. These are now going to be introduced. The three linearly independent weight 2 modular forms of level 3, $Y_{\mathbf{3}}^{(2)} = (Y_1, Y_2, Y_3)$, form a triplet of A_4 and can be expressed in terms of the Dedekind eta functions. The modular forms of higher weight are generated starting from these modular forms of weight 2.

4.2. Models with two modular A_4 symmetries

We started by constructing one model where it is assumed that neutrinos get their mass through the WO (an effective term of the type $\frac{1}{\Lambda} Y L^2 H_u^2$), and afterwards another model where the SS mechanism is used (the effective term from the superpotential that gives rise to a Dirac mass matrix being of the form $\frac{1}{\Lambda} L Y^\nu \nu^c H_u$). At high energies, these models are based in two modular symmetries, A_4^l and A_4^ν, with modulus fields denoted by τ_l and τ_ν, respectively, which are then broken to the diagonal subgroup A_4^D. After the modulus fields acquire different VEV's, different mass textures are realised in the charged lepton and neutrino sectors, in such a way that the PMNS gets the TM_2 form.

The superpotential for these models can be separated into one part containing the mass terms for the charged leptons, w_e, and the other the neutrino mass terms, w_ν: $w = w_e + w_\nu$. For all three models, w_e is

$$w_e = \Big(\alpha(LY^l(\tau_l))_{\mathbf{1}} e^c + \\ + \beta(LY^l(\tau_l))_{\mathbf{1}'} \mu^c + \\ + \gamma(LY^l(\tau_l))_{\mathbf{1}''} \tau^c\Big) H_d, \quad (17)$$

where L is a triplet containing the leptonic left-handed doublets, e^c, μ^c and τ^c are singlets representing the right-handed charged leptons and α, β and γ are arbitrary complex constants.

These models will obviously differ in their neutrino terms. For the WO model,

$$w_\nu = \frac{1}{\Lambda}\Big((L^2)_{\mathbf{1}} Y_{\mathbf{1}}(\tau_l, \tau_\nu) + \\ + (L^2)_{\mathbf{1}''} Y_{\mathbf{1}'}(\tau_l, \tau_\nu) + \\ + \frac{1}{\Lambda}(L^2)_{\mathbf{3}} \Phi Y_{\mathbf{3}}(\tau_l, \tau_\nu)\Big) H_u^2, \quad (18)$$

while that for the first SS model,

$$w_\nu = \frac{Y^\nu}{\Lambda} L \Phi \nu^c H_u + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{1}} + \\ + \frac{1}{2} M_{\mathbf{1}'}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{1}''} + \frac{1}{2} M_{\mathbf{3}}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{3}}, \quad (19)$$

where the Yukawa coupling Y^ν is simply a constant, and for the second SS model,

$$w_\nu = \frac{1}{\Lambda} L \Phi Y_{\mathbf{1}}^\nu(\tau_\nu) \nu^c H_u + \frac{1}{\Lambda} L \Phi Y_{\mathbf{3}}^\nu(\tau_\nu) \nu^c H_u + \\ + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{1}} + \frac{1}{2} M_{\mathbf{1}'}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{1}''} + \\ + \frac{1}{2} M_{\mathbf{1}''}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{1}'} + \frac{1}{2} M_{\mathbf{3}}(\tau_\nu)(\nu^c \nu^c)_{\mathbf{3}}. \quad (20)$$

For all the three models that were constructed, the breaking from two modular symmetries A_4^l and A_4^ν to a single A_4^D is achieved when the bi-triplet Φ that couples L^2 in the WO model, or L and ν^c in the SS to a triplet modular form of A_4^ν, acquires the VEV $\langle\Phi\rangle = v_\Phi P_{23}$. This happens because, given that the same transformation γ can be performed in A_4^l and A_4^ν simultaneously, there is still a single modular symmetry, the diagonal subgroup A_4^D that is conserved.

Now, we must consider the symmetry breaking of this A_4^D. We assume that the charged lepton modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T = \frac{3}{2} + \frac{i}{2\sqrt{3}}$, which is a stabiliser of T_τ. At this stabiliser, a residual modular Z_3^T symmetry is preserved in the charged lepton sector. This implies that the modular form Y^l, which has weight $+6$, gets the direction $(1, 0, 0)$. This direction leads to a diagonal charged lepton mass matrix when the Higgs field H_d acquires a VEV $\langle H_d\rangle = (0, v_d)$. The masses for the charged leptons can be reproduced by adjusting the parameters α, β and γ.

For the other modular field τ_ν, if it acquires the VEV $\langle\tau_\nu\rangle = \tau_S = i$, a residual modular Z_2^S symmetry is preserved in the neutrino sector. The direction of Y for WO model or Y^ν for second SS model, and $M_{\mathbf{3}}$ for both SS models at this stabiliser is going to be $(1, 1, 1)$.

We obtain a mass sum rule for the neutrino masses. For the WO model and the first SS model, this sum rule can be written as:

$$m_2^\eta = f_1(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\; m_1^\eta + \\ + f_3(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\; m_3^\eta \quad (21)$$

where

$$f_1(\theta, \alpha_1, \alpha_2, \alpha_3) = \frac{1}{2} e^{2i\alpha_3}\Big(e^{-2i\alpha_1} \cos^2\theta - \\ - e^{-2i\alpha_2} \sin^2\theta - \sqrt{3} e^{-i(\alpha_1+\alpha_2)} \sin 2\theta\Big) \quad (22)$$

$$f_3(\theta, \alpha_1, \alpha_2, \alpha_3) = -\frac{1}{2} e^{2i\alpha_3}\Big(e^{2i\alpha_1} \cos^2\theta - \\ - e^{2i\alpha_2} \sin^2\theta - \sqrt{3} e^{i(\alpha_1+\alpha_2)} \sin 2\theta\Big). \quad (23)$$

With these definitions, we can say that for the model where we use the WO, we choose $\eta = +1$, and choosing $\eta = -1$ we will obtain the sum rule for the first SS model.

For the second SS model, the sum rule is more complicated given that additional parameters arise from the consideration of an additional triplet modular form $Y_{\mathbf{3}}^\nu$ and a singlet $M_{\mathbf{1}''}$ for the right-handed neutrinos:

$$
\begin{aligned}
\frac{1}{m_2} = \Bigg| &-3M_{\mathbf{1}''} + \frac{1}{8m_1}\Big(e^{2i\alpha_1}\left(4h_2^2+12h_2h_3-3h_3^2+8h_2+12h_3+4\right)\cos^2\theta - \\
&- e^{2i\alpha_2}\left(4h_2^2+12h_2h_3-3h_3^2-8h_2-12h_3+4\right)\sin^2\theta - \sqrt{3}e^{i(\alpha_1+\alpha_2)}\left(4h_2^2-4h_2h_3-3h_3^2-4\right)\sin 2\theta\Big) \\
&- \frac{1}{8m_3}\Big(e^{-2i\alpha_1}\left(4h_2^2+12h_2h_3-3h_3^2-8h_2-12h_3+4\right)\cos^2\theta - \\
&- e^{-2i\alpha_2}\left(4h_2^2+12h_2h_3-3h_3^2+8h_2+12h_3+4\right)\sin^2\theta - \sqrt{3}e^{-i(\alpha_1+\alpha_2)}\left(4h_2^2-4h_2h_3-3h_3^2-4\right)\sin 2\theta\Big)\Bigg|,
\end{aligned}
\tag{24}
$$

where $M_{\mathbf{1}''}$ and h_i are arbitrary complex constants. Note that when these extra parameters vanish, $h_2 = h_3 = M_{\mathbf{1}''} = 0$, the sum rule for model 1 is recovered.

These sum rules and the equations that relate the model parameters with the neutrino mixing parameters provide what is needed to do a numerical minimisation using the χ^2 function:

$$
\chi^2 = \sum_i \left(\frac{P_i(\{x\}) - BF_i}{\sigma_i}\right)^2, \tag{25}
$$

where P_i are the values provided by the considered model, BF the best fit value from NuFit [9] and σ_i is also provided by NuFit, when averaging the upper and lower σ provided. For the fitting, six variables were considered: the three mixing angles, the atmospheric and solar neutrino squared mass differences, and the Dirac neutrino CP violation phase.

For all the models, all of the best fit points (bfp) are within their 3σ ranges for both orderings. In fact, for NO, all the observables except θ_{12}, near the upper limit of the 3σ range, are compatible with their 1σ ranges. For IO, additionally to θ_{12}, θ_{23}, for the WO model, and δ, for the SS models, are also outside their 1σ region. For all three models, NO provides the best fit, with $\chi^2/6 = 1.57$, which is the same value for all the three models. This is not surprising given the contribution to the χ^2 is coming not from the masses, but from the mixing angles, and both models give TM_2 mixing.

It is also possible to obtain the expected $m_{\beta\beta}$ for neutrinoless beta decay doing a numerical computation. The allowed regions of m_{lightest} vs $m_{\beta\beta}$ of Figure 1 (for NO, $m_{\text{lightest}} = m_1$ and, for IO, $m_{\text{lightest}} = m_3$) were obtained, using again as constraints the data from [9]. In both figures it is also shown the current upper limit provided by KamLAND-Zen, $m_{\beta\beta} < 61 - 165$ meV [10]. Results from PLANCK 2018 also constrain the sum of neutrino masses, although different constrains can be obtained depending on the data considered (for more details, see [11]). In the figures are plotted two shadowed regions, a very disfavoured region $\sum m_i > 0.60$ eV (considering the limit 95%C.L.,Planck lensing+BAO+θ_{MC}) and a disfavoured region $\sum m_i > 0.12$ eV (considering the limit 95%C.L.,Planck TT,TE,EE+lowE+lensing+BAO+θ_{MC}).

For NO, there are some points compatible with the 1σ ranges of the observables other than θ_{12}. These points were plotted with a darker red colour. For IO, at least one of the other observables is incompatible with its 1σ region, as happened for the bfp, hence only the 3σ compatible points are shown for IO.

We conclude then that only the bfp's for NO for the three models that were constructed are outside the disfavoured region. For the WO model, only for normal mass ordering do we have points outside the disfavoured region. For the first SS model, both mass orderings have points outside the disfavoured region, although the non-disfavoured region for IO is smaller. For the second SS model, the 1σ points for NO do not form a characteristic structure as happened for model 1 but are dispersed within the other points that have at least an observable other than θ_{12} outside its 1σ region.

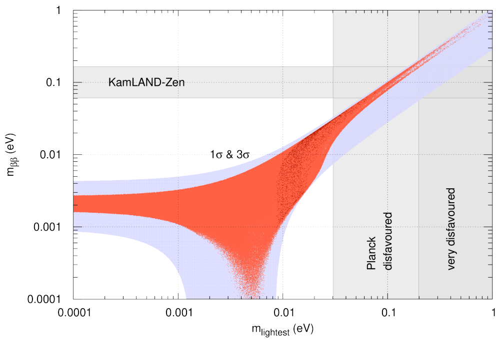
However, the second SS model is much less restrictive than the first one, since m_{lightest} covers all orders of magnitude and almost all the available region for m_{lightest} vs $m_{\beta\beta}$ and, more importantly, the minimum value for $m_{\beta\beta}$ also approaches zero. More specifically, model 1 is simply a special case of model 2 when we neglect all the extra parameters that where introduced in model 2 due to the new terms that appear when we assign a higher weight to the modular forms Y^{ν}.

From what has been written, it is inferred that NO is hence the preferred mass ordering, although this means that smaller orders of magnitude for both m_1 and $m_{\beta\beta}$, which are harder to access experimentally, are still compatible with experimental values.

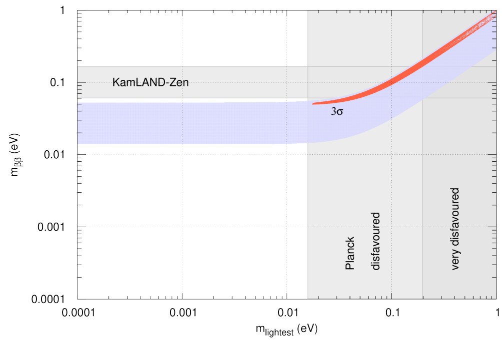
5. Two A_5 modular symmetries for Golden Ratio 2 Mixing

We constructed also two models that use two A_5 modular symmetries in order to obtain the golden ratio mixing plus a rotation among the first and the third columns. We note once again that [12] already employs a single A_5 modular symmetry and two moduli in models using the Weinberg operator to generate the neutrino masses. The model that uses some fixed points of the modular fields lead to the same mixing we are going to discuss here, although that is not explicit in [12].

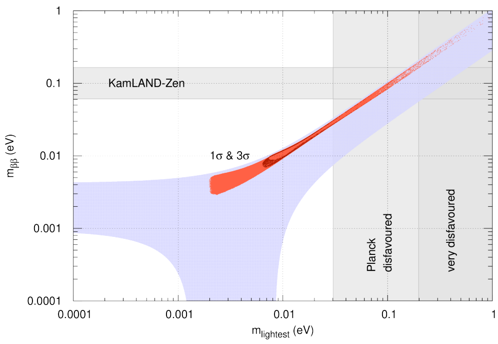
The golden ratio (GR) mixing is a mixing associated in previous works with models based in the A_5 symmetry, and this is not different for models using multiple modular A_5. This mixing has the same problem as the TBM mixing: it is incompati-



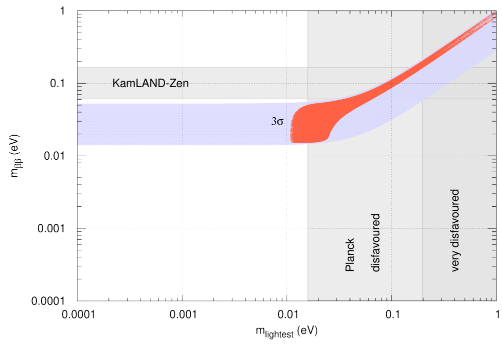
(a) Normal Ordering for WO model



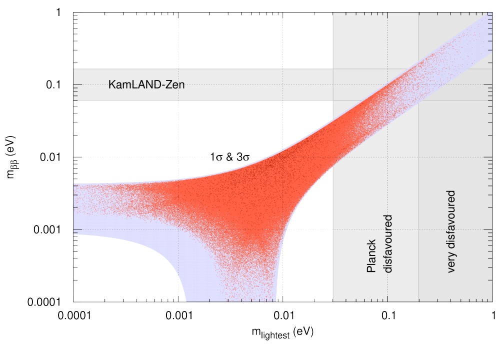
(b) Inverted Ordering for WO model



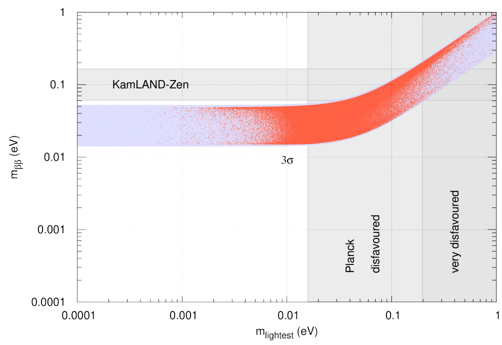
(c) Normal Ordering for first SS model



(d) Inverted Ordering for first SS model



(e) Normal Ordering for second SS model



(f) Inverted Ordering for second SS model

Figure 1: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both orderings of neutrino masses for models using two A_4 modular symmetries.

ble with the experimental results for θ_{13}, and thus we want to work with models that preserve only the first or the second columns of the GR mixing matrix, that can be written as the GR matrix times a rotation between the other two columns.

For a model where the second column of the GR mixing, $(\frac{1}{\sqrt{2+\phi}}, \frac{\phi}{\sqrt{4+2\phi}}, \frac{\phi}{\sqrt{4+2\phi}})$, being ϕ the golden ratio: $\phi = \frac{1+\sqrt{5}}{2}$, is preserved, we can use the 3σ C.L. range of $\sin^2\theta_{13}$ [9] to obtain the allowed ranges for the other mixing angles. We obtain that the 1σ NuFit region is within the interval found for $\sin^2\theta_{23}$, which overlaps with the 3σ region for this parameter, with our result extending below 0.407(0.411) for NO(IO) and not reaching its upper limit. The range of allowed values for $\sin^2\theta_{12}$ is near the lowest limit of the 1σ region although outside.

For a model where the first column is preserved instead, we conclude that the range of allowed values for $\sin^2\theta_{12}$ is outside the 3σ region and thus the class of models that preserve the first column of the golden ratio mixing matrix, which we call GR_1 mixing, are disfavoured by experiment. Consequently, we were only interested in models that preserve the second column of the golden ratio mixing, which we call GR_2, although, as pointed out previously, even for these models $\sin^2\theta_{12}$ is outside the experimental 1σ interval.

5.1. Modular A_5 symmetry and residual symmetries

The group A_5 is the group of even permutations of 5 objects and has 60 elements. It is generated by two operators S_τ and T_τ obeying

$$S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^5 = 1. \tag{26}$$

This group has one singlet $\mathbf{1}$, two triplets $\mathbf{3}$ and $\mathbf{3'}$, one quadruplet $\mathbf{4}$ and one quintuplet $\mathbf{5}$ as its irreducible representations.

Similarly to what was done for $\Gamma_3 \sim A_4$, the Yukawa couplings in a theory that is invariant under a $\Gamma_5 \sim A_5$ symmetry are also going to be modular forms, but in this case of level 5. The eleven linearly independent weight 2 modular forms of level 5 form a quintuplet $Y_{\mathbf{5}}^{(2)} = (Y_1, Y_2, Y_3, Y_4, Y_5)$ of A_5, a triplet $\mathbf{3}$ $Y_{\mathbf{3}}^{(2)} = (Y_6, Y_7, Y_8)$ and a triplet $\mathbf{3'}$ $Y_{\mathbf{3'}}^{(2)} = (Y_9, Y_{10}, Y_{11})$. These modular functions can be expressed in terms of the third theta function and the modular forms of higher weight are generated starting from these eleven modular forms of weight 2.

5.2. Models with two modular A_5 symmetries

We constructed one model where it is assumed that neutrinos get their mass through the WO and afterwards another model where the SS mechanism is used. At high energies, these models are based in two modular symmetries, A_5^l and A_5^ν, with modulus fields denoted by τ_l and τ_ν, respectively. These will be broken to the diagonal subgroup A_5^D, and, after the modulus fields acquire different VEV's, different mass textures are realised in the charged lepton and neutrino sectors, in such a way that the GR_2 mixing is recovered for the PMNS.

The superpotential for these models can be separated into $w = w_e + w_\nu$. For both models, w_e is

$$\begin{aligned} w_e = \Big(&\alpha_1 Y_{\mathbf{1}}^l(\tau_l)(LE^c)_{\mathbf{1}} + \\ &+ \alpha_2 Y_{\mathbf{3}^{(\prime)}}^l(\tau_l)(LE^c)_{\mathbf{3}^{(\prime)}} + \\ &+ \alpha_3 Y_{\mathbf{5}}^l(\tau_l)(LE^c)_{\mathbf{5}}\Big) H_d, \end{aligned} \tag{27}$$

where L and E^c are triplets containing the leptonic left-handed doublets and right-handed charged leptons and α_i are arbitrary complex constants.

These models will obviously differ in their neutrino terms. For the WO model,

$$w_\nu = \frac{1}{\Lambda} L^2 \left[Y_{\mathbf{1}}^\nu(\tau_\nu) + \Phi\left(Y_{\mathbf{5}_1}^\nu(\tau_\nu) + Y_{\mathbf{5}_2}^\nu(\tau_\nu)\right)\right] H_u^2, \tag{28}$$

while for the SS model,

$$\begin{aligned} w_\nu = &\frac{Y^\nu}{\Lambda} L\Phi\nu^c H_u + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \\ &+ \frac{1}{2}\left(M_{\mathbf{5}_1}(\tau_\nu) + M_{\mathbf{5}_2}(\tau_\nu)\right)(\nu^c\nu^c)_{\mathbf{5}}, \end{aligned} \tag{29}$$

where the Yukawa coupling Y^ν is simply a constant.

For the WO model, a bi-quintuplet Φ, which is a quintuplet under both A_5^l and A_5^ν, was introduced to mediate the breaking from two A_5 to a single one, while for the SS model a bi-triplet Φ was used instead. For the WO model, if Φ acquires the VEV $\langle\Phi\rangle = v_\Phi P_{(25)(34)}$ the symmetry $A_5^l \times A_5^\nu$ is broken but the diagonal subgroup A_5^D is still conserved. The same considerations are valid for the SS model if Φ acquires the VEV $\langle\Phi\rangle = v_\Phi P_{23}$.

The flavour structure after A_5^D symmetry breaking now follows. We assume that the charged lepton modular field τ_l acquires the VEV $\langle\tau_l\rangle = \tau_T = i\infty$, which means that a residual modular Z_5^T symmetry is preserved in the charged lepton sector. This VEV leads to an almost diagonal charged lepton mass matrix when the Higgs field H_d acquires a VEV $\langle H_d\rangle = (0, v_d)$. The masses for the charged leptons can be reproduced by adjusting the parameters α_i, and the mass matrix for the charged leptons can be diagonalized by multiplying on the left by the identity matrix and on the right by P_{23} and thus the PMNS matrix is simply the matrix that diagonalizes the mass matrix for the neutrinos. These considerations are valid whether we choose the triplets in both models to be $\mathbf{3}$ or $\mathbf{3'}$.

For the other modular field τ_ν, if $\rho_L \sim \mathbf{3'}$, when the modular field acquires the VEV $\langle\tau_\nu\rangle = \tau_S = i$ and k_ν is even, a residual modular Z_2^S symmetry is preserved in the neutrino sector and he PMNS matrix gets the GR_2 form.

The sum rules for these models can be put in a simpler expression:

$$\begin{aligned} m_2^\eta = &f_1(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\ m_1^\eta + \\ &+ f_3(\eta\theta, \eta\alpha_1, \eta\alpha_2, \eta\alpha_3)\ m_3^\eta \end{aligned} \tag{30}$$

where

$$\begin{aligned} f_1(\theta, \alpha_1, \alpha_2, \alpha_3) = &\frac{1}{10} e^{2i\alpha_3}\Big(\left(5 - \sqrt{5}\right) e^{-2i\alpha_1}\cos^2\theta - \\ &- \left(5 + \sqrt{5}\right) e^{-2i\alpha_2}\sin^2\theta + 4\sqrt{5} e^{-i(\alpha_1+\alpha_2)}\sin 2\theta\Big) \end{aligned} \tag{31}$$

$$\begin{aligned} f_3(\theta, \alpha_1, \alpha_2, \alpha_3) = &-\frac{1}{10} e^{2i\alpha_3}\Big(\left(5 + \sqrt{5}\right) e^{2i\alpha_1}\cos^2\theta - \\ &- \left(5 - \sqrt{5}\right) e^{2i\alpha_2}\sin^2\theta + 4\sqrt{5} e^{i(\alpha_1+\alpha_2)}\sin 2\theta\Big). \end{aligned} \tag{32}$$

With these definitions, for the WO model we choose $\eta = +1$ and for the SS model we chose $\eta = -1$, and both the sum rules are recovered.

We turn now to the agreement between the models using $\rho_L \sim \mathbf{3'}$ and experiment. Using these sum rules and the equations that relate the neutrino mixing parameters and the parameters of the GR_2 mixing, we can do a numerical minimisation using the χ^2 function Eq.(25).

For all models and orderings, all the bfp's are within their 3σ ranges. For NO, all the observables except θ_{12}, near the lower limit of the 1σ range, are compatible with their 1σ ranges. For IO, additionally to θ_{12}, θ_{23}, for the WO model, and θ_{23} and δ, for the SS model, are also outside its 1σ region. NO provides the best fit for all models, with $\chi^2/6 = 0.55$, which is the same value for both models. This is not surprising given the contribution to the χ^2 is coming from the mixing angles, and both models give GR_2 mixing.

Doing a numerical computation, the allowed regions of m_{lightest} vs $m_{\beta\beta}$ fou neutrinoless beta decay of Figure 2 were obtained. For the WO model, we conclude then that both the bfp's are in the disfavoured region, and for the SS model, only the bfp for NO is outside the disfavoured region.

For NO, there are some points compatible with the 1σ ranges of the observables other than θ_{12} (which is, as already said, always near the lower 1σ limit although outside), which are inside a larger group containing the points that have $\chi^2/6 < 1$. These points were plotted with a darker red colour. For IO, at least one of the other observables is incompatible with its 1σ region, hence only the 3σ compatible points are shown for IO. For both models, only for normal mass orderings do we have points outside the disfavoured region.

Taking also into account the inferior limits of the darker red region, we conclude that NO is once again the preferred mass ordering, although, when comparing the SS model with the WO model, smaller values of $m_{\beta\beta}$, which are harder to access experimentally, are compatible with the available experimental results.

6. Conclusions

In this work, we employed the framework of multiple modular symmetries to build models with minimal field content that are able to reproduce viable mixings. For the models using two A_4 modular symmetries, the tri-maximal 2 mixing was obtained, and, for the models using two A_5 modular symmetries, a variation of the golden ratio mixing where only the second column is preserved, which was called GR_2, was obtained instead.

We described how the multiple A_4 and A_5 modular symmetries can be broken to a single symmetry group and showed possible assignments of fields and weights under these two modular symmetries leading to the desired mixing scheme. Three explicit models for A_4 and two for A_5 were built (with different weights and using the Weinberg operator or the seesaw mechanism to generate the neutrino masses) and shown to be predictive and to reproduce the observed mixing angles and mass differences with good fits.

Neutrinoless double beta decay is expected, with the inverted ordering possibility almost entirely disfavoured by cosmological observations and less compatible with the 1σ best fit intervals for the experimental observables than the normal ordering of neutrino masses. This occurs for all the models, independent of the mechanism that generates the masses. Furthermore, the χ^2 values obtained for all the models, which depended mainly on the $\sin^2\theta_{12}$ deviation from the best fit point, favour the GR_2 mixing scheme more than the TM_2 mixing.

It should be noted that this work is possible to be continued and will be continued. First of all, in October 2021, new results from NuFit were published at `http://www.nu-fit.org/` which seems to mean that the connection between our results and the results from this global fit needs to be updated. The results differ more significantly from the July 2020 data in the best fit points for $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$, and also on their 3σ range, but these are still not much significant differences. Thus, we expect that no noticeable changes seem to apply. Nevertheless, it would be a good idea to update the analysis considering these more recent confidence intervals, which can be easily done.

Secondly, for the bi-quintuplet Φ for the models using A_5, the vacuum alignments are still being studied and should be improved in the near future. All the solutions were not obtained fully for the alignment of the bi-quintuplet, and for the bi-triplet, no equations that can be fully solved were obtained so far. We conclude that more driving fields of different nature need to be added to the present model to account for the Φ VEV when using A_5.

In conclusion, the models that were constructed during this dissertation maintain their valid results and prove to be in agreement with experiment, and so, despite the present incompleteness of the A_5 alignments in its present version, this work is a useful addendum to the field of modular field symmetries.

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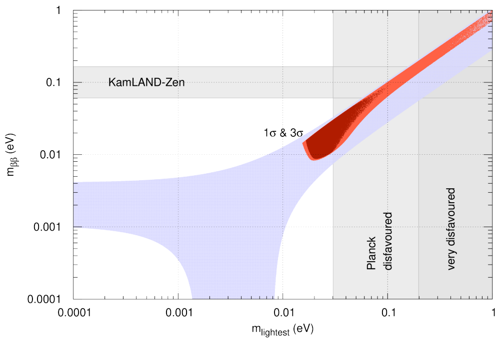
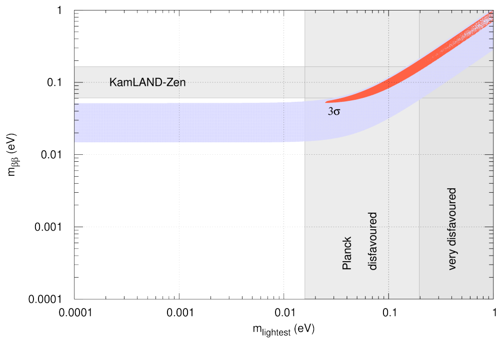
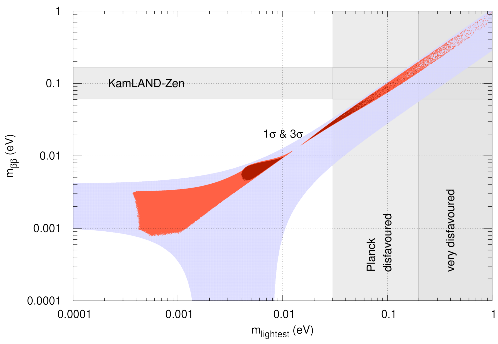


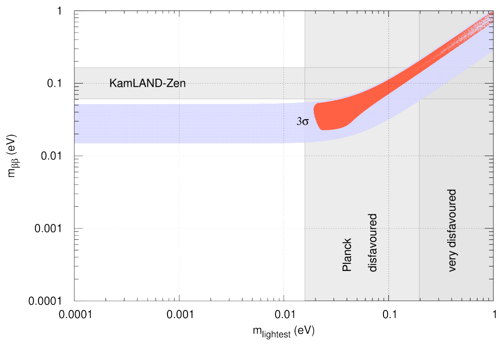
(a) Normal Ordering for WO model



(b) Inverted Ordering for WO model



(c) Normal Ordering for SS model



(d) Inverted Ordering for SS model

Figure 2: Predictions of m_{lightest} vs $m_{\beta\beta}$ for both ordering of neutrino masses for models using two A_5 modular symmetries.

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