

# STOCHASTIC DUALITY FOR SYMMETRIC SIMPLE EXCLUSION AND INCLUSION IN CONTACT WITH RESERVOIRS

BEATRIZ DA COSTA SALVADOR

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ABSTRACT.

We consider the Symmetric Simple Exclusion,  $SEP(\alpha)$ ,  $\alpha \in \mathbb{N}$ , and Inclusion,  $SIP(\alpha)$ ,  $\alpha \in \mathbb{R}^+$ , with open boundary. These processes have duals that substitute the open boundary by absorbing boundary, where each process and its dual are linked by their corresponding classical duality functions. As a consequence of duality, starting all processes from their invariant measure, we obtain explicit formulas for the density profile and 2-points correlation function for  $SEP(\alpha)$  and  $SIP(\alpha)$ , and also for the 3-points correlation function for  $SEP(1)$ . These are found by computing absorption probabilities for the dual processes. The limit of such functions is shown to be connected with Green functions that are solutions of an initial value problem with different boundary conditions depending on the value of  $\theta$ .

**Keywords:** Symmetric Simple Exclusion Process, Symmetric Simple Inclusion Process, Duality, Absorption Probabilities, Stationary Correlations.

## 1. INTRODUCTION

Introduced in the mathematics community by Spitzer [9] in the 70's and further developed by Liggett [7], Interacting Particle Systems (IPS) focus on understanding the time evolution of physical quantities observed at the macroscopic level - for example, temperature, density, pressure - by analyzing the long time behavior of particle systems whose dynamics is defined at the microscopic level and with assumed random behavior. At a first level, every time a continuous time Markov process is defined it is natural to start by characterizing, if they exist, its invariant measures. These measures are also called stationary and are characterized by, starting at time zero from any of them, at anytime  $t$ , the system will have the same distribution. When this invariant measure is unique, it represents the distribution for which the system converge, as time goes to infinity. From here, studying physical quantities of the system, becomes a two side way problem: finding their profile once (that we will follow here) or before the stationary state is attained.

Motivated by these and many other questions, IPS combines different mathematical tools, such as stochastic duality, or simply, duality, to extract information about its objects of study. Introduced in 1948 by Levy [8] and in 1957 by Karlin and McGregor for birth and death processes [6], duality theory allows us to connect some Markov processes to others that in general are of simpler analysis. Two important simplifications that can be done when the duality property is satisfied are: we can relate particle systems with boundary reservoirs that allow creation and annihilation of particles - *open boundary* - with processes that only allow death of particles - *absorbing boundary*; and we can put in a duality relationship systems that can have numerous particles, with processes that evolve with only a few. Besides these simplifications, duality also provides

an algebraic approach to study Interacting Particle Systems, due to its strong connection with Lie algebra's theory. This new point of view about duality has been developed in [1], [3] and [5] and many other articles, providing a useful way to obtain new duality functions for two dual processes.

Here is a description of the content of these notes. In Section 2, we define the Symmetric Simple Exclusion Process with open boundary,  $SEP(\alpha)$ , with  $\alpha \in \mathbb{N}$ , and the Symmetric Simple Inclusion Process also with open boundary,  $SIP(\alpha)$ , with  $\alpha \in \mathbb{R}^+$ , and discuss the existence and uniqueness of stationary measure for the defined models. Section 3 is dedicated to duality theory. We recall the basic tools to construct the classical duality function that provides a connection between our models with their corresponding versions conserving the bulk's dynamics and changing the boundary into absorbing boundary points. In Section 4, we provide applications of the duality relationships given in Section 3. For  $SEP(\alpha)$ , the case  $\alpha = 1$  has been very well studied in the literature since here there is available other tools besides duality, namely, the Matrix Product Ansatz Method (MPA). Also in this notes, in Section 4, we give it a special attention from a duality point of view. We obtain explicit formulas and study the corresponding limits, depending on the value of the parameter  $\theta$ , for the stationary density and 2-points stationary correlation functions for the models  $SEP(\alpha)$  and  $SIP(\alpha)$  (for  $SEP(1)$ , also the 3-points stationary correlation function), by obtaining first explicit formulas for the absorption probabilities for at most three dual particles. Finally, in Section 5, we compare the obtained results in Section 4, summarize the conclusions of the present work and leave some remarks for future work.

## 2. THE MODELS: SEP( $\alpha$ ) AND SIP( $\alpha$ )

On the microscopic space  $\Lambda_N := \{1, \dots, N-1\}$ , called bulk, we define two different models - the SEP( $\alpha$ ) and the SIP( $\alpha$ ), where the *boundary points* 0 and  $N$ , called *left* and *right reservoir*, respectively, are considered as reservoirs with an arbitrary number of particles. Both models are continuous time Markov Process where particles jump with the same elementary rate only to nearest left and right neighbors - this is why they are called *symmetric* and *simple*. For SEP( $\alpha$ ), the state space is  $\Omega_N^{Ex} := \{0, 1, \dots, \alpha\}^{\Lambda_N}$ , while, for SIP( $\alpha$ ), is  $\Omega_N^{In} := \mathbb{N}_0^{\Lambda_N}$ . These models are characterized by the superposition of two dynamics: the *Kawasaki dynamics*, that describe the interaction of the particles inside the bulk; and the *Glauber dynamics*, that describes the interaction of the reservoirs with the particles on the bulk (allows creation and annihilation of particles from the endpoints of the bulk) - this is why we say that the considered processes have *open boundary*. In both models, the points in  $\Lambda_N$  are occupation sites for particles that have to wait a certain time before a jump can occur. The waiting time is exponentially distributed with parameter 1, to have the Markov property of memory loss. Between any two sites  $x$  and  $x+1$ , with  $x \in \{0, \dots, N-1\}$ , we allocate an independent *Poisson clock* such that, every time the clock rings, a particle that is placed at site  $x$  (resp.  $x+1$ ) can jump to the site  $x+1$  (resp.  $x$ ). For SEP( $\alpha$ ), this jump only occurs if the arrival site is not occupied with more than  $\alpha-1$  particles - *exclusion rule*. This means that there is an upper bound,  $\alpha$ , for the number of particles allowed per site for the SEP( $\alpha$ ). In contrast, for SIP( $\alpha$ ), on the bulk, the jump rate of particles increases as much as more particles are at the arrival site and any number of particles is allowed at each site - *inclusion rule*. In SIP( $\alpha$ ), particles tend to create piles and stay close to each other, while in SEP( $\alpha$ ), they tend to be apart and be repelled by each other. The infinitesimal Markov generator of each of the models, that we will denote by  $\mathcal{L}_\alpha^A$ , where  $A \in \{Ex, In\}$  (if  $A = Ex$ , we have the generator for SEP( $\alpha$ ), and, if  $A = In$ , we have the generator for SIP( $\alpha$ )), is defined, for all  $\eta \in \Omega_N^A$  and  $f \in \mathcal{D}(\mathcal{L}_\alpha^A) \subset \mathcal{F}(\Omega_N^A)$ <sup>1</sup>, as  $\mathcal{L}_\alpha^A f(\eta) = \mathcal{L}_{l,\alpha}^A f(\eta) + \mathcal{L}_{bulk,\alpha}^A f(\eta) + \mathcal{L}_{r,\alpha}^A f(\eta)$ , where, denoting, for every  $x, y \in \{0, \dots, N\}$ ,

$$\nabla_{x,y} f(\eta) := f(\eta^{x,y}) - f(\eta),$$

$$\mathcal{L}_{l,\alpha}^A f(\eta) = \frac{\gamma}{N\theta} \eta(1) \nabla_{1,0} f(\eta) + \frac{\epsilon}{N\theta} [\alpha \pm \eta(1)] \nabla_{0,1} f(\eta),$$

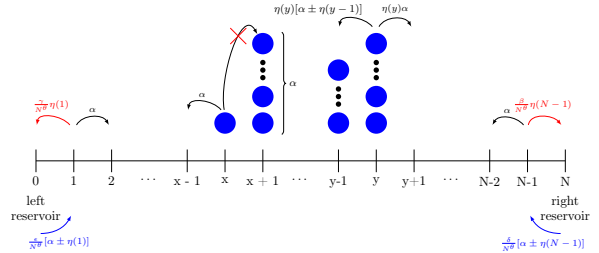
$$\mathcal{L}_{bulk,\alpha}^A f(\eta) = \sum_{x=1}^{N-2} c_{x,x+1}(\eta) \nabla_{x,x+1} f(\eta) + c_{x+1,x}(\eta) \nabla_{x+1,x} f(\eta),$$

with  $c_{x,y}(\eta) := \eta(x)[\alpha \pm \eta(y)]$  and

$$\mathcal{L}_{r,\alpha}^A f(\eta) = \frac{\beta}{N\theta} \eta(N-1) \nabla_{N-1,N} f(\eta) + \frac{\delta}{N\theta} [\alpha \pm \eta(N-1)] \nabla_{N,N-1} f(\eta),$$

<sup>1</sup>For any set  $A$  we define  $\mathcal{F}(A)$  as the set of functions  $f: A \rightarrow \mathbb{R}$ .

where, when  $A = Ex$ , the rates for SEP( $\alpha$ ) must be taken with the minus sign, while for SIP( $\alpha$ ) we take the plus sign. The parameters  $\gamma, \epsilon, \beta$  and  $\delta$  are positives and the parameter  $\theta \in \mathbb{R}$  tunes the strength of the interaction between the bulk and the boundary. The elements of  $\Omega_N^A$  given by  $\eta^{1,0}, \eta^{0,1}, \eta^{x,x+1}, \eta^{x+1,x}, \eta^{N-1,N}$  and  $\eta^{N,N-1}$  represent the change of configuration after the jump, injection or extraction of one particle. The following picture represents the dynamics of the models.



A natural question that we should ask every time we define an interacting particle system, is about its invariant measures: do they exist? Since any reversible probability measure  $\mu$  with respect to the Markov generator  $\mathcal{L}$  defined in a countable space  $\Omega$  is a stationary measure for  $\mathcal{L}$ , in order to find the invariant measures for the models, it is easier to start by looking for reversible measures - Proposition 3.2 of [1].

**Theorem 2.1.** *If*

$$\frac{\epsilon}{\epsilon + \gamma} = \frac{\delta}{\delta + \beta} = \rho, \quad (1)$$

*then, the reversible measure for SEP( $\alpha$ ) is given by*

$$\mu^{SEP(\alpha)}(\eta) = \prod_{x=1}^{N-1} \binom{\alpha}{\eta(x)} \rho^{\eta(x)} (1-\rho)^{\alpha-\eta(x)}.$$

Recalling that every irreducible continuous time Markov chain with finite state space has a unique non-zero invariant probability measure, independently if (1) is satisfied or not, there always exists a unique non-zero invariant measure for SEP( $\alpha$ ), which is only completely characterized under the conditions of Theorem 2.1.

**Theorem 2.2.** *If*

$$\frac{\epsilon}{\gamma} = \frac{\delta}{\beta} = \rho \quad (2)$$

*with  $\epsilon < \gamma$  and  $\delta < \beta$ , then, the reversible measure for SIP( $\alpha$ ) is given by*

$$\mu^{SIP(\alpha)}(x) = \prod_{x=1}^{N-1} \frac{\Gamma(\alpha+x)}{\Gamma(\alpha)x!} \rho^x (1-\rho)^\alpha.$$

Even if condition (2) is not satisfied, as pointed out in Appendix A of [2], there still exists a unique

non-zero invariant measure for  $\text{SIP}(\alpha)$ . When considering  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$  without reservoirs, the measures presented above describe a family of reversible measures with free parameter  $\rho \in (0, 1)$ .

When conditions (1), for  $\text{SEP}(\alpha)$ , and (2), for  $\text{SIP}(\alpha)$  (both with open boundary) are satisfied, we say that the system is in equilibrium, otherwise, we say the system to be out of equilibrium. Defining  $\rho_0$  and  $\rho_N$ , that are called the ‘‘densities of the left and right reservoirs’’, respectively, by the values given in the following table (taken from [1]), saying that the system is in equilibrium (resp. out of equilibrium) is the same as asking the density of the left and right reservoirs to be equal (resp. different).

Model	$\rho_0$	$\rho_N$
$\text{SEP}(\alpha)$	$\alpha \frac{\epsilon}{\gamma + \epsilon}$	$\alpha \frac{\delta}{\beta + \delta}$
$\text{SIP}(\alpha)$	$\alpha \frac{\epsilon}{\gamma - \epsilon}$	$\alpha \frac{\delta}{\beta - \delta}$

Even though it is not known a complete characterization of the unique invariant measure of any of our models out of equilibrium, using duality, we can still find explicit expressions and the corresponding limit of functions of interest.

### 3. DUALITY

**Definition 3.1.** *Two continuous time Markov processes  $\mathbf{X}$  and  $\mathbf{Y}$  with state spaces  $\Omega$  and  $\Omega^{\text{dual}}$ , respectively, are said to be dual if there exists a function  $D : \Omega \times \Omega^{\text{dual}} \rightarrow \mathbb{R}$ , the **duality function**, such that, for all  $(\eta, \hat{\eta}) \in \Omega \times \Omega^{\text{dual}}$  and  $t \geq 0$ ,*

$$\mathbb{E}_\eta[D(X_t, Y)] = \mathbb{E}_{\hat{\eta}}[D(X, Y_t)]. \quad (3)$$

When everything is well defined, duality can also be defined using the infinitesimal Markov generators of the processes. If  $\mathcal{L}$  and  $\mathcal{L}^{\text{dual}}$  represent the infinitesimal Markov generators of two continuous time Markov processes  $\mathbf{X}$  and  $\mathbf{Y}$ , respectively, then duality can be defined by replacing condition (3) by, for every  $\eta \in \Omega$  and  $\hat{\eta} \in \Omega^{\text{dual}}$ ,

$$[\mathcal{L}D(\cdot, \hat{\eta})](\eta) = [\mathcal{L}^{\text{dual}}D(\eta, \cdot)](\hat{\eta}).$$

The goal of this chapter is to recall basic tools for the construction of the classical duality function that puts in a duality relation  $\text{SEP}(\alpha)$  (resp.  $\text{SIP}(\alpha)$ ) with open boundary with its corresponding version keeping the bulk dynamics and replacing the boundary reservoirs by only absorbing points. This construction requires a series of steps.

**Theorem 3.1.** *A continuous time Markov process  $\mathbf{X}$  with countable state space  $\Omega$  that admits a reversible measure  $\mu$  is always self-dual with self-duality function  $D$ , called the **cheap self-duality function**, given by  $D(\eta, \hat{\eta}) = (\mu(\hat{\eta}))^{-1} \mathbb{1}_{\eta=\hat{\eta}}$ , for all  $\eta, \hat{\eta} \in \Omega$ , where  $\mathbb{1}_{\eta=\hat{\eta}}$  is one if  $\eta = \hat{\eta}$  and zero otherwise.*

Using the fact that  $\text{SEP}(\alpha)$  (resp.  $\text{SIP}(\alpha)$ ) without reservoirs has a family of reversible measures

that are independent from the choice of  $\gamma, \epsilon, \beta$  and  $\delta$ , we can prove self-duality for  $\text{SEP}(\alpha)$  (resp.  $\text{SIP}(\alpha)$ ) without reservoirs using Theorem 3.1. From the cheap self-duality functions, for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$ , one can construct new duality functions inspired on the next result.

**Theorem 3.2.** *Let  $d$  be a duality function for a self-dual process  $\mathbf{X}$  with state space  $\Omega$  and infinitesimal Markov generator  $\mathcal{L}$ . Let  $S$  be a symmetry of  $\mathcal{L}$ , i.e.,  $\mathcal{L}S = S\mathcal{L}$ . Then,  $D = Sd$  and  $D^T = dS^T$  are again self-duality functions for  $\mathbf{X}$ .*

This means that new duality functions for our processes, when considered without reservoirs, can be constructed by using, for example, the cheap self-duality function, and finding symmetries of the infinitesimal Markov generator, when this is an element of the universal enveloping algebra of some Lie algebra. For our models, the Lie algebras considered are  $su(2)$  and  $su(1, 1)$ , for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$ , respectively. By looking at representation of these Lie algebras, we can obtain the classical self-duality function for  $\text{SEP}(\alpha)$  (resp.  $\text{SIP}(\alpha)$ ), that can be modified to provide the desired duality relationship between our models with open boundary and themselves but now with only absorbing boundary points.

**Theorem 3.3.**  *$\text{SEP}(\alpha)$  (resp.  $\text{SIP}(\alpha)$ ) with open boundary and  $\text{SEP}(\alpha)$  (resp.  $\text{SIP}(\alpha)$ ) with only absorbing boundary points are dual processes with classical duality function  $D$  given, for every  $\eta \in \Omega_N^A$  and  $\hat{\eta} \in \Omega_N^{\text{dual}}$ , with  $A \in \{Ex, In\}$ , by*

$$D(\eta, \hat{\eta}) = \rho_0^{\hat{\eta}(0)} \prod_{x=1}^{N-1} d^{\text{bulk}}(\eta(x), \hat{\eta}(x)) \rho_N^{\hat{\eta}(N)}, \quad (4)$$

where  $d^{\text{bulk}}$  is defined by

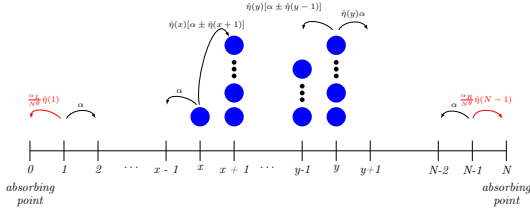
$$d^{\text{bulk}}(n, m) = \begin{cases} \frac{n!}{(n-m)!} \frac{(\alpha-m)!}{\alpha!} \mathbb{1}_{n \geq m}, & \text{for } \text{SEP}(\alpha) \\ \frac{n!}{(n-m)!} \frac{\Gamma(\alpha)}{\Gamma(\alpha+m)} \mathbb{1}_{n \geq m}, & \text{for } \text{SIP}(\alpha) \end{cases}$$

for every  $n, m \in \{0, \dots, \alpha\}$ , for  $\text{SEP}(\alpha)$ , and  $n, m \in \mathbb{N}_0$ , for  $\text{SIP}(\alpha)$ , where

$$\Omega_N^{\text{dual}} = \begin{cases} \mathbb{N}_0 \times \{0, \dots, \alpha\}^{|\Lambda_N|} \times \mathbb{N}_0, & \text{for } \text{SEP}(\alpha), \\ \mathbb{N}_0 \times \mathbb{N}_0^{|\Lambda_N|} \times \mathbb{N}_0, & \text{for } \text{SIP}(\alpha). \end{cases}$$

The jump rates associated to the dynamics for the dual processes are illustrated on the following picture, where  $\alpha_L = \gamma \pm \epsilon$  and  $\alpha_R = \beta \pm \delta$  (for  $\text{SEP}(\alpha)$ , we should take the plus sign, and, for  $\text{SIP}(\alpha)$ , the minus).

In the next section, we will use these duality relations to find explicit formulas for some functions of interest.



#### 4. DENSITY AND CORRELATIONS VIA DUALITY

Let  $\eta \in \Omega_N^A$ , with  $A \in \{Ex, In\}$  represent the initial configuration of each of the process and  $\mu_{ss}$  be the unique stationary measure of each of the studied processes. Define the discrete stationary density profile  $\rho_{ss}^N$ :

$$\rho_{ss}^N(x) = \mathbb{E}_{\mu_{ss}}[\eta(x)]. \quad (5)$$

For  $k \in \mathbb{N}_2$ , define the  $k$ -points stationary correlation function  $\varphi_{ss}^N$  by

$$\varphi_{ss}^N(x_1, \dots, x_k) = \mathbb{E}_{\mu_{ss}}[\bar{\eta}(x_1) \dots \bar{\eta}(x_k)], \quad (6)$$

where  $\bar{\eta}(x) = \eta(x) - \mathbb{E}_{\mu_{ss}}[\eta(x)]$  and  $x_j \in \Lambda_N$ , for any  $j \in \{1, \dots, k\}$ . We are now interested in finding explicit expressions for each one of these functions. To do that, we follow a duality approach that allow us to prove the identity (that is true for all our models), if  $x_1 < \dots < x_k$ ,

$$\mathbb{E}_{\mu_{ss}}[\eta(x_1) \dots \eta(x_k)] = \sum_{m=0}^k \rho_0^m \rho_N^{k-m} p_{x_1, \dots, x_k}^N(m), \quad (7)$$

where  $p_{x_1, \dots, x_k}^N(m)$  represents the probability that, starting from the dual configuration  $\hat{\eta} = \delta_{x_1} + \dots + \delta_{x_k}$  (meaning that, we start the dual process with one particle at each site  $x_1, \dots, x_k \in \Lambda_N$ )  $m$  of the  $k$  dual particles are absorbed on the left reservoir and the remaining  $k - m$  are absorbed on the right reservoir on a lattice of size  $N$ . For SEP( $\alpha$ ) and SIP( $\alpha$ ), using the expression of the the duality function, we obtain

$$\begin{aligned} & \mathbb{E}_{\mu_{ss}}[\eta(x_1) \dots \eta(x_k)] \pm \mathbb{E}_{\mu_{ss}}[\eta(x_2) \dots \eta(x_k)] \\ &= \frac{\alpha \pm 1}{\alpha} \sum_{m=0}^k \rho_0^m \rho_N^{k-m} \underbrace{\mathbb{P}_{\eta}(\hat{\eta}_{\infty}(0)=m, \hat{\eta}_{\infty}(N)=k-m)}_{=p_{x_1, \dots, x_k}^N(m)}, \end{aligned}$$

where we take the minus sign for SEP( $\alpha$ ) and the plus sign for SIP( $\alpha$ ), if  $x_1 = x_2 < x_3 < \dots < x_k$ . It becomes clear now that to find the functions that we are interested in, we need to compute absorption probabilities on the dual process. Here, we will choose the parameters  $\gamma, \epsilon, \delta$  and  $\beta$  satisfying: for SEP( $\alpha$ ),  $\gamma + \epsilon = \beta + \delta = \alpha$ , and, for SIP( $\alpha$ ),  $\gamma - \epsilon = \beta - \delta = \alpha$ . The previous choices, called Liggett condition, still gives us enough freedom to choose two of the four parameters and, therefore, find the stationary density profile and the 2-points

<sup>2</sup>By this we mean that, when we extend the bulk to the points 0 and  $N$ , the point with higher coordinate is  $N$  even though we have  $N + 1$  possible positions to place a particle on the dual system.

(for SEP(1), also 3-points) stationary correlation functions out of equilibrium.

### 4.1. SEP(1) - Absorption Probabilities

**4.1.1 Case  $k = 1$ :** This case not only gives us the absorption probabilities for SEP(1) with one dual particle, but also for SEP( $\alpha$ ) and SIP( $\alpha$ ) since they describe the same model, with the jump rates on the bulk that we have for SEP(1) rescaled by  $\alpha$  for SEP( $\alpha$ ) and SIP( $\alpha$ ). This rescale does not affect the computations of the absorption probabilities.

Choosing  $k = 1$  means that our dual process will start with just one dual particle that can either be absorbed at 0 or  $N$ , therefore  $m \in \{0, 1\}$ . Let  $x \in \Lambda_N$  represent the site where the dual particle is initially placed, i.e., we start from the dual configuration  $\hat{\eta} = \delta_x$ . Observe that, if we extend the possible values of  $x$  to include 0 and  $N$ , then  $p_0^N(m)$  and  $p_N^N(m)$  represent boundary terms.

To compute  $p_x^N(m)$ , for each fixed  $m$ , we condition on the first jump and use the Markov property to get a system of linear equations that  $p_x^N(m)$  solves:

$$\begin{cases} \mathfrak{B}_N^\theta p_x^N(m) = 0, & \text{if } x \in \Lambda_N, \\ p_0^N(m) = \mathbb{1}_{m=1}, \\ p_N^N(m) = \mathbb{1}_{m=0}, \end{cases} \quad (8)$$

where the operator  $\mathfrak{B}_N^\theta$  is defined, for every function  $f \in \mathcal{F}(\{0, \dots, N\})$  and for all  $x \in \Lambda_N$ , as

$$\mathfrak{B}_N^\theta f(x) = a_x[f(x-1) - f(x)] + b_x[f(x+1) - f(x)], \quad (9)$$

with  $a_x = \frac{N^\theta}{N^\theta} \mathbb{1}_{x=1} + N^2 \mathbb{1}_{x \neq 1}$  and  $b_x = \frac{N^\theta}{N^\theta} \mathbb{1}_{x=N-1} + N^2 \mathbb{1}_{x \neq N-1}$ . For every  $x \in \{2, \dots, N-2\}$ ,  $p_x^N(m)$  is solution to the one-dimensional discrete Laplace equation, i.e.,  $\Delta_N^{1D} p_x^N(m) = 0$  where the one-dimensional discrete Laplace operator  $\Delta_N^{1D}$  is defined, for every  $f \in \mathcal{F}(\{0, \dots, N\})$  and for all  $x \in \Lambda_N$ , as

$$\Delta_N^{1D} f(x) = N^2[f(x-1) - 2f(x) + f(x+1)]. \quad (10)$$

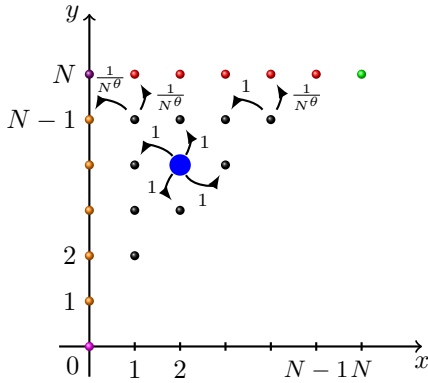
Then, for every  $x \in \Lambda_N$ ,  $p_x^N(m) = A^m x + B^m$ , with  $A^m, B^m \in \mathbb{R}$  to be determined. Note that  $A^m$  and  $B^m$  do not represent the power  $m$  of some real numbers  $A$  and  $B$ , but it is just a notation to recall that the coefficients depend on  $m$ . Solving the previous system and including the boundary conditions, for every  $x \in \{0, \dots, N\}$ , we get

$$\begin{cases} p_x^N(1) = \frac{N^\theta - 1 + N - x}{N + 2N^\theta - 2} + \frac{N^\theta - 1}{N + 2N^\theta - 2} [\mathbb{1}_{\{x=0\}} - \mathbb{1}_{\{x=N\}}], \\ p_x^N(0) = \frac{N^\theta - 1 + x}{N + 2N^\theta - 2} - \frac{N^\theta - 1}{N + 2N^\theta - 2} [\mathbb{1}_{\{x=0\}} - \mathbb{1}_{\{x=N\}}]. \end{cases} \quad (11)$$

**4.1.2 Case  $k = 2$ :** Now the dual process starts with 2 particles that can be absorbed at 0 or  $N$ . So, we can have both particles absorbed at 0, one absorbed at 0 and the other at  $N$  or both absorbed at  $N$ , i.e.,  $m \in \{0, 1, 2\}$ . Let  $x, y \in \Lambda_N$ , with  $x < y$  where now the initial configuration of the dual process is  $\hat{\eta} = \delta_x + \delta_y$ . Again, if we extend the possible

values for  $x$  and  $y$  to the boundary points, then  $p_{0,y}^N(m)$  and  $p_{x,N}^N(m)$  represent boundary terms.

Observe that the dual of SEP(1) with just two particles represents a particle performing a two-dimensional random walk on a triangle (jumps can only occur to points that are left, right, upward or downward nearest-neighbor of the initial position of the random walk) with absorbing cathetuses, meaning that, once a particle reaches a cathetus, it can not leave it, restricting the particle to continue to perform a one-dimensional random walk, as in the case  $k = 1$  (on a line segment with absorbing boundary). Due to the exclusion rule, for SEP(1), we can not consider on the triangle the points of the form  $(x, x)$  with  $x \in \Lambda_N$ . These points have to be considered when repeating this strategy for SEP( $\alpha$ ) and SIP( $\alpha$ ). The particles' jump rates are given as in the following picture.



In what follows, the 2-dimensional discrete Laplacian operator, that we will denote by  $\Delta_N^{2D}$ , will be a central object. This operator is defined, for every  $f \in \mathcal{F}(\mathcal{BT}^N)$  and  $(x, y) \in \mathcal{T}^N$ , where  $\mathcal{BT}^N = \{(x, y) \in \{0, \dots, N\}^2 \mid x < y\}$  and  $\mathcal{T}^N = \{(x, y) \in (\Lambda_N)^2 \mid x < y\}$ , as

$$\Delta_N^{2D} f(x, y) = \begin{cases} \Delta_{N,full}^{2D} f(x, y), & \text{if } y \neq x + 1, \\ \Delta_{N,ref}^{2D} f(x, x + 1), & \text{if } y = x + 1. \end{cases}$$

If  $y \neq x + 1$ ,  $\Delta_{N,full}^{2D} f(x, y) := (\Delta_N^{1D})_x f(x, y) + (\Delta_N^{1D})_y f(x, y)$ , and, if  $y = x + 1$ ,  $\Delta_{N,ref}^{2D} f(x, x + 1) := N^2 [f(x - 1, x + 1) + f(x, x + 2) - 2f(x, x + 1)]$ , where  $(\Delta_N^{1D})_x$  and  $(\Delta_N^{1D})_y$  denote the 1-dimensional Laplacian operator acting on the first and second variable, respectively.

As in the case  $k = 1$ , to compute  $p_{x,y}^N(m)$ , we condition on the first jump and get a system of equations for which  $p_{x,y}^N(m)$  is the unique solution:

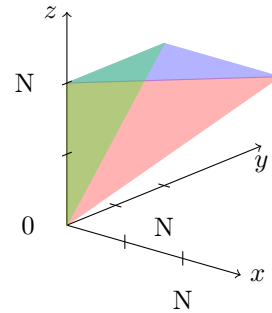
$$\begin{cases} \mathfrak{D}_N^0 p_{x,y}^N(m) = 0, & \text{if } (x, y) \in \mathcal{T}^N \\ p_{0,y}^N(m) = p_y^N(m - 1) \mathbb{1}_{\{m \neq 0\}}, & \text{if } y = 0, \dots, N, \\ p_{x,N}^N(m) = p_x^N(m) \mathbb{1}_{\{m \neq 2\}}, & \text{if } x = 0, \dots, N. \end{cases}$$

where the operator  $\mathfrak{D}_N^0$  is defined, for every  $f \in \mathcal{F}(\mathcal{BT}^N)$  and for all  $(x, y) \in \mathcal{T}^N$ , by  $\mathfrak{D}_N^0 f(x, y) = a_x [f(x - 1, y) - f(x, y)] + b_y [f(x, y + 1) - f(x, y)] + c_{x,y} [f(x + 1, y) +$

$f(x, y - 1) - 2f(x, y)]$ , where  $a_x = \frac{N^2}{N^\theta} \mathbb{1}_{x=1} + N^2 \mathbb{1}_{x \neq 1}$ ,  $b_y = \frac{N^2}{N^\theta} \mathbb{1}_{y=N-1} + N^2 \mathbb{1}_{y \neq N-1}$  and  $c_{x,y} = N^2 \mathbb{1}_{y \neq x+1}$ . If  $x, y \in \{2, \dots, N - 2\}$  with  $x < y$ , then  $p_{x,y}^N(m)$  is solution to the two-dimensional discrete Laplace equation, meaning that  $p_{x,y}^N(m) = A^m x + B^m y + C^m xy + D^m$ . Solving the previous system with this ansatz, defining  $\tilde{p}_x^{N-1}(0) := \frac{N^\theta - 1 + x}{N + 2N^\theta - 3}$  and  $\tilde{p}_x^{N-1}(1) := \frac{N^\theta - 2 + N - x}{N + 2N^\theta - 3}$ , recalling the formulas for  $p_x^N(0)$  and  $p_y^N(1)$  given in (11), we get

$$\begin{cases} p_{x,y}^N(0) = p_x^N(0) - p_y^N(1) \tilde{p}_x^{N-1}(0), & \text{if } (x, y) \in \mathcal{T}^N, \\ p_{x,y}^N(1) = p_x^N(1) + p_y^N(1) [\tilde{p}_x^{N-1}(1) - \tilde{p}_x^{N-1}(0)], & \text{if } (x, y) \in \mathcal{T}^N, \\ p_{x,y}^N(2) = p_y^N(1) \tilde{p}_x^{N-1}(1), & \text{if } (x, y) \in \mathcal{T}^N, \\ p_{0,y}^N(m) = p_y^N(m - 1) \mathbb{1}_{\{m \neq 0\}}, & \text{if } y = 0, \dots, N \text{ and } m = 0, 1, 2, \\ p_{x,N}^N(m) = p_x^N(m) \mathbb{1}_{\{m \neq 2\}}, & \text{if } x = 0, \dots, N \text{ and } m = 0, 1, 2. \end{cases} \quad (12)$$

**4.1.3 Case  $k = 3$ :** Now the dual process starts with  $k = 3$  particles that can be absorbed at 0 or  $N$ . We can have  $k$  possible scenarios: three particles are absorbed at 0, one is absorbed at 0 and two at  $N$ , two at 0 and one at  $N$  or all three absorbed at  $N$ , i.e.,  $m \in \{0, 1, 2, 3\}$ . Let  $x, y, z \in \Lambda_N$ , where  $x < y < z$ , represent the position where each particle started. If we extend the possible values for  $x$  and  $z$  to 0 or  $N$ , then  $p_{0,y,z}^N(m)$  and  $p_{x,y,N}^N(m)$  represent boundary terms. It is very important to remark that, since, in SEP(1), we do not allow particles to be on top of each other, it does not make sense to define  $p_{x,y,z}^N(m)$  on the line  $x = y = z$  nor on the planes  $x = y$  and  $y = z$ . Remark that SEP(1) with 3 dual particles can be thought as a random walk of just one particle on the discrete simplex  $\mathcal{S}^N := \{(x, y, z) \in (\Lambda_N)^3 \mid x < y < z\}$  and planes  $\mathcal{S}_0^N := \{(0, y, z) \in (\Lambda_N)^3 \mid y < z\}$  and  $\mathcal{S}_N^N := \{(x, y, N) \in (\Lambda_N)^3 \mid x < y\}$ . In the following picture, the set  $\mathcal{S}^N$  correspond to the interior points (excluding the boundary) of a discretization (in the points with positive integer coordinates) of the simplex here presented and  $\mathcal{S}_0^N$  and  $\mathcal{S}_N^N$  may be interpreted as the points of the same discretization that lie on the face painted in green and in blue, respectively. Here, the 3-dimensional discrete



Laplacian operator, denoted by  $\Delta_N^{3D}$ , will play a

central role. It is defined, for every  $f \in \mathcal{F}(\mathcal{BS}^N)$ , where  $\mathcal{BS}^N = S^N \cup S_0^N \cup S_N^N$ , as

$$\Delta_N^{3D} f(x,y,z) = \begin{cases} \Delta_{N,full}^{3D} f(x,y,z), & \text{if } y \neq x+1 \text{ and } z \neq y+1, \\ (\Delta_{N,ref}^{2D})_{x,y} f(x,y,z) + (\Delta_N^{1D})_z f(x,y,z), & \text{if } y = x+1 \text{ and } z \neq y+1, \\ (\Delta_{N,ref}^{2D})_{y,z} f(x,y,z) + (\Delta_N^{1D})_x f(x,y,z), & \text{if } y \neq x+1 \text{ and } z = y+1, \\ \Delta_{N,ref}^{3D} f(x,y,z), & \text{if } y = x+1 \text{ and } z = x+2, \end{cases}$$

where,  $\Delta_{N,full}^{3D} := (\Delta_N^{1D})_x + (\Delta_N^{1D})_y + (\Delta_N^{1D})_z$ ,  $\Delta_{N,ref}^{3D} f(x, x+1, x+2) := N^2[f(x-1, x+1, x+2) + f(x, x+1, x+3) - 2f(x, x+1, x+2)]$ , and  $(\Delta_{N,ref}^{2D})_{x,y}$  (resp.  $(\Delta_{N,ref}^{2D})_{y,z}$ ) represents the 2-dimensional reflected discrete Laplacian acting on the first and second (resp. second and third) arguments of  $f$ .

As above, conditioning on the first jump, we get a system of linear equations for  $p_{x,y,z}^N(m)$ , that can be compacted in

$$\begin{cases} \mathfrak{R}_N^\theta p_{x,y,z}^N(m) = 0, & \text{for } (x,y,z) \in S^N \\ p_{0,y,z}^N(m) = p_{y,z}^N(m-1) \mathbb{1}_{\{m \neq 0\}}, & \text{if } (y,z) \in \mathcal{BT}^N, \\ p_{x,y,N}^N(m) = p_{x,y}^N(m) \mathbb{1}_{\{m \neq 3\}}, & \text{if } (x,y) \in \mathcal{BT}^N, \end{cases}$$

where the operator  $\mathfrak{R}_N^\theta$  is defined, for every function  $f \in \mathcal{F}(\mathcal{BS}^N)$  and for all  $(x,y,z) \in S^N$ , by

$$\begin{aligned} \mathfrak{R}_N^\theta f(x,y,z) = & a_x [f(x-1,y,z) - f(x,y,z)] + b_z [f(x,y,z+1) - f(x,y,z)] \\ & + c_{y,z} [f(x,y+1,z) + f(x,y,z-1) - 2f(x,y,z)] \\ & + c_{x,y} [f(x+1,y,z) + f(x,y-1,z) - 2f(x,y,z)], \end{aligned}$$

where  $a_x = \frac{N^2}{N^\theta} \mathbb{1}_{x=1} + N^2 \mathbb{1}_{x \neq 1}$ ,  $b_z = \frac{N^2}{N^\theta} \mathbb{1}_{z=N-1} + N^2 \mathbb{1}_{z \neq N-1}$ ,  $c_{y,z} = N^2 \mathbb{1}_{y \neq z-1}$  and  $c_{x,y} = N^2 \mathbb{1}_{x \neq y-1}$ . Since  $p_{x,y,z}^N(m)$  is solution of the three-dimensional discrete Laplace equation if  $2 \leq x < y < z \leq N-2$ , it can be written again in polynomial form. Solving the previous system and, for every  $(x,y) \in \mathcal{T}^N$ , defining  $\tilde{p}_{x,y}^{N-1}(0) := \frac{(N^\theta - 1 + x)(N^\theta - 2 + y)}{(N + 2N^\theta - 3)(N + 2N^\theta - 4)}$ ,  $\tilde{p}_{x,y}^{N-1}(2) := \frac{(N^\theta - 2 + N - y)(N^\theta - 3 + N - x)}{(N + 2N^\theta - 3)(N + 2N^\theta - 4)}$  and  $\tilde{p}_{x,y}^{N-1}(1) := 1 - \tilde{p}_{x,y}^{N-1}(0) - \tilde{p}_{x,y}^{N-1}(2)$  and recalling the expressions of  $p_{x,y}^N(0)$  and  $p_{y,z}^N(1)$  given in (12), we get

$$\begin{cases} p_{x,y,z}^N(0) = p_{x,y}^N(0) - p_z^N(1) \tilde{p}_{x,y}^{N-1}(0), & \text{if } (x,y,z) \in S^N, \\ p_{x,y,z}^N(1) = p_{x,y}^N(1) + p_z^N(1) [\tilde{p}_{x,y}^{N-1}(1) - \tilde{p}_{x,y}^{N-1}(0)], & \text{if } (x,y,z) \in S^N, \\ p_{x,y,z}^N(2) = p_{x,y}^N(2) + p_z^N(1) [\tilde{p}_{x,y}^{N-1}(2) - \tilde{p}_{x,y}^{N-1}(1)], & \text{if } (x,y,z) \in S^N, \\ p_{x,y,z}^N(3) = p_z^N(1) \tilde{p}_{x,y}^{N-1}(2), & \text{if } (x,y,z) \in S^N, \\ p_{0,y,z}^N(m) = p_{y,z}^N(m-1) \mathbb{1}_{\{m \neq 0\}}, & \text{if } (y,z) \in \mathcal{BT}^N \text{ and } m \in \{0,1,2,3\}, \\ p_{x,y,N}^N(m) = p_{x,y}^N(m) \mathbb{1}_{\{m \neq 3\}}, & \text{if } (x,y) \in \mathcal{BT}^N \text{ and } m \in \{0,1,2,3\}. \end{cases} \quad (13)$$

## 4.2. SEP(1) - Density and correlations

**4.2.1 Discrete stationary density profile:** Using equation (7) with  $k = 1$  (and, therefore,  $m \in \{0,1\}$ ), we can use the absorption probabilities above with

one dual particle to obtain an explicit expression for (5) as:

$$\rho_{ss}^N(x) = \rho_0 p_x^N(1) + \rho_N p_x^N(0). \quad (14)$$

Substituting in (14) the results obtained in (11), we get that, for any  $x \in \Lambda_N$ ,

$$\rho_{ss}^N(x) = \frac{(\rho_0 + \rho_N)(N^\theta - 1) + \rho_0(N - x) + \rho_N x}{N + 2N^\theta - 2}. \quad (15)$$

Then, if, for any  $x \in \Lambda_N$ , we assume  $\frac{x}{N} \rightarrow u \in [0,1]$ , as  $N$  goes to infinity, the limit in  $\frac{x}{N}$  of (11), defining  $\rho_1 := \rho_N$ , is given by

$$\lim_{N \rightarrow \infty} \rho_{ss}^N(x) = \bar{\rho}(u) := \begin{cases} \frac{\rho_0 + \rho_1}{2}, & \text{if } \theta > 1, \\ \frac{\rho_0 + \rho_1 + \rho_0(1-u) + \rho_N u}{3}, & \text{if } \theta = 1, \\ \rho_0(1-u) + \rho_N u, & \text{if } \theta < 1. \end{cases}$$

Remark that, independently from the value of  $\theta$ , the limit  $\lim_{N \rightarrow \infty} \rho_{ss}^N(x)$  is a linear function of  $u$  with different coefficients depending on the value of  $\theta$  and

$$\lim_{N \rightarrow \infty} \max_{x \in \Lambda_N} \left| \rho_{ss}^N(x) - \bar{\rho}\left(\frac{x}{N}\right) \right| = 0. \quad (16)$$

**4.2.2 2-points stationary correlation function:** Combining equation (6) with  $k = 1$ , the identity in (7) and equations (12) and (11), a simple, but long, computation gives us, for  $(x,y) \in \mathcal{T}^N$ ,

$$\varphi_{ss}(x,y) = -\frac{(\rho_N - \rho_0)^2}{N + 2N^\theta - 3} p_x^N(0) p_y^N(1).$$

Observe that, for any  $(x,y) \in \mathcal{T}_N$ , since  $p_x^N(0), p_y^N(1) > 0$ , if  $\rho_N \neq \rho_0$ , then  $\varphi_{ss}^N(x,y)$  is negative. This is due to the repelling of the particles caused by the exclusion rule. Also, for any  $\theta \in \mathbb{R}$ ,  $\lim_{N \rightarrow \infty} \varphi_{ss}^N(x,y) = 0$ , meaning that this function decays to zero when we pass to the macroscopic space. The question now is, what is the order of its decay? For any  $x,y \in \Lambda_N$  with  $x < y$ , assuming  $\frac{x}{N} \rightarrow u$  and  $\frac{y}{N} \rightarrow v$  as  $N \rightarrow \infty$ , the limit  $\lim_{N \rightarrow \infty} N \varphi_{ss}^N(x,y)$  now depends on the value of  $\theta$ , which means that, depending on the strength of the reservoirs, we will obtain different limit functions of parameters  $u$  and  $v$ , which are non-identically zero if  $\theta \leq 1$ .

Case 1 -  $\theta < 1$ :

$$\lim_{N \rightarrow \infty} N \varphi_{ss}^N(x,y) = -(\rho_1 - \rho_0)^2 G^{2,Dir}(u,v), \quad (17)$$

where  $\rho_1$  is the same as for the limit of the stationary density profile and  $G^{2,Dir}(u,v) := u(1-v)$  is the Green function of the 2-dimensional Laplacian on the upper triangle

$$T = \{(u,v) \in [0,1]^2 \mid u \leq v\}, \quad (18)$$

which is reflected on the diagonal  $u = v$ , and with homogeneous Dirichlet boundary conditions, i.e.,  $G^{2,Dir}$  is solution of the initial value problem

$$\begin{cases} \Delta^{2D} G^{2,Dir}(u,v) = \delta_{u=v}, & \text{if } (u,v) \in \text{int } T, \\ G^{2,Dir}(0,v) = G^{2,Dir}(u,1) = 0, & \text{if } u,v \in [0,1], \end{cases} \quad (19)$$

with  $\text{int } T = \{(u, v) \in (0, 1)^2 \mid u \leq v\}$  and

$$\Delta^{2D} = \begin{cases} \frac{d^2}{du^2} + \frac{d^2}{dv^2}, & \text{in } (\text{int } T) \setminus DT \\ \frac{\partial}{\partial u} - \frac{\partial}{\partial v}, & \text{in } DT \end{cases}, \text{ where}$$

$$DT = \{(u, u) \mid u \in (0, 1)\}.$$

Case 2 -  $\theta = 1$ :

$$\lim_{N \rightarrow \infty} N \varphi_{ss}^N(x, y) = -\frac{(\rho_1 - \rho_0)^2}{9} G^{2,Rob}(u, v),$$

where  $G^{2,Rob}(u, v) := \frac{(u+1)(2-v)}{3}$  is the Green function of the 2-dimensional Laplacian (as above) on the upper triangle  $T$ , defined in (18), which is reflected on the line  $u = v$  and with homogeneous Robin boundary conditions, i.e.,  $G^{2,Rob}$  is solution of the initial value problem

$$\begin{cases} \Delta^{2D} G^{2,Rob}(u, v) = \delta_{u=v}, & \text{if } (u, v) \in \text{int } T, \\ \frac{\partial}{\partial u} G^{2,Rob}(0, v) = G^{2,Rob}(0, v), & \text{if } v \in [0, 1], \\ \frac{\partial}{\partial v} G^{2,Rob}(u, 1) = -G^{2,Rob}(u, 1) & \text{if } u \in [0, 1], \end{cases}$$

where  $\text{int } T$  and  $\Delta^{2D}$  are defined as above. In this regime, the limit  $\lim_{N \rightarrow \infty} N \varphi_{ss}^N(x, y)$  is again a multiple of the solution of the 2-dimensional Laplace equation on  $T$  but with the homogeneous Dirichlet boundary conditions replaced by homogeneous Robin boundary conditions.

Case 3 -  $\theta > 1$ :

$$\lim_{N \rightarrow \infty} N \varphi_{ss}^N(x, y) = 0.$$

This means that, for slow boundary ( $\theta > 1$ ), the 2-points stationary correlation function has an order of decay higher than a linear function of  $N$ . In fact, the correct order to see a non-trivial limit in this case is  $N^\theta$ . Then, if instead of assuming  $\frac{x}{N} \rightarrow u$  and  $\frac{y}{N} \rightarrow v$  as  $N \rightarrow \infty$ , we assume  $\frac{x}{N^\theta} \rightarrow u$  and  $\frac{y}{N^\theta} \rightarrow v$  as  $N \rightarrow \infty$ , we obtain

$$\lim_{N \rightarrow \infty} N^\theta \varphi_{ss}^N(x, y) = -\frac{(\rho_N - \rho_0)^2}{8} (u+1)(1-v).$$

A summary of these results can be found in [4].

**4.2.3 3-points stationary correlation function:** Combining the results obtained in (12) and equation (13), performing long computations, we get an explicit formula for the 3-points stationary correlation function that is written in terms of the one particle absorption probabilities as we had for the 2-points stationary correlation function:

$$\varphi_{ss}^N(x, y, z) = 2 \frac{\rho_N - \rho_0}{N + 2N^\theta - 4} [\varphi_{ss}^N(x, y) p_x^N(1) - p_x^N(0) \varphi_{ss}^N(y, z)].$$

Observe that, for any  $x, z \in \Lambda_N$ ,  $p_x^N(0), p_z^N(1) > 0$ , then, the sign of  $\varphi_{ss}^N(x, y, z)$  will depend on  $y$  and the difference between  $\rho_N$  and  $\rho_0$ . Also, for any  $\theta \in \mathbb{R}$ ,  $\lim_{N \rightarrow \infty} N \varphi_{ss}^N(x, y, z) = 0$ , meaning that, at the macroscopic level, the 3-points stationary correlation function decays to zero faster than a linear function of  $N$ , and, therefore, faster than the 2-points stationary correlation function. We expect

that, as  $k$  increases, the corresponding  $k$ -th point stationary centered correlation function converges faster to zero. In the current case, its order of decay is quadratic, i.e.,  $N^2$ , if  $\theta \leq 1$ , and  $N^{2\theta}$ , if  $\theta > 1$ . For any  $x, y, z \in \Lambda_N$  with  $x < y < z$ , assuming  $\frac{x}{N} \rightarrow u \in [0, 1]$ ,  $\frac{y}{N} \rightarrow v \in [0, 1]$  and  $\frac{z}{N} \rightarrow w \in [0, 1]$  as  $N \rightarrow \infty$ , then, the limit  $\lim_{N \rightarrow \infty} N^2 \varphi_{ss}^N(x, y, z)$ , similarly to what to have for the 2-points stationary correlation function, depends on the value of  $\theta$ .

Case 1 -  $\theta < 1$ :

$$\lim_{N \rightarrow \infty} N^2 \varphi_{ss}^N(x, y, z) = -2(\rho_1 - \rho_0)^3 G^{3,Dir}(u, v, w),$$

where  $G^{3,Dir}(u, v, w) = u(1-2v)(1-w)$  is the Green function of the 3-dimensional Laplacian on the simplex

$$S = \{(u, v, w) \in [0, 1]^3 \mid u \leq v \leq w\}, \quad (20)$$

which is reflected on the planes  $S_1, S_2$  and on the line  $S_3$ , where  $S_1 = \{(u, v, w) \in S \mid u = v\}$ ,  $S_2 = \{(u, v, w) \in S \mid v = w\}$  and  $S_3 = \{(u, v, w) \in S \mid u = v = w\}$ , and with homogeneous Dirichlet boundary conditions, i.e.,  $G^{3,Dir}$  is solution of the initial value problem

$$\begin{cases} \Delta^{3D} G^{3,Dir}(u, v, w) = (1-w)\delta_{u=v} + u\delta_{w=v}, & \text{if } (u, v, w) \in \text{int } S, \\ G^{3,Dir}(0, v, w) = G^{3,Dir}(u, v, 1) = 0, & \text{if } (u, v), (v, w) \in Q, \end{cases}$$

where  $\text{int } S = \{(u, v, w) \in (0, 1)^3 \mid u \leq v \leq w\}$ ,

$$\Delta^{3D} = \begin{cases} \frac{d^2}{du^2} + \frac{d^2}{dv^2} + \frac{d^2}{dw^2}, & \text{in } (\text{int } S) \setminus ((\text{int } S_1 \cup \text{int } S_2)), \\ \frac{d^2}{du^2} + \frac{d}{dv} - \frac{d}{dw}, & \text{in } (\text{int } S_1) \setminus (\text{int } S_3), \\ \frac{d^2}{dw^2} + \frac{d}{du} - \frac{d}{dv}, & \text{in } (\text{int } S_2) \setminus (\text{int } S_3), \\ \frac{d}{du} - \frac{d}{dw}, & \text{in } \text{int } S_3, \end{cases}$$

and  $\text{int } S_j = (\text{int } S) \cap S_j$ , for  $j \in \{1, 2, 3\}$ .

Case 2 -  $\theta = 1$ :

$$\lim_{N \rightarrow \infty} N^2 \varphi_{ss}^N(x, y, z) = -\frac{2(\rho_1 - \rho_0)^3}{9} G^{3,Rob}(u, v, w),$$

where  $G^{3,Rob}(u, v, w) = \frac{(1+u)(1-2v)(2-w)}{27}$  is the Green function of the 3-dimensional Laplacian (as above) on the simplex  $S$ , defined in (20), which is reflected on the same planes and line and with homogeneous Robin boundary conditions, i.e.,  $G^{3,Rob}$  is solution of the initial value problem

$$\begin{cases} \Delta^{3D} G^{3,Rob}(u, v, w) = (1-w)\delta_{u=v} + u\delta_{w=v}, & \text{if } (u, v, w) \in \text{int } S, \\ \frac{\partial}{\partial u} G^{3,Rob}(0, v, w) = G^{3,Rob}(0, v, w), & \text{if } (v, w) \in T, \\ \frac{\partial}{\partial w} G^{3,Rob}(u, v, 1) = -G^{3,Rob}(u, v, 1) & \text{if } (u, v) \in T, \end{cases}$$

Case 3 -  $\theta > 1$ :

$$\lim_{N \rightarrow \infty} N^2 \varphi_{ss}^N(x, y, z) = 0.$$

As we had for the 2-points correlation function, this means that, for slow boundary, the 3-points stationary correlation function has an order of decay higher than a quadratic function of  $N$ . In fact, the correct order to see a non-trivial limit is  $N^{2\theta}$ .



Then, if instead of assuming  $\frac{x}{N} \rightarrow u$ ,  $\frac{y}{N} \rightarrow v$  and  $\frac{z}{N} \rightarrow w$  as  $N \rightarrow \infty$ , we assume  $\frac{x}{N^\theta} \rightarrow u$ ,  $\frac{y}{N^\theta} \rightarrow v$  and  $\frac{z}{N^\theta} \rightarrow w$  as  $N \rightarrow \infty$ , we obtain

$$\lim_{N \rightarrow \infty} N^{2\theta} \varphi_{ss}^N(x, y, z) = -\frac{(\rho_1 - \rho_0)^3 (1+u)v(1-w)}{2 \cdot 8}.$$

**Remark 4.1.** *By using the forward Kolmogorov's equation, one can also find a system of equations for which the 2-points and 3-points stationary correlation function are solution and, passing to the limit in  $N$ , recover the solutions above.*

### 4.3. SEP( $\alpha$ ) with $\alpha \geq 2$ - Absorption Probabilities

*4.3.1 Case  $k = 2$ :* Let us keep the same notation we used in SEP(1) for the absorption probabilities. Since in SEP( $\alpha$ ), with  $\alpha \geq 2$ , we allow more than one particle per site, we are also interested in computing  $p_{x,x}^N(m)$ . Here  $p_{x,x}^N(m)$  denotes the probability that, starting from a dual configuration with 2 particles at site  $x$  (in a pile),  $m$  particles being absorbed at the left reservoir. Analogous to SEP(1), conditioning on the first jump, we get a system of ten equations with two boundary conditions for  $p_{x,y}^N(m)$  that can be compacted in

$$\begin{cases} \mathfrak{U}_N^\theta p_{x,y}^N(m) = 0, \text{ for } x=1, \dots, N-2, \ y=x+1, \dots, N-1, \\ p_{0,y}^N(m) = p_y^N(m-1) \mathbb{1}_{\{m \neq 0\}}, \text{ if } y=0, \dots, N, \\ p_{x,N}^N(m) = p_x^N(m) \mathbb{1}_{\{m \neq 2\}}, \text{ if } x=0, \dots, N, \end{cases} \quad (21)$$

where the operator  $\mathfrak{U}_N^\theta$  is defined, for every function  $f \in \mathcal{F}(\mathcal{BCT}^N)$  and for every  $(x, y) \in \mathcal{CT}^N$ , as

$$\begin{aligned} \mathfrak{U}_N^\theta f(x, y) = & a_x [f(x-1, y) - f(x, y)] + b_y [f(x, y+1) - f(x, y)] \\ & + d_{x,y} [f(x+1, y) + f(x, y-1) - 2f(x, y)], \end{aligned} \quad (22)$$

where  $a_x = \frac{N^2}{N^\theta} \mathbb{1}_{x=1} + N^2 \mathbb{1}_{x \neq 1}$ ,  $b_y = \frac{N^2}{N^\theta} \mathbb{1}_{y=N-1} + N^2 \mathbb{1}_{y \neq N-1}$  and  $d_{x,y} = -\frac{N^2}{\alpha} \mathbb{1}_{y=x+1} + N^2 \mathbb{1}_{y \neq x, x+1}$ , with  $\mathcal{BCT}^N := \{(x, y) \in \{0, \dots, N\}^2 \mid x \leq y\}$  and  $\mathcal{CT}^N := \{(x, y) \in (\Lambda_N)^2 \mid x \leq y\}$ . For every  $x, y \in \{2, \dots, N-2\}$ , if  $x < y$  and  $|x-y| \geq 2$ ,  $\mathfrak{U}_N^\theta = \Delta_{N,full}^{2D}$ , and if  $y = x$ ,  $\mathfrak{U}_N^\theta = \Delta_{N,ref}^{2D}$ , where  $\Delta_{N,full}^{2D}$  and  $\Delta_{N,ref}^{2D}$  have the exact same meaning as in SEP(1). Also, if  $x \in \{2, \dots, N-2\}$  and  $y = x+1$ ,

$$\mathfrak{U}_N^\theta p_{x,x+1}^N(m) = \Delta_{N,full}^{2D} p_{x,x+1}^N(m) + \frac{1}{\alpha-1} \Delta_{N,ref}^{2D} p_{x,x+1}^N(m).$$

This shows that if  $\alpha \geq 2$ , the operator that we obtain for  $x \in \{2, \dots, N-2\}$  and  $y = x+1$  is no longer the 2-dimensional reflected Laplacian as we had for SEP(1). Instead, over the line  $\{(x, x+1) \mid x \in \Lambda_N\}$ , we will observe a super position of the operators that act above and over this line.

As we did for SEP(1), can we find  $p_{x,y}^N(m)$  by starting with an ansatz? The natural ansatz to take

is, for every  $(x, y) \in \mathcal{CT}^N$ ,

$$p_{x,y}^N(m) = \begin{cases} A^m x + B^m y + C^m xy + D^m, & \text{if } |x-y| \geq 2, \\ \tilde{A}_m x + \tilde{B}_m y + \tilde{C}_m xy + \tilde{D}_m, & \text{if } y = x+1, \\ \bar{A}_m x + \bar{B}_m x^2 + \bar{C}_m, & \text{if } y = x, \end{cases} \quad (23)$$

with  $A^m, B^m, C^m, D^m, \bar{A}_m, \bar{B}_m, \bar{C}_m, \tilde{D}_m, \tilde{A}_m, \tilde{B}_m, \tilde{C}_m, \tilde{D}_m \in \mathbb{R}$  to be determined.

If  $\theta = 0$ , using the ansatz in equation (23) to solve the system in (21), performing long (but, again, simple) computations, for every  $x, y \in \{0, \dots, N\}$  with  $x \leq y$ , we obtain

$$\begin{cases} p_{x,y}^N(0) = \frac{-1+\alpha y}{-1+\alpha N} p_x^N(0) + \frac{1}{2N(-1+\alpha N)} \mathbb{1}_{y=x}, \\ p_{x,y}^N(1) = \frac{(\alpha N+1)x + (\alpha N-1)y - 2\alpha xy}{N(-1+\alpha N)} \frac{1}{N(-1+\alpha N)} \mathbb{1}_{y=x}, \\ p_{x,y}^N(2) = \frac{-1+\alpha(N-x)}{-1+\alpha N} p_y^N(1) + \frac{1}{2N(-1+\alpha N)} \mathbb{1}_{y=x}. \end{cases} \quad (24)$$

Remark that, for  $\theta = 0$ , we don't need to consider different coefficients for the diagonal  $y = x+1$ , but only for the points on the diagonal  $y = x$ . We could not obtain the analogous expressions for  $\theta \neq 0$  since in this case we have more "discontinuities of the coefficients" of  $p_{x,y}^N(m)$ , meaning that the ansatz in (23) do not solve the system in (21).

### 4.4. SEP( $\alpha$ ) - Density and correlations

*4.4.1 Discrete stationary density profile:* Since the absorption probabilities for one dual particle are the same for SEP(1) and SEP( $\alpha$ ), by equation (7), the stationary density profile for SEP( $\alpha$ ) is  $\rho_{ss}^N$ , where  $\rho_{ss}^N$  is as in (15) but now  $\rho_0$  and  $\rho_1$  are the density of the left and right reservoirs of SEP( $\alpha$ ).

*4.4.2 2-points stationary correlation function:* Observe that  $\varphi_{ss}^N(x, y)$  is symmetric on the discretized square  $(\Lambda_N)^2$ , so we only care about finding its value on  $x, y \in \Lambda_N$  with  $x \leq y$ . As in SEP(1), combining equation (6), the identity in (7) with  $k = 2$  and also the previous formulas for the absorption probabilities, if  $\theta = 0$ , then, for every  $(x, y) \in \mathcal{CT}^N$ ,

$$\begin{aligned} \varphi_{ss}^N(x, y) = & -\frac{(\rho_0 - \rho_N)^2}{-1+\alpha N} p_x^N(0) p_y^N(1) + \frac{(\rho_0 - \rho_N)^2}{2N(-1+\alpha N)} \mathbb{1}_{y=x} \\ & + \left\{ \rho_{ss}^N(x) - \frac{\rho_0^2}{\alpha} p_{x,x}^N(2) - \frac{\rho_0 \rho_N}{\alpha} p_{x,x}^N(1) - \frac{\rho_N^2}{\alpha} p_{x,x}^N(0) \right\} \mathbb{1}_{y=x}. \end{aligned} \quad (25)$$

Observe that, for any  $x, y \in \Lambda_N$ , since  $p_x^N(0), p_y^N(1) > 0$ , if  $\rho_N \neq \rho_0$ , then, for  $x \neq y$ ,  $\varphi_{ss}^N(x, y) < 0$ . For  $y = x$ , we have that  $\varphi_{ss}^N(x, x) < \rho_{ss}^N(x)$ , if  $N \geq 2$  (which is clearly the case). As we had for SEP( $\alpha$ ),  $\lim_{N \rightarrow \infty} \varphi_{ss}^N(x, y) = 0$ , meaning that the 2-points stationary correlation function for SEP( $\alpha$ ) decays to zero when we pass to the macroscopic space. The question now is, what is the order of its decay? Like we had for SEP(1), it has decay of order  $N$ . For any  $(x, y) \in \mathcal{CT}^N$ , assuming that  $\frac{x}{N} \rightarrow u$  and  $\frac{y}{N} \rightarrow v$  as  $N \rightarrow \infty$ , then

$$\begin{aligned} \lim_{N \rightarrow \infty} N \left[ \varphi_{ss}^N(x, y) - \rho_{ss}^N(x) \frac{\alpha - \rho_{ss}^N(x)}{\alpha} \mathbb{1}_{y=x} \right] \\ = -\left(1 - \frac{1}{\alpha} \mathbb{1}_{v=u}\right) \frac{(\rho_0 - \rho_1)^2}{\alpha} G^{2,Dir}(u, v), \end{aligned}$$



where  $G^{2,Dir}$  is the same as in (19). Observe that the difference between this case and the previous ( $\alpha = 1$ ) is the multiplicative constant  $1/\alpha$ . This means that, macroscopically, changing the parameter  $\alpha$  from 1 to any natural number, only rescales the limit function of  $N\varphi_{ss}^N(x, y)$  obtain in (17) by a factor of  $1/\alpha$ . The  $\rho_0$  and  $\rho_N = \rho_1$  appearing in these sections are as in the table on Section 2.

In the next section, we will apply the same strategies for  $SIP(\alpha)$ .

#### 4.5. $SIP(\alpha)$ - Absorption Probabilities

*4.5.1 Case  $k = 2$ :* Like in  $SEP(\alpha)$ , in  $SIP(\alpha)$ , we allow more than one particle per site, and therefore, it makes sense to consider the absorption probability  $p_{x,y}^N(m)$  with  $y = x$ , with the same meaning as in  $SEP(\alpha)$ , with  $\alpha \geq 2$ . Then, conditioning on the first jump, we get a system of 10 equations for  $p_{x,y}^N(m)$  with two boundary conditions that can be compacted in:

$$\begin{cases} \mathfrak{W}_N^\theta p_{x,y}^N(m) = 0, \text{ for } x=1, \dots, N-2, y=x+1, \dots, N-1, \\ p_{0,y}^N(m) = p_y^N(m-1) \mathbb{1}_{\{m \neq 0\}}, \text{ if } y=0, \dots, N, \\ p_{x,N}^N(m) = p_x^N(m) \mathbb{1}_{\{m \neq 2\}}, \text{ if } x=0, \dots, N, \end{cases} \quad (26)$$

where the operator  $\mathfrak{W}_N^\theta$  is defined, for every function  $f \in \mathcal{F}(\mathcal{BCT}^N)$  and for all  $(x, y) \in \mathcal{CT}^N$ , by

$$\begin{aligned} \mathfrak{W}_N^\theta f(x, y) = & a_x [f(x-1, y) - f(x, y)] + b_y [f(x, y+1) - f(x, y)] \\ & + e_{x,y} [f(x+1, y) + f(x, y-1) - 2f(x, y)], \end{aligned}$$

where  $a_x = \frac{N^2}{N^\theta} \mathbb{1}_{x=1} + N^2 \mathbb{1}_{x \neq 1}$ ,  $b_y = \frac{N^2}{N^\theta} \mathbb{1}_{y=N-1} + N^2 \mathbb{1}_{y \neq N-1}$  and  $e_{x,y} = \frac{N^2}{\alpha} \mathbb{1}_{y=x+1} + N^2 \mathbb{1}_{y \neq x, x+1}$ . For every  $x, y \in \{2, \dots, N-2\}$ , if  $x < y$  and  $|x-y| \geq 2$ ,  $\mathfrak{W}_N^\theta = \Delta_{N,full}^{2D}$ , and if  $y = x$ ,  $\mathfrak{W}_N^\theta = \Delta_{N,ref}^{2D}$ . Also, if  $x \in \{2, \dots, N-2\}$  and  $y = x+1$ ,

$$\mathfrak{W}_N^\theta p_{x,x+1}^N(m) = \Delta_{N,full}^{2D} p_{x,x+1}^N(m) + \frac{1}{\alpha+1} \Delta_{N,ref}^{2D} p_{x,x+1}^N(m).$$

This shows that, like in  $SEP(\alpha)$ , over the line  $\{(x, x+1) \mid x \in \Lambda_N\}$ , we observe an overlap of the operators that act above and over this line. Then, the natural ansatz to take to solve the system of equations is the same as in (23).

**If  $\theta = 0$ ,** using that ansatz to solve the system in (26), if  $x, y \in \{0, \dots, N\}$  with  $x \leq y$ , we get

$$\begin{cases} p_{x,y}^N(0) = \frac{1+\alpha y}{1+\alpha N} p_x^N(0) - \frac{1}{2N(1+\alpha N)} \mathbb{1}_{y=x}, \\ p_{x,y}^N(1) = \frac{(\alpha N-1)x + (1+\alpha N)y - 2\alpha xy}{N(1+\alpha N)} + \frac{1}{N(1+\alpha N)} \mathbb{1}_{y=x}, \\ p_{x,y}^N(2) = \frac{1+\alpha[N-x]}{1+\alpha N} p_y^N(1) - \frac{1}{2N(1+\alpha N)} \mathbb{1}_{y=x}. \end{cases} \quad (27)$$

Like in  $SEP(\alpha)$ , this means that on the diagonal  $y = x$ , the expression of  $p_{x,y}^N(m)$  changes from the one inside the triangle, i.e., for  $(x, y) \in \mathcal{T}^N$ , by a constant that depends on  $\alpha$ ,  $N$  and  $m$ . Also, in this case,  $\tilde{A}_m = A^m$ ,  $\tilde{B}_m = B^m$ ,  $\tilde{C}_m = C^m$  and  $\tilde{D}_m = D^m$ . We could not obtain the explicit expressions for the absorption probabilities for  $\theta \neq 0$ , by the same reasons presented in  $SEP(\alpha)$ .

#### 4.6. $SIP(\alpha)$ - Density and correlations

*4.6.1 Discrete stationary density profile:*  $SEP(\alpha)$  and  $SIP(\alpha)$ , with only one dual particle, describe the same model - equal density profile.

*4.6.2 2-points stationary correlation function:* As for  $SEP(\alpha)$ , if  $\theta = 0$ , for every  $(x, y) \in \mathcal{CT}^N$ ,

$$\begin{aligned} \varphi_{ss}^N(x, y) = & \frac{(\rho_0 - \rho_N)^2}{-1 + \alpha N} p_x^N(0) p_y^N(1) - \frac{(\rho_0 - \rho_N)^2}{2N(-1 + \alpha N)} \mathbb{1}_{y=x} \\ & + \left\{ -\rho_{ss}^N(x) + \frac{\rho_0^2}{\alpha} p_{x,x}^N(2) + \frac{\rho_0 \rho_N}{\alpha} p_{x,x}^N(1) + \frac{\rho_N^2}{\alpha} p_{x,x}^N(0) \right\} \mathbb{1}_{y=x}. \end{aligned} \quad (28)$$

Observe that, for any  $x, y \in \Lambda_N$ , since  $p_x^N(0), p_y^N(1) > 0$ , then, if  $\rho_N \neq \rho_0$ , for  $x \neq y$ ,  $\varphi_{ss}^N(x, y) > 0$ . Also, we have that  $\varphi_{ss}^N(x, x) > \rho_{ss}^N$ , if  $N \geq 2$ . We observe that the expression obtain in (25) and in (28) are very similar with a small change of signs.

For  $SIP(\alpha)$ , we still have that  $\lim_{N \rightarrow \infty} \varphi_{ss}^N(x, y) = 0$ , meaning that the 2-points stationary correlation function for  $SIP(\alpha)$  decays to zero when we pass to the macroscopic space. Its order of decay of order is  $N$ . Finally, for any  $x, y \in \Lambda_N$ , assuming  $\frac{x}{N} \rightarrow u$  and  $\frac{y}{N} \rightarrow v$  as  $N \rightarrow \infty$ , then

$$\begin{aligned} \lim_{N \rightarrow \infty} N \left[ \varphi_{ss}^N(x, y) + \rho_{ss}^N(x) \frac{\alpha + \rho_{ss}^N(x)}{\alpha} \mathbb{1}_{y=x} \right] \\ = \left( 1 + \frac{1}{\alpha} \mathbb{1}_{v=u} \right) \frac{(\rho_0 - \rho_1)^2}{\alpha} G^{2,Dir}(u, v) \end{aligned}$$

where  $G^{2,Dir}$  has the same interpretation as in  $SEP(1)$  and  $SEP(\alpha)$ . Observe that the difference from the analogous result obtained for  $SEP(\alpha)$  is the change of sign from a minus to a plus.

#### 5. CONCLUSIONS AND COMPARING RESULTS

Here is a summary of the main conclusion of the present work. In Section 2, we concluded that, for each model, there exists a unique invariant measure, which is also reversible and of homogeneous product form when the left and right reservoirs have equal densities. In Section 3, we saw that both models introduced in Section 2 have a dual process that conserves the bulk dynamics exchanging the reservoir's dynamics to absorbing boundary points. We recalled that, to construct the classical duality function connecting the models with their duals, it is very important to remark that, once a reversible measure is available for a given continuous time Markov process with a countable state space, self-duality comes for free using the cheap self-duality function and that we can generate duality functions by applying symmetries of the Markov generator to a known duality function. At last, in Section 4, we obtained explicit expressions for the absorption probabilities with  $k = 1, 2, 3$  dual particles for  $SEP(1)$  with  $\theta \in \mathbb{R}$  and with  $k = 1, 2$  dual particles for  $SEP(\alpha)$  and  $SIP(\alpha)$  with  $\theta = 0$ . Let us now compare the results. For  $m \in \{0, 1, 2\}$ , for every  $(x, y) \in \mathcal{T}^N$ ,  $p_{x,y}^N(m)$ , for all models, can be

factorized, as  $p_{x,y}^N(m) = p_x^N(m) + p_y^N(1)F_x(m)$ , with

$$F_x(m) = \begin{cases} p_x^{N-1}(m) - p_x^{N-1}(m-1), & \text{for SEP}(1) \\ p_x^{N-\alpha^{-1}}(m) - p_x^{N-\alpha^{-1}}(m-1), & \text{for SEP}(\alpha) \\ p_x^{N+\alpha^{-1}}(m) - p_x^{N+\alpha^{-1}}(m-1), & \text{for SIP}(\alpha) \end{cases} \quad (29)$$

where  $p_x^{N+r}(m) := \frac{x}{N+r} \mathbb{1}_{m=0} + \frac{N+r-x}{N+r} \mathbb{1}_{m=1}$ , for every  $r \in \mathbb{R}$ . So, if  $\alpha^{-1} \in \mathbb{N}$ , then  $p_x^{N+\alpha^{-1}}(1)$  (resp.  $p_x^{N+\alpha^{-1}}(0)$ ) represents the probability that, starting from one dual particle at site  $x$  on a lattice of size  $N+\alpha^{-1}$  (the size increases), one particle is absorbed at 0 (resp.  $N$ ). On the other hand, if  $\alpha^{-1} \notin \mathbb{N}$ , then  $p_x^{N\pm\alpha^{-1}}(1)$  and  $p_x^{N\pm\alpha^{-1}}(0)$  are no longer absorption probabilities since to see them as that we would need to consider a lattice of non-integer size (the size  $N$  decreases by  $\alpha^{-1}$ , for  $\text{SEP}(\alpha)$ , and increases by  $\alpha^{-1}$ , for  $\text{SIP}(\alpha)$ ) that, by our construction of the microscopic space from the macroscopic space, is not allowed.

Comparing now the absorption probabilities on the main diagonal points, i.e.,  $p_{x,x}^N(m)$ , with  $x \in \Lambda_N$ : taking  $\theta = 0$ , for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$  (recall that it does not make sense to talk about  $p_{x,x}^N(m)$  for  $\text{SEP}(1)$ , since only one particle is allowed per site), we see that, for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$ , for  $m \in \{0, 1, 2\}$ ,  $p_{x,x}^N(m)$  can be factorized, for every  $x \in \Lambda_N$  as  $p_{x,x}^N(m) = p_x^N(m) + p_x^N(1)F_x(m) + H(m)$ , where  $F_x(m)$  is as in (29) and  $H(m)$  is a real number that depends on  $m$ , but not on  $x$ , and is defined, for  $m \in \{0, 1, 2\}$ , as

$$H(m) := \begin{cases} \frac{(-1)^m \alpha^{-1}}{2N(N-\alpha^{-1})} [\mathbb{1}_{m \neq 0} + \mathbb{1}_{m \neq 2}], & \text{for SEP}(\alpha), \\ \frac{(-1)^{m+1} \alpha^{-1}}{N(N+\alpha^{-1})} [\mathbb{1}_{m \neq 0} + \mathbb{1}_{m \neq 2}], & \text{for SIP}(\alpha). \end{cases} \quad (30)$$

Summarizing, for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$ , if  $\theta = 0$ , for  $m \in \{0, 1, 2\}$ ,  $p_{x,y}^N(m)$  can be factorized, for every  $x, y \in \Lambda_N$  with  $x \leq y$ , as

$$p_{x,y}^N(m) = p_x^N(m) + p_y^N(1)F_x(m) + H(m)\mathbb{1}_{y=x}, \quad (31)$$

where  $F_x(m)$  and  $H(m)$  are the ones defined above.

As we saw in Section 4, the stationary density profiles of  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$  are given by the same expression, since with only one particle, these processes describe the same dynamics. Comparing the results for the 2-points stationary correlation function of both processes, for the choice  $\theta = 0$ , we observe that  $\varphi_{ss}^N(x, y)$ , with  $x, y \in \Lambda_N$ , factorizes as

$$\begin{aligned} & \varphi_{ss}^N(x, y) + C_\alpha \rho_{ss}^N(x) \frac{\alpha + C_\alpha \rho_{ss}^N(x)}{\alpha} \mathbb{1}_{y=x} \\ &= \frac{(\rho_0 - \rho_N)^2}{\alpha} \left\{ NH(1)p_x^N(0)p_y^N(1) + H(0)\mathbb{1}_{y=x} \right\} + \\ &+ \frac{C_\alpha(\rho_0 - \rho_N)^2}{\alpha^2} \left[ NH(1)p_x^N(0)p_y^N(1) + H(0) \right] \mathbb{1}_{y=x}, \end{aligned} \quad (32)$$

where  $H(0)$  and  $H(1)$  are the same as in (30) and  $C_\alpha = -1$ , for  $\text{SEP}(\alpha)$ , and  $C_\alpha = 1$  for  $\text{SIP}(\alpha)$ . Also, for  $\text{SEP}(\alpha)$ , we have  $\varphi_{ss}^N(x, y) < 0$ , while for  $\text{SIP}(\alpha)$ ,  $\varphi_{ss}^N(x, y) > 0$ , for all  $x, y \in \Lambda_N$  with  $y \neq x$ . Finally, by the methods presented in this section, we concluded that solving systems of equations with some boundary conditions, with patience (due to the extensive computations that are involved), we

can obtain the same results known for  $\text{SEP}(1)$  with  $\theta = 0$  and with general  $\theta$  that were obtained in the literature using Matrix Product Ansatz Method (MPA). Moreover, note that, even though MPA is only available for  $\text{SEP}(1)$  and not for  $\text{SEP}(\alpha)$  or  $\text{SIP}(\alpha)$ , our strategy also works for these models and we expect to work for other interacting particle systems. The results obtained in Section 4 that involved extensive computations were checked using Mathematica.

As future work, we want to obtain the solution for the system of equations that the absorption probabilities for 2 dual particles with  $\theta \in \mathbb{R}$  for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$  solve and from here obtain explicit expressions for the 2-points stationary correlation function for  $\text{SEP}(\alpha)$  and  $\text{SIP}(\alpha)$  and study its corresponding limiting function at the macroscopic level. An alternative approach could be to apply Kolmogorov's equation to obtain (and then solve) the equation satisfied by the 2-points correlation function for these models. We expect that the operators that will appear will be related to the ones obtained for the absorption probabilities  $p_{x,y}^N(m)$  with an extra error function taking values only on the diagonal  $y = x$ . From here, we would also like to see if the stationary correlation functions for higher order can also be written in a factorized form as in (32) using absorption probabilities of just one dual particle or even recursively like it is known for  $\text{SEP}(1)$ .

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INSTITUTO SUPERIOR TÉCNICO, LISBOA, PORTUGAL  
Email address: beatriz.salvador@tecnico.ulisboa.pt