TÉCNICO LISBOA

Stochastic Duality for Symmetric Simple Exclusion and Inclusion in contact with reservoirs

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To all the people who, living or
dead, left me a little bit of themselves ...

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Resumo

A dinâmica dos processos de Exclusão, SEP(α), $\alpha \in \mathbb{N}$, e de Inclusão, SIP(α), $\alpha \in \mathbb{R}^+$, Simples Simétricos com fronteira aberta, consiste em partículas efetuando um passeio aleatório em tempo contínuo no espaço $\{1, \ldots, N-1\}$ (para $N \in \mathbb{N}_2$ fixado) com os pontos 0 e N identificados como reservatórios. Para SEP(α), apenas α partículas são permitidas por sítio, enquanto que, para SIP(α), não há limitação. Na fronteira esquerda (respetivamente, direita), particulas são injetadas e absorvidas com taxas dependentes de N e parâmetros $\epsilon, \gamma > 0$ (respetivamente, $\delta, \beta > 0$) e $\theta \in \mathbb{R}$.

Estes modelos têm processos duais que substituem a fronteira aberta por fronteira absorvente e que estão ligados pela sua correspondente função de dualidade clássica. Como consequência da dualidade, começando ambos os processos da sua medida invariante, obtemos fórmulas explícitas para a densidade e função de correlação centrada de dois pontos para SEP(α) e SIP(α), e também para a função de correlação centrada de três pontos para SEP(1). Estas são encontradas calculando probabilidades de absorção para uma e duas partículas nos duais do SEP(α) e do SIP(α) e também para três particulas no dual do SEP(1). O limite destas funções mostra-se estar relacionado com funções de Green que são solução de um probema de valor initial com diferentes condições de fronteira dependendo do valor de θ.

Os resultados mais importantes desta dissertação são: Teorema 3.4.1 e Teorema 3.5.1, as identidades (**??**) e (4.65) e as funções acima referidas obtidas no Capítulo 4.

Palavras-chave: Processo de Exclusão Simples Simétrico, Processo de Inclusão Simples Simétrico, Dualidade, Probabilidades de Absorção, Correlações Estacionárias.

Abstract

The dynamics of the Symmetric Simple Exclusion, SEP(α), $\alpha \in \mathbb{N}$, and Inclusion, SIP(α), $\alpha \in \mathbb{R}^+$, processes with open boundary, consists of particles performing continuous time random walks on the space $\{1, \ldots, N-1\}$ (for $N \in \mathbb{N}_2$ fixed) with points 0 and N identified as boundary reservoirs. For SEP(α), no more than α particles are allowed per site, while, for SIP(α), there is no upper bound. At the left (respectively, right) boundary, particles are injected or absorbed with rates that depend on N and the parameters $\epsilon, \gamma > 0$ (respectively, $\delta, \beta > 0$) and $\theta \in \mathbb{R}$.

These models have dual processes that substitute the open boundary by absorbing boundary points and that are linked by their corresponding classical duality function. As consequence of duality, starting all processes from their invariant measure, we obtain explicit formulas for the density profile and 2-points centered correlation function for SEP(α) and SIP(α), and also for the 3-points centered correlation function for SEP(1). These are found by computing absorption probabilities for one and two particles on the dual SEP(α) and SIP(α) and also for three particles on the dual SEP(1). The corresponding limit of such functions is shown to be connected with Green functions that are solution of an initial value problem with different boundary conditions depending on the value of θ.

The most important results of this thesis are: Theorem 3.4.1 and Theorem 3.5.1, identities (**??**) and (4.65) and the functions mentioned above presented in Chapter 4.

Keywords: Symmetric Simple Exclusion Process, Symmetric Simple Inclusion Process, Duality, Absorption Probabilities, Stationary Correlations.

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Nomenclature

$(\Omega, \mathcal{F}, P)$ Probability space with σ-algebra $\mathcal{F}$ and probability measure P.

$\mathfrak{gl}(U)$ Set of all linear maps from the real or complex vector space U to itself.

i.o. Infinitely often.

MPA Matrix Product Ansatz Method.

Greek symbols

α Parameter which labels the family of processes.

β Parameter associated to the rate to get out of the system from the right reservoir.

δ Parameter associated to the rate to get in the system from the right reservoir.

ϵ Parameter associated to the rate to get in the system from the left reservoir.

η Configuration for the initial process.

Γ Gamma function.

γ Parameter associated to the rate to get out of the system from the left reservoir.

$\hat{\eta}$ Configuration for the dual process.

μ Probability measure.

Ω State space of a continuous time Markov process.

θ Strength of the reservoir.

Number Sets

$\mathbb{C}$ Complex numbers.

$\mathbb{N}$ Natural numbers.

$\mathbb{N}_0$ Natural numbers including zero.

$\mathbb{N}_2$ Natural numbers greater or equal than 2.

$\mathbb{R}$ Positive real numbers.

$\mathbb{R}$ Real numbers.

Roman symbols

e Napier's number.

Subscripts

∞ Infinity.

Supercripts

^{T} Matrix transpose.

* Adjoint operator.

Math Symbols

$\times$ Cartesian product.

Chapter 1

Introduction

1.1 Motivation

Applying statistical methods and probability theory, Statistical Mechanics tries to characterize equilibrium and non-equilibrium states of particle systems that evolve at a microscopic level with the goal of explaining macroscopic physical phenomena. Depending on the type of dynamics prescribed between particles, this characterization can be a very hard task, especially if the system is out of equilibrium. Motivated by the intrinsic interesting mathematical properties of such microscopic systems, Interacting Particle Systems emerged as a branch of Statistical Mechanics. Introduced in the mathematics community by Spitzer in the 70's [19] and further developed by Liggett [14], Interacting Particle Systems have been in growing development and are subject of vast research.

Studying continuous time Markov processes, this field focus on understanding the time evolution of physical quantities observed at the macroscopic level - for example, temperature, density, pressure - by analyzing the long time behavior of systems of particles whose dynamics is defined at the microscopic level and with assumed random behavior. At a first level, every time a continuous time Markov process is defined it is natural to start by characterizing, if they exist, what are its invariant measures. Meaning that, starting at time zero from that distribution, at anytime t, we will have the same distribution. Or even, when this invariant measure is unique, represent the distribution that, as time goes to infinity, the system will be closer and closer to. From here, studying, for example, the density, temperature or pressure of the system, becomes a two side way problem: finding the profile of these physical quantities once the stationary state is attained (meaning that there is no longer time evolution) or before that state is reached. In this thesis, we will be following the first path.

Motivated by answering to these and many other questions, Interacting Particle Systems combines different mathematical tools to extract useful information about its models of study. Among the powerful probabilistic tools that are used is stochastic duality, or simply, duality. Introduced in 1948 by Levy [16] and in 1957 by Karlin and McGregor for birth and death processes [12], duality theory allows us to connect some Markov processes to others that in general are of simpler analysis. Here are two important simplifications that can be done when the duality property is satisfied by a given process and

that we will use in this thesis:

1. *From open to absorbing boundaries:* using duality, we can relate particle systems with boundary reservoirs that allow creation and annihilation of particles - *open boundary* - with processes that only allow death of particles - *absorbing boundary*. This simplification allows us to compute exact formulas of some relevant functions, such as the density profile and the $k-$th point centered correlation functions, for $k \in \mathbb{N}_2$, of the initial process using the absorption probabilities of the dual process. Some examples are the Symmetric Simple Exclusion Process, which appeared in 1985 in Liggett's book [14] where duality is explored when only one particle is allowed per site, and the Symmetric Simple Inclusion Process with open boundary. They will be object of study on this thesis.

2. *From many to few:* we can put in a duality relationship systems that can have numerous particles, with processes that evolve with only a few. For example, the case when we convert open to absorbing boundaries. Also, using self-duality (when a process is dual to itself), we can use a copy of the initial process to compute the density and $k-$th point centered correlations, for $k \in \mathbb{N}_2$, using only one and k dual particles, respectively. This allows a huge simplification on the study of correlations for self-dual processes that have a very large number of particles.

Besides these simplifications in terms of applications, duality also provides an algebraic approach to study Interacting Particle Systems, due to its strong connection with Lie algebra's Theory and its representations. This new point of view about duality has been developed in [1], [4] and [9] and many others, providing a useful way to obtain new duality functions for two dual processes that we will recall in Chapter 3.

1.2 Topic Overview, Objectives and Deliverables

In this thesis, we will study the consequences of having duality between Markov processes defined on a lattice with boundary reservoirs and the "same" Markov processes defined on the "same" lattice extended to the boundary points that now are only absorbing. Namely, we will focus on establishing duality between the Symmetric Simple Exclusion Process with open boundary, SEP(α), with $\alpha \in \mathbb{N}$, and with absorbing boundary points, of particular interest, the special case where only one particle is allowed per site, SEP(1), and the Symmetric Simple Inclusion Process with boundary reservoirs, SIP(α), with $\alpha \in \mathbb{R}^+$, and absorbing boundary points. In both models, we choose boundary rates that depend on the size of the lattice and on the value of an additional parameter $\theta \in \mathbb{R}$, that tunes the boundary reservoirs' strength. Our aim is to obtain, using the duality property, explicit formulas for the stationary density profile and the $k-$th point stationary centered correlation functions, with k at most three (Chapter 4) by computing absorption probabilities of the dual processes (starting with at most three dual particles), that are solutions of systems of equations obtained conditioning on the first jump. We also want to obtain a factorized form for these absorption probabilities and correlation functions, motivated by what it is known

for SEP(1) where the Matrix Product Ansatz Method (MPA) is available - see, for example, [2], [18] and [17]. Up to our knowledge, MPA is not available for SEP(α), with $\alpha \geq 2$, nor SIP(α), with $\alpha \in \mathbb{R}^+$.

1.3 Thesis Outline

Here is a description of the content and main results of this thesis. In Chapter 2, we give the construction of how to define, starting from the macroscopic space (continuous space), the microscopic space (discrete space). In the last one, we define two different interacting particle systems: the Symmetric Simple Exclusion Process with open boundary, SEP(α), with $\alpha \in \mathbb{N}$, the special case $\alpha = 1$, SEP(1), and the Symmetric Simple Inclusion Process also with open boundary, SIP(α), where here α belongs to $\mathbb{R}^+$. We present a detailed proof of the characterization of the unique reversible measure for SEP(α) and state the analogous results for the reversible measure of SIP(α), both in the context of equilibrium (equal density on the left and right reservoirs) and as a closed system (meaning, without reservoirs). We also remark the differences, in terms of reversible measures, between the models when considered with or without reservoirs. The chapter ends with the discussion of existence and uniqueness of stationary measure for SEP(α) and SIP(α) out of equilibrium (different density on the left and right reservoirs).

Chapter 3 is dedicated to duality theory. Here, we review two different ways of defining duality between two continuous time Markov processes, in terms of expectations and via infinitesimal Markov generators. We recall in Theorem 3.1.1 the connection between reversibility proved in Chapter 2 and a first duality relation for processes that are dual to themselves via the so-called cheap self-duality function. We also give a quick overview on Lie algebra theory, focusing on representations of the Lie algebras $su(2)$ and $su(1,1)$. We finish the chapter by using the previous general algebraic results to construct the classical duality functions, of homogeneous product form, providing, via duality, a connection between our models with their corresponding versions conserving the bulk's dynamics and changing the boundary into absorbing boundary points.

In Chapter 4, we provide applications of the duality relations proved in Chapter 3. Namely, we obtain explicit formulas and study the corresponding limits, depending on the value of θ, for the stationary density and 2-points stationary centered correlation functions for the models SEP(α) and SIP(α) (for SEP(1), also the 3-points stationary centered correlation function), by obtaining first explicit formulas for the absorption probabilities for at most three dual particles (whose results were checked using Mathematica). We provide for SEP(1) a different argument for the case $\theta = 0$, motivated by the aim of obtaining a factorized formula.

Last, in Appendix A, we include the link for the Mathematica files with the code used to obtain the coefficients for the absorption probabilities of both SEP(α), SEP(1) and SIP(α); in Appendix B we write the formulas for the absorption probabilities that are also valid for the boundary points 0 and N; and, finally, in Appendix C, we give a different way of obtaining the 2-points and 3-points stationary centered correlation functions based on Kolmogov's equation.

Chapter 2

Description of the models and preliminary results

2.1 From macrospace (continuous) to microspace (discrete)

In Interacting Particle Systems, we are interested in understanding the evolution in time of some physical quantities, such as density, temperature, pressure, in a system that is defined, macroscopically, in time and space, by looking at the evolution of the interaction of particles at the microscopic level. We will consider here that the macroscopic space is the closed interval $[0,1]$ and define the microscopic space as the rescaled end points of the intervals of a partition of the macroscopic space, excluding the boundary points. We will now describe the full construction.

Let $N \in \mathbb{N}$ be a scaling parameter and consider a discretization of the interval $[0,1]$ in N subintervals I_i ($i = 0, 1, ..., N-1$) where $I_i = \left[\frac{i}{N}, \frac{i+1}{N}\right)$ for $i = 0, ..., N-2$ and $I_{N-1} = \left[\frac{N-1}{N}, 1\right]$. We will denote by Λ_N the set of points $\Lambda_N := \{1, ..., N-1\}$, which we will call *bulk* and that corresponds to the extreme points of the intervals I_i rescaled by N, neglecting the boundary points 0 and N ($i = 0, 1, \ldots, N-1$). The bulk will be our microscopic space. For the points 0 and N, that we will call *boundary points* , we will consider them to be reservoirs with an arbitrary number of particles. We will also frequently call the point 0, *left reservoir*, and N, *right reservoir*. See Figure 2.1 for an illustration of the constructions described above.

Once defined the microscopic space, we will select one type of dynamics to describe the interaction of the particles inside the bulk, called the *Kawasaki dynamics* [1], and another for the interaction of the reservoirs with the particles on the bulk, called the *Glauber dynamics* . As we will see, the Glauber dynamics will allow creation and annihilation of particles from the points 1 and $N-1$ of the bulk, and, to remind us of this, we will say that the considered models have *open boundary*.

We will describe in the next sections the two models that we will be focusing on, and consequently, these two different types of dynamics.

[1] Here we will be using an abuse of terminology, since usually *Kawasaki dynamics* stands for the bulk dynamics in SEP(1).

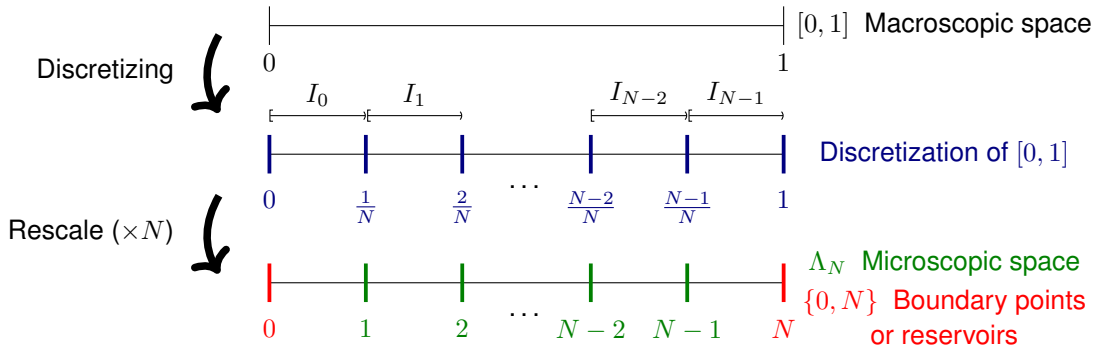

Figure 2.1: Macroscopic space, partition of the interval $[0,1]$ and the corresponding microscopic space.

2.2 Symmetric Simple Exclusion with open boundary - SEP(α)

Let $\alpha \in \mathbb{N}$. We define the space of configurations

$$\Omega_N^{Ex} := \{\eta = (\eta_1, ..., \eta_{N-1}) \mid \eta_i = 0, 1, ..., \alpha \text{ for all } i = 1, ..., N-1\} = \{0, 1, ..., \alpha\}^{\Lambda_N}$$

and interpret $\eta \in \Omega_N^{Ex}$ also as a function $\eta : \Lambda_N \to \{0, 1, ..., \alpha\}$ where, for every $x \in \Lambda_N$ and $j \in \{0, 1, ..., \alpha\}$, $\eta(x) = j$, if at site x there exists j particles. For every $x \in \Lambda_N$, we will call $\eta(x)$ the occupation variable at site x. Given a configuration $\eta \in \Omega_N^{Ex}$, we define, for all $x, y \in \Lambda_N$ with $|x - y| = 1$,

$$\eta^{x,y}(z) = \begin{cases} \eta(z), \text{ if } z \in \Lambda_N \setminus \{x, y\} \\ \eta(x), \text{ if } \eta(x) = 0 \text{ and } z = x \\ \eta(y), \text{ if } \eta(y) = \alpha \text{ and } z = y \\ \eta(x) - 1, \text{ if } z = x \text{ and } \eta(x) \geq 1 \\ \eta(y) + 1, \text{ if } z = y \text{ and } \eta(y) \leq \alpha - 1 \end{cases}, \tag{2.1}$$

$$\eta^{0,1}(z) = \begin{cases} \eta(z), \text{ if } z \in \{2, ..., N-1\} \text{ or if } \eta(1) = \alpha \text{ and } z = 1 \\ \eta(1) + 1, \text{ if } z = 1 \text{ and } \eta(1) \leq \alpha - 1 \end{cases}, \tag{2.2}$$

$$\eta^{1,0}(z) = \begin{cases} \eta(z), \text{ if } z \in \{2, ..., N-1\} \text{ or if } \eta(1) = 0 \text{ and } z = 1 \\ \eta(1) - 1, \text{ if } z = 1 \text{ and } \eta(1) \geq 1 \end{cases}, \tag{2.3}$$

$$\eta^{N-1,N}(z) = \begin{cases} \eta(z), \text{ if } z \in \{1, ..., N-2\} \text{ or if } \eta(N-1) = 0 \text{ and } z = 1 \\ \eta(N-1) - 1, \text{ if } z = N-1 \text{ and } \eta(N-1) \geq 1 \end{cases}, \tag{2.4}$$

$$\eta^{N,N-1}(z) = \begin{cases} \eta(z), \text{ if } z \in \{1, ..., N-2\} \text{ or if } \eta(N-1) = \alpha \text{ and } z = N-1 \\ \eta(N-1) + 1, \text{ if } z = N-1 \text{ and } \eta(N-1) \leq \alpha - 1 \end{cases}. \tag{2.5}$$

The elements of Ω_N^{Ex} defined in (2.1), (2.2), (2.3), (2.4), (2.5) represent a change of configuration induced by the jump, injection of extraction of one particle.

The Symmetric Simple Exclusion Process (sometimes also called in the literature generalized SEP(α) or Symmetric Simple Partial Exclusion Process) with open boundary - SEP(α) - is a continuous time Markov Process which is *simple* (jumps only occur to nearest neighbors) and has an *exclusion rule* (we say it is partial if $\alpha \geq 2$, meaning that, even though there is an upper bound, α, for the number of particles allowed per site, we still can have piles with more than one particle). The points in Λ_N are occupation sites for particles that have to wait a certain time, which is exponentially distributed with parameter 1 (to have the Markov property of memory loss), before a jump can occur. To each interval I_i ($i = 0, \ldots, N-1$), we allocate an independent *Poisson clock* such that, every time the clock rings, a particle that is placed at one of the extremes of I_i can jump to the other if it is not occupied with more than $\alpha - 1$ particles. Since these clocks are all independent, the probability that two of them ring at the same time is zero, meaning that a configuration η can only change when only one jump, injection or extraction of a particle occurs.

For the jump rates: fixing a configuration $\eta \in \Omega_N^{Ex}$, a particle at site $x \in \Lambda_N \setminus \{1, N-1\}$ has rate $p(-1)\eta(x)[\alpha - \eta(x-1)]$ to jump to site $x-1$ and $p(1)\eta(x)[\alpha - \eta(x+1)]$ to jump to site $x+1$. To define the complete dynamics at sites 1 and $N-1$, we will also consider the boundary points 0 and N, as left and right reservoir, respectively, with an arbitrary number of particles and that can inject or extract particles at/from the bulk through the sites 1 and $N-1$. The inclusion of these reservoirs adds to the model a new dynamic (that no longer conserves the total number of particles at each time t) for the sites 1 and $N-1$. For the boundary rates, we introduce a parameter $\theta \in \mathbb{R}$ that indicates the strength of the interaction of the reservoirs with the system: the higher the value of θ, the weaker the interaction. So, given a configuration $\eta \in \Omega_N^{Ex}$, a particle at site 1 (resp. $N-1$) has rate $p(1)\eta(1)[\alpha - \eta(2)]$ (resp. $p(-1)\eta(N-1)[\alpha - \eta(N-2]$) to jump to site 2 (resp. $N-2$), and rate $p(-1)\frac{\gamma}{N^\theta}\eta(1)$ (resp. $p(1)\frac{\beta}{N^\theta}\eta(N-1)$), with $\gamma, \beta > 0$, to get out of the system from the left (resp. right) reservoir. The left (resp. right) reservoir has injection rate $p(1)\frac{\epsilon}{N^\theta}[\alpha - \eta(1)]$ (resp. $p(-1)\frac{\delta}{N^\theta}[\alpha - \eta(N-1)]$), with $\epsilon, \delta > 0$. We say that this process is *symmetric* when $p(1) = p(-1) = 1$, i.e, a particle at site $x \in \Lambda_N \setminus \{1, N-1\}$ has the same elementary rate p (rate when there is only one particle on the system) to jump to its left or to its right. Thus, the SEP(α) dynamics is the result of a superposition of the two dynamics explained above. See Figure 2.2 for a summary.

Mathematically, Kawasaki and Glauber's dynamics are described by the infinitesimal Markov generator of the process. Recall that, given a continuous time Markov process **X** with countable state space Ω, its infinitesimal Markov generator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{D}(\mathcal{L})$, is defined as the linear operator that satisfies

$$\mathcal{L}f = \lim_{t \to 0^+} \frac{S_t f - f}{t}, \quad \text{for all } f \in \mathcal{D}(\mathcal{L}), \tag{2.6}$$

where, if Ω is finite, $\mathcal{D}(\mathcal{L})$ is $\mathcal{F}(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is a function}\}$[2]; and, if Ω is not finite, $\mathcal{D}(\mathcal{L})$ is the set of local functions for a fixed probability measure ν_ρ in Ω, i.e, functions $f : \Omega \to \mathbb{R}$ that are only different

[2] For any set A, we will always define $\mathcal{F}(A)$ by $\mathcal{F}(A) := \{f : A \to \mathbb{R} \mid f \text{ is a function }\}$.

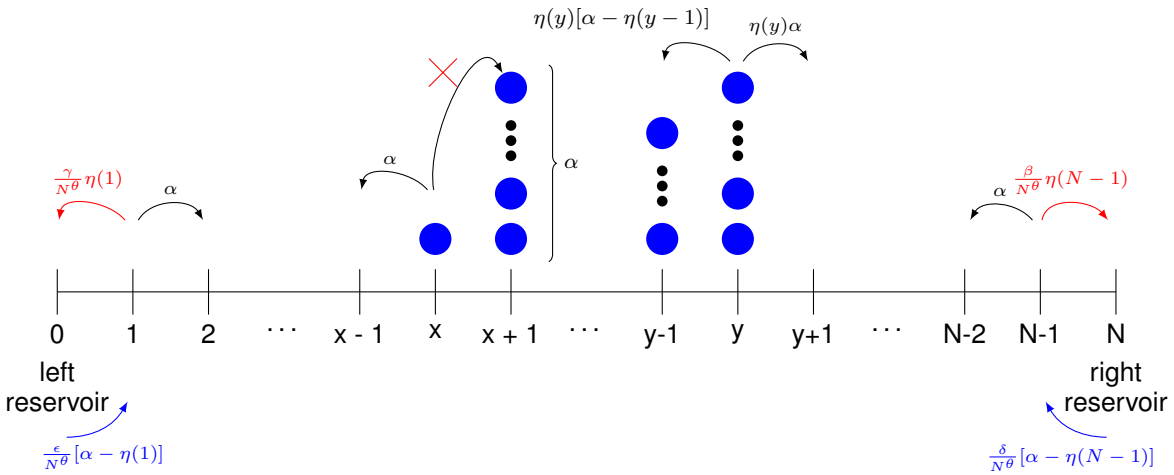

Figure 2.2: Illustration of Kawasaki and Glauber's dynamics for SEP(α) with the rates associated to each particle jump.

from zero in a finite subset of Ω. Above $\{S_t\}_{t\geq 0}$ denotes the Markov semigroup associated to **X** which is a family of linear operators from $\mathcal{F}(\Omega)$ to itself that satisfy the following properties:

1. $S_0 = I$, where I denotes the identity operator in $\mathcal{F}(\Omega)$;

2. $t \to S_t f$ is right continuous, for all $f \in \mathcal{F}(\Omega)$;

3. $S_{t+s} = S_t S_s$, for all $s, t \geq 0$;

4. $S_t id = id$, for all $t \geq 0$ where id is defined as, for all $\eta \in \Omega$, $id(\eta) = \eta$;

5. $S_t f \geq S_t g$, for all $t \geq 0$ and for all $f, g \in \mathcal{F}(\Omega)$ such that, for all $\eta \in \Omega$, $f(\eta) \geq g(\eta)$.

In our case $\Omega = \Omega_N^{Ex}$ and we define the infinitesimal Markov Generator for SEP(α), that we will denote by $\mathcal{L}_\alpha^{Ex}$, as

$$\mathcal{L}_\alpha^{Ex} f(\eta) = \mathcal{L}_{l,\alpha}^{Ex} f(\eta) + \mathcal{L}_{bulk,\alpha}^{Ex} f(\eta) + \mathcal{L}_{r,\alpha}^{Ex} f(\eta), \tag{2.7}$$

for all $\eta \in \Omega_N^{Ex}$ and $f \in \mathcal{F}(\Omega_N^{Ex})$, where

$$\begin{aligned}
\mathcal{L}_{l,\alpha}^{Ex} f(\eta) &= \frac{\gamma}{N^\theta}\eta(1)\Big\{f(\eta^{1,0}) - f(\eta)\Big\} + \frac{\epsilon}{N^\theta}[\alpha - \eta(1)]\Big\{f(\eta^{0,1}) - f(\eta)\Big\}, \\
\mathcal{L}_{bulk,\alpha}^{Ex} f(\eta) &= \sum_{x=1}^{N-2} \{\mathcal{L}_{x,x+1} f(\eta) + \mathcal{L}_{x+1,x} f(\eta)\} \\
&= \sum_{x=1}^{N-2} \Big[c_{x,x+1}(\eta)\Big\{f(\eta^{x,x+1}) - f(\eta)\Big\} + c_{x+1,x}(\eta)\Big\{f(\eta^{x+1,x}) - f(\eta)\Big\}\Big],
\end{aligned} \tag{2.8}$$

where $c_{x,x+1}(\eta) := \eta(x)[\alpha - \eta(x+1)]$ and $c_{x+1,x}(\eta) := \eta(x+1)[\alpha - \eta(x)]$, and

$$\mathcal{L}_{r,\alpha}^{Ex} f(\eta) = \frac{\beta}{N^\theta}\eta(N-1)\Big\{f(\eta^{N-1,N}) - f(\eta)\Big\} + \frac{\delta}{N^\theta}[\alpha - \eta(N-1)]\Big\{f(\eta^{N,N-1}) - f(\eta)\Big\}.$$

2.2.1 Reversible Measure

A natural question that we should ask every time we define an interacting particle system, is about its invariant measures: do they exist? In the next definition, we recall what is a stationary (also called invariant) measure.

Definition 2.2.1 (Stationary Measure)**.** *Let **X** be a continuous time Markov process with state space Ω, Markov semigroup $(S_t)_{t\geq 0}$ and infinitesimal Markov generator $\mathcal{L}$ with dense domain $\mathcal{D}(\mathcal{L})$. We say that a probability measure μ is a stationary measure if it is invariant under the action of S_t, meaning that, for all $t \geq 0$, $\mu = \mu S_t$, where μS_t is the distribution at time t of the process **X**, or equivalently, if $\mu\mathcal{L} = 0$, i.e.*

$$\forall f \in \mathcal{D}(\mathcal{L}) \quad \int_\Omega \mathcal{L}f(\eta)d\mu(\eta) = 0. \tag{2.9}$$

One important subset of the set of all probability measures is the set of the reversible measures.

Definition 2.2.2 (Reversible Measure)**.** *Let **X** be a continuous time Markov process with state space Ω and infinitesimal Markov generator $\mathcal{L}$ with dense domain $\mathcal{D}(\mathcal{L})$. We say that a probability measure μ defined in Ω is reversible if, for all $f, g \in \mathcal{D}(\mathcal{L})$,*

$$\int_\Omega \mathcal{L}f(\eta)g(\eta)d\mu = \int_\Omega f(\eta)\mathcal{L}g(\eta)d\mu, \quad \textit{or equivalently,} \quad \langle \mathcal{L}f, g\rangle_\mu = \langle f, \mathcal{L}g\rangle_\mu, \tag{2.10}$$

where $\langle\cdot,\cdot\rangle_\mu$ is the usual inner product in $L^2(\Omega,\mu)$.

If Ω is countable, then the previous condition is equivalent to the detailed balance equation

$$\mu(\eta)\mathcal{L}(\eta,\xi) = \mu(\xi)\mathcal{L}(\xi,\eta), \quad \textit{for every } \eta,\xi \in \Omega \textit{ and } \eta \neq \xi. \tag{2.11}$$

Remark 2.2.1. *From (2.10), it is clear that saying that a probability measure is reversible is the same as asking $\mathcal{L}$ to be a symmetric operator on $L^2(\Omega,\mu)$.*

In the next proposition (Remark 2.1 of [4]), we recall what is the relationship between invariant and reversible measures.

Proposition 2.2.1 (Reversibility implies invariance)**.** *Any reversible probability measure μ defined in a countable space Ω with respect to the Markov generator $\mathcal{L}$ is a stationary measure for $\mathcal{L}$.*

Proof. Since Ω is countable, in order to prove invariance we need to check that, for every $\eta \in \Omega$, $\sum_{\xi\in\Omega} \mu(\xi)\mathcal{L}(\xi,\eta) = 0$. Since μ is reversible,

$$\sum_{\xi\in\Omega} \mu(\xi)\mathcal{L}(\xi,\eta) = \mu(\eta)\mathcal{L}(\eta,\eta) + \sum_{\substack{\xi\in\Omega\\ \xi\neq\eta}} \mu(\eta)\mathcal{L}(\eta,\xi) = \mu(\eta)\underbrace{\sum_{\xi\in\Omega} \mathcal{L}(\eta,\xi)}_{=0} = 0,$$

where the first equality follows from equation (2.11) and the last is a consequence that the generator of

a continuous time Markov chain $\{X_t\}_{t\geq 0}$ with a countable state space has the form

$$\mathcal{L}f(\eta) = \sum_{\xi\in\Omega} c(\eta,\xi)[f(\xi) - f(\eta)] = \sum_{\xi\in\Omega} L(\eta,\xi)f(\xi),$$

for every $\eta \in \Omega$, where $c(\eta,\xi)$ represent the transition rate of going from η to ξ and has the properties

$$c(\eta,\xi) = L(\eta,\xi) \geq 0 \text{ and } \sum_{\substack{\xi\in\Omega \\ \xi\neq\eta}} c(\eta,\xi) = c(\eta,\eta).$$

(see Subsection 2.1.1 of [4]). □

So, to find the invariant measures of the SEP(α), it is easier to start by looking for reversible measures. In the next theorem, for a special choice of the parameters $\epsilon, \gamma, \beta, \delta$, we characterize the unique reversible measure for SEP(α) (Proposition 3.2 of [1]). The proof we present here follows a similar strategy to the one presented in [6], for the case $\alpha = 1$. An alternative proof can be done using the detailed balance equations, as in [4], Lemma 2.1, where it is demonstrated for the bulk's dynamics only. Remark that, since we are working with a countable (in fact, finite) state space, the integrals that will appear below should be interpreted as sums in Ω_N^{Ex}.

Theorem 2.2.1. *If*

$$\frac{\epsilon}{\epsilon+\gamma} = \frac{\delta}{\delta+\beta} = \rho, \tag{2.12}$$

then the reversible measure for SEP(α), denoted by $\mu^{SEP(\alpha)}$, is given by homogeneous product measures with marginals Binomial distributions with parameter $\alpha \in \mathbb{N}$ and $\rho \in (0,1)$, i.e.,

$$\mu^{SEP(\alpha)}(\eta) = \prod_{x=1}^{N-1} \binom{\alpha}{\eta(x)} \rho^{\eta(x)}(1-\rho)^{\alpha-\eta(x)}. \tag{2.13}$$

Proof. To simplify notation, we will use μ instead of $\mu^{SEP(\alpha)}$.

We want to show that the (generalized) SEP(α) generator is self-adjoint with respect to μ, namely: for every $f, g \in \mathcal{F}(\Omega_N^{Ex})$, $\langle \mathcal{L}_\alpha^{Ex} f, g\rangle_\mu = \langle f, \mathcal{L}_\alpha^{Ex} g\rangle_\mu$. Expanding the left-hand side of last identity, we get

$$\langle \mathcal{L}_\alpha^{Ex} f, g\rangle_\mu = \underbrace{\int_{\eta\in\Omega_N^{Ex}} \mathcal{L}_{l,\alpha}^{Ex} f(\eta) g(\eta) d\mu}_{(\star)} + \underbrace{\int_{\eta\in\Omega_N^{Ex}} \mathcal{L}_{bulk,\alpha}^{Ex} f(\eta) g(\eta) d\mu}_{(\star\star)} + \underbrace{\int_{\eta\in\Omega_N^{Ex}} \mathcal{L}_{r,\alpha}^{Ex} f(\eta) g(\eta) d\mu}_{(\star\star\star)}.$$

To prove reversibility, it is enough to show that $\mathcal{L}_{l,\alpha}^{Ex}$, $\mathcal{L}_{r,\alpha}^{Ex}$ and $\mathcal{L}_{bulk,\alpha}^{Ex}$ are self-adjoint with respect to μ. First, let us compute $\mu(\eta^{0,1})$, $\mu(\eta^{1,0})$, $\mu(\eta^{x,x+1})$, $\mu(\eta^{x+1,x})$, $\mu(\eta^{N-1,N})$ and $\mu(\eta^{N-1,N})$ (for $x = 1, ..., N-2$).

Regarding the definition of $\eta^{0,1}$ in equation (2.2), we have:

- if $\eta(1) \leq \alpha - 1$, then $\mu(\eta^{0,1}) = \rho^{1+\sum_{i=1}^{N-1}\eta(i)}(1-\rho)^{(N-1)\alpha-1-\sum_{i=1}^{N-1}\eta(i)} \frac{\alpha-\eta(1)}{\eta(1)+1} \prod_{i=1}^{N-1} \binom{\alpha}{\eta(i)}$. Rearranging the previous expression, we get that

$$(1-\rho)\eta^{0,1}(1)\mu(\eta^{0,1}) = \rho(\alpha - \eta(1))\mu(\eta). \tag{2.14}$$

- if $\eta(1) = \alpha$, then $\mu(\eta^{0,1}) = \mu(\eta)$.

Analogously, using (2.5),

- if $\eta(N-1) \leq \alpha - 1$, then $(1-\rho)\eta^{N,N-1}(N-1)\mu(\eta^{N,N-1}) = \rho(\alpha - \eta(N-1))\mu(\eta)$.

- if $\eta(N-1) = \alpha$, then $\mu(\eta^{N,N-1}) = \mu(\eta)$.

For every $x \in \{1, ..., N-2\}$, taking into account equation (2.1), we know that

- if $\eta(x) \neq 0$ and $\eta(x+1) \neq \alpha$, then

$$\mu(\eta^{x,x+1}) = \rho^{\sum_{i=1}^{N-1}\eta(i)}(1-\rho)^{(N-1)\alpha - \sum_{i=1}^{N-1}\eta(i)} \frac{[\alpha - \eta(x+1)]\eta(x)}{[\alpha - \eta(x)+1][\eta(x+1)+1]} \prod_{i=1}^{N-1} \binom{\alpha}{\eta(i)},$$

and, therefore,

$$[\alpha - \eta^{x,x+1}(x)]\eta^{x,x+1}(x+1)\mu(\eta^{x,x+1}) = [\alpha - \eta(x+1)]\eta(x)\mu(\eta). \tag{2.15}$$

- if $\eta(x) = 0$ or $\eta(x+1) = \alpha$, then $\mu(\eta^{x,x+1}) = \mu(\eta)$.

By similar computations, for any $x \in \{1, ..., N-2\}$:

- if $\eta(x+1) \neq 0$ and $\eta(x) \neq \alpha$, then

$$[\alpha - \eta^{x+1,x}(x+1)]\eta^{x,x+1}(x)\mu(\eta^{x+1,x}) = [\alpha - \eta(x)]\eta(x+1)\mu(\eta). \tag{2.16}$$

- if $\eta(x+1) = 0$ or $\eta(x) = \alpha$, then $\mu(\eta^{x+1,x}) = \mu(\eta)$.

For $\mu(\eta^{1,0})$, recalling the definition of $\eta^{1,0}$ in (2.3), we have:

- if $\eta(1) \neq 0$, then $\mu(\eta^{1,0}) = \rho^{\sum_{i=1}^{N-1}\eta(i)-1}(1-\rho)^{(N-1)\alpha+1-\sum_{i=1}^{N-1}\eta(i)} \frac{\eta(1)}{\alpha-\eta(1)+1} \prod_{i=1}^{N-1} \binom{\alpha}{\eta(i)}$. Thus,

$$\rho[\alpha - \eta^{1,0}(1)]\mu(\eta^{1,0}) = (1-\rho)\eta(1)\mu(\eta). \tag{2.17}$$

- if $\eta(1) = 0$, then $\mu(\eta^{1,0}) = \mu(\eta)$.

Similarly, for $\mu(\eta^{N-1,N})$, using (2.4), we get:

- if $\eta(N-1) \neq 0$, then $\rho[\alpha - \eta^{N-1,N}(N-1)]\mu(\eta^{N-1,N}) = (1-\rho)\eta(N-1)\mu(\eta)$.

- if $\eta(N-1) = 0$, then $\mu(\eta^{N-1,N}) = \mu(\eta)$.

Starting by ($\star$), let us decompose Ω_N^{Ex} in to the hyper-planes Ω^i, for $i \in \{0, \ldots, \alpha\}$, i.e., $\Omega_N^{Ex} = \bigcup_{i=0}^{\alpha} \Omega^i$, where $\Omega^i = \{\eta \in \Omega_N^{Ex} : \eta(1) = i\}$. Each hyper-plane Ω^i fixes the number of particles at site 1 to be equal to i, where $i \in \{0, \ldots, \alpha\}$.

- If $i \in \{1, \dots, \alpha - 1\}$, then

$$\begin{aligned}\int_{\eta\in\Omega^i} \mathcal{L}^{Ex}_{l,\alpha} f(\eta)g(\eta)d\mu &= \int_{\eta\in\Omega^i} \frac{\gamma}{N^\theta}\eta(1)f(\eta^{1,0})g(\eta)d\mu + \int_{\eta\in\Omega^i} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta^{0,1})g(\eta)d\mu \\ &\quad - \int_{\eta\in\Omega^i} \frac{\gamma}{N^\theta}\eta(1)f(\eta)g(\eta)d\mu - \int_{\eta\in\Omega^i} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu \\ &= \int_{\xi=\eta^{1,0}\in\Omega^{i-1}} \frac{\gamma}{N^\theta}\frac{\rho}{1-\rho}[\alpha-\xi(1)]f(\xi)g(\xi^{0,1})d\mu(\xi) + \int_{\xi=\eta^{0,1}\in\Omega^{i+1}} \frac{\epsilon}{N^\theta}\frac{1-\rho}{\rho}\xi(1)f(\xi)g(\xi^{1,0})d\mu(\xi) \\ &\quad - \int_{\eta\in\Omega^i} \frac{\gamma}{N^\theta}\eta(1)f(\eta)g(\eta)d\mu - \int_{\eta\in\Omega^i} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu,\end{aligned}$$

  where the last equality is obtained by changing variables on the first two integrals of the right hand-side of the first equality, noting that $\{\eta^{0,1}\}^{1,0} = \eta$ and using (2.14) and (2.17). Since $\rho = \frac{\epsilon}{\epsilon+\gamma}$, then, $\frac{\epsilon[1-\rho]}{\rho} = \gamma$, $\frac{\gamma\rho}{1-\rho} = \epsilon$ and

$$\begin{aligned}\int_{\eta\in\Omega^i} \mathcal{L}^{Ex}_{l,\alpha} f(\eta)g(\eta)d\mu &= \int_{\xi\in\Omega^{i-1}} \frac{\epsilon}{N^\theta}[\alpha-\xi(1)]f(\xi)g(\xi^{0,1})d\mu(\xi) + \int_{\xi\in\Omega^{i+1}} \frac{\gamma}{N^\theta}\xi(1)f(\xi)g(\xi^{1,0})d\mu(\xi) \\ &\quad - \int_{\eta\in\Omega^i} \frac{\gamma}{N^\theta}\eta(1)f(\eta)g(\eta)d\mu - \int_{\eta\in\Omega^i} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu.\end{aligned} \tag{2.18}$$

- if $i = 0$, for the same choice of ρ, we get

$$\begin{aligned}\int_{\eta\in\Omega^0} \mathcal{L}^{Ex}_{l,\alpha} f(\eta)g(\eta)d\mu &= \int_{\eta\in\Omega^0} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta^{0,1})g(\eta)d\mu - \int_{\eta\in\Omega^0} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu \\ &= \int_{\xi=\eta^{0,1}\in\Omega^1} \frac{\epsilon}{N^\theta}\frac{1-\rho}{\rho}\xi(1)f(\xi)g(\xi^{1,0})d\mu(\xi) - \int_{\eta\in\Omega^0} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu \\ &= \int_{\xi\in\Omega^1} \frac{\gamma}{N^\theta}\xi(1)f(\xi)g(\xi^{1,0})d\mu(\xi) - \int_{\eta\in\Omega^0} \frac{\epsilon}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu.\end{aligned} \tag{2.19}$$

- if $i = \alpha$, also using ρ as above, we get

$$\begin{aligned}\int_{\eta\in\Omega^\alpha} \mathcal{L}^{Ex}_{l,\alpha} f(\eta)g(\eta)d\mu &= \int_{\eta\in\Omega^\alpha} \frac{\gamma}{N^\theta}\eta(1)f(\eta^{1,0})g(\eta)d\mu - \int_{\eta\in\Omega^\alpha} \frac{\gamma}{N^\theta}\eta(1)f(\eta)g(\eta)d\mu \\ &= \int_{\xi=\eta^{1,0}\in\Omega^{\alpha-1}} \frac{\gamma}{N^\theta}\frac{\rho}{1-\rho}[\alpha-\xi(1)]f(\xi)g(\xi^{0,1})d\mu(\xi) - \int_{\eta\in\Omega^\alpha} \frac{\gamma}{N^\theta}\eta(1)f(\eta)g(\eta)d\mu \\ &= \int_{\xi\in\Omega^{\alpha-1}} \frac{\epsilon}{N^\theta}[\alpha-\xi(1)]f(\xi)g(\xi^{0,1})d\mu(\xi) - \int_{\eta\in\Omega^\alpha} \frac{\gamma}{N^\theta}[\alpha-\eta(1)]f(\eta)g(\eta)d\mu.\end{aligned} \tag{2.20}$$

Since

$$\int_{\xi\in\Omega^\alpha} \frac{\epsilon}{N^\theta}[\alpha-\xi(1)]f(\xi)g(\xi^{0,1})d\mu(\xi) = \int_{\xi\in\Omega^0} \frac{\gamma}{N^\theta}\xi(1)f(\xi)g(\xi^{1,0})d\mu(\xi) = 0, \tag{2.21}$$

then, putting (2.18), (2.19), (2.20) and (2.21) together in ($\star$) and noting that the decomposition of Ω is with disjoint sets, we get that

$$\int_{\eta\in\Omega} \mathcal{L}^{Ex}_{l,\alpha} f(\eta)g(\eta)d\mu = \int_{\eta\in\Omega} f(\eta)\mathcal{L}^{Ex}_{l,\alpha} g(\eta)d\mu. \tag{2.22}$$

Analogously, expanding ($\star\star$) and decomposing Ω_N^{Ex}, for each fixed $x \in \{1, \dots, N-2\}$, as $\Omega_N^{Ex} = \bigcup_{i,j=0}^{\alpha} \Omega_x^{i,j}$, where $\Omega_x^{i,j} = \{\eta \in \Omega_N^{Ex} \mid \eta(x) = i \text{ and } \eta(x+1) = j\}$ (these sets, for each $i, j \in \{0, \dots, \alpha\}$, fix the number of particles at x to be i and at $x+1$ to be j), we have that

$$\begin{aligned}
&\int_{\eta \in \Omega_N^{Ex}} \mathcal{L}_{bulk,\alpha}^{Ex} f(\eta) g(\eta) d\mu \\
&= \sum_{x=1}^{N-2} \sum_{i,j=0}^{\alpha} \Big\{ \int_{\eta \in \Omega_x^{i,j}} \eta(x)[\alpha - \eta(x+1)] f(\eta^{x,x+1}) g(\eta) d\mu + \int_{\eta \in \Omega_x^{i,j}} \eta(x+1)[\alpha - \eta(x)] f(\eta^{x+1,x}) g(\eta) d\mu \Big\} \\
&\quad - \sum_{x=1}^{N-2} \Big\{ \int_{\eta \in \Omega_N^{Ex}} \eta(x)[\alpha - \eta(x+1)] f(\eta) g(\eta) d\mu - \int_{\eta \in \Omega_N^{Ex}} \eta(x+1)[\alpha - \eta(x)] f(\eta) g(\eta) d\mu \Big\}. \qquad (2.23)
\end{aligned}$$

For every $x \in \{1, \dots, N-2\}$:

- if $i \in \{1, \dots, \alpha\}$ and $j \in \{0, \dots, \alpha-1\}$, then, changing variables to $\xi = \eta^{x,x+1}$ and using (2.15) and (2.1), we obtain

$$\int_{\eta \in \Omega_x^{i,j}} \eta(x)[\alpha - \eta(x+1)] f(\eta^{x,x+1}) g(\eta) d\mu = \int_{\xi = \eta^{x,x+1} \in \Omega_x^{i-1,j+1}} \xi(x+1)[\alpha - \xi(x)] f(\xi) g(\xi^{x+1,x}) d\mu.$$

  Since, for every $i, j \in \{0, \dots, \alpha\}$,

$$\int_{\eta \in \Omega_x^{0,j}} \eta(x)[\alpha - \eta(x+1)] f(\eta^{x,x+1}) g(\eta) d\mu = \int_{\eta \in \Omega_x^{i,\alpha}} \eta(x)[\alpha - \eta(x+1)] f(\eta^{x,x+1}) g(\eta) d\mu = 0$$

  and

$$\int_{\xi \in \Omega_x^{i,0}} \xi(x+1)[\alpha - \xi(x)] f(\xi) g(\xi^{x+1,x}) d\mu = \int_{\xi \in \Omega_x^{\alpha,j}} \xi(x+1)[\alpha - \xi(x)] f(\xi) g(\xi^{x+1,x}) d\mu = 0,$$

  it is clear that

$$\int_{\eta \in \Omega_N^{Ex}} \eta(x)[\alpha - \eta(x+1)] f(\eta^{x,x+1}) g(\eta) d\mu = \int_{\xi = \eta^{x,x+1} \in \Omega_N^{Ex}} \xi(x+1)[\alpha - \xi(x)] f(\xi) g(\xi^{x+1,x}) d\mu. \qquad (2.24)$$

- if $i \in \{0, \dots, \alpha-1\}$ and $j \in \{1, \dots, \alpha\}$, then, changing variables to $\xi = \eta^{x+1,x}$ and using (2.16) and (2.1), we get that

$$\int_{\eta \in \Omega_x^{i,j}} \eta(x+1)[\alpha - \eta(x)] f(\eta^{x+1,x}) g(\eta) d\mu = \int_{\xi = \eta^{x+1,x} \in \Omega_x^{i+1,j-1}} \xi(x)[\alpha - \xi(x+1)] f(\xi) g(\xi^{x,x+1}) d\mu.$$

  Since, for every $i, j \in \{0, \dots, \alpha\}$,

$$\int_{\eta \in \Omega_x^{\alpha,j}} \eta(x+1)[\alpha - \eta(x)] f(\eta^{x+1,x}) g(\eta) d\mu = \int_{\eta \in \Omega_x^{i,0}} \eta(x+1)[\alpha - \eta(x)] f(\eta^{x+1,x}) g(\eta) d\mu = 0$$

  and

$$\int_{\xi\in\Omega_x^{0,j}} \xi(x)[\alpha-\xi(x+1)]f(\xi)g(\xi^{x,x+1})d\mu = \int_{\xi\in\Omega_x^{i,\alpha}} \xi(x)[\alpha-\xi(x+1)]f(\xi)g(\xi^{x,x+1})d\mu = 0,$$

we obtain

$$\int_{\eta\in\Omega_N^{Ex}} \eta(x+1)[\alpha-\eta(x)]f(\eta^{x+1,x})g(\eta)d\mu = \int_{\xi=\eta^{x+1,x}\in\Omega_N^{Ex}} \xi(x)[\alpha-\xi(x+1)]f(\xi)g(\xi^{x,x+1})d\mu. \tag{2.25}$$

Therefore, combining (2.24) and (2.25) and replacing in (2.23), we get that

$$\int_{\eta\in\Omega_N^{Ex}} \mathcal{L}_{bulk,\alpha}^{Ex} f(\eta)g(\eta)d\mu. = \int_{\eta\in\Omega_N^{Ex}} f(\eta)\mathcal{L}_{bulk,\alpha}^{Ex} g(\eta)d\mu. \tag{2.26}$$

Observe that last computation holds independently of the parameter ρ. At last, expanding $(\star\star\star)$ and using the same kind of arguments as we did for the operator $\mathcal{L}_{l,\alpha}^{Ex}$, since $\rho = \frac{\delta}{\delta+\beta}$, then

$$\int_{\eta\in\Omega_N^{Ex}} \mathcal{L}_{r,\alpha}^{Ex} f(\eta)g(\eta)d\mu = \int_{\eta\in\Omega_N^{Ex}} f(\eta)\mathcal{L}_{r,\alpha}^{Ex} g(\eta)d\mu. \tag{2.27}$$

Finally, putting together (2.22), (2.26) and (2.27), the proof is completed. □

Remark 2.2.2. *In the proof of Theorem 2.2.1, we also showed that:*

- *for any $\rho \in (0,1)$, SEP(α) without reservoirs always admits reversible measures which are homogeneous products of Binomial(α,ρ) distributions - Lemma 2.1, [4];*
- *$\mathcal{L}_{l,\alpha}^{Ex}$ (resp. $\mathcal{L}_{r,\alpha}^{Ex}$) is a self-adjoint operator for the inner product taken with respect a product form measure with marginals Binomial(α, $\frac{\epsilon}{\epsilon+\gamma}$) (resp. Binomial($\alpha$, $\frac{\delta}{\delta+\beta}$)) distributions.*

In the previous result, we have characterized the unique reversible measure of SEP(α) for a specific choice of the parameters $\epsilon,\gamma,\delta,\beta \geq 0$. But what can we say about the invariant measures for the other possible values for ϵ,γ,δ and β? The existence and uniqueness of these measures will be discussed at the end of this chapter.

Motivated by this lack of information on the invariant measures, we will use stochastic duality to obtain results on Chapter 4 about some relevant functions (this will be made clear later on) with the aim of obtaining important information on the invariant measure.

2.2.2 The special case $\alpha = 1$

The case $\alpha = 1$ has been intensively study in the literature (see [13] for a scaling limit perspective) and here additional tools are available, such as the Matrix Product Ansatz Method (MPA) (see for example [17], [18] and [2]). In this context, it is important to note that now we only allow one particle per site, and therefore, the exclusion rule does not permit the system to have piles of particles at any site $x \in \Lambda_N$. Also, if a particle at site x starts on the left-hand side of another particle at site y (with $x < y$),

as time evolves, they cannot switch places. Thus, given an initial configuration, there is an order for the particles on the bulk that cannot be changed through time while they are on the bulk. We will use the binary notation, 0 or 1, for the occupation variables, to say if, given a configuration $\eta \in \Omega_N^{Ex}$ (taking $\alpha = 1$), a point in Λ_N is empty or full, respectively.

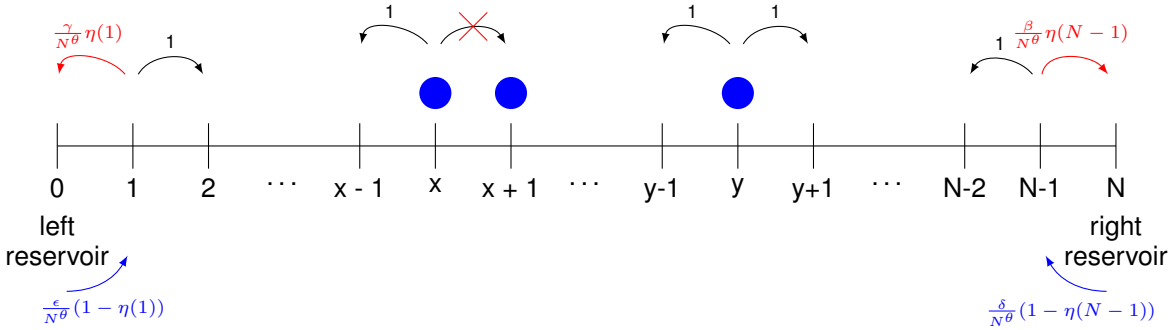

Figure 2.3: SEP(1) dynamics.

Observe that, for $\alpha = 1$, for every $x \in \{1, \dots, N-1\}$, $\eta(x)^2 = \eta(x)$, $\eta^{x,x+1}(x) = \eta(x+1) = \eta^{x+1,x}(x)$, $\eta^{x,x+1}(x+1) = \eta(x) = \eta^{x+1,x}(x+1)$, $\eta^{0,1}(1) = 1-\eta(1) = \eta^{1,0}(1)$, $\eta^{N,N-1}(1) = 1-\eta(N-1) = \eta^{N-1,N}(1)$. This implies that here it does not matter the order of the jump since the resulting configuration is going to be the same. So, even though, we defined SEP(1) with a jump dynamics, by the previous observation, the same process can be defined with a flip dynamics (see Section 2.3 of [6]).

It is an immediate consequence of Theorem 2.2.1 that SEP(1) admits a unique reversible measure, $\mu^{SEP(1)}$, which is given by homogeneous product measures with marginals Binomial(1,ρ) $\equiv$ Bernoulli(ρ) distributions, where $\rho \in (0,1)$ depends on ϵ, γ, δ and β.

One of the advantages of using the flip dynamics for this model is that we no longer have to distinguish between a jump to the left or to the right, simplifying a lot the computations for the proof of Theorem 2.2.1 in this special case.

2.3 Symmetric Simple Inclusion in contact with reservoirs - SIP(α)

Let $\alpha \in \mathbb{R}^+$. In contrast to what we had in the first model, in the Symmetric Simple Inclusion Process, SIP(α), we do not have an upper bound on the total number of particles per site on the *bulk*. Therefore, for this model, we define a new space of configurations

$$\Omega_N^{In} := \{\eta = (\eta_1, ..., \eta_{N-1}) \mid \eta_i \in \mathbb{N}_0 \text{ for all } i = 1, ..., N-1\} = {\mathbb{N}_0}^{\Lambda_N}$$

and interpret $\eta \in \Omega_N^{In}$ also as a function $\eta : \Lambda_N \to \mathbb{N}_0$ where, for every $x \in \Lambda_N$ and $j \in \mathbb{N}_0$, $\eta(x) = j$, if on the bulk at site x exists j particles. As above, we will call, for every $x \in \Lambda_N$, $\eta(x)$ the occupation variable at site x. We also define the change of configurations (not fearing confusion, we will keep the

same notation that we used in SEP(α)[3]): given $\eta \in \Omega_N^{In}$, for all $x, y \in \Lambda_N$ with $|x-y|=1$,

$$\eta^{x,y}(z) = \begin{cases} \eta(z), \text{ if } z \in \Lambda_N \setminus \{x,y\}, \text{ or , if } \eta(x)=0 \text{ and } z=x, \\ \eta(x)-1, \text{ if } z=x \text{ and } \eta(x) \geq 1, \\ \eta(y)+1, \text{ if } z=y, \end{cases}$$

$$\eta^{0,1}(z) = \begin{cases} \eta(z), \text{ if } z \in \{2,...,N-1\}, \\ \eta(1)+1, \text{ if } z=1, \end{cases} \qquad \eta^{N,N-1}(z) = \begin{cases} \eta(z), \text{ if } z \in \{1,...,N-2\}, \\ \eta(N-1)+1, \text{ if } z=N-1, \end{cases}$$

$$\eta^{1,0}(z) = \begin{cases} \eta(z), \text{ if } z \in \{2,...,N-1\}, \\ \eta(1), \text{ if } z=1 \text{ and } \eta(1)=0, \\ \eta(1)-1, \text{ if } z=1 \text{ and } \eta(1)\geq 1, \end{cases} \qquad \eta^{N-1,N}(z) = \begin{cases} \eta(z), \text{ if } z \in \{1,...,N-2\}, \\ \eta(N-1), \text{ if } z=N-1 \text{ and } \eta(N-1)=0, \\ \eta(N-1)-1, \text{ if } z=N-1 \text{ and } \eta(N-1)\geq 1. \end{cases}$$

The Symmetric Simple Inclusion Process in contact with reservoirs, SIP(α), is a continuous time Markov Process which is *symmetric* and *simple* (with the same meaning as for SEP(α)). The points in Λ_N are occupation sites that can have any number of particles and, as above, all particles have to wait a certain time, which is exponentially distributed with parameter 1, to jump to its nearest left or right neighbor. We use here the same construction with the *Poisson clocks* that we used for SEP(α). The main difference from SEP(α) is that we replace the *exclusion rule* by an *inclusion rule*, so that, on the bulk, the jump rate of particles increases as much as more particles are at the arrival site. Thus, they tend to create piles and stay close to each other, while in SEP(α), particles tended to be apart and be repelled by each other.

Given a configuration $\eta \in \Omega_N^{In}$, a particle at site $x \in \Lambda_N \setminus \{1, N-1\}$ has rate $p(1)\eta(x)[\alpha+\eta(x+1)]$ to jump to site $x+1$ and rate $p(-1)\eta(x)[\alpha+\eta(x-1)]$ to jump to site $x-1$. Remark that these rates differ from the rates for SEP(α) by the change of a minus sign into a plus. As above, we will also consider the (boundary) points 0 and N as left and right reservoirs, respectively, that can absorb and inject particles on the system at the points 1 and $N-1$. We define the jump rates for a particle starting at 1 (respectively, $N-1$) as, fixing a configuration $\eta \in \Omega_N^{In}$, the rate to jump out of the system from the left (respectively, right) reservoir is $p(-1)\frac{\gamma}{N^\theta}\eta(1)$ (respectively, $p(1)\frac{\beta}{N^\theta}\eta(N-1)$), with $\gamma, \beta > 0$, and to jump to 2 (respectively, $N-2$) is $p(1)\eta(1)[\alpha+\eta(2)]$ (respectively, $p(-1)\eta(N-1)[\alpha+\eta(N-2)]$). We define the injection rates of the left and right reservoirs as $p(1)\frac{\epsilon}{N^\theta}[\alpha+\eta(1)]$ and $p(-1)\frac{\delta}{N^\theta}[\alpha+\eta(N-1)]$, respectively, with $\epsilon, \delta > 0$. Since we are considering the process to be symmetric, this means that we choose $p(1)=p(-1)=1$. See Figure 2.4 for a summary.

We can also describe this dynamics by the infinitesimal Markov generator for SIP(α), that we denote by $\mathcal{L}_\alpha^{In}$. We define $\mathcal{L}_\alpha^{In}$ with domain $\mathcal{D}(\mathcal{L}_\alpha^{In}) \subset \mathcal{F}(\Omega_N^{In})$ as

$$\mathcal{L}_\alpha^{In} f(\eta) = \mathcal{L}_{l,\alpha}^{In} f(\eta) + \mathcal{L}_{bulk,\alpha}^{In} f(\eta) + \mathcal{L}_{r,\alpha}^{In} f(\eta), \tag{2.28}$$

[3]Let us keep in mind that every time we want to compare SEP(α) with SIP(α), we must think that we are restricting α to the natural numbers.

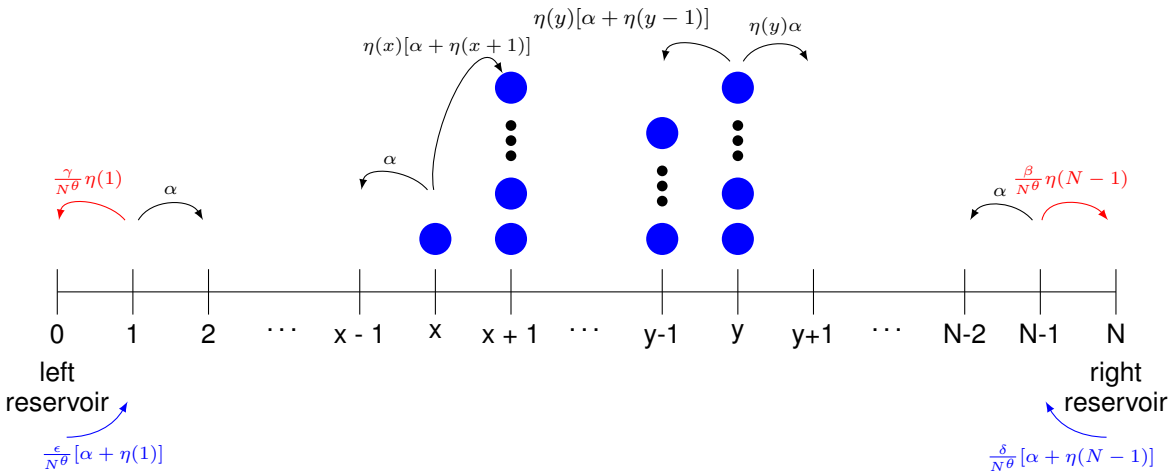

Figure 2.4: SIP(α) dynamics.

for all $\eta \in \Omega_N^{In}$ and $f \in \mathcal{D}(\mathcal{L}_\alpha^{In})$, where

$$\begin{aligned}\mathcal{L}_{l,\alpha}^{In} f(\eta) &= \frac{\gamma}{N^\theta}\eta(1)\Big\{f(\eta^{1,0}) - f(\eta)\Big\} + \frac{\epsilon}{N^\theta}[\alpha+\eta(1)]\Big\{f(\eta^{0,1}) - f(\eta)\Big\},\\ \mathcal{L}_{bulk,\alpha}^{In} f(\eta) &= \sum_{x=1}^{N-2} c_{x,x+1}\Big\{f(\eta^{x,x+1}) - f(\eta)\Big\} + c_{x+1,x}\Big\{f(\eta^{x+1,x}) - f(\eta)\Big\},\end{aligned} \tag{2.29}$$

defining here $c_{x,x+1} = \eta(x)[\alpha+\eta(x+1)]$ and $c_{x+1,x} = \eta(x+1)[\alpha+\eta(x)]$,

$$\mathcal{L}_{r,\alpha}^{In} f(\eta) = \frac{\beta}{N^\theta}\eta(N-1)\Big\{f(\eta^{N-1,N}) - f(\eta)\Big\} + \frac{\delta}{N^\theta}[\alpha+\eta(N-1)]\Big\{f(\eta^{N,N-1}) - f(\eta)\Big\}.$$

Here θ has the same meaning as before, namely, it tunes the strength of the interaction between the bulk and the boundary. We remark that there is a change of sign in $c_{x,x+1}$ and in $c_{x+1,x}$ from SEP(α) to SIP(α).

2.3.1 Reversible Measure

Analogously to what we have for SEP(α), SIP(α) also admits a unique reversible stationary measure when $\frac{\epsilon}{\gamma} = \frac{\delta}{\beta}$. The proof of reversibility for the generator of the bulk's dynamics can be found in [4], Lemma 2.2, with the approach using detailed balance equations. For the proof including the reservoirs' dynamics, one could follow the same strategy as in the proof of Theorem 2.2.1 or Proposition 3.2 of [1].

Theorem 2.3.1. *If*

$$\frac{\epsilon}{\gamma} = \frac{\delta}{\beta} = \rho \tag{2.30}$$

with $\epsilon < \gamma$ and $\delta < \beta$, then, the reversible stationary measure for SIP(α), $\mu^{SIP(\alpha)}$, is given by homogeneous product measures with marginals Negative Binomials distributions with parameters $\alpha > 0$ and $\rho \in (0,1)$, i.e.,

$$\mu^{SIP(\alpha)}(x) = \prod_{x=1}^{N-1} \frac{\Gamma(\alpha+x)}{\Gamma(\alpha)x!}\rho^x(1-\rho)^\alpha. \tag{2.31}$$

2.4 Equilibrium and non-equilibrium invariant measures

As we saw above, for a special choice of the parameters ϵ, γ, β and δ, the reversible measure that we found for SEP(α), in particular SEP(1), and for SIP(α) is of product form and it is completely characterized. Since every reversible measure is invariant, then we know that for this same special choice, there is an invariant measure of product form for each one of our models. But, what if we choose different values for ϵ, γ, β and δ? Can we guarantee existence and uniqueness of an invariant measure for SEP(α) and SIP(α) with open boundary?

2.4.1 Existence and uniqueness for SEP(α)

In what follows, we will give a general sufficient condition for the existence and uniqueness of invariant measures for continuous time Markov processes, that we will apply to SEP(α). Recall the definitions of irreducibility and recurrence (Definition 2.47 and 2.49 of [15]) of a continuous time Markov chain.

Definition 2.4.1 (Irreducibility)**.** *A continuous time Markov chain $\{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with state space Ω is **irreducible** if, for all $\eta, \xi \in \Omega$ and $t > 0$, $P^\eta[X_t = \xi] > 0$, where $P^\eta[X_t = \xi]$ denotes the probability that, starting at time $t = 0$ from the configuration η, the chain has the configuration ξ at time t.*

Definition 2.4.2 (Recurrence)**.** *Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov chain defined on a probability space $(\Omega, \mathcal{F}, P)$ with state space Ω. We say that a state $\eta \in \Omega$ is recurrence if $P^\eta[X_t = \eta \text{ i.o. }] = 1$, where $P^\eta[X_t = \eta \text{ i.o. }]$ represents the probability of, starting at time $t = 0$ from the configuration η, the chain returns to the configuration η an infinite number of times. If this condition is not satisfied, we say that $\eta \in \Omega$ is a transient state. If all states of $\{X_t\}_{t\geq 0}$ are recurrent we say that the chain is recurrent and, if all are transient, we call it transient.*

Remark 2.4.1. *It is known that for continuous time Markov chains, either all states are recurrent or all are transient (Proposition 2.52 of [15]).*

The next result (Proposition 2.59 and 2.61 of [15]) gives us sufficient conditions for the existence and uniqueness of invariant measures for a given continuous time Markov process.

Theorem 2.4.1. *Every irreducible recurrent continuous time Markov chain has a unique non-zero invariant probability measure.*

An irreducible continuous time Markov chain with finite state space Ω is always recurrent. In fact, if it was transient, that would mean that it could only visit each state a finite number of times. In other words, for every $\eta \in \Omega$, there would exist $I_1^\eta, \ldots, I_{k_\eta}^\eta$ finite intervals of time, for some $k_\eta \in \mathbb{N}$, such that for every $t \in \mathbb{R}_0^+ \setminus \bigcup_{i=1}^{k_\eta} I_i^\eta$, $X_t \neq \eta$. Since Ω is finite, if we took $t \in \mathbb{R}_0^+ \setminus \bigcup_{\eta\in\Omega} \bigcup_{i=1}^{k_\eta} I_i^\eta$ (that exists because if not, then this would mean that $\mathbb{R}_0^+$ could be covered with a finite union of intervals, which is impossible since $\mathbb{R}_0^+$ is non-compact for the usual topology), then, we would have that, for every $\eta \in \Omega$, $X_t \neq \eta$. This is clearly impossible, since $\{X_t\}_{t\geq 0}$ has state space Ω and because the chain is irreducible, there

always exists a non-zero probability of starting from any configuration, at time t the chain to be in any other state. Therefore, the chain has to be recurrent.

Corollary 2.4.1. *Every irreducible continuous time Markov chain with finite state space has a unique non-zero invariant probability measure.*

Note that, for each fixed $N \in \mathbb{N}$, SEP(α) defined in Λ_N can be seen as describing a continuous time Markov process $\{X_t\}_{t\geq 0}$ with finite state space Ω_N^{Ex}. Since $\{X_t\}_{t\geq 0}$ is clearly irreducible, then, by Corollary 2.4.1, we know that SEP(α) admits a unique non-zero invariant measure. This means that independently of γ, ϵ, β and δ (that we ask to be non-zero), there always exists a unique non-zero invariant measure for SEP(α).

2.4.2 Existence and uniqueness for SIP(α)

For SIP(α), we will apply a different argument, following Appendix A of [3]. The proof uses a strategy based on duality and on the knowledge of what is the classical duality function between SIP(α) with open boundary and SIP(α) with absorbing boundary. Therefore, we encourage now the reader to continue reading this chapter and, at the end of Chapter 3, with all the tools available, see the proof of this fact in Proposition 3.5.1.

2.4.3 Density of the reservoirs

Let μ_{ss}^l (resp. μ_{ss}^r) be the unique reversible measure for which $\mathcal{L}_{l,\alpha}$ (resp. $\mathcal{L}_{r,\alpha}$) is a self-adjoint operator. As we saw in the proof of Theorem 2.2.1, even if $\frac{\epsilon}{\gamma+\epsilon} \neq \frac{\delta}{\beta+\delta}$, for SEP(α), and $\frac{\epsilon}{\gamma} \neq \frac{\delta}{\beta}$, for SIP($\alpha$), μ_{ss}^l (resp. μ_{ss}^r) is of product form with marginals with Binomial$\left(\alpha, \frac{\epsilon}{\gamma+\epsilon}\right)$ $\left(\text{resp. Binomial } \left(\alpha, \frac{\delta}{\beta+\delta}\right)\right)$ distribution, for SEP(α), and Binomial$\left(\alpha, \frac{\epsilon}{\gamma-\epsilon}\right)$ $\left(\text{resp. Binomial } \left(\alpha, \frac{\delta}{\beta-\delta}\right)\right)$ distribution, for SIP(α).

Denote by $\mathbb{E}_{\mu_{ss}^l}$ (resp. $\mathbb{E}_{\mu_{ss}^r}$) the expectation in Ω_N taken with respect to μ_{ss}^l (resp. $\mathbb{E}_{\mu_{ss}^r}$). Let us define

$$\rho_0 := \mathbb{E}_{\mu_{ss}^l}[\eta(1)] \quad \text{and} \quad \rho_N := \mathbb{E}_{\mu_{ss}^r}[\eta(N-1)], \tag{2.32}$$

where ρ_0 and ρ_N are called "left and right densities", respectively, and represent the average number of particles at sites 1 and $N-1$. The following table, taken from [1], determine ρ_0 and ρ_N for SEP(α) and SIP(α).

Model	ρ_0	ρ_N
SEP(α)	$\alpha\frac{\epsilon}{\gamma+\epsilon}$	$\alpha\frac{\delta}{\beta+\delta}$
SIP(α)	$\alpha\frac{\epsilon}{\gamma-\epsilon}$	$\alpha\frac{\delta}{\beta-\delta}$

Table 2.1: Definition of ρ_0 and ρ_N for SEP(α) and SIP(α).

From (2.32) and Table 2.1, two immediate questions arise:

1. How can we compute ρ_0 and ρ_N?

2. Neglecting the factor α, why do these quantities are related with the ones that restricted the possible values of ϵ, γ, β and δ in Theorems 2.2.1 and 2.3.1 (remark that asking $\rho_0 = \rho_N$ in SIP(α) is the same as asking $\frac{\gamma}{\epsilon} = \frac{\beta}{\delta}$)?

For the first question: to obtain ρ_0 and ρ_N one only needs to observe that ρ_0 and ρ_N are just the expectation of a discrete random variable whose law is μ^l_{ss} and μ^r_{ss}, respectively. To answer the second question, we will introduce two important concepts: equilibrium and non-equilibrium in the stationary state. Physically, we say that our system is in equilibrium if the density of the left and right reservoirs are equal. For us, these densities represent the average number of particles at sites 1 and $N-1$, respectively. Therefore, saying that a system is in equilibrium is the same as saying that $\rho_0 = \rho_N$. On the other hand, if the densities are different, then, physically, this means that there is a flux of particles in the system from the reservoir with higher density to the one with lower density. In this case, the system is said to be out of equilibrium, or equivalently, in non-equilibrium. Keeping the physical nomenclature, we will say that our system is in the non-equilibrium stationary state if $\rho_0 \neq \rho_N$ and the system is taken starting from the stationary measure. For us, stationary will always represent that we are considering the expectation with respect to the invariant measure, that as we saw, always exists. So, recalling Theorems 2.2.1 and 2.3.1, the conditions that restricted the possible values of ϵ, γ, β and δ are equivalent to requiring the system to be in equilibrium. From now on, we will use this nomenclature with the meaning explained above.

Chapter 3

Stochastic Duality

Stochastic duality, or simply, duality, is a probabilistic tool that allows studying some Markov processes. This very useful property is based on extracting information about a given Markov process X by analysing another, Y - its dual - that, in general, is simpler. The link between these two processes is given by what is called a **duality function** (that is not unique). Besides making use of probabilistic concepts, duality also has a strong connection with the Lie Algebras' theory, giving us an algebraic approach to study Markov processes.

In this chapter we will introduce the basic definitions regarding duality, recall some Lie algebra's important definitions and results that we will use to study and prove duality for SEP(α) and SIP(α). We finish this chapter obtaining the expressions for the classical duality functions connecting SEP(α) and SIP(α) with open boundary with their corresponding duals that have absorbing boundary and use the classical duality function to prove existence and uniqueness of an invariant measure for SIP(α).

3.1 Basic Definitions and Results

In this section, we follow [4]. For a good survey on duality, see also [11].

Definition 3.1.1 (Duality via expectation)**.** *Let* $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ *and* $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$ *be two continuous time Markov processes with state spaces* Ω *and* Ω^{dual}*, respectively. We say that the process* $\boldsymbol{Y}$ *is dual of* $\boldsymbol{X}$ *if there exists a function* $D : \Omega \times \Omega^{dual} \to \mathbb{R}$*, which is called* ***duality function****, such that*

$$\mathbb{E}_\eta[D(X_t, Y)] = \mathbb{E}_{\hat{\eta}}[D(X, Y_t)] \tag{3.1}$$

for all $(\eta, \hat{\eta}) \in \Omega \times \Omega^{dual}$ *and for all* $t \geq 0$*. If* $\boldsymbol{X}$ *has as dual process itself, we say that* $\boldsymbol{X}$ *is self-dual.*

From the definition above, it is immediate that, if $\boldsymbol{Y}$ is dual to $\boldsymbol{X}$, then $\boldsymbol{X}$ is dual to $\boldsymbol{Y}$. Using (3.1), we can relate the Markov semigroups and the infinitesimal Markov generators of two dual continuous time Markov processes (recall the definitions of Chapter 2).

Proposition 3.1.1. *Let* $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ *and* $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$ *be two dual continuous time Markov processes with state spaces* Ω *and* Ω^{dual}*, respectively, and with duality function* D*. Let* $\{S_t\}_{t\geq 0}$ *and* $\{S_t^{dual}\}_{t\geq 0}$ *be*

*the Markov semigroups of **X** and **Y**, respectively. Then,*

$$[S_t D(\cdot, \hat{\eta})](\eta) = \left[S_t^{dual} D(\eta, \cdot)\right](\hat{\eta}), \text{ for all } t \geq 0, \eta \in \Omega \text{ and } \hat{\eta} \in \Omega^{dual}. \tag{3.2}$$

*Also, if $\mathcal{L}$ and $\mathcal{L}^{dual}$ are the infinitesimal Markov generators of **X** and **Y** and have dense domain $\mathcal{D}(\mathcal{L})$ and $\mathcal{D}(\mathcal{L}^{dual})$, respectively, we have that*

$$[\mathcal{L} D(\cdot, \hat{\eta})](\eta) = \left[\mathcal{L}^{dual} D(\eta, \cdot)\right](\hat{\eta}), \text{ for all } \eta \in \Omega \text{ and } \hat{\eta} \in \Omega^{dual}, \tag{3.3}$$

assuming that everything is well-defined, i.e.

$$\text{for all } \eta \in \Omega,\ D(\eta, \cdot) \in \mathcal{D}(\mathcal{L}^{dual}) \text{ and, for all } \hat{\eta} \in \Omega^{dual},\ D(\cdot, \hat{\eta}) \in \mathcal{D}(\mathcal{L}). \tag{3.4}$$

Usually, the relationship in equation (3.3) is helpful to find duality relations; while equation (3.1) will be useful for applications.

Under some conditions (that we state in Proposition 3.1.2), we can also define duality for two continuous time Markov processes using condition (3.3).

Definition 3.1.2 (Duality of processes via Markov generators)**.** *Let **X** $= \{X_t\}_{t \geq 0}$ and **Y** $= \{Y_t\}_{t \geq 0}$ be two continuous time Markov processes with state spaces Ω and Ω^{dual}, respectively, and Markov generator $\mathcal{L}$ and $\mathcal{L}^{dual}$, respectively. We say that the process **Y** is dual of **X** if there exists a function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$, a **duality function**, such that (3.3) is satisfied.*

The following proposition gives us the conditions where we have equivalence between the two definitions of duality.

Proposition 3.1.2. *Let **X** $= \{X_t\}_{t \geq 0}$ and **Y** $= \{Y_t\}_{t \geq 0}$ be two continuous time Markov processes with state spaces Ω and Ω^{dual}, respectively. Consider the Markov semigroups of **X** and **Y**, $(S_t)_{\{t \geq 0\}}$ and $(S_t^{dual})_{\{t \geq 0\}}$, respectively, and its Markov generators, $\mathcal{L}$ and $\mathcal{L}^{dual}$, respectively. Suppose that for all $\eta \in \Omega, \hat{\eta} \in \Omega^{dual}$ and $t \geq 0$, $S_t D(\cdot, \hat{\eta}) \in \mathcal{D}(\mathcal{L})$ and $S_t^{dual} D(\eta, \cdot) \in \mathcal{D}(\mathcal{L}^{dual})$. Then, Definitions 3.1.1 and 3.1.2 are equivalent.*

The proof of the previous result can be found in Proposition 1.2 of [11].

From the definitions above, some natural questions emerge:

1. Given a continuous time Markov process **X**, can we find a continuous time Markov process **Y** and a function D that satisfies (3.3)?
2. Given two dual processes, how can we generate different duality functions?

Until the end of this chapter, we will be concerned with answering to these questions for SEP(α) and SIP(α).

Remark 3.1.1. *If Ω and Ω^{dual} are countable sets, then condition (3.3) can be written as:*

$$\begin{aligned} & \forall(\eta,\hat{\eta}) \in \Omega \times \Omega^{dual} \quad \sum_{\xi\in\Omega} \mathcal{L}(\eta,\xi)D(\xi,\hat{\eta}) = \sum_{\hat{\xi}\in\Omega} \mathcal{L}^{dual}(\hat{\eta},\hat{\xi})D(\eta,\hat{\xi}) \\ \Leftrightarrow \quad & \forall(\eta,\hat{\eta}) \in \Omega \times \Omega^{dual} \quad \sum_{\xi\in\Omega} \mathcal{L}(\eta,\xi)D(\xi,\hat{\eta}) = \sum_{\hat{\xi}\in\Omega} D(\eta,\hat{\xi})(\mathcal{L}^{dual})^T(\hat{\xi},\hat{\eta}) \Leftrightarrow \mathcal{L}D = D(\mathcal{L}^{dual})^T, \end{aligned} \tag{3.5}$$

where $\mathcal{L}$ is the rate matrix for the initial process and $\mathcal{L}^{dual}$ is the transpose of the rate matrix of the dual process.

In the next proposition, we give a sufficient condition for a Markov process to be self-dual (Lemma 5.1 of [4]).

Theorem 3.1.1. *Suppose Ω is countable. Then, a continuous time Markov process $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ with state space Ω that admits a reversible measure μ is always self-dual with self-duality function D given by*

$$D(\eta,\hat{\eta}) = \frac{\mathbb{1}_{\eta=\hat{\eta}}}{\mu(\hat{\eta})} \tag{3.6}$$

*for all $\eta,\hat{\eta} \in \Omega$, where $\mathbb{1}_{\eta=\hat{\eta}}$ is one if $\eta = \hat{\eta}$ and zero otherwise. The function defined in (3.6) is called **the cheap self-duality function**.*

Proof. Since μ is reversible for $\boldsymbol{X}$, we have that, for all $\eta,\hat{\eta} \in \Omega$,

$$\begin{aligned} \mathcal{L}D(\cdot,\hat{\eta})(\eta) &= \sum_{\xi\in\Omega} \mathcal{L}(\eta,\xi)D(\xi,\hat{\eta}) = \sum_{\xi\in\Omega} \mathcal{L}(\eta,\xi)\frac{\mathbb{1}_{\xi}(\hat{\eta})}{\mu(y)} = \frac{\mathcal{L}(\eta,\hat{\eta})}{\mu(\hat{\eta})} = \frac{\mathcal{L}(\hat{\eta},\eta)}{\mu(\eta)} = \frac{\mathcal{L}^T(\eta,\hat{\eta})}{\mu(\eta)} \\ &= \sum_{\xi\in\Omega} \frac{\mathbb{1}_{\eta}(\xi)}{\mu(\xi)}\mathcal{L}^T(\xi,\hat{\eta}) = \sum_{\xi\in\Omega} D(\eta,\xi)\mathcal{L}^T(\xi,\hat{\eta}) = D\mathcal{L}^T(\eta,\cdot)(\hat{\eta}) \end{aligned}$$

where we used the detailed balance equation (2.11) on the fourth equality. This proves that $\boldsymbol{X}$ is a self-dual process with respect to D. □

The previous theorem shows us that, at least for the subclass of continuous time Markov processes that admit a reversible measure, the answer to question 1. is affirmative.

Remark 3.1.2. *Recall that, if we choose $\epsilon,\gamma,\delta,\beta$ such that they do not satisfy the condition* (2.12)*, for SEP(α), and* (2.30)*, for SIP(α), then, the product measures in* (2.13)*, for SEP(α), and* (2.13)*, for SIP(α), are no longer reversible and therefore we can not prove self-duality for SEP(α) and SIP(α) with open boundary. But, as we observed, without reservoirs, there is always available a collection of reversible measures that are of product form independently from the choice of the parameters $\epsilon,\gamma,\delta,\beta$. This observation suggests us to study self-duality for SEP(α) and SIP(α) without reservoirs. The idea now will be to use the self-duality of SEP(α) and SIP(α) without reservoirs that is available due to Theorem 3.1.1, and try to find a new duality function, constructed from the cheap self-duality function, that not only satisfies (3.3) with $\mathcal{L} = \mathcal{L}^{dual} = \mathcal{L}_{bulk}$ where, $\mathcal{L}_{bulk}$ stands for the bulk generator of each model, but also with $\mathcal{L} = \mathcal{L}_{l,\alpha}$ (resp. $\mathcal{L}_{r,\alpha}$) and $\mathcal{L}^{dual} = \mathcal{L}^{dual}_{l,\alpha}$ (resp. $\mathcal{L}^{dual}_{r,\alpha}$), for some operators $\mathcal{L}^{dual}_{l,\alpha}$ (resp. $\mathcal{L}^{dual}_{r,\alpha}$) that describe the dynamics of the left (resp. right) reservoirs on the dual processes of SEP(α) and SIP(α).*

Our goal will be to construct such a duality function, as described above, connecting SEP(α) and SIP(α) with open boundary to SEP(α) and SIP(α) with only absorbing boundary points, respectively.

Remark 3.1.3. *Observe that, if there exists a matrix S such that $S\mathcal{L}_{bulk} = \mathcal{L}_{bulk}S$, where here $\mathcal{L}_{bulk}$ denotes the rate matrix associated to the infinitesimal Markov generator of the bulk, then, denoting by D the cheap self-duality function associated to $\mathcal{L}_{bulk}$, then SD is again a self-duality function, since $\mathcal{L}_{bulk}SD = S\mathcal{L}_{bulk}D = SD\mathcal{L}_{bulk}$. As already pointed out in [5], the previous observation can be made for any infinitesimal Markov generator and gives us a way to find new duality functions for a self-dual process: by finding symmetries of the infinitesimal Markov generator, when this is an element of the universal enveloping algebra of some Lie algebra, and apply these symmetries to a known self-duality function - the general case can be seen in Theorem 3.3.1. Therefore, duality theory is strongly connected to Lie algebras' theory. At this point, it is crucial to remind the reader some basic notions on Lie algebras' theory so we can clearly understand the previous statements.*

3.2 Lie Algebras and its representations

For this section, we will be following [7] and [10].

Definition 3.2.1. *Let V a finite dimensional linear vector space under some field $\mathbb{K}$ (which we will consider to be $\mathbb{R}$ or $\mathbb{C}$) and define in V a bilinear operation $[\cdot,\cdot] : V \times V \to V$, called **Lie bracket** or simply **bracket**, that satisfies*

1. $[X,Y] = -[Y,X]$ *(anti-symmetry)*
2. $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ *(Jacobi's Identity).*

*We will denote the pair $(V,[\cdot,\cdot])$, by $\mathfrak{g}$, and we call $\mathfrak{g}$ a real or complex Lie algebra, depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, respectively. The dimension of the vector space V under the field $\mathbb{K}$ is called the **dimension** of the Lie Algebra $\mathfrak{g}$.*

Example 3.2.1.

1. *Denoting by $[\cdot,\cdot]$ the usual matrix commutator operator and by tr the trace of a matrix, we have that $(\mathrm{sl}(2,\mathbb{R}),[\cdot,\cdot])$ is a real Lie algebra, given by $\mathrm{sl}(2,\mathbb{R}) = \{A \in M_2(\mathbb{R}) : trA = 0\}$, where $M_2(\mathbb{R})$ denotes the usual set of two by two matrices with real entries; and $(\mathrm{sl}(2,\mathbb{C}),[\cdot,\cdot])$ is a real or complex Lie algebra, where $\mathrm{sl}(2,\mathbb{C}) = \{A \in M_2(\mathbb{C}) : trA = 0\}$, and $M_2(\mathbb{C})$ denotes the set of two by two matrices with complex entries;*
2. *$\mathrm{su}(2)$ is a three-dimensional real Lie algebra with basis $\{s_x, s_y, s_z\}$ and commutation relations given by $[s_x,s_y] = 2s_z$, $[s_x,s_z] = 2s_y$ and $[s_y,s_z] = 2s_x$.*
3. *$\mathrm{su}(1,1)$ is a three-dimensional real Lie algebra with basis $\{t_x, t_y, t_z\}$ and commutation relations given by $[t_x,t_y] = 2t_z$, $[t_x,t_z] = 2t_y$ and $[t_y,t_z] = -2t_x$.*

Remark 3.2.1. *As we will see through this chapter, the Lie algebras* $\mathrm{su}(2)$ *and* $\mathrm{su}(1,1)$ *will play a central role. Namely, we will be interested in specific representations for each one of these Lie algebras that will allow us to write* $\mathcal{L}_{bulk}$ *as products and sums of elements of these two Lie algebras. In the next definition we introduce a very important concept in Lie algebra's theory - Lie algebra's representation.*

Definition 3.2.2 (Lie Algebra Representation)**.** *Let* $\mathfrak{g}$ *be a real (resp. complex) Lie algebra. We say that* (π, U) *is a representation of* $\mathfrak{g}$ *if* U *is a real (resp. complex) vector space and* $\pi : \mathfrak{g} \to \mathfrak{gl}(U)$ *is a Lie algebra homomorphism, i.e., for every* $X, Y \in \mathfrak{g}$*,* $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$*, where* $\mathfrak{gl}(U)$ *represents the set of all linear maps from* U *to* U*.*

Remark 3.2.2.

- *In concrete, consider the following two by two matrices, known as Pauli Matrices,*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  If we consider the real vector space A *generated by* $\{i\sigma_1, i\sigma_2, i\sigma_3\}$*, then, the linear map* $\pi : \mathrm{su}(2) \to A$ *defined by* $\pi(s_w) = -i\sigma_w$*, for* $w \in \{x, y, z\}$*, is a faithfull (injective) representation of* $\mathrm{su}(2)$*. In fact, by direct computations, we have that* $[\sigma_1, \sigma_2] = 2i\sigma_3$*,* $[\sigma_1, \sigma_3] = -2i\sigma_2$ *and* $[\sigma_2, \sigma_3] = 2i\sigma_1$*, where here the bracket should be interpret as the commutator operator between matrices, then* $\pi([s_x, s_y]) = \pi(2s_z) = -2i\sigma_3 = [-i\sigma_1, -i\sigma_2] = [\pi(s_x), \pi(s_y)]$*, and, similarly, one shows that* $\pi([s_y, s_z]) = [\pi(s_y), \pi(s_z)]$ *and* $\pi([s_x, s_z]) = [\pi(s_x), \pi(s_z)]$*.*

- *Similarly, consider*

$$\tau_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \qquad \tau_y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1 \qquad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3.$$

  Taking V *as the complex vector space generated by* $\{i\tau_x, i\tau_y, i\tau_z\}$*, then* $\pi : \mathrm{su}(1,1) \to V$ *defined by* $\pi(t_x) = -i\tau_z$*,* $\pi(t_y) = -i\tau_y$ *and* $\pi(t_z) = -i\tau_x$*, is a faithful representation of* $\mathrm{su}(1,1)$*. Using the commutation relations above for* σ_1*,* σ_2 *and* σ_3 *and the bi-linearity of the Lie bracket, then* $[\tau_x, \tau_y] = 2i\tau_z$*,* $[\tau_x, \tau_z] = 2i\tau_y$ *and* $[\tau_y, \tau_z] = -2i\tau_x$*, we easily observe that* $\pi([t_x, t_y]) = \pi(2t_z) = -2i\tau_x = [-i\tau_z, -i\tau_y] = [\pi(t_x), \pi(t_y)]$*, and, similarly,* $\pi([t_y, t_z]) = [\pi(t_y), \pi(t_z)]$ *and* $\pi([t_x, t_z]) = [\pi(t_x), \pi(t_z)]$*.*

To study representations of a real semisimple Lie algebra $\mathfrak{g}$ (which is the case of $su(2)$ and $su(1,1)$), it is convenient to consider its complexification, detonated by $\mathfrak{g}_{\mathbb{C}}$. Then, from the basis of $su(2)$ and $su(1,1)$, we want now to construct a basis of $(su(2))_{\mathbb{C}}$ and $(su(1,1))_{\mathbb{C}}$. From the matrices introduced in the previous remark, set

$$J^0 = \tfrac{1}{2}\sigma_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J^+ = \tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J^- = \tfrac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.7}$$

and

$$K^0 = \tfrac{1}{2}t_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K^+ = \tfrac{1}{2}(t_2 + it_1) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad K^- = \tfrac{1}{2}(t_2 - it_1) = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}. \tag{3.8}$$

The representations introduced before can be naturally extended to the complexification of each of the corresponding Lie algebras. Since these new representations are faithful and the dimension of their domain is finite and equal to the dimension of their image (both three), then, it is well defined their inverse. Not fearing confusion, let us identify J^0, J^+ and J^- (resp. K^0, K^+ and K^-) with the corresponding image of the matrices in (3.7) (resp. (3.8)) through the inverse function of π.

Since $\{s_x, s_y, s_z\}$ is a basis for $su(2)$, then $\{J^0, J^+, J^-\}$ is a basis for $(su(2))_{\mathbb{C}}$ with the commutation relations given by

$$[J^0, J^+] = J^+, \quad [J^0, J^-] = -J^-, \quad [J^+, J^-] = 2J^0. \tag{3.9}$$

Analogously, since $\{t_x, t_y, t_z\}$ is a basis for $su(1,1)$, then $\{K^0, K^+, K^-\}$ is a basis for $(su(1,1))_{\mathbb{C}}$ with the commutation relations given by

$$[K^0, K^+] = K^+, \quad [K^0, K^-] = -K^-, \quad [K^+, K^-] = -2K^0. \tag{3.10}$$

Observe that, for these new basis, the backet between any two elements of the basis gives a real multiple of another basis element. This implies that we can talk about real representations of $(su(2))_{\mathbb{C}}$ and $(su(1,1))_{\mathbb{C}}$ that are completely determined if we define its value for the basis that we constructed. Remark that $\{J^0, J^+, J^-\}$ is also basis for the real Lie algebra $sl(2,\mathbb{R})$ and for the complex Lie algebra $sl(2,\mathbb{C})$. On the other hand, $\{K^0, K^+, K^-\}$ is a basis for the complex Lie algebra $sl(2,\mathbb{C})$ but not for the real Lie algebra $sl(2,\mathbb{R})$. In fact, $su(2)$ and $su(1,1)$ are two real forms of $sl(2,\mathbb{C})$ that are not isomorphic ($su(2)$ and $sl(2,\mathbb{R})$ are isomorphic but $su(1,1)$ and $sl(2,\mathbb{R})$ are not).

Recalling that our aim is to show how to construct a new duality function starting from one that it is known, we will need to recall the definition of the center of a Lie algebra and symmetries of a Lie algebra element.

Definition 3.2.3 (Center of a Lie Algebra)**.** *Given a Lie algebra $(V, [\cdot,\cdot])$, we define its **center** as the set*

$$\{A \in V : [A, X] = 0 \text{ for all } X \in V\}.$$

In other words, it is the set of all the elements that commute with every element of $(V, [\cdot,\cdot])$.

Definition 3.2.4 (Symmetry)**.** *Let $(V, [\cdot,\cdot])$ be a Lie algebra and consider two elements $A, B \in V$. We say that A is a symmetry of B if $[A, B] = 0$.*

Remark 3.2.3.

- *Elements in the center of a Lie algebra are symmetries of all the elements of the Lie algebra.*
- *If $(V, [\cdot,\cdot])$ is a Lie algebra of matrices with the commutator as the corresponding Lie bracket, then, an element of V is a symmetry of another one if they commute.*

Finding symmetries of a given element A of a Lie algebra $\mathfrak{g}$,in general, is not easy, specially if the Lie bracket is not the usual commutator operator. So, can we avoid working with brackets that are not the

commutator operator? This would no longer be a question if one can embed $\mathfrak{g}$ as a subspace of some associative algebra $\mathcal{A}$ in such a way that the bracket on $\mathfrak{g}$ may be computed as $[X,Y] = XY - YX$, where XY and YX are computed in $\mathcal{A}$. In other words, we would like to find an injective map $i : \mathfrak{g} \to \mathcal{A}$ such that, for every $X, Y \in \mathfrak{g}$, $i[X,Y] = i(X)i(Y) - i(Y)i(X)$. For matrix Lie groups $G \subset Gl(n;\mathbb{C})$, where $Gl(n;\mathbb{C})$ denotes the set of all invertible square matrices of dimension $n \times n$ as coefficients in $\mathbb{C}$, it is known that their corresponding Lie algebra $\mathfrak{g}$ is a subspace of the associative algebra $M_n(\mathbb{C})$ and the bracket on $\mathfrak{g}$ is indeed given by $[X,Y] = XY - YX$ (see Section 9.3 of [10]). The universal enveloping algebra is exactly this associative algebra that we are trying to find.

Definition 3.2.5 (Universal Enveloping Algebra)**.** *For any Lie algebra $\mathfrak{g}$, the **universal enveloping algebra** of $\mathfrak{g}$ is an associative algebra $U(\mathfrak{g})$ with identity and an injective map $i : \mathfrak{g} \to U(\mathfrak{g})$ such that:*

(1) The Lie algebra $\mathfrak{g}$ is embedded in $U(\mathfrak{g})$ by i;

(2) For every $X, Y \in \mathfrak{g}$, $i[X,Y] = i(X)i(Y) - i(Y)i(X)$, meaning that the Lie bracket in $U(\mathfrak{g})$ can be identified with the usual commutator operator and the map i is an homomorphism of Lie algebras. Indeed, the universal envelopping algebra $U(\mathfrak{g})$ is a Lie algebra with Lie bracket given by the commutator operator.

With this, we can extend the definition of symmetry given in Definition 3.2.4 by asking to exist an element $B \in \mathcal{A}$ such that B and A commute. We will now introduce a special element, when exists, of the universal enveloping algebra.

Definition 3.2.6 (Casimir)**.** *Suppose that $\mathfrak{g}$ is an n-dimensional Lie algebra. Let B be a nondegenerate bilinear form on $\mathfrak{g}$ for which the adjoint action of $\mathfrak{g}$ on itself is skew-symmetric, meaning that, for all $X, Y, Z \in \mathfrak{g}$, $B(ad_X Y, Z) = -B(Y, ad_X Z)$, where $ad_X : \mathfrak{g} \to \mathfrak{g}$ is defined, for every $U, W \in \mathfrak{g}$, by $ad_W U = [W,U]$. Let $\{X_i\}_{i=1}^n$ be a basis of $\mathfrak{g}$ and $\{X^i\}_{i=1}^n$ be the corresponding dual basis with respect to B, i.e., for every $i, j \in \{1,\dots,n\}$, $B(X_i, X^j) = \mathbb{1}_i(j)$, where $\mathbb{1}_i(j)$ is the indicator function associated to i. Then, the Casimir element $\mathfrak{C}$ is an element of the Universal Enveloping Algebra of $\mathfrak{g}$ that is defined by*

$$\mathfrak{C} := \sum_{i=1}^{n} X_i X^i. \tag{3.11}$$

We remark that the definition of the Casimir is independent from the choice of basis, and it is a central element of the Universal Enveloping Algebra of $\mathfrak{g}$ (Proposition 10.5 of [10]). Before going any further, we provide two examples below, where the Casimir is explicitly computed. This will be useful in the algebraic description of our interacting particle systems.

Example 3.2.2.

1. *Casimir for $(su(2))_{\mathbb{C}}$:*

   Take $\mathfrak{g} = (su(2))_{\mathbb{C}}$ and the bilinear form B, called the Killing form, in $\mathfrak{g}$ defined, for every $X, Y \in \mathfrak{g}$, by

$$B(X,Y) = \frac{1}{4} tr(ad_X ad_Y). \tag{3.12}$$

Since $\{J^0, J^+, J^-\}$ is a basis for $(su(2))_{\mathbb{C}}$, to completely determine B, we only have to compute $B(J^0, J^0), B(J^0, J^+), B(J^0, J^-), B(J^+, J^+), B(J^+, J^-)$ and $B(J^-, J^-)$.

First, remark that, for $X \in \{J^0, J^+, J^-\}$, ad_X can be identified as a 3×3 matrix $\widehat{ad_X}$, if we use the bijection $\phi : \mathfrak{g} \to \mathbb{C}^3$ which is given, for any $Z \in \mathfrak{g}$, by $\phi(Z) = (a, b, c)$, where $Z = aJ^0 + bJ^+ + cJ^-$, for unique $a, b, c \in \mathbb{C}$, in the sense that, $ad_X = \phi^{-1} \circ \widehat{ad_X} \circ \phi$. Then,

$$ad_{J^0} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ad_{J^+} \equiv \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ad_{J^-} \equiv \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.13}$$

By direct computations,

$$\begin{aligned} &B(J^0, J^0) = \frac{1}{2}, \quad B(J^0, J^+) = 0, \quad B(J^0, J^-) = 0, \\ &B(J^+, J^0) = 0, \quad B(J^+, J^+) = 0, \quad B(J^+, J^-) = 1, \\ &B(J^-, J^0) = 0, \quad B(J^-, J^+) = 1, \quad B(J^-, J^-) = 0. \end{aligned}$$

Therefore, choosing the basis $\{J^0, J^+, J^-\}$ of $\mathfrak{g}$, the corresponding dual basis with respect to the Killing form is $\{2J^0, J^-, J^+\}$, and the Casimir element of $su(2)$ is

$$\mathfrak{C} = 2J^0J^0 + J^+J^- + J^-J^+. \tag{3.14}$$

2. *Casimir for $(su(1,1))_{\mathbb{C}}$:*

Consider now in $\mathfrak{g} = (su(1,1))_{\mathbb{C}}$ the Killing form B in $\mathfrak{g}$ with the same definition as in (3.12)*. Since $\{K^0, K^+, K^-\}$ is a basis for $(su(1,1))_{\mathbb{C}}$, as before, to completely determine B, we only have to determine $B(K^0, K^0), B(K^0, K^+), B(K^0, K^-), B(K^+, K^+), B(K^+, K^-)$ and $B(K^-, K^-)$. By direct computations,*

$$\begin{aligned} &B(K^0, K^0) = \frac{1}{2}, \quad B(K^0, K^+) = 0, \quad B(K^0, K^-) = 0 \\ &B(K^+, K^0) = 0, \quad B(K^+, K^+) = 0, \quad B(K^+, K^-) = -1 \\ &B(K^-, K^0) = 0, \quad B(K^-, K^+) = -1, \quad B(K^-, K^-) = 0, \end{aligned}$$

and, therefore, B is non-degenerate. Therefore, choosing the basis $\{K^0, K^+, K^-\}$ of $\mathfrak{g}$, the corresponding dual basis with respect to the Killing form is $\{2K^0, -K^-, -K^+\}$, and the Casimir element of $su(1,1)$ is

$$\mathfrak{C} = 2K^0K^0 - K^+K^- - K^-K^+. \tag{3.15}$$

It will be also helpful for our construction of the classical duality function to recall the definition of coproduct of a Lie algebra element.

Definition 3.2.7 (Coproduct)**.** *The coproduct of a Lie algebra element X is denoted by $\Delta(X)$ and defined*

via the tensor product $\otimes$, as

$$\Delta(X) = id \otimes X + X \otimes id, \tag{3.16}$$

and that it can be extended as an algebra homomorphism to the universal enveloping algebra, i.e., for every elements X, Y of the Lie algebra,

$$\Delta(XY) = \Delta(X)\Delta(Y) = (id \otimes X + X \otimes id)(id \otimes Y + Y \otimes id) = X \otimes Y + id \otimes (YX) + id \otimes (XY) + Y \otimes X. \tag{3.17}$$

The next result shows us how to construct symmetries for the coproduct of two elements (Lemma 5.2 of [4]).

Lemma 3.2.1. *Given a Lie algebra $\mathfrak{g}$, if S is a symmetry of C, where C can be an element in $\mathfrak{g}$ or in the universal enveloping algebra of $\mathfrak{g}$, then $\Delta(S)$ is a symmetry for $\Delta(C)$.*

Proof. Since S is a symmetry for C, then $[S, C] = 0$ (here $[\cdot, \cdot]$ is interpreted as the commutator operator on the level of the universal enveloping algebra). We want to prove that $[\Delta(S), \Delta(C)] = 0$. Then, using that the coproduct is an algebra homomorphism, we get that $[\Delta(S), \Delta(C)] = \Delta(S)\Delta(C) - \Delta(C)\Delta(S) = \Delta(SC - CS) = \Delta([S, C]) = \Delta(0) = 0$. □

With this last result, we can finally comeback to understanding the connection between duality and Lie algebras.

3.3 Self-duality and symmetries: an overview

As we saw in Theorem 3.1.1, every continuous time Markov process with a countable state space that admits a reversible measure is always self-dual via the cheap self-duality function. But starting from the cheap self-duality function, can we construct other non-trivial duality functions? Yes! The next theorem shows that the composition of a symmetry of the generator with any duality function of a self-dual process is always a self-duality function for the same process. Therefore, taking in particular the cheap duality function, using different symmetries, one can construct new duality functions.

Theorem 3.3.1. *Let d be a duality function for a self-dual process $\boldsymbol{X}$ defined in Ω with infinitesimal Markov generator $\mathcal{L}$. Let S be a symmetry of $\mathcal{L}$. Then,*

$$D = Sd \tag{3.18}$$

is again a self-duality function for $\boldsymbol{X}$.

Proof. Using (3.18), we get $\mathcal{L}D = \mathcal{L}Sd = S\mathcal{L}d = Sd\mathcal{L}^T = D\mathcal{L}^T$, which completes the proof. □

Remark 3.3.1. *For the applications in the cases of SEP(α) and SIP(α), we will see that their corresponding infinitesimal Markov generators will not be written as linear combinations, but sum of products, of elements of a given Lie algebra, which, in general, do not belong to the Lie algebra. Yet, we will be able to identify them as elements of the center of the universal enveloping algebra of some Lie algebras.*

Since each of these elements is central, they commute with all the elements of the Lie algebra, and therefore, every element of the Lie algebra will be a symmetry, in the more general definition, of the infinitesimal Markov generators.

3.4 Duality for SEP(α)

We will always work under the assumption that we want to find a duality function of product form, i.e.

$$D(\eta, \hat{\eta}) = \prod_{i=1}^{N-1} d(\eta(x), \hat{\eta}(x))$$

where d will be called the single site duality function. Using the fact that SEP(α) without reservoirs has a family of reversible measures that are independent from the choice of γ, ϵ, β and δ (recall Remark 2.2.2), we can prove self-duality for SEP(α) without reservoirs using Theorem 3.1.1.

Corollary 3.4.1. *SEP(α) without reservoirs is a self-dual process with self-duality function d_{bulk}^{cheap} given, for every $\eta, \hat{\eta} \in \Omega_N^{Ex}$, by*

$$d_{bulk}^{cheap}(\eta, \hat{\eta}) = \prod_{x=1}^{N-1} d_{bulk}(\eta(x), \hat{\eta}(x)), \tag{3.19}$$

where, $d_{bulk} : \{0, \dots, \alpha\} \times \{0, \dots, \alpha\} \to \mathbb{R}$ is defined, for every $n, m \in \{0, \dots, \alpha\}$, as

$$d_{bulk}(n, m) = \frac{n![\alpha - n]!}{\alpha!} \mathbb{1}_{n=m}, \tag{3.20}$$

where $\mathbb{1}_{n=m}$ is one if $n = m$ and zero otherwise.

Proof. By Theorem 3.1.1, if μ_ρ denotes the reversible measure of product form for SEP(α) without reservoirs associated to the parameter ρ, then

$$D(\eta, \hat{\eta}) = \frac{\mathbb{1}_\eta(\hat{\eta})}{\mu_\rho(\hat{\eta})} = \prod_{x=1}^{N-1} \left(\frac{1-\rho}{\rho}\right)^{\eta(x)} \frac{\eta(x)![\alpha - \eta(x)]!}{\alpha!} (1-\rho)^{-\alpha} \mathbb{1}_\eta(\hat{\eta})$$

is a self-duality function. Choosing $\rho = \frac{1}{2}$ and noting that independently from η, in SEP(α) without reservoirs, the total quantity of particles is conserved $\sum_{x=1}^{N-1} \eta(x) = k$ for fixed $k \in \mathbb{N}$, we can neglect the constant terms $\prod_{x=1}^{N-1} \left(\frac{1-\rho}{\rho}\right)^{\eta(x)}$ and $(1-\rho)^{-\alpha}$ to obtain (3.19) and (3.20). □

Why do we need the previous result? As we already mentioned, we want to prove that SEP(α) with open boundary has SEP(α) with absorbing boundary points as dual process. To do that, we will find a self-duality function using a symmetry for the bulk generator, as in [4], proving (3.1), or equivalently, (3.5) with $\mathcal{L}$ substituted by the respective generator of the bulk. Then, we will modify the obtained self-duality function to extend it to the reservoirs.

Self-duality for SEP(α) without reservoirs using symmetries and real representations of $(su(2))_{\mathbb{C}}$:

Consider the following real representation of $(su(2))_{\mathbb{C}}$:

$$\pi : (su(2))_{\mathbb{C}} \to \mathfrak{gl}(\mathcal{F}(\{0,\dots,\alpha\}))$$
$$\pi(aJ^0 + bJ^+ + cJ^-) = -aJ^0 + bJ^- + cJ^+ \text{ for every } a,b,c \in \mathbb{R},$$

where, for every $f \in \mathcal{F}(\{0,\dots,\alpha\})$ and for all $n \in \{0,\dots,\alpha\}$, defining $f(-1) := 0$ and $f(\alpha+1) := 0$,

$$[J^0 f](n) = \frac{1}{2}(2n-\alpha)f(n), \quad [J^- f](n) = [\alpha - n]f(n+1), \quad [J^+ f](n) = nf(n-1).$$

Indeed, by direct computations, recalling (3.9), we have that

$$J^-J^+ - J^+J^- = -2J^0 \Leftrightarrow [\pi(J^+),\pi(J^-)] = 2\pi(J^0) = \pi[J^+,J^-],$$
$$J^+J^0 - J^0J^+ = J^+ \Leftrightarrow [\pi(J^-),\pi(-J^0)] = \pi(J^-) = \pi[J^-,-J^0],$$
$$J^0J^- - J^-J^0 = -J^- \Leftrightarrow [\pi(-J^0),\pi(J^+)] = \pi(-J^+) = \pi[-J^0,J^+],$$

which shows that π is a Lie algebra homomorphism, and, therefore, a representation. We should also remark that when we write, for $a,b \in \{0,+,-\}$, $[\pi(J^a),\pi(J^b)]$, we are making an abuse of notation, and we should interpret this as $[\pi(J^a),\pi(J^b)] := \pi(J^a)\pi(J^b) - \pi(J^b)\pi(J^a)$ (composition of functions).

Now we want to extend the previous representation to act on functions in $\mathcal{F}(\{0,\dots,\alpha\}^{\Lambda_N})$ using the tensor product. For every $x \in \Lambda_N$ and for every $f \in \mathcal{F}(\{0,\dots,\alpha\}^{\Lambda_N})$, we define

$$\begin{aligned} J_x^0 f(y_1,\dots,y_{N-1}) &:= [id\otimes\cdots\otimes\pi(-J^0)\otimes\cdots\otimes id]f(y_1,\dots,y_x,\dots,y_{N-1}) \\ &= \frac{1}{2}(2y_x-\alpha)f(y_1,\dots,y_x,\dots,y_{N-1}) \\ J_x^- f(y_1,\dots,y_{N-1}) &:= [id\otimes\cdots\otimes\pi(J^+)\otimes\cdots\otimes id]f(y_1,\dots,y_x,\dots,y_{N-1}) \\ &= (\alpha-y_x)f(y_1,\dots,y_x+1,\dots,y_{N-1}) \\ J_x^+ f(y_1,\dots,y_{N-1}) &:= [id\otimes\cdots\otimes\pi(J^-)\otimes\cdots\otimes id]f(y_1,\dots,y_x,\dots,y_{N-1}) \\ &= y_x f(y_1,\dots,y_x-1,\dots,y_{N-1}). \end{aligned}$$

Using J_x^0, J_x^+ and J_x^-, for $x \in \Lambda_N$, we can write

$$\mathcal{L}^{Ex}_{bulk,\alpha} = \sum_{x=1}^{N-2} \underbrace{J_x^+J_{x+1}^- + J_x^-J_{x+1}^+ + 2J_x^0J_{x+1}^0 - \frac{\alpha^2}{2}id}_{=\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}}. \tag{3.21}$$

In fact, for every $f \in \mathcal{F}(\Omega_N^{SEP})$ and $\eta = (\eta(1), \ldots, \eta(N-1)) \in \Omega_N^{SEP}$, if $x \in \{1, \ldots, N-2\}$,

$$
\begin{aligned}
&[J_x^+ J_{x+1}^- + J_x^- J_{x+1}^+ + 2J_x^0 J_{x+1}^0 - \frac{\alpha^2}{2} id] f(\eta) \\
=&\eta(x)[\alpha - \eta(x+1)] f(\eta - \delta_x + \delta_{x+1}) + [\alpha - \eta(x)]\eta(x+1) f(\eta + \delta_x - \delta_{x+1}) + \\
&\qquad + 2\left(\eta(x) - \frac{\alpha}{2}\right)\left(\eta(x+1) - \frac{\alpha}{2}\right) f(\eta) - \frac{\alpha^2}{2} f(\eta) \\
=&\eta(x)[\alpha - \eta(x+1)][f(\eta^{x,x+1}) - f(\eta)] + \eta(x+1)[\alpha - \eta(x)][f(\eta^{x+1,x}) - f(\eta)] = [\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}] f(\eta),
\end{aligned}
$$

where $\eta \pm \delta_x := (\eta(1), \ldots, \eta(x) \pm 1, \eta(x+1), \ldots, \eta(N-1))$. Summing over x, we obtain (3.21).

Let us now find a symmetry for $\mathcal{L}_{bulk,\alpha}^{Ex}$. We start by finding, for each $x \in \{1, \ldots, N-2\}$, a symmetry for $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$. Observe that,

$$
J_x^+ J_{x+1}^- + J_x^- J_{x+1}^+ + 2J_x^0 J_{x+1}^0 - \frac{\alpha^2}{2} id = \underbrace{id \otimes \cdots \otimes id}_{x-1 \text{ times}} \otimes [J^+ \otimes J^- + J^- \otimes J^+ + 2J^0 \otimes J^0 - \frac{\alpha^2}{2} id \otimes id] \otimes \underbrace{id \otimes \cdots \otimes id}_{N-x-2 \text{ times}}.
$$

Therefore, finding a symmetry for $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$ is equivalent to find a symmetry A for $J^+ \otimes J^- + J^- \otimes J^+ + 2J^0 \otimes J^0 - \frac{\alpha^2}{2} id \otimes id$, since, then, $\underbrace{id \otimes \cdots \otimes id}_{x-1 \text{ times}} \otimes A \otimes \underbrace{id \otimes \cdots \otimes id}_{N-x-2 \text{ times}}$ is again a symmetry for $J_x^+ J_{x+1}^- + J_x^- J_{x+1}^+ + 2J_x^0 J_{x+1}^0 - \frac{\alpha^2}{2} id$. Recall from (3.14) that the Casimir element for $su(2)$ is $\mathfrak{C} = 2J^0 J^0 + J^- J^+ + J^+ J^-$, and its coproduct, $\Delta(\mathfrak{C}) = id \otimes \mathfrak{C} + \mathfrak{C} \otimes id + 4J^0 \otimes J^0 + 2J^+ \otimes J^- + 2J^- \otimes J^+$. Since, for every $f \in \mathcal{F}(\{0, \ldots, \alpha\} \times \{0, \ldots, \alpha\})$ and $n, m \in \{0, \ldots, \alpha\}$,

$$
\begin{aligned}
[id \otimes \mathfrak{C} + \mathfrak{C} \otimes id] f(n,m) &= [\alpha - m + 1] m f(n,m) + [m+1][\alpha - m] f(n,m) + 2[m - \frac{1}{2}\alpha]^2 f(n,m) + \\
&\qquad [\alpha - n + 1] n f(n,m) + (n+1)[\alpha - n] f(n,m) + 2[n - \frac{1}{2}\alpha]^2 f(n,m) \\
&= \alpha\left(\frac{\alpha}{2} + 1\right) f(n,m),
\end{aligned}
$$

then,

$$
\Delta(\mathfrak{C}) = 2\left[2J^0 \otimes J^0 + J^+ \otimes J^- + J^- \otimes J^+ + \frac{\alpha^2}{2} id \otimes id\right] + \alpha\left(1 - \frac{\alpha}{2}\right) id \otimes id. \tag{3.22}
$$

Equation (3.22) implies that, A is a symmetry of $2J^0 \otimes J^0 + J^+ \otimes J^- + J^- \otimes J^+ + \frac{\alpha^2}{2} id \otimes id$ if, and only if, A is a symmetry of $\Delta(\mathfrak{C})$. Since the co-product is a Lie algebra homomorphism that can be extended, still respecting the bracket, to elements of the universal enveloping algebra, if B is a symmetry for the element of the universal enveloping algebra of $(su(2))_{\mathbb{C}}$, $\mathfrak{C} = J^- J^+ + J^+ J^- + 2J^0 J^0$, then, by Lemma 3.2.1, $A = \Delta(B)$ is a symmetry for $\Delta(\mathfrak{C})$. Because $\mathfrak{C}$ is a central element, this means that all the elements of $(su(2))_{\mathbb{C}}$ are symmetries for $\mathfrak{C}$. Therefore, we can take $B = J^+, J^-$ or J^0, compute its coproduct to construct a symmetry for $\Delta(\mathfrak{C})$ and, finally, obtain a symmetry for $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$. We choose $B = J^+$ so we can obtain the classical single site self-duality function for SEP(α) without reservoirs, as in [4], Section 5.3, Proposition 5.1. Therefore, $\Delta(J^+) = J^+ \otimes id + id \otimes J^+$ is a symmetry of $\Delta(\mathfrak{C})$, and so, every power of $\Delta(J^+)$ is also a symmetry of $\Delta(\mathfrak{C})$. We consider the exponential to have

a duality result in product form. Thus,

$$
\begin{aligned}
e^{\Delta(J^+)} &= e^{J^+\otimes id + id\otimes J^+} = e^{J^+\otimes id} e^{id\otimes J^+} \\
&= \left[\sum_{j=0}^{\infty} \frac{(J^+\otimes id)^j}{j!}\right]\left[\sum_{j=0}^{\infty} \frac{(id\otimes J^+)^j}{j!}\right] = \left[\sum_{j=0}^{\infty} \frac{(J^+)^j}{j!}\otimes id\right]\left[id\otimes \sum_{j=0}^{\infty} \frac{(J^+)^j}{j!}\right] \\
&= \left[e^{J^+}\otimes id\right]\left[id\otimes e^{J^+}\right] = e^{J^+}\otimes e^{J^+}.
\end{aligned} \tag{3.23}
$$

Therefore, $e^{\Delta(J^+)}$ is a symmetry for $2J^0\otimes J^0 + J^+\otimes J^- + J^-\otimes J^+ + \frac{\alpha^2}{2} id\otimes id$, where the second equality in (3.23) is a consequence of $[J^+\otimes id][id\otimes J^+] - [id\otimes J^+][J^+\otimes id] = J^+\otimes J^+ - J^+\otimes J^+ = 0$.

Now, denoting

$$
\underbrace{id\otimes\cdots\otimes id}_{x-1 \text{ times}}\otimes e^{J^+}\otimes e^{J^+}\otimes \underbrace{id\otimes\cdots\otimes id}_{N-x-2 \text{ times}}
$$

by $e_x^{J^+} e_{x+1}^{J^+}$, we know that $e_x^{J^+} e_{x+1}^{J^+}$ is a symmetry for $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$, for each $x \in \{1,\dots,N-2\}$.

Remark 3.4.1.

- *The operator e^{J^+} is well-defined, because, for every $f \in \mathcal{F}(\{0,\dots,\alpha\})$ and $n \in \{0,\dots,\alpha\}$, ${}^{J^+}f(n) = \underbrace{\sum_{j=0}^{n} \frac{(J^+)^j f(n)}{j!}}_{\text{finite sum}}$, since, $f(m) = 0$, for every $m \in \mathbb{Z}\setminus\{0,\dots,\alpha\}$.*

- *The following property holds:*

$$
e^{J_y^+} = \sum_{j=0}^{n} \frac{(id\otimes\cdots\otimes id\otimes J^+\otimes id\otimes\dots id)^j}{j!} = \sum_{j=0}^{n} \frac{id\otimes\cdots\otimes id\otimes (J^+)^j\otimes id\otimes\dots id}{j!} = e_y^{J^+}
$$

  where J^+ is placed in position y, with $y \in \Lambda_N$.

Remark 3.4.2. *Observe that, for every $x, y \in \{1,\dots,N-1\}$, if $y \neq x$, then, for every $a, b \in \{0,+,-\}$, $J_x^a J_y^b = J_y^b J_x^a$, which implies that, if $y \neq x, x+1$, J_y^+ is a symmetry for $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$ and so $e^{J_y^+} = e_y^{J^+}$ is a symmetry of $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$. Therefore, for every $x \in \{1,\dots,N-2\}$,*

$$
S^+ = \prod_{y=1}^{N-1} e_y^{J^+} = \underbrace{e^{J^+}\otimes\cdots\otimes e^{J^+}}_{N-1 \text{ times}} \tag{3.24}
$$

is a symmetry of $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$ and also of $\mathcal{L}_{bulk,\alpha}^{In}$. Because of this homogeneous factorized form of the symmetry of $\mathcal{L}_{bulk,\alpha}^{In}$ using the tensor product and that d_{bulk}^{cheap} is also of homogeneous product form, we can obtain a new self-duality function for SEP(α) without reservoirs that keeps this property.

Corollary 3.4.2. *SEP(α) without reservoirs is self-dual with duality function D_{bulk}^{SEP} given, for every $\eta, \hat{\eta} \in \Omega_N^{SEP}$, by,*

$$
D_{bulk}^{SEP}(\eta,\hat{\eta}) = \prod_{x=1}^{N-1} d_{bulk}^{SEP}(\eta(x),\hat{\eta}(x)) \tag{3.25}
$$

where $d_{bulk}^{SEP} : \{0,\dots,\alpha\} \times \{0,\dots,\alpha\} \to \mathbb{R}$ *is defined, for every* $n, m \in \{0,\dots,\alpha\}$*, by*

$$d_{bulk}^{SEP}(n,m) = \frac{n!(\alpha-m)!}{(n-m)!\alpha!}\mathbb{1}_{n\geq m}. \tag{3.26}$$

Proof. Since S^+ is a symmetry for $\mathcal{L}_{bulk,\alpha}^{Ex}$ and d_{bulk}^{cheap} is a duality function for SEP(α) without reservoirs, this result is a consequence of Theorem 3.3.1. Indeed, for every $\eta, \hat{\eta} \in \Omega_N$,

$$[S^+ d_{bulk}^{cheap}(\cdot,\hat{\eta})](\eta) = \prod_{x=1}^{N-1}\left[e^{J^+} d_{bulk}(\cdot,\hat{\eta}(x))\right][\eta(x)] \tag{3.27}$$

is a self-duality function for SEP(α) without reservoirs. Now, for each $x \in \{1,\dots,N-1\}$,

$$\begin{aligned}\left[e^{J^+} d_{bulk}(\cdot,\hat{\eta}(x))\right][\eta(x)] &= \sum_{j=0}^{\infty}\frac{1}{j!}(J^+)^j d_{bulk}(\cdot,\hat{\eta}(x))[\eta(x)] = \sum_{j=0}^{\eta(x)}\frac{1}{j!}\frac{\eta(x)!}{[\eta(x)-j]!}d_{bulk}(\eta(x)-j,\hat{\eta}(x)) \\ &= \sum_{j=0}^{\eta(x)}\underbrace{\frac{1}{j!}\eta(x)!\frac{[\alpha-\eta(x)+j]!}{\alpha!}\mathbb{1}_{\eta(x)-j}(\hat{\eta}(x))}_{\text{is only different from 0 if } 0\leq j=\eta(x)-\hat{\eta}(x)} = \frac{\eta(x)!(\alpha-\hat{\eta}(x))!}{(\eta(x)-\hat{\eta}(x))!\alpha!}\mathbb{1}_{\eta(x)\geq\hat{\eta}(x)} \\ &= d_{bulk}^{SEP}(\eta(x),\hat{\eta}(x)).\end{aligned}$$

Therefore, $D_{bulk}^{SEP}(\eta,\hat{\eta}) = [S^+ d_{bulk}^{cheap}(\cdot,\hat{\eta})](\eta)$ is a self-duality function for SEP(α) without reservoirs. □

Remark 3.4.3. *Denoting* e^{J^-} *by* S^-*, since* $[S^+ d_{bulk}^{cheap}(\cdot,\hat{\eta})](\eta) = [S^- d_{bulk}^{cheap}(\eta,\cdot)](\hat{\eta})$*, we could get the same duality function using* e^{J^-} *as a symmetry for* $\mathcal{L}_{bulk,\alpha}^{In\ dual} = \mathcal{L}_{bulk,\alpha}^{In}$*. Indeed,*

$$\begin{aligned}[S^- d_{bulk}^{cheap}(\eta,\cdot)](\hat{\eta}) &= \prod_{x=1}^{N-1}\left[e^{J^-} d_{bulk}(\eta(x),\cdot)\right][\hat{\eta}(x)] = \prod_{x=1}^{N-1}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\frac{[\alpha-\hat{\eta}(x)]!\eta(x)!}{\alpha!}\mathbb{1}_{\eta(x),\hat{\eta}(x)+j}\right] \\ &= \prod_{x=1}^{N-1}\left[\sum_{j=0}^{\infty}\frac{1}{j!}\frac{[\alpha-\eta(x)+j]!\eta(x)!}{\alpha!}\mathbb{1}_{\eta(x)-j,\hat{\eta}(x)}\right] = [S^+ d_{bulk}^{cheap}(\cdot,\hat{\eta})](\eta).\end{aligned}$$

The choice of the duality function D_{bulk}^{SEP} *given in* (3.25) *forces, for each configuration* η *of the initial process, the configurations* $\hat{\eta}$ *of the dual process to have at each site at most the same number of particles that were at that site on the configuration* η*. This duality function allows converting the study of the dynamics of SEP(*α*) without reservoirs, starting from configurations with many particles, to study the same dynamics but starting with only few particles. This simplification will turn out to be very important specially for SEP(*α*) with open boundary. In particular, this will tell us that, to understand the evolution of some quantities, like the density function, we can consider configurations of SEP(*α*) with absorbing boundary points with only few particles and, instead of a possible increase of the number of particles on the system that could happen in SEP(*α*) with open boundary, in the dual, the number of particles can only decrease with time. Also remark that, if we had chosen* $B = e^{J^-}$ *and had considered as duality function*

$$[S^- d_{bulk}^{cheap}(\cdot,\hat{\eta})](\eta) = \prod_{x=1}^{N-1}\frac{[\alpha-\eta(x)]!\hat{\eta}(x)!}{[\hat{\eta}(x)-\eta(x)]!\alpha!}\mathbb{1}_{\hat{\eta}(x)\geq\eta(x)},$$

(which is the transpose self-duality function) we would be relating SEP(α) without reservoirs with itself, but the configurations for the dual process would have, at each site, at least, the same number of particles as the configurations in the initial process, which make the dynamics' study even harder since we can be considering more particles on the dual system. This is because J^- increases the number of particles at the site it is acting. On the other hand, if we had considered $B = e^{J^0}$, we would obtain a duality function that it is just a multiple of the cheap duality function. Indeed, since we are considering SEP(α) without reservoirs, fixing the initial number of particles, $m \in \mathbb{N}_0$, on the system independently from the initial configuration η, we get that

$$\begin{aligned}[S^0 d_{bulk}^{cheap}(\cdot,\hat{\eta})](\eta) &= \prod_{x=1}^{N-2}\sum_{j=0}^{\infty}\frac{(\alpha-\frac{\eta(x)}{2})^j}{j!}d_{bulk}(\eta(x),\hat{\eta}(x)) \\ &= \prod_{x=1}^{N-2} e^{\alpha-\frac{\eta(x)}{2}} d_{bulk}(\eta(x),\hat{\eta}(x)) = e^{(N-1)\alpha-\sum_{x=1}^{N-1}\frac{\eta(x)}{2}}\prod_{x=1}^{N-2} d_{bulk}(\eta(x),\hat{\eta}(x)) \\ &= \underbrace{e^{(N-1)\alpha-\frac{m}{2}}}_{\text{constant}} d_{bulk}^{cheap}(\eta(x),\hat{\eta}(x)).\end{aligned}$$

This is because J^0 doesn't change the number of particles at the site it is acting. This duality function does not give us any new information that we did not know already about the system.

Now, we can state and prove duality between SEP(α) with open and with absorbing boundary.

Theorem 3.4.1. *SEP(α) with absorbing boundary points, with infinitesimal Markov generator $\mathcal{L}_\alpha^{Ex,dual} : \mathcal{F}(\Omega_N^{dual}) \to \mathcal{F}(\Omega_N^{dual})$, given by*

$$\mathcal{L}_\alpha^{Ex,dual} = \mathcal{L}_{l,\alpha}^{dual} + \mathcal{L}_{bulk,\alpha}^{Ex} + \mathcal{L}_{r,\alpha}^{dual}, \tag{3.28}$$

where

$$\Omega_N^{dual} = \mathbb{N}_0 \times \{0,\dots,\alpha\}^{\Lambda_N} \times \mathbb{N}_0, \tag{3.29}$$

and, for every $f \in \mathcal{F}(\Omega_N^{dual})$ and $\hat{\eta} \in \Omega_N^{dual}$,

$$\mathcal{L}_{l,\alpha}^{dual} f(\hat{\eta}) = \frac{\alpha_L}{N^\theta}\hat{\eta}(1)[f(\hat{\eta}^{1,0}) - f(\hat{\eta})] \tag{3.30}$$

$$\mathcal{L}_{r,\alpha}^{dual} f(\hat{\eta}) = \frac{\alpha_R}{N^\theta}\hat{\eta}(N-1)[f(\hat{\eta}^{N-1,N}) - f(\hat{\eta})], \tag{3.31}$$

and $\mathcal{L}_{bulk(\alpha)}^{Ex}$ is extended to $\mathcal{F}(\Omega_N^{dual})$ with the same expression as in (2.8)*, is dual to SEP(α) with open boundary defined in Section 2.2 with classical duality function $D^{SEP(\alpha)}$ given, for every $(\eta,\hat{\eta}) \in \Omega_N \times \Omega_N^{dual}$, by*

$$D^{SEP(\alpha)}(\eta,\hat{\eta}) = \left[\frac{\epsilon}{\epsilon+\gamma}\right]^{\hat{\eta}(0)} \underbrace{\prod_{x=1}^{N-1} d_{bulk}^{SEP}(\eta(x),\hat{\eta}(x))}_{=D_{bulk}^{SEP(\alpha)}} \left[\frac{\delta}{\delta+\beta}\right]^{\hat{\eta}(N)} \tag{3.32}$$

where d_{bulk}^{SEP} is the same as in (3.26), if $\alpha_L = \epsilon+\gamma$ and $\alpha_R = \delta+\beta$. Here, $\hat{\eta}^{1,0}$ and $\hat{\eta}^{N-1,N}$ are given, for every $x \in \Lambda_N \cup \{0,N\}$, by

$$\hat{\eta}^{1,0}(x) = \begin{cases} \hat{\eta}(x), \text{ if } x = 2, \dots, N \\ \hat{\eta}(1) - 1, \text{ if } x = 1 \\ \hat{\eta}(0) + 1 \text{ if } x = 0, \end{cases} \quad \text{and} \quad \hat{\eta}^{N-1,N}(x) = \begin{cases} \hat{\eta}(x), \text{ if } x = 0, \dots, N-2 \\ \hat{\eta}(N-1) - 1, \text{ if } x = N-1 \\ \hat{\eta}(N) + 1 \text{ if } x = N. \end{cases} \tag{3.33}$$

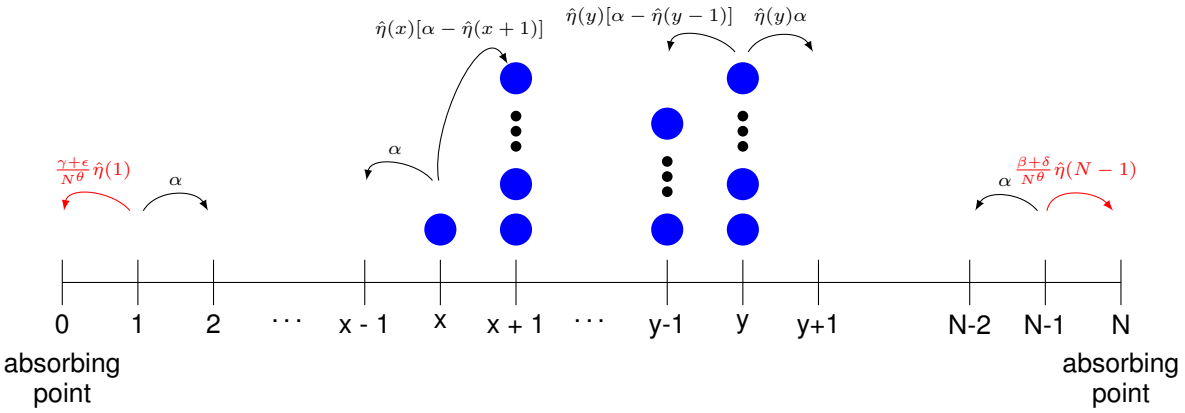

Figure 3.1: The dual of SEP(α) with open boundary is SEP(α) with only absorbing boundary.

Proof. Since, $\mathcal{L}^{In}_{bulk,\alpha}$ does not change the value of $\hat{\eta}(0)$ or $\hat{\eta}(N)$, then

$$\mathcal{L}^{Ex}_{bulk,\alpha} D^{SEP(\alpha)}(\cdot, \hat{\eta})(\eta) = \left[\frac{\epsilon}{\epsilon+\gamma}\right]^{\hat{\eta}(0)} \mathcal{L}^{Ex}_{bulk,\alpha} D^{SEP}_{bulk}(\cdot, \hat{\eta})(\eta) \left[\frac{\delta}{\delta+\beta}\right]^{\hat{\eta}(N)} \tag{3.34}$$

and we already know that the duality function in (3.25) is a self-duality function for SEP(α) without reservoirs. Then, it comes for free, from Corollary 3.4.2, that $\mathcal{L}^{Ex}_{bulk,\alpha} D^{SEP(\alpha)}(\cdot, \hat{\eta})(\eta) = \mathcal{L}^{Ex}_{bulk,\alpha} D^{SEP(\alpha)}(\eta, \cdot)(\hat{\eta})$.

For the generator of the left reservoir, we have

$$\begin{aligned}
&\mathcal{L}_{l,\alpha} D^{SEP(\alpha)}(\cdot, \hat{\eta})(\eta) \\
=&\frac{\gamma}{N^\theta}\eta(1)[D^{SEP(\alpha)}(\eta^{1,0}, \hat{\eta}) - D^{SEP(\alpha)}(\eta, \hat{\eta})] + \frac{\epsilon}{N^\theta}[\alpha - \eta(1)][D^{SEP(\alpha)}(\eta^{0,1}, \hat{\eta}) - D^{SEP(\alpha)}(\eta, \hat{\eta})] \\
=&\frac{\gamma}{N^\theta} D^{SEP(\alpha)}(\eta, \hat{\eta}) \left[[\eta(1) - \hat{\eta}(1)]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)+1} - \eta(1)\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)}\right] \\
&\quad + \frac{\epsilon}{N^\theta}\frac{[\alpha - \eta(1)]}{\eta(1) - \hat{\eta}(1) + 1} D^{SEP(\alpha)}(\eta, \hat{\eta}) \left[[\eta(1) + 1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1} - [\eta(1) - \hat{\eta}(1) + 1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)}\right].
\end{aligned}$$

Since $[\eta(1) - \hat{\eta}(1)]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)+1} - \eta(1)\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)} = -\hat{\eta}(1)\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)}$ and, similarly, $[\eta(1) + 1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1} - [\eta(1) - \hat{\eta}(1) + 1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)} = \hat{\eta}(1)\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}$, then,

$$\begin{aligned}
&\mathcal{L}_{l,\alpha} D^{SEP(\alpha)}(\cdot, \hat{\eta})(\eta) = \frac{1}{N^\theta} D^{SEP(\alpha)}(\eta, \hat{\eta}) \left[-\gamma\hat{\eta}(1)\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)} + \epsilon\frac{[\alpha - \eta(1)]\hat{\eta}(1)}{\eta(1) - \hat{\eta}(1) + 1}(\eta, \hat{\eta})\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}\right] \\
=&\frac{\hat{\eta}(1)}{N^\theta[\eta(1) - \hat{\eta}(1) + 1]} D^{SEP(\alpha)}(\eta, \hat{\eta})[-[\gamma + \epsilon][\eta(1) - \hat{\eta}(1) + 1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)} + \\
&\qquad \epsilon[\eta(1) - \hat{\eta}(1) + 1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)} + \epsilon[\alpha - \eta(1)]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}].
\end{aligned}$$

Since $[\eta(1)-\hat{\eta}(1)+1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)}+[\alpha-\eta(1)]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}=[\alpha-\hat{\eta}(1)+1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}$, combining the previous results, we get

$$
\begin{aligned}
&\mathcal{L}_{l,\alpha}D^{SEP(\alpha)}(\cdot,\hat{\eta})(\eta)\\
=&\frac{\hat{\eta}(1)}{N^{\theta}[\eta(1)-\hat{\eta}(1)+1]}D^{SEP(\alpha)}(\eta,\hat{\eta})[-[\gamma+\epsilon][\eta(1)-\hat{\eta}(1)+1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)}+\epsilon[\alpha-\hat{\eta}(1)+1]\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}]\\
=&-\frac{[\epsilon+\gamma]\hat{\eta}(1)}{N^{\theta}}D^{SEP(\alpha)}(\eta,\hat{\eta})+\frac{[\epsilon+\gamma]\hat{\eta}(1)}{N^{\theta}}\underbrace{\frac{\epsilon}{\epsilon+\gamma}\frac{[\alpha-\hat{\eta}(1)+1]}{\eta(1)-\hat{\eta}(1)+1}\mathbb{1}_{\eta(1)\geq\hat{\eta}(1)-1}D^{SEP(\alpha)}(\eta,\hat{\eta})}_{=D^{SEP(\alpha)}(\eta,\hat{\eta}^{1,0})}\\
=&\frac{\epsilon+\gamma}{N^{\theta}}\hat{\eta}(1)[D^{SEP(\alpha)}(\eta,\hat{\eta}^{1,0})-D^{SEP(\alpha)}(\eta,\hat{\eta})]=\mathcal{L}^{dual}_{l,\alpha}D^{SEP(\alpha)}(\eta,\cdot)(\hat{\eta}).
\end{aligned}
$$

By similar arguments, we obtain $\mathcal{L}_{r,\alpha}D^{SEP(\alpha)}(\cdot,\hat{\eta})(\eta)=\mathcal{L}^{dual}_{r,\alpha}D^{SEP(\alpha)}(\eta,\cdot)(\hat{\eta})$.

Combining the previous results, we can finally write $\mathcal{L}^{Ex}_{\alpha}D^{SEP(\alpha)}(\cdot,\hat{\eta})(\eta)=\mathcal{L}^{Ex,dual}_{\alpha}D^{SEP(\alpha)}(\eta,\cdot)(\hat{\eta})$, namely, SEP($\alpha$) with open boundary and SEP(α) with absorbing boundary points are dual processes with duality function $D^{SEP(\alpha)}$. □

Remark 3.4.4. *Observe that the duality function in* (3.35) *does not dependent on* θ*.*

3.4.1 The special case $\alpha=1$

Remark 3.4.5. *If* $\alpha=1$*, then, for every* $\eta\in\Omega_N$ *and* $x\in\{1,\dots,N-1\}$, $\eta(x)\in\{0,1\}$*, and* $\frac{\eta(x)![1-\eta(x)]!}{1!}=1$

As we did for SEP(α), we can also get a self-duality result for SEP(1) without reservoirs.

Corollary 3.4.3. *SEP(1) without reservoirs is a self-dual process with cheap self-duality function given, for any* $\eta,\hat{\eta}\in\Omega_N$*, by*

$$\prod_{x=1}^{N-1}\mathbb{1}_{\{\eta(x)=\hat{\eta}(x)\}}.$$

Proof. Follows immediately from Remark 3.4.5 and Corollary 3.4.1 taking $\alpha=1$. □

Corollary 3.4.4. *The open SEP(1) defined as in* (2.7)*, taking* $\alpha=1$*, and absorbing SEP(1) defined as in* (3.28)*, setting* $\alpha=1$*, are dual processes with duality function* $D^{SEP(\alpha)}$ *given, for every* $\eta,\hat{\eta}\in\Omega_N$*, by,*

$$D^{SEP(1)}(\eta,\hat{\eta})=\left[\frac{\epsilon}{\epsilon+\gamma}\right]^{\hat{\eta}(0)}\prod_{x=1}^{N-1}\mathbb{1}_{\{\eta(x)\geq\hat{\eta}(x)\}}\left[\frac{\delta}{\delta+\beta}\right]^{\hat{\eta}(N)}, \tag{3.35}$$

if $\alpha_L=\epsilon+\gamma$ *and* $\alpha_R=\delta+\beta$*.*

Proof. This result follows by combining Remark 3.4.5 with Theorem 3.4.1 taking $\alpha=1$. □

3.5 Duality for SIP(α)

Analogously to what we did for SEP(α), we will always work under the assumption that we want to find a duality function of product form, i.e,

$$D(\eta, \hat{\eta}) = \prod_{i=1}^{N-1} d(\eta(x), \hat{\eta}(x)),$$

where d is called the single site duality function. We want to obtain a similar duality result connecting SIP(α) with boundary reservoirs and SIP(α) with absorption boundary points, as we did for SEP(α). Following the same strategy as in the previous sections, we start by proving self-duality for SIP(α) without reservoirs, then, finding a symmetry for the bulk's generator using a representation of a given Lie algebra and, with it, find a new self-duality function for SIP(α) without reservoirs. Finally, we use the obtained results for SIP(α) without reservoirs to construct a duality function that is now connecting, via duality, SIP(α) with boundary reservoirs and SIP(α) with absorption boundary points. Recalling that, for SIP(α) without reservoirs, the reversible measures were of product form with marginals Negative Binomial(α, ρ) with arbitrary $\rho \in (0,1)$, the next result is a consequence of Theorem 3.1.1.

Corollary 3.5.1. *SIP(α) without reservoirs is a self-dual process with duality function d_{Bulk}^{cheap} given, for any $\eta, \hat{\eta} \in \Omega_N^{In}$, by*

$$d_{Bulk}^{cheap}(\eta, \hat{\eta}) = \prod_{x=1}^{N-1} d_{Bulk}(\eta(x), \hat{\eta}(x)), \tag{3.36}$$

where $d_{Bulk} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ is defined, for every $n, m \in \mathbb{N}_0$, as

$$d_{Bulk}(n, m) = \frac{\Gamma(\alpha) n!}{\Gamma(\alpha + n)} \mathbb{1}_{n=m}. \tag{3.37}$$

Proof. By Theorem 3.1.1, $D(\eta, \hat{\eta}) = \prod_{x=1}^{N-1} \frac{\Gamma(\alpha)\eta(x)!}{\Gamma(\alpha+\eta(x))} \rho^{-\eta(x)} (1-\rho)^{-\alpha} \mathbb{1}_{\eta(x)=\hat{\eta}(x)}$ is a self-duality function. Since, independently of η, in SIP(α) without reservoirs, the total number of particles is conserved, i.e., $\sum_{x=1}^{N-1} \eta(x) = k$ for fixed $k \in \mathbb{N}$, we can neglect the constant term $\prod_{x=1}^{N-1} \rho^{-\eta(x)} (1-\rho)^{-\alpha}$ to obtain (3.36) and (3.37). □

Self-duality for SIP(α) without reservoirs using symmetries and real representations of $(su(1,1))_{\mathbb{C}}$:

Consider the following representation of $(su(1,1))_{\mathbb{C}}$:

$$\sigma : (su(1,1))_{\mathbb{C}} \to \mathfrak{gl}(\mathcal{F}(\mathbb{N}_0))$$
$$\sigma(aK^0 + bK^+ + cK^-) = aK^0 + bK^+ + cK^- \text{ for every } a, b, c \in \mathbb{R},$$

where, for every $f \in \mathcal{F}(\mathbb{N}_0)$ and for all $n \in \mathbb{N}_0$,

$$[K^0 f](n) = \frac{1}{2}(2n + \alpha) f(n), \quad [K^- f](n) = [\alpha + n] f(n+1), \quad [K^+ f](n) = n f(n-1).$$

Indeed, by direct computations, we have that

$$K^-K^+ - K^+K^- = 2K^0 \Leftrightarrow [\sigma(K^-), \sigma(K^+)] = 2\sigma(K^0) = \sigma[K^-, K^+],$$
$$K^+K^0 - K^0K^+ = -K^+ \Leftrightarrow [\sigma(K^+), \sigma(K^0)] = -\sigma(K^+) = \sigma[K^+, K^0],$$
$$K^0K^- - K^-K^0 = K^- \Leftrightarrow [\sigma(K^0), \sigma(K^-)] = -\sigma(K^-) = \sigma[K^0, K^-],$$

which shows that, in fact, π is a Lie algebra homomorphism, and, therefore, a representation. As for SEP(α), we extend the previous representation to act on functions in $\mathcal{F}(\mathbb{N}_0^{\Lambda_N})$ using the tensor product. For every $x \in \Lambda_N$, we define, for every $f \in \mathcal{F}(\mathbb{N}_0^{\Lambda_N})$,

$$\begin{aligned}
K_x^0 f(y_1, \dots, y_{N-1}) &:= [id \otimes \cdots \otimes \sigma(K^0) \otimes \cdots \otimes id] f(y_1, \dots, y_x, \dots, y_{N-1}) \\
&= \frac{1}{2}(2y_x + \alpha) f(y_1, \dots, y_x, \dots, y_{N-1}) \\
K_x^- f(y_1, \dots, y_{N-1}) &:= [id \otimes \cdots \otimes \sigma(K^-) \otimes \cdots \otimes id] f(y_1, \dots, y_x, \dots, y_{N-1}) \\
&= (\alpha + y_x) f(y_1, \dots, y_x + 1, \dots, y_{N-1}) \\
K_x^+ f(y_1, \dots, y_{N-1}) &:= [id \otimes \cdots \otimes \sigma(K^+) \otimes \cdots \otimes id] f(y_1, \dots, y_x, \dots, y_{N-1}) \\
&= y_x f(y_1, \dots, y_x - 1, \dots, y_{N-1}).
\end{aligned}$$

Using K_x^0, K_x^+ and K_x^-, for $x \in \Lambda_N$, we can write

$$\mathcal{L}_{bulk,\alpha}^{In} = \sum_{x=1}^{N-2} \underbrace{K_x^+K_{x+1}^- + K_x^-K_{x+1}^+ - 2K_x^0K_{x+1}^0 + \frac{\alpha^2}{2}id}_{=\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}}. \tag{3.38}$$

In fact, for every $f \in \mathcal{F}(\Omega_N)$ and $\eta = (\eta(1), \dots, \eta(N-1)) \in \Omega_N$, if $x \in \{1, \dots, N-2\}$,

$$\begin{aligned}
&[K_x^+K_{x+1}^- + K_x^-K_{x+1}^+ - 2K_x^0K_{x+1}^0 + \frac{\alpha^2}{2}id] f(\eta(1), \dots, \eta(N-1)) \\
=&\eta(x)[\alpha + \eta(x+1)] f(\eta(1), \dots, \eta(x) - 1, \eta(x+1) + 1, \dots, \eta(N-1)) + \\
&\quad [\alpha + \eta(x)]\eta(x+1) f(\eta(1), \dots, \eta(x) + 1, \eta(x+1) - 1, \dots, \eta(N-1)) - \\
&\quad - 2\left(\eta(x) + \frac{\alpha}{2}\right)\left(\eta(x+1) + \frac{\alpha}{2}\right) f(\eta(1), \dots, \eta(N-1)) + \frac{\alpha^2}{2} f(\eta(1), \dots, \eta(N-1)) \\
=&\eta(x)[\alpha + \eta(x+1)][f(\eta^{x,x+1}) - f(\eta)] + \eta(x+1)[\alpha + \eta(x)][f(\eta^{x+1,x}) - f(\eta)] = [\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}] f(\eta).
\end{aligned}$$

Summing over x, we obtain (3.38).

Let us now find a symmetry for $\mathcal{L}_{bulk,\alpha}^{In}$. First, for each $x \in \{1, \dots, N-2\}$, finding a symmetry for $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$. Note that,

$$\begin{aligned}
&K_x^+K_{x+1}^- + K_x^-K_{x+1}^+ - 2K_x^0K_{x+1}^0 + \frac{\alpha^2}{2}id \\
=&\underbrace{id \otimes \cdots \otimes id}_{x-1 \text{ times}} \otimes [K^+ \otimes K^- + K^- \otimes K^+ - 2K^0 \otimes K^0 + \frac{\alpha^2}{2} id \otimes id] \otimes \underbrace{id \otimes \cdots \otimes id}_{N-x-2 \text{ times}}.
\end{aligned}$$

Therefore, finding a symmetry A for $\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}$ is equivalent to find a symmetry A for $K^+\otimes K^- + K^-\otimes K^+ - 2K^0\otimes K^0 + \frac{\alpha^2}{2}id\otimes id$, since $\underbrace{id\otimes\cdots\otimes id}_{x-1 \text{ times}}\otimes A\otimes\underbrace{id\otimes\cdots\otimes id}_{N-x-2 \text{ times}}$ is again a symmetry for $K_x^+K_{x+1}^- + K_x^-K_{x+1}^+ - 2K_x^0K_{x+1}^0 + \frac{\alpha^2}{2}id$. Recall from (3.15) that the Casimir element for $(su(1,1))_{\mathbb{C}}$ is $\mathfrak{C} = 2K^0K^0 - K^-K^+ - K^+K^-$, and its coproduct, $\Delta(\mathfrak{C}) = id\otimes\mathfrak{C} + \mathfrak{C}\otimes id + 4K^0\otimes K^0 - 2K^+\otimes K^- - 2K^-\otimes K^+$. Since, for every $f\in\mathcal{F}(\mathbb{N}_0\times\mathbb{N}_0)$ and $n,m\in\mathbb{N}_0$,

$$\begin{aligned}[id\otimes\mathfrak{C}+\mathfrak{C}\otimes id]f(n,m) &= -[\alpha+m][m+1]f(n,m) - m[\alpha+m-1]f(n,m) + 2[m+\frac{1}{2}\alpha]^2f(n,m) - \\ &\quad - [\alpha+n][n+1]f(n,m) - n[\alpha+n-1]f(n,m) + 2[n+\frac{1}{2}\alpha]^2f(n,m) \\ &= \alpha\left(\frac{\alpha}{2}-1\right)f(n,m).\end{aligned}$$

Then,

$$\Delta(\mathfrak{C}) = -2\left[K^+\otimes K^- + K^-\otimes K^+ - 2K^0\otimes K^0 + \frac{\alpha^2}{2}id\otimes id\right] + \alpha\left(\frac{3\alpha}{2}-1\right)id\otimes id. \tag{3.39}$$

Equation (3.39) implies that, A is a symmetry of $K^+\otimes K^- + K^-\otimes K^+ - 2K^0\otimes K^0 + \frac{\alpha^2}{2}id\otimes id$ if, and only if, A is a symmetry of $\Delta(\mathfrak{C})$. Therefore, by Lemma 3.2.1, if B is a symmetry for the element of the universal enveloping algebra of $(su(1,1))_{\mathbb{C}}$, $\mathfrak{C} = 2K^0K^0 - K^-K^+ - K^+K^-$, then $A=\Delta(B)$ is a symmetry for $\Delta(\mathfrak{C})$. Since $\mathfrak{C}$ is a central element, this means that all the elements of $(su(1,1))_{\mathbb{C}}$ are symmetries for $\mathfrak{C}$. Therefore, we can take $B=K^+, K^-$ or K^0 to compute its coproduct and construct a symmetry for $\Delta(\mathfrak{C})$ to finally obtain a symmetry for $\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}$. As in SEP($\alpha$), we chose $B=K^+$ so we can obtain the classical single site duality function for SIP(α) without reservoirs, as in [4], Section 5.3, Proposition 5.2. Therefore, $\Delta(K^+) = K^+\otimes id + id\otimes K^+$ is a symmetry of $\Delta(\mathfrak{C})$, and so, every power of $\Delta(K^+)$ is also a symmetry of $\Delta(\mathfrak{C})$. Thus, by similar computations as the ones we did for SEP(α),

$$e^{\Delta(K^+)} = e^{K^+\otimes id + id\otimes K^+} = e^{K^+\otimes id}e^{id\otimes K^+} = e^{K^+}\otimes e^{K^+}, \tag{3.40}$$

is a symmetry for $2K^0\otimes K^0 - K^+\otimes K^- - K^-\otimes K^+ + \frac{\alpha^2}{2}id\otimes id$, where the second equality in (3.40) is a consequence of $[K^+\otimes id][id\otimes K^+] - [id\otimes K^+][K^+\otimes id] = K^+\otimes K^+ - K^+\otimes K^+ = 0$. Now, denoting

$$\underbrace{id\otimes\cdots\otimes id}_{x-1 \text{ times}}\otimes e^{K^+}\otimes e^{K^+}\otimes\underbrace{id\otimes\cdots\otimes id}_{N-x-2 \text{ times}}$$

by $e_x^{K^+}e_{x+1}^{K^+}$, we know that $e_x^{K^+}e_{x+1}^{K^+}$ is a symmetry for $\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}$, for each $x\in\{1,\dots,N-2\}$.

Remark 3.5.1. *The operator e^{K^+} is well-defined, because, e^{J^+} is also well-defined and $K^+ = iJ^+$.*

Remark 3.5.2. *Observe that, for every $x,y\in\{1,\dots,N-1\}$, if $y\neq x$, then, for every $a,b\in\{0,+,-\}$, $K_x^aK_y^b = K_y^bK_x^a$, which implies that, if $y\neq x,x+1$, K_y^+ is also a symmetry for $\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}$ and so $e^{K_y^+} = e_y^{K^+}$ (replace J^+ with K^+ in point 2 of Remark 3.4.1) is a symmetry of $\mathcal{L}_{x,x+1}+\mathcal{L}_{x+1,x}$. Therefore,*

for every $x \in \{1, \dots, N-2\}$,

$$C^+ = \prod_{y=1}^{N-1} e_y^{K^+} = \underbrace{e^{K^+} \otimes \cdots \otimes e^{K^+}}_{N-1 \text{ times}} \tag{3.41}$$

is a symmetry of $\mathcal{L}_{x,x+1} + \mathcal{L}_{x+1,x}$ *and also of* $\mathcal{L}^{In}_{bulk,\alpha}$*. Like in SEP(*$\alpha$*), because of this homogeneous factorized form of the symmetry of* $\mathcal{L}^{In}_{bulk,\alpha}$*, using the tensor product and also because* d^{cheap}_{Bulk} *is of homogeneous product form, we can obtain a new-duality function for SIP(*α*) without reservoirs that keeps this property.*

Corollary 3.5.2 (Self-duality for SIP(α) without reservoirs using symmetry)**.** *SIP(*α*) without reservoirs is self-dual with duality function* D^{SIP}_{Bulk} *given, for every* $\eta, \hat{\eta} \in \Omega_N$*, by,*

$$D^{SIP}_{Bulk}(\eta, \hat{\eta}) = \prod_{x=1}^{N-1} d^{SIP}_{Bulk}(\eta(x), \hat{\eta}(x)) \tag{3.42}$$

where $d^{SIP}_{Bulk} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ *is defined, for every* $n, m \in \mathbb{N}_0$*, by*

$$d^{SIP}_{Bulk}(n, m) = \frac{n!}{(n-m)!} \frac{\Gamma(\alpha)}{\Gamma(\alpha + m)} \mathbb{1}_{n \geq m}, \tag{3.43}$$

and it is called the single site classical duality function for SIP(α*) without reservoirs.*

Proof. Since C^+, defined in (3.41), is a symmetry for $\mathcal{L}^{In}_{bulk,\alpha}$ and d^{cheap}_{Bulk} is a duality function for SIP(α) without reservoirs, this result is a consequence of Theorem 3.3.1. Indeed, for every $\eta, \hat{\eta} \in \Omega_N$,

$$[C^+ d^{cheap}_{Bulk}(\cdot, \hat{\eta})](\eta) = \prod_{x=1}^{N-1} \left[e^{K^+} d_{Bulk}(\cdot, \hat{\eta}(x)) \right] [\eta(x)] \tag{3.44}$$

is a self-duality function for SIP(α) without reservoirs. Now, for each $x \in \Lambda_N$,

$$\begin{aligned}
\left[e^{K^+} d_{Bulk}(\cdot, \hat{\eta}(x)) \right] [\eta(x)] &= \sum_{j=0}^{\infty} \frac{1}{j!} (K^+)^j d_{Bulk}(\cdot, \hat{\eta}(x))[\eta(x)] = \sum_{j=0}^{\eta(x)} \frac{1}{j!} \frac{\eta(x)!}{[\eta(x)-j]!} d_{Bulk}(\eta(x) - j, \hat{\eta}(x)) \\
&= \sum_{j=0}^{\eta(x)} \underbrace{\frac{1}{j!} \eta(x)! \frac{\Gamma(\alpha)}{\Gamma(\alpha + \eta(x) - j)} \mathbb{1}_{\eta(x)-j, \hat{\eta}(x)}}_{\text{is only different from 0 if } 0 \leq j = \eta(x) - \hat{\eta}(x)} \\
&= \frac{\eta(x)!}{(\eta(x) - \hat{\eta}(x))!} \frac{\Gamma(\alpha)}{\Gamma(\alpha + \hat{\eta}(x))} \mathbb{1}_{\eta(x) \geq \hat{\eta}(x)} = d^{SIP}_{Bulk}(\eta(x), \hat{\eta}(x)).
\end{aligned}$$

Therefore, $D^{SIP}_{Bulk}(\eta, \hat{\eta}) = [C^+ d^{cheap}_{Bulk}(\cdot, \hat{\eta})](\eta)$ is a duality function for the self-dual process SIP(α) without reservoirs. □

Finally, we can state duality between SIP(α) with open boundary and SIP(α) with only absorbing boundary points, whose proof, for $\theta = 0$, can be found in Theorem 4.1 of [1] noticing that, for $x \in \Lambda_N$, if $\eta(x) < \hat{\eta}(x)$, then $\frac{1}{(\eta(x) - \hat{\eta}(x))!} := 0$. The proof for general θ is analogous to what we did in Theorem 3.4.1 for SEP(α), and we leave the details to the reader.

Theorem 3.5.1. *The SIP(α) with only absorbing boundary points, with infinitesimal Markov generator, $\mathcal{L}_{\alpha}^{In,dual} : L^2(\Omega_N^{In,dual}) \to L^2(\Omega_N^{In,dual})$, given by*

$$\mathcal{L}_{\alpha}^{In,dual} = \mathcal{L}_{l,\alpha}^{dual} + \mathcal{L}_{bulk,\alpha}^{In} + \mathcal{L}_{r,\alpha}^{dual}, \tag{3.45}$$

where $L^2(\Omega_N^{In,dual})$ represents the set of local L^2 functions with domain $\Omega_N^{In,dual} = \mathbb{N}_0^{\bar{\Lambda}_N}$ where $\bar{\Lambda}_N = \{0, \dots, N\}$, and $\mathcal{L}_{bulk,\alpha}^{In}$, $\mathcal{L}_{l,\alpha}^{dual}$ and $\mathcal{L}_{r,\alpha}^{dual}$ are extended to $L^2(\Omega_N^{In,dual})$ with the same definition as in (2.29), (3.30) *and* (3.31)*, respectively, is dual to SIP(α) with open boundary, defined as in* (2.28)*, with classical duality function $D^{SIP(\alpha)}$ given, for every $(\eta, \hat{\eta}) \in \Omega_N^{In} \times \Omega_N^{In,dual}$, by,*

$$D^{SIP(\alpha)}(\eta, \hat{\eta}) = \left[\frac{\epsilon}{\gamma - \epsilon}\right]^{\hat{\eta}(0)} \underbrace{\prod_{x=1}^{N-1} d_{Bulk}^{SIP}(\eta(x), \hat{\eta}(x))}_{=D_{bulk}^{SIP(\alpha)}} \left[\frac{\delta}{\beta - \delta}\right]^{\hat{\eta}(N)}, \tag{3.46}$$

where d_{Bulk}^{SIP} is the same as in (3.43), if $\alpha_L = \gamma - \epsilon$ and $\alpha_R = \beta - \delta$.

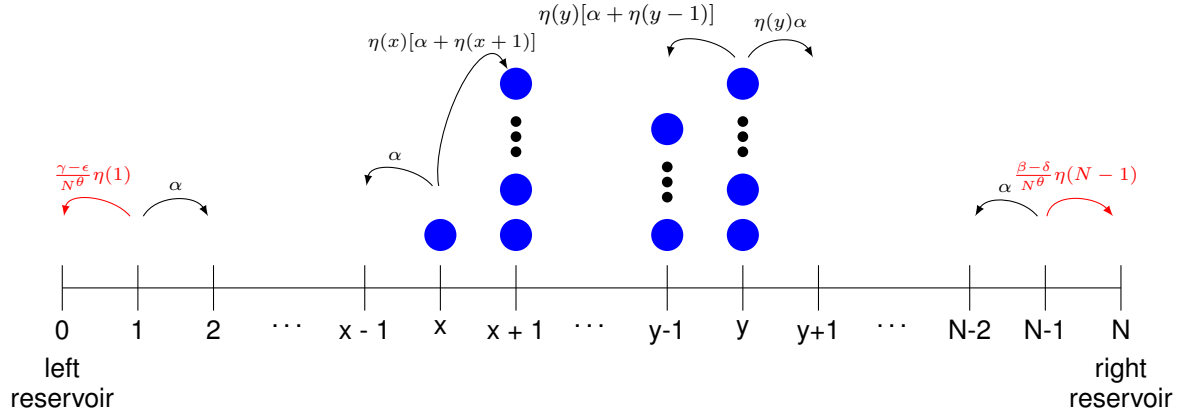

Figure 3.2: The dual of SIP(α) with open boundary is SIP(α) with only absorbing boundary.

Remark 3.5.3. *As in SEP(α), the duality function in* (3.46) *does not dependent on θ.*

Remark 3.5.4. *Observe that, in Theorems 3.4.1 and 3.5.1, we do not ask any restrict on the parameters γ, ϵ, δ and β. So, duality between SEP(α) (respectively, SIP(α)) with open boundary and SEP(α) (respectively, SIP(α)) with absorbing boundary points is valid for equilibrium and non-equilibrium states. This will be very important because it will allow finding the stationary density profile and stationary centered correlations out of equilibrium.*

We are now in position to prove the existence and uniqueness of invariant measure for SIP(α), following Appendix A of [3].

Proposition 3.5.1. *SIP(α) with open boundary has a unique invariant measure.*

Proof. To show that a probability measure μ on a countable space Ω is the unique stationary measure for the particle system $\{\eta_t\}_{t\geq 0}$, it is enough to prove that, for all probability measures μ' on Ω, $\mu' S_t$

converge weakly to μ as t goes to infinity, i.e, for all bounded functions $f : \Omega \to \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}_{\mu'}[f(\eta_t)] = \mathbb{E}_{\mu}[f(\eta)] \tag{3.47}$$

(in fact, this is an equivalent characterization of a unique invariant probability measure of an irreducible Markov chain - see Theorem 2.66. of [15]). The idea is to prove that there exists a unique probability measure μ that is characterized by its factorial moments and, consequently, characterized by the integrals

$$\int_{\Omega_N} D^{SIP(\alpha)}(\eta, \hat{\eta}) d\mu(\eta), \tag{3.48}$$

for every $\hat{\eta} \in \Omega_N^{dual}$ (to a measure that has this property we will call a *tempered* measure). As we saw above,

$$D^{SIP(\alpha)}(\eta, \hat{\eta}) = \left[\frac{\epsilon}{\gamma-\epsilon}\right]^{\hat{\eta}(0)} \prod_{x=1}^{N-1} \frac{\eta(x)!}{(\eta(x)-\hat{\eta}(x))!} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\hat{\eta}(x))} \mathbb{1}_{\eta(x)\geq\hat{\eta}(x)} \left[\frac{\delta}{\beta-\delta}\right]^{\hat{\eta}(N)}$$

is the classical duality function between SIP(α) with open boundary and SIP(α) with absorbing boundary, $\Omega_N^{dual} = \mathbb{N}_0^{1+|\Lambda_N|+1}$ is the state space of SIP(α) with absorbing boundary and, for every fixed $\hat{\eta}$, $D^{SIP(\alpha)}(\eta, \hat{\eta})$ is written in terms of products of factorial moments of the variables $\{\eta(x) : x \in \Lambda_N\}$ with some weights (since $\hat{\eta}$ is fixed, then $\left[\frac{\epsilon}{\gamma-\epsilon}\right]^{\hat{\eta}(0)} \left[\frac{\delta}{\beta-\delta}\right]^{\hat{\eta}(N)} \prod_{x=1}^{N-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\hat{\eta}(x))}$ is constant). To finish the proof, we will need to show that the same probability measure will also satisfy (3.47), and, therefore, we have found the unique invariant measure of SIP(α). By duality, for every $\eta \in \Omega_N$ and $\hat{\eta} \in \Omega_N^{dual}$ such that $|\hat{\eta}| := \sum_{x=0}^{N} \hat{\eta}(x) = k$ (the total number of dual particles is k),

$$\lim_{t\to\infty} \mathbb{E}_{\eta}[D^{SIP(\alpha)}(\eta_t, \hat{\eta})] = \lim_{t\to\infty} \mathbb{E}_{\hat{\eta}}[D^{SIP(\alpha)}(\eta, \hat{\eta}_t)] = \sum_{m=0}^{k} (\rho_N)^{k-m} (\rho_0)^m p_{\hat{\eta}}^N(m) \tag{3.49}$$

where the last equality is a consequence that, as time goes to infinity, all the particles on the bulk will eventually be absorbed by one of the boundary points. Here $p_{\hat{\eta}}^N(m)$ represents the probability that, starting from the dual configuration $\hat{\eta}$, m of the k particles are absorbed at 0, $\mathbb{E}_\eta$ and $\mathbb{E}_{\hat{\eta}}$ represent the expectation in Ω_N and Ω_N^{dual} taken with respect to the measures δ_η and $\delta_{\hat{\eta}}$, respectively, and ρ_0 and ρ_N have the interpretation as in Chapter 2. Now, taking $C = \max\{\rho_N, \rho_0\} \in (0, \infty)$, we have

$$\sum_{m=0}^{k} (\rho_N)^{k-m} (\rho_0)^m p_{\hat{\eta}}^N(m) \leq \sum_{m=0}^{k} C^{k-m} C^m p_{\hat{\eta}}^N(m) = C^k \underbrace{\sum_{m=0}^{k} p_{\hat{\eta}}^N(m)}_{=1} = C^{|\hat{\eta}|}. \tag{3.50}$$

Now applying Lemma A.1 of [3], defining μ^* by $\lim_{t\to\infty} \delta_\eta S_t$, then, μ^* is tempered and therefore, it is the unique probability measure for which

$$\mathbb{E}_{\mu^*}[D^{SIP(\alpha)}(\eta_t, \hat{\eta})] = \sum_{m=0}^{k} (\rho_N)^{k-m} (\rho_0)^m p_{\hat{\eta}}^N(m). \tag{3.51}$$

We remark that, since μ^* is uniquely determined by (3.51), choosing a different configuration η, we would obtain exactly same measure.

Since everything we did until now was independent from $\hat{\eta}$ and μ^* is uniquely characterized by the limiting moments in (3.49), then, for every bounded function $f : \Omega_N \to \mathbb{R}$ and $\eta \in \Omega_N$, $\lim_{t\to\infty} \mathbb{E}_\eta[f(\eta_t)] = \mathbb{E}_{\mu^*}[f]$. Finally, for every probability measure μ in Ω_N, $\mathbb{E}_\mu[f(\eta_t)] = \mathbb{E}_\mu[\mathbb{E}_\eta[f(\eta_t)]]$ and, since, for every $\eta \in \Omega_N$, $\lim_{t\to\infty} \mathbb{E}_\eta[f(\eta_t)] = \mathbb{E}_{\mu^*}[f]$ and $|\mathbb{E}_\eta[f(\eta_t)]| \leq \sup_{\eta\in\Omega_N} |f(\eta)| \leq C$, for some $C = C(f) > 0$ and for every $t \in \mathbb{R}_0^+$, by the dominated convergence theorem,

$$\lim_{t\to\infty} \mathbb{E}_\mu[f(\eta_t)] = \lim_{t\to\infty} \mathbb{E}_\mu[\mathbb{E}_\eta[f(\eta_t)]] = \mathbb{E}_\mu[\lim_{t\to\infty} \mathbb{E}_\eta[f(\eta_t)]] = \mathbb{E}_\mu[\mathbb{E}_{\mu^*}[f]] = \mathbb{E}_{\mu^*}[f],$$

where the last equality follows from the fact that μ is a probability measure. Therefore, μ^* is the unique invariant measure of SIP(α). □

Chapter 4

Applications of Duality

As we saw in the last chapter, SEP(α) and SIP(α) with open boundary have as dual processes SEP(α) and SIP(α) with absorbing boundary, respectively – recall Theorems 3.4.1, 3.4.4 (for the special case $\alpha = 1$) and 3.5.1. Once a particle reaches any of the boundary points, it can not leave that state (and we say that the particle has been absorbed at that point). So, it makes sense to talk about absorption probabilities in the dual processes, i.e., starting from a given configuration with k dual particles, what is the probability that m of those k will be absorbed at 0 and the remaining $k - m$ will be absorbed at N?

In this chapter, we will start by computing absorption probabilities for the dual processes of SEP(α), with special interest for the case $\alpha = 1$, and SIP(α), starting from a configuration with k particles (in our computations, k will be at most 3). It will become clearer in Section 4.2.1 why do we need these absorption probabilities and how can duality make use of them to obtain other important results.

We will make the simplification of choosing the parameters γ, ϵ, δ and β satisfying:

- for SEP(α), with $\alpha \in \mathbb{N}$, $\gamma + \epsilon = \beta + \delta = \alpha$;

- for SIP(α), with $\alpha \in \mathbb{R}^+$, $\gamma - \epsilon = \beta - \delta = \alpha$.

The previous choices, called Liggett conditions, still gives us enough freedom to choose two of the four parameters (γ, ϵ, δ and β) and, therefore, find the stationary density profile and the 2-points (for SEP(1), also 3-points) stationary correlation functions out of equilibrium. This choice simplifies the computations that will be done in the following sections and it allows us to obtain explicit expressions for the absorption probabilities for SEP(α) and SIP(α). We could drop this restriction in all the models for the case $k = 1$ and still obtain analogous results with similar computations, but, as k increases, dropping this restriction would create an extra asymmetry on the models that is reflected on the lost of "continuity" (even in the case $\theta = 0$) in the coefficients of the polynomials that will describe the absorption probabilities that we want to find.

4.1 SEP(1) - Absorption Probabilities

4.1.1 Case k = 1

- **Notation and conventions**

Recall that choosing $k = 1$ means that our dual process will start with just one dual particle that can either be absorbed at 0 or N. Therefore, the number of particles absorbed at 0, represented here by m, can either be 0 or 1, i.e., $m \in \{0, 1\}$. Let $x \in \Lambda_N$ (the definition of Λ_N was introduced in Section 2.1) that represents the site where the dual particle is initially placed, i.e., we start from the dual configuration $\hat{\eta} = \delta_x := (0, \underbrace{0, \ldots, 0}_{x-1 \text{ zeros}}, 1, 0, \ldots, 0, 0)$. Denote by $p_x^N(1)$ (resp. $p_x^N(0)$) the probability that, started from the configuration with one particle placed at site x, one (resp. zero) dual particle (resp. particles) being absorbed at 0. Observe that, if we extend the possible values of x to include 0 and N, then p_0^N and p_N^N represent boundary terms. Let us keep in mind that, for each m, $p_x^N(m)$ can be though as the image at x of a function $p^N(m) : \{0, \ldots, N\} \to [0, 1]$ where $p^N(m)(x) := p_x^N(m)$.

We will write $\mathbb{P}(x \to y)$, with $x \in \Lambda_N$ and $y \in \{x - 1, x + 1\}$, to represent the transition probability of a particle going from x to y in one step. In what follows, the 1-dimensional discrete Laplacian operator, that we will denote by Δ_N^{1D}, will be a central object. This operator $\Delta_N^{1D} : \mathcal{F}(\{0, \ldots, N\})$[1] $\to \mathcal{F}(\Lambda_N)$ is defined, on a function $f \in \mathcal{F}(\{0, \ldots, N\})$, for every $x \in \Lambda_N$, as

$$\Delta_N^{1D} f(x) = N^2[f(x - 1) - 2f(x) + f(x + 1)]. \tag{4.1}$$

- **Conditioning and results**

Observe that SEP(1) with absorbing boundary points with just one particle represents a particle performing a continuous time one-dimensional random walk (that is symmetric on the bulk) with absorbing boundary and jump rates given as in Figure 4.1. This picture is a simplification of Figure 3.2, for SEP(1) with absorbing boundary points and just one particle.

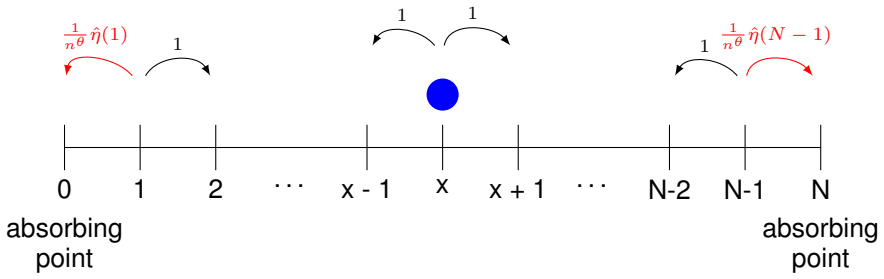

Figure 4.1: Jump rates for SEP(1) with absorbing boundary points with just one particle.

To compute $p_x^N(m)$, for each fixed m, we will condition on the first jump to get a system of equations that $p_x^N(m)$ has to satisfy. Then:

[1] Recall that for any set A we defined $\mathcal{F}(A)$ as the set of functions $f : A \to \mathbb{R}$.

- If x = 1, meaning that we start with a particle at site $x = 1$, then, for its first jump, the particle has two possibilities: either it jumps to the right, i.e., to the point 2 with rate 1, or it jumps to the left, i.e., to the boundary point 0 with rate $\frac{1}{N^\theta}$. Therefore, we get the following formula

$$\begin{aligned} p_1^N(m) =& \mathbb{P}(m \text{ particles are absorbed at 0 starting with one particle at } 1 | 1 \to 0)\mathbb{P}(1 \to 0) + \\ & \mathbb{P}(m \text{ particles are absorbed at 0 starting with one particle at } 1 | 1 \to 2)\mathbb{P}(1 \to 2) \end{aligned}$$

  To obtain the transition probability $\mathbb{P}(1 \to 0)$ (resp. $\mathbb{P}(1 \to 2)$), one has to recall that we only have to make the ratio between the rate of going from 1 to 0 (resp. from 1 to 2) and the sum of the rates for every possible jumps at the present position.

  As a consequence of the Markov property, the probability that, starting with one particle at site 1, that particle is absorbed at 0 given that the first jump was to the right, i.e., to 2, is equal to the probability that, starting with one particle at 2, that one is absorbed at 0. Analogously, the probability that, starting with one particle at site $N-1$, that particle is absorbed at N given that the first jump was to the left, i.e., to $N-2$, is equal to the probability that, starting with one particle at $N-2$, that one is absorbed at N. Also, the probability that, starting with one particle at 1, that particle is absorbed at 0 (resp. at N) given that the first jump was to the left, i.e., to 0 (resp. to N), is interpreted as the boundary value $p_0^N(1)$ (resp.$p_0^N(0)$). Here, $p_0^N(1)$ represents the probability that a particle, that is at the point 0, being absorbed at 0 and $p_0^N(0)$ represents the probability that a particle, that is at the point 0, being absorbed at N. Since, once a particle reaches any of the points 0 or N, it has to stay there forever, we have that $p_0^N(0) = 0$ and $p_0^N(1) = 1$. Therefore, we obtain the following equation

$$p_1^N(m) = \frac{\frac{1}{N^\theta}}{1+\frac{1}{N^\theta}} p_0^N(m) + \frac{1}{1+\frac{1}{N^\theta}} p_2^N(m) \quad \Leftrightarrow \quad (1+N^\theta)p_1^N(m) = p_0^N(m) + N^\theta p_2^N(m).$$

- Analogously, if $x \in \{2, ..., N-2\}$, then $2p_x^N(m) = p_{x+1}^N(m) + p_{x-1}^N(m)$, i.e., $\Delta_N^{1D} p_x^N(m) = 0$.

- Finally, if $x = N-1$, then $(1+N^\theta)p_{N-1}^N(m) = p_N^N(m) + N^\theta p_{N-2}^N(m)$, where here, analogously to the case $x = 1$, $p_N^N(1)$ represents the probability that a particle that is at the point N being absorbed at 0 and $p_N^N(0)$ represents the probability that a particle that is at the point N being absorbed at N. By the same arguments as in the case $x = 1$, we conclude that $p_N^N(0) = 1$ and $p_N^N(1) = 0$.

Putting together the boundary conditions for each value of m and the previous results, we obtain a system of equations that can be summarized, like in [7], as:

$$\begin{cases} \mathfrak{B}_N^\theta p_x^N(m) = 0, \text{ for } x = 1, \dots, N-1, \\ p_0^N(m) = \mathbb{1}_{m=1}, \\ p_N^N(m) = \mathbb{1}_{m=0}, \end{cases} \tag{4.2}$$

where the operator $\mathfrak{B}_N^\theta : \mathcal{F}(\{0,\dots,N\}) \to \mathcal{F}(\Lambda_N)$ is defined, for every function $f \in \mathcal{F}(\{0,\dots,N\})$, as

$$\mathfrak{B}_N^\theta f(x) = a_x[f(x-1) - f(x)] + b_x[f(x+1) - f(x)], \tag{4.3}$$

with $a_x = \frac{N^2}{N^\theta}\mathbb{1}_{x=1} + N^2\mathbb{1}_{x\neq 1}$ and $\quad b_x = \frac{N^2}{N^\theta}\mathbb{1}_{x=N-1} + N^2\mathbb{1}_{x\neq N-1}$.

For every $x \in \{2,\dots,N-2\}$, $p_x^N(m)$ is solution to the one-dimensional discrete Laplace equation ($\Delta_N^{1D} p_x^N(m) = 0$). Then, for every $x \in \Lambda_N$,

$$p_x^N(m) = A^m x + B^m, \tag{4.4}$$

with $A^m, B^m \in \mathbb{R}$ to be determined. Note that A^m and B^m do not represent the power m of some real numbers A and B, but it is just a notation to recall that the coefficients depend on m. Using the remaining equations (that use the boundary conditions) given in (4.2) by taking $x = 1$ and $x = N-1$, we get, for m = 1,

$$\begin{cases} (1+N^\theta)(A^1+B^1) = N^\theta(2A^1+B^1)+1 \\ (1+N^\theta)[A^1(N-1)+B^1] = N^\theta[A^1(N-2)+B^1] \end{cases} \Leftrightarrow \begin{cases} A^1 = -\frac{1}{N+2N^\theta-2} \\ B^1 = \frac{N^\theta+N-1}{N+2N^\theta-2} \end{cases}.$$

We could easily adapt the previous computations to $m = 0$, however, by the total law, $p_x^N(0) = 1 - p_x^N(1)$, we do not need to compute the remaining coefficients. Replacing in (4.4), we get

$$\begin{cases} p_x^N(0) = \frac{N^\theta-1+x}{N+2N^\theta-2}, \quad \text{if } x \in \Lambda_N, \\ p_x^N(1) = \frac{N^\theta-1+N-x}{N+2N^\theta-2}, \quad \text{if } x \in \Lambda_N, \\ p_0^N(m) = \mathbb{1}_{m=1}, \\ p_N^N(m) = \mathbb{1}_{m=0}. \end{cases} \tag{4.5}$$

Remark that, if $\theta \neq 0$, the first and second equations in (4.5) can not include the points $x = 0$ or $x = N$, since what we obtain does not coincide with the boundary conditions $p_x^N(1)$ and $p_x^N(0)$. This means that we lose "continuity" of the coefficients [2] when we get to the boundary points 0 and N. This is due to the fact that at $x = 1$ and $x = N-1$, $p_x^N(m)$ is no longer a solution to the one-dimensional discrete Laplace equation, since the factor N^θ creates a distortion on the operator (which is only eliminated when $\theta = 0$). In any case, using indicator functions at $x = 0$ and $x = N$, we can rewrite (4.5) in a compact form that is valid for every $x \in \{0,\dots,N\}$ (see Appendix B).

If $\theta = 0$, for every $x \in \{0,\dots,N\}$,

$$p_x^N(1) = 1 - \frac{x}{N} \quad \text{and} \quad p_x^N(0) = \frac{x}{N}, \tag{4.6}$$

where here we no longer have the "continuity" until the boundary issue, since, in this case, p_x^N is solution to the one-dimensional discrete Laplace equation in the whole Λ_N.

[2] We will use the term "continuity" to say that the coefficients A^m and B^m do not have dependence in x, i.e., they are the same for every $x \in \{0,\dots,N\}$.

Remark 4.1.1. *The computations with only one dual particle are relatively simple. As the number of particles increase, we get more and more extensive computations. Therefore, we will make use of Mathematica to verify our results for 2 and 3 particles (see Appendix A for the code).*

4.1.2 Case k = 2

- **Notation and conventions**

Note that now the dual process starts with $k = 2$ particles that can be absorbed at 0 or N. So, we can have both particles absorbed at 0, one absorbed at 0 and the other at N or both absorbed at N, i.e., $m \in \{0, 1, 2\}$. Let $x, y \in \Lambda_N$, with $x < y$ represent the position where each particle started, i.e., the initial configuration of the dual process is $\hat{\eta} = \delta_x + \delta_y$. Denote by $p^N_{x,y}(m)$ the probability that m dual particles are absorbed at the left reservoir knowing that one of the $k = 2$ particles started at position x and the other at position y. Observe that, if we extend the possible values for x and y to $x = 0$ or $y = N$, then $p^N_{0,y}(m)$ and $p^N_{x,N}(m)$ represent boundary terms, and we can also think about $p^N_{0,0}(m)$ and $p^N_{N,N}(m)$ with the natural meaning. Let us also keep in mind here that, for each m, $p^N_{x,y}(m)$ can be though as the image at (x, y) of a function $p^N(m) : \mathcal{T}^N \to [0, 1]$ where $p^N(m)(x, y) := p^N_{x,y}(m)$ which can be extended to the cathetus $\mathcal{T}^N_0$ and $\mathcal{T}^N_N$. Here $\mathcal{T}^N$ is a discretized triangle defined as $\mathcal{T}^N = \{(x, y) \in \Lambda^2_N \mid x < y\}$, and the cathetus $\mathcal{T}^N_0$ and $\mathcal{T}^N_N$ are defined by $\mathcal{T}^N_0 = \{(0, y) \mid y \in \{0, \dots, N\}\}$ and $\mathcal{T}^N_N = \{(x, N) \mid x \in \{0, \dots, N\}\}$.

It is very important to remark here that it does not make sense to consider $p^N_{x,y}(m)$ at the diagonal points (x, x) with $x \in \Lambda_N$ (we will denote by $Diag$ the set of these diagonal points). This is due to the fact that, in SEP(1), we do not allow particles to be on top of each other. This simple observation clearly distinguishes the special case $\alpha = 1$ in the analysis of SEP(α). As we will see, if $\alpha \geq 2$, we will have to look at what happens to $p^N_{x,y}(m)$ at these diagonal points, creating an extra difficulty. See Figure 4.5 for a geometric representation of the sets defined above.

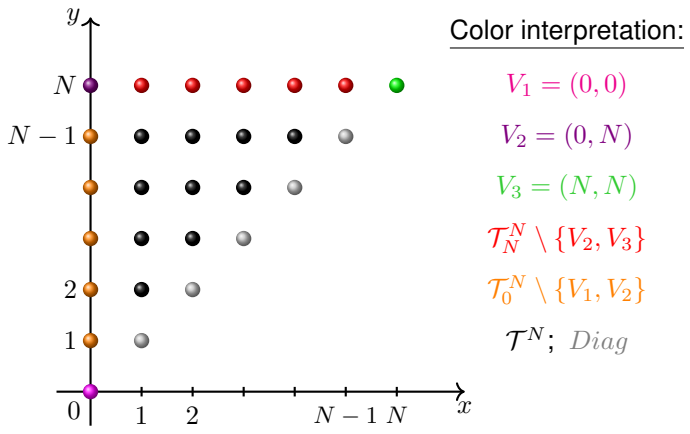

Figure 4.2: Geometric representation of $\mathcal{T}$, $\mathcal{T}_0$, $\mathcal{T}_N$ and $Diag$.

In what follows, the 2-dimensional discrete Laplacian operator, that we will denote by Δ^{2D}_N, will be a central object. This operator $\Delta^{2D}_N : \mathcal{F}(\mathcal{BT}^N) \to \mathcal{F}(\mathcal{T}^N)$, where $\mathcal{BT}^N = \{(x, y) \in \{0, \dots, N\}^2 \mid x < y\}$, is

defined, for every $f \in \mathcal{F}(\mathcal{BT}^N)$ and $(x,y) \in \mathcal{T}^N$, as

$$\Delta_N^{2D} f(x,y) = \begin{cases} \Delta_{N,full}^{2D} f(x,y), & \text{if } y \neq x+1, \\ \Delta_{N,ref}^{2D} f(x,x+1), & \text{if } y = x+1, \end{cases} \text{, where}$$

$$\begin{cases} \Delta_{N,full}^{2D} f(x,y) := N^2[f(x-1,y) + f(x+1,y) + f(x,y-1) + f(x,y+1) - 4f(x,y)], & \text{if } y \neq x+1, \\ \Delta_{N,ref}^{2D} f(x,x+1) := N^2[f(x-1,x+1) + f(x,x+2) - 2f(x,x+1)], & \text{if } y = x+1. \end{cases}$$

To $\Delta_{N,ref}^{2D}$ we call the 2-dimensional discrete reflected Laplacian. Remark that $\Delta_{N,full}^{2D} = (\Delta_N^{1D})_x + (\Delta_N^{1D})_y$, where $(\Delta_N^{1D})_x$ and $(\Delta_N^{1D})_y$ denote the 1-dimensional Laplacian operator acting on the first and second variable, respectively, of a function in $\mathcal{F}(\mathcal{BT}^N)$. This notation will be helpful later on.

- **Conditioning and results**

Observe that SEP(1) with absorbing boundary points with just two particles represents a particle performing a two-dimensional random walk on a triangle (jumps can only occur to points that are left, right, upward or downward nearest-neighbor of the initial position of the random walk) with absorbing cathetuses, meaning that, once a particle reaches a cathetus, it can not leave it, restricting the particle to continue to perform a one-dimensional random walk, as in the case $k = 1$ (on a line segment with absorbing boundary) and jump rates given as in Figure 4.3.

As we did for the case $k = 1$, to compute $p_{x,y}^N(m)$, we condition on the first jump and we get a system of equations that $p_{x,y}^N(m)$ has to satisfy, for each fixed m. Then:

- If $x = 1$ and $y = 2$, because of the exclusion rule, there are only two possible jumps: either the particle at 1 is absorbed at 0 or the particle at 2 jumps to the right, i.e., to site 3. In Figure 4.3, this represents on the triangle either a jump from the point $(1,2)$ to the point $(0,2)$, which occurs with rate $\frac{1}{N^\theta}$, or, from the point $(1,2)$ to the point $(1,3)$, which occurs with rate 1.

  Then, again by the Markov property, we obtain the identity $(1+N^\theta)p_{1,2}^N(m) = p_{0,2}^N(m) + N^\theta p_{1,3}^N(m)$.

- If $x = 1$ and $y \in \{3, ..., N-2\}$, following the same type of arguments as before, we get

$$(3N^\theta + 1)p_{1,y}^N(m) = p_{0,y}^N(m) + N^\theta p_{2,y}^N(m) + N^\theta p_{1,y+1}^N(m) + N^\theta p_{1,y-1}^N(m).$$

- If $x = 1$ and $y = N-1$,

$$(2N^\theta + 2)p_{1,N-1}^N(m) = p_{0,N-1}^N(m) + N^\theta p_{2,N-1}^N(m) + N^\theta p_{1,N-2}^N(m) + p_{1,N}^N(m).$$

- If $x \in \{2, ..., N-3\}$ and $y = x+1$,

$$2p_{x,x+1}^N(m) = p_{x-1,x+1}^N(m) + p_{x,x+2}^N(m), \text{ i.e., } \Delta_{N,ref}^{2D} p_{x,x+1}^N(m) = 0.$$

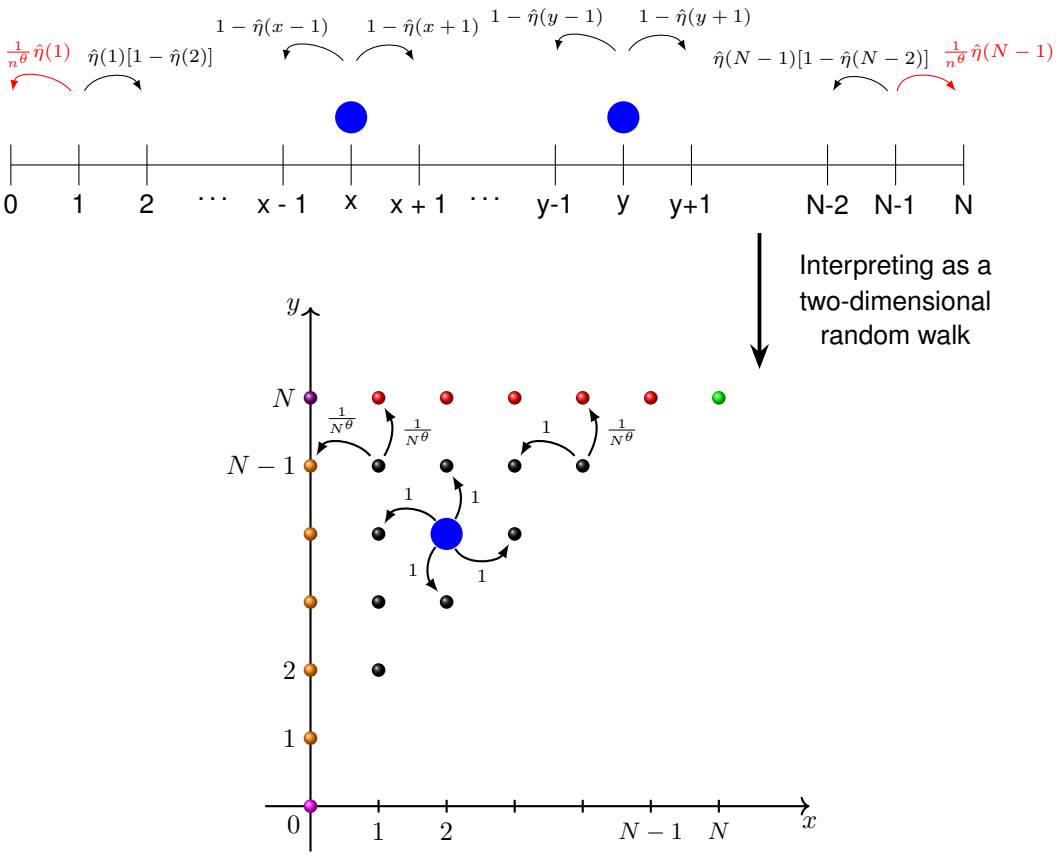

Figure 4.3: Geometric interpretation of SEP(1) with absorbing boundary points with two particles. Representation of some of the jump rates (in blue, we represent the random walk).

- If $x \in \{2, ..., N-4\}$ and $y \in \{x+2, ..., N-2\}$,

$$4p^N_{x,y}(m) = p^N_{x-1,y}(m) + p^N_{x+1,y}(m) + p^N_{x,y-1}(m) + p^N_{x,y+1}(m), \text{ i.e., } \Delta^{2D}_{N,full}p^N_{x,y}(m) = 0.$$

- If $x \in \{2, ..., N-3\}$ and $y = N-1$,

$$(3N^\theta + 1)p^N_{x,N-1}(m) = N^\theta p^N_{x-1,N-1}(m) + N^\theta p^N_{x+1,N-1}(m) + N^\theta p^N_{x,N-2}(m) + p^N_{x,N}(m).$$

- Finally, if $x = N-2$ and $y = N-1$, $(1+N^\theta)p^N_{N-2,N-1}(m) = N^\theta p^N_{N-3,N-1}(m) + p^N_{N-2,N}(m)$.

Thus, we get a system of equations with boundary conditions given, for every $(x,y) \in \mathcal{T}^N_0$, by $p^N_{x,y}(m) = p^N_y(m-1)\mathbb{1}_{\{m\neq 0\}}$ and, for every $(x,y) \in \mathcal{T}^N_N$, $p^N_{x,y}(m) = p^N_x(m)\mathbb{1}_{\{m\neq 2\}}$.

Let us briefly explain the expression for $p^N_{x,y}(m)$, when $(x,y) \in \mathcal{T}^N_0$ (for $(x,y) \in \mathcal{T}^N_N$, it is analogous). Recall that $p^N_{0,y}(m)$ represents the probability that m particles are absorbed at 0 knowing that one particle is already at site 0 and the other one is at site y. So, if $m = 0$, it means that all particles were absorbed at site N. Since one of the dual particles is already at zero and can not leave that state, then $p^N_{0,y}(0) = 0$. If $m \in \{1,2\}$, this means that at least one particle was absorbed at 0. Again, since one of the dual particles is already at zero, that same particle already counts as an absorbed particle. Then, the value of $p^N_{0,y}(m)$, for $m = 1$ and $m = 2$, only depends on what happens to the particle at y. Therefore, $p^N_{0,y}(1) = p^N_y(0)$

and $p^N_{0,y}(2) = p^N_y(1)$.

The previous equations jointly with the boundary conditions can be summed up, like in [7], as:

$$\begin{cases} \mathfrak{O}^\theta_N p^N_{x,y}(m) = 0, \text{ for } x = 1, \dots, N-2 \text{ and } y = x+1, \dots, N-1, \\ p^N_{0,y}(m) = p^N_y(m-1)\mathbb{1}_{\{m\neq 0\}}, \text{ if } y = 0, \dots, N, \\ p^N_{x,N}(m) = p^N_x(m)\mathbb{1}_{\{m\neq 2\}}, \text{ if } x = 0, \dots, N. \end{cases} \tag{4.7}$$

where the operator $\mathfrak{O}^\theta_N : \mathcal{F}(\mathcal{BT}^N) \to \mathcal{F}(\mathcal{T}^N)$ is defined, for every function $f \in \mathcal{F}(\mathcal{BT}^N)$ and for every $(x, y) \in \mathcal{T}^N$, i.e., $x, y \in \Lambda_N$ with $x < y$, by

$$\mathfrak{O}^\theta_N f(x,y) = a_x[f(x-1,y) - f(x,y)] + b_y[f(x,y+1) - f(x,y)] + c_{x,y}[f(x+1,y) + f(x,y-1) - 2f(x,y)], \tag{4.8}$$

where $a_x = \frac{N^2}{N^\theta}\mathbb{1}_{x=1} + N^2\mathbb{1}_{x\neq 1}$, $b_y = \frac{N^2}{N^\theta}\mathbb{1}_{y=N-1} + N^2\mathbb{1}_{y\neq N-1}$ and $c_{x,y} = N^2\mathbb{1}_{y\neq x+1}$.

For every $x, y \in \{2, \dots, N-2\}$ with $x < y$, $\Delta^{2D}_N p^N_{x,y}(m) = 0$. Thus, $p^N_{x,y}(m)$ is solution to the discrete Laplace equation on the triangle $\mathcal{T}^N$ and, therefore, can be written in polynomial form

$$p^N_{x,y} = A^m x + B^m y + C^m xy + D^m \tag{4.9}$$

with $A^m, B^m, C^m, D^m \in \mathbb{R}$ to be determined[3] by the remaining equations and boundary conditions compacted in (4.7). One possible way of finding these coefficients is by solving it with the ansatz given in (4.9). Here are the results that we checked using Mathematica (see Appendix A for Mathematica's code): denoting by $\tilde{p}^{N-1}_x(0) := \frac{N^\theta - 1 + x}{N + 2N^\theta - 3}$ and $\tilde{p}^{N-1}_x(1) := \frac{N^\theta - 2 + N - x}{N + 2N^\theta - 3}$, we get

$$\begin{cases} p^N_{x,y}(0) = p^N_x(0) - p^N_y(1)\tilde{p}^{N-1}_x(0), \text{ if } (x,y) \in \mathcal{T}^N, \\ p^N_{x,y}(1) = p^N_x(1) + p^N_y(1)\left[\tilde{p}^{N-1}_x(1) - \tilde{p}^{N-1}_x(0)\right], \text{ if } (x,y) \in \mathcal{T}^N, \\ p^N_{x,y}(2) = p^N_y(1)\tilde{p}^{N-1}_x(1), \text{ if } (x,y) \in \mathcal{T}^N, \\ p^N_{0,y}(m) = p^N_y(m-1)\mathbb{1}_{\{m\neq 0\}}, \text{ if } y = 0, \dots, N \text{ and } m = 0, 1, 2, \\ p^N_{x,N}(m) = p^N_x(m)\mathbb{1}_{\{m\neq 2\}}, \text{ if } x = 0, \dots, N \text{ and } m = 0, 1, 2, \end{cases} \tag{4.10}$$

where $p^N_x(0)$ and $p^N_y(1)$ correspond to the expressions given in (4.5). We observe that, if $\theta = 0$, then $\tilde{p}^{N-1}_x(0) = p^{N-1}_x(0)$ and $\tilde{p}^{N-1}_x(1) = p^{N-1}_x(1)$.

Again, if we try to replace x and y in the expressions for $p^N_{x,y}(0)$, $p^N_{x,y}(1)$ and $p^N_{x,y}(2)$ given in (4.10) by 0 or N, we immediately see that what we get does not match with the correct boundary conditions. This is related to the fact that if $x = 1$ or $y = N-1$, $p^N_{x,y}(m)$ is no longer a solution for the two-dimensional discrete Laplace equation, since, like for the one particle case, the factor N^θ creates a distortion on the operator. However, using indicator functions, we can still get compacted expressions for $p^N_{x,y}(0)$, $p^N_{x,y}(1)$ and $p^N_{x,y}(2)$ that are valid for any $x, y \in \{0, \dots, N\}$ with $x < y$ (see Appendix B).

[3]Remark here that A^m and B^m are not the same coefficients as the ones in the case $k = 1$, but we will use the same notation to remind the reader that they are new coefficients that still depend on m.

For $\theta = 0$, we can use a different argument: instead of starting with the ansatz (4.9), we write $p^N_{x,y}(m)$, for every $(x, y) \in \mathcal{BT}^N$, as

$$p^N_{x,y}(m) = A^m_x B^m_y + C^m_x, \tag{4.11}$$

where $W^m_x := W^m(x)$ and $W^m : \{0, \dots, N\} \to \mathbb{R}$, for $W \in \{A, C\}$, and $B^m_y := B^m(y)$ and $B^m : \{0, \dots, N\} \to \mathbb{R}$. Instead of finding 12 coefficients, we now have to find the expression of 9 functions. This strategy allow us to find a recursive (in m and N) formula for $p^N_{x,y}(m)$ and also write $p^N_{x,y}(m)$ in terms of the absorption probabilities of just one particle.

This strategy is particularly useful in this case, because we have "continuity" of the coefficients of $p^N_{x,y}(m)$ until the boundary and the system in (4.7) is reduced to two different types of equations:

$$\begin{cases} \Delta^{2D}_{N,full} p^N_{x,y}(m) = 0, \quad x = 1, \dots, N-3 \text{ and } y = x+2, \dots, N-1 \\ \Delta^{2D}_{N,ref} p^N_{x,y}(m) = 0, \quad x = 1, \dots, N-2 \text{ and } y = x+1 \end{cases}. \tag{4.12}$$

This means that $p^N_{x,y}(m)$ is a solution to the discrete Laplace equation (that is reflected at $y = x+1$), but now with $x, y \in \Lambda_N$ and $x < y$. Since we have "continuity" until the boundary, the expression for $p^N_{x,y}(m)$ given in (4.11) is valid even if $x = 0$ and $y = N$. So, using the boundary condition for $y = N$ and defining $p^N_x(2) := 0$ [4], we get that $C^m_x = -A^m_x B^m_N + p^N_x(m)$. Therefore, $p^N_{x,y}(m) = A^m_x[B^m_y - B^m_N] + p^N_x(m)$. Observe that if $\theta \neq 0$, this first step is no longer valid since $p^N_{x,y}(m)$ given in (4.11) would not be valid with the same coefficients in $x = 0$ and $y = N$. Using the other boundary condition, for $x = 0$, we get that

$$A^m_0[B^m_y - B^m_N] + p^N_0(m) = \begin{cases} p^N_y(m-1), \text{ if } m = 1, 2 \\ 0, \text{ if } m = 0 \end{cases}.$$

If $m = 1, 2$, then, $A^m_0 \neq 0$ (since $p^N_y(m-1) \neq p^N_0(m)$, if $y \neq N$) and $B^m_y - B^m_N = \frac{p^N_y(m-1) - p^N_0(m)}{A^m_0}$.

Then,

$$p^N_{x,y}(m) = \begin{cases} A^m_x \frac{p^N_y(m-1) - p^N_0(m)}{A^m_0} + p^N_x(m), \text{ if } m = 1, 2 \\ A^0_x[B^0_y - B^0_N] + p^N_x(0), \text{ if } m = 0 \end{cases}.$$

Now using (4.12) and that $\Delta^{1D}_N p^N_x(m) = 0$ (the same holds replacing x by y), we get, for $m = 1, 2$,

$$\begin{cases} (\Delta^{1D}_N)_x p^N_{x,y}(m) + (\Delta^{1D}_N)_y p^N_{x,y}(m) = 0 \\ \Delta^{2D}_{ref} p^N_{x,y}(m) = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \Delta^{1D}_N A^m_x = 0, \ x = 1, \dots, N-3 \\ A^m_x = \frac{A^m_{x-1}[p^N_{x+1}(m-1) - p^N_0(m)] + A^m_0[p^N_{x-1}(m) - p^N_x(m)]}{p^N_x(m-1) - p^N_0(m)}, \ x \in \Lambda_N \setminus \{N-1\} \end{cases}. \tag{4.13}$$

Then, using recursively (4.13), we get:

[4] From now on, every time the reader sees $p^N_x(m)$ for some value of m that is not 0 or 1, it should be interpreted as zero.

- For $m = 2$

$$A_x^2 = A_{x-1}^2 \frac{p_{x+1}^N(1)}{p_x^N(1)} = A_0^2 \frac{p_{x+1}^N(1)}{p_x^N(1)} \frac{p_x^N(1)}{p_{x-1}^N(1)} \frac{p_{x-1}^N(1)}{p_{x-2}^N(1)} \cdots \frac{p_2^N(1)}{p_1^N(1)} = A_0^2 \frac{p_{x+1}^N(1)}{p_1^N(1)}.$$

  Using (4.6), we see that $\frac{p_{x+1}^N(1)}{p_1^N(1)} = 1 - \frac{x}{N-1} = p_x^{N-1}(1)$, which corresponds to the absorption probability for a particle, starting at site x, being absorbed at 0 on a lattice of size $N-1$.

- For $m = 1$, since, for every $x \in \{1, \dots, N-1\}$, $\Delta_N^{1D} p_x^N(m) = 0$, applying recursively the second equation in (4.13), we get

$$\begin{aligned}
\frac{A_x^1}{A_0^1} &= \frac{A_{x-1}^1}{A_0^1} \frac{p_{x+1}^N(1)}{p_x^N(1)} + \frac{p_x^N(1) - p_{x-1}^N(1)}{p_x^N(1)} \\
&= \frac{p_{x+1}^N(1)}{p_x^N(1)} \left[\frac{A_{x-2}^1}{A_0^1} \frac{p_x^N(1)}{p_{x-1}^N(1)} + \frac{p_{x-1}^N(1) - p_{x-2}^N(1)}{p_{x-1}^N(1)} \right] + \frac{p_x^N(1) - p_{x-1}^N(1)}{p_x^N(1)} \\
&= \frac{A_{x-2}^1}{A_0^1} \frac{p_{x+1}^N(1)}{p_{x-1}^N(1)} + \frac{[2p_x^N(1) - p_{x-1}^N(1)][p_{x-1}^N(1) - p_{x-2}^N(1)] + p_{x-1}^N(1)[p_{x-1}^N(1) - p_{x-2}^N(1)]}{p_x^N(1) p_{x-1}^N(1)} \\
&= \frac{A_{x-2}^1}{A_0^1} \frac{p_{x+1}^N(1)}{p_{x-1}^N(1)} + 2\frac{p_{x-1}^N(1) - p_{x-2}^N(1)}{p_{x-1}^N(1)} \\
&= \cdots = \frac{A_{x-x}^1}{A_0^1} \frac{p_{x+1}^N(1)}{p_{x-(x-1)}^N(1)} + x\frac{p_{x-(x-1)}^N(1) - p_{x-x}^N(1)}{p_{x-(x-1)}^N(1)} = \frac{p_{x+1}^N(1)}{p_1^N(1)} + x\frac{p_1^N(1) - 1}{p_1^N(1)}.
\end{aligned}$$

  Again, using (4.6) and the fact that $p_1^N(1) = 1 - p_1^N(0)$, we obtain

$$\frac{A_x^1}{A_0^1} = \underbrace{1 - \frac{x}{N-1}}_{p_x^{N-1}(1)} - \underbrace{\frac{x}{N-1}}_{p_x^{N-1}(0)} = p_x^{N-1}(1) - p_x^{N-1}(0). \tag{4.14}$$

  We are only missing now the expression of $p_{x,y}^N(0)$.

- If $m = 0$, then $A_0^0 = 0$ otherwise B_y^0 was constant ($B_y^m = B_N^m$), which is absurd since that would imply that $p_{x,y}^N(0)$ would not depend on y (we would get that $p_{x,y}^N(0) = p_x^N(0)$). To obtain the expression for $p_{x,y}^N(0)$, we can try a similar (but harder) approach as before, but it is much simpler to use total law of probability. Then,

$$p_{x,y}^N(0) = \underbrace{1 - p_x^N(1)}_{= p_x^N(0)} + [p_x^{N-1}(1) - p_x^{N-1}(0)] p_y^N(1) - p_x^{N-1}(1) p_y^N(1) = -p_x^{N-1}(0) p_y^N(1) + p_x^N(0).$$

Summarizing, for every $(x, y) \in \mathcal{BT}^N$, we have

$$\begin{cases}
p_{x,y}^N(0) = -p_y^N(1) p_x^{N-1}(0) + p_x^N(0), \\
p_{x,y}^N(1) = -p_y^N(1)[p_x^{N-1}(1) - p_x^{N-1}(0)] + p_x^N(1), \\
p_{x,y}^N(2) = p_y^N(1) p_x^{N-1}(1),
\end{cases} \tag{4.15}$$

or equivalently, for $m = 0, 1, 2$ and $\theta = 0$,

$$p^N_{x,y}(m) = p^N_x(m) + p^N_y(1)[p^{N-1}_x(m-1) - p^{N-1}_x(m)]. \tag{4.16}$$

Remark 4.1.2. *Observing the solutions in* (4.10)*, even for $\theta \neq 0$, the absorption probabilities factorize as a function of y times a function of x. Therefore, we expect that the argument that we did for $\theta = 0$ can be generalized for $\theta \neq 0$, but now the boundary conditions will have to be taken into account on the system of equations and not used directly on the ansatz, as we did.*

4.1.3 Case k = 3

- **Notation and conventions**

Now the dual process starts with $k = 3$ particles that can be absorbed at 0 or N. So, we can have k possible scenarios: three particles are absorbed at 0, one is absorbed at 0 and two at N, two at 0 and one at N or all three absorbed at N, i.e., $m \in \{0, 1, 2, 3\}$. Let $x, y, z \in \Lambda_N$, where $x < y < z$, represent the position where each particle started, i.e., the initial configuration of the dual process is $\hat{\eta} = \delta_x + \delta_y + \delta_z$. Denote by $p^N_{x,y,z}(m)$ the probability that, knowing that the three particles started, each one, on positions x, y and z, m of them being absorbed at the left reservoir on a lattice of size N[5]. Observe that, if we extend the possible values for x and z to $x = 0$ or $z = N$, then $p^N_{0,y,z}(m)$ and $p^N_{x,y,N}(m)$ represent boundary terms. Let us keep in mind here that $p^N_{x,y,z}(m)$ can be though as the image at (x, y, z) of a function $p^N(m) : \mathcal{S}^N \to [0, 1]$ ($p^N(m)(x, y, z) := p^N_{x,y,z}(m)$), where $\mathcal{S}^N$ is a discretized simplex defined as $\mathcal{S}^N = \{(x, y, z) \in \Lambda^3_N \mid x < y < z\}$. Also, $p^N_{x,y}(m)$ can be extended to the faces $\mathcal{S}^N_0$ and $\mathcal{S}^N_N$, i.e., $\mathcal{S}^N_0 = \{(0, y, z) \mid (y, z) \in \mathcal{BT}^N\}$ and $\mathcal{S}^N_N = \{(x, y, N) \mid (x, y) \in \mathcal{BT}^N\}$, where $\mathcal{BT}^N = \{(x, y) \in \{0, \dots, N\}^2 \mid x < y\}$ as it was in the case $k = 2$.

It is very important to remark that, since, in SEP(1), we do not allow particles to be on top of each other, it does not make sense to define $p^N_{x,y}(m)$ on the line $x = y = z$ nor on the planes $x = y$ and $y = z$. These sets have to be considered when repeating this strategy for SEP(α) and SIP(α). The discrete simplex $\mathcal{S}^N$ corresponds to the interior points (excluding the boundary) of a discretization (in the points with positive integer coordinates) of the simplex presented in Figure 4.4; the sets $\mathcal{S}^N_0$ and $\mathcal{S}^N_N$ may be interpreted as the points of the same discretization that lie on the face painted in green and the face painted in blue, respectively, also in Figure 4.4.

In what follows, the 3-dimensional discrete Laplacian operator, that we will denote by Δ^{3D}_N, will be a central object. This operator $\Delta^{3D}_N : \mathcal{F}(\mathcal{BS}^N) \to \mathcal{F}(\mathcal{S}^N)$, is defined, for every $f \in \mathcal{F}(\mathcal{BS}^N)$, where

[5] By this, we mean here that, when we extend the bulk to the points 0 and N, the point with higher coordinate is N even thought we have $N + 1$ possible positions to place a particle on the dual system.

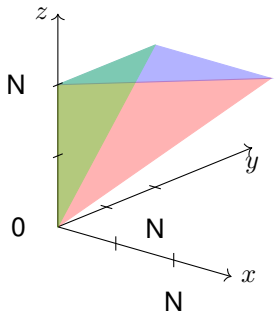

Figure 4.4: Geometric representation of the simplex.

$\mathcal{BS}^N = \{(x,y,z) \in (\Lambda_N)^3 \mid 0 \le x < y < z \le N\}$, and $(x,y,z) \in \mathcal{S}^N$, as

$$\Delta_N^{3D} f(x,y,z) = \begin{cases} \Delta_{N,full}^{3D} f(x,y,z), & \text{if } y \neq x+1 \text{ and } z \neq y+1, \\ (\Delta_{N,ref}^{2D})_{x,y} f(x,y,z) + (\Delta_N^{1D})_z f(x,y,z), & \text{if } y = x+1 \text{ and } z \neq y+1, \\ (\Delta_{N,ref}^{2D})_{y,z} f(x,y,z) + (\Delta_N^{1D})_x f(x,y,z), & \text{if } y \neq x+1 \text{ and } z = y+1, \\ \Delta_{N,ref}^{3D} f(x,y,z), & \text{if } y = x+1 \text{ and } z = x+2, \end{cases}$$

where, $\Delta_{N,full}^{3D} f(x,y,z) := (\Delta_N^{1D})_x f(x,y,z) + (\Delta_N^{1D})_y f(x,y,z) + (\Delta_N^{1D})_z f(x,y,z)$, $\Delta_{N,ref}^{3D} f(x,x+1,x+2) := N^2[f(x-1,x+1,x+2) + f(x,x+1,x+3) - 2f(x,x+1,x+2)]$, and $(\Delta_{N,ref}^{2D})_{x,y}$ (resp. $(\Delta_{N,ref}^{2D})_{y,z}$) represents the 2-dimensional reflected discrete Laplacian acting on the first and second (resp. second and third) arguments of f. We call $\Delta_{N,ref}^{3D}$ the 3-dimensional discrete reflected Laplacian.

- **Conditioning and results**

As above, conditioning on the first jump, we get a system of linear equations, that can be compacted in the following system:

$$\begin{cases} \mathfrak{R}_N^\theta p_{x,y,z}^N(m) = 0, \text{ for } x = 1,\dots,N-3, y = x+1,\dots,N-2 \text{ and } z = y+1,\dots,N-1, \\ p_{0,y,z}^N(m) = p_{y,z}^N(m-1)\mathbb{1}_{\{m\neq 0\}}, \text{ if } (y,z) \in \mathcal{BT}^N, \\ p_{x,y,N}^N(m) = p_{x,y}^N(m)\mathbb{1}_{\{m\neq 3\}}, \text{ if } (x,y) \in \mathcal{BT}^N, \end{cases} \tag{4.17}$$

where $\mathcal{BT}^N$ is the set that we introduced in the case $k=2$ and the operator $\mathfrak{R}_N^\theta : \mathcal{F}(\mathcal{BS}^N) \to \mathcal{F}(S^N)$ is defined, for every function $f \in \mathcal{F}(\mathcal{BS}^N)$ and for every $x,y,z \in \Lambda_n^3$ with $x < y < z$, by

$$\mathfrak{R}_N^\theta f(x,y,z) = a_x[f(x-1,y,z) - f(x,y,z)] + b_z[f(x,y,z+1) - f(x,y,z)] + c_{y,z}[f(x,y+1,z) + f(x,y,z-1) - 2f(x,y,z)] + c_{x,y}[f(x+1,y,z) + f(x,y-1,z) - 2f(x,y,z)],$$

where $a_x = \frac{N^2}{N^\theta}\mathbb{1}_{x=1} + N^2\mathbb{1}_{x\neq 1}$, $b_z = \frac{N^2}{N^\theta}\mathbb{1}_{z=N-1} + N^2\mathbb{1}_{z\neq N-1}$, $c_{y,z} = N^2\mathbb{1}_{y\neq z-1}$ and $c_{x,y} = N^2\mathbb{1}_{x\neq y-1}$.

For every $x,y,z \in \{2,\dots,N-2\}$ such that $x<y<z$, $\mathfrak{R}_N^\theta p_{x,y,z}^N(m) = \Delta_N^{3D} p_{x,y,z}^N(m) = 0$. Therefore, $p_{x,y,z}^N(m)$ is solution to the discrete Laplace equation on the simplex $\mathcal{S}$ and can be described by a

polynomial of the form

$$p^N_{x,y,z}(m) = A^m x + B^m y + C^m z + D^m xy + E^m xz + F^m yz + G^m xyz + H^m, \tag{4.18}$$

with $A^m, B^m, C^m, D^m, E^m, F^m, G^m, H^m \in \mathbb{R}$ to be determined by the remaining equations and boundary conditions.

Again, we can use the ansatz (4.18) to solve the system compacted in (4.17) and compare our results with Mathematica. The solution is the following (see Appendix A for the links for the code): denoting by $\tilde{p}^{N-1}_{x,y}(0) := \frac{(N^\theta-1+x)(N^\theta-2+y)}{(N+2N^\theta-3)(N+2N^\theta-4)}$, $\tilde{p}^{N-1}_{x,y}(2) := \frac{(N^\theta-2+N-y)(N^\theta-3+N-x)}{(N+2N^\theta-3)(N+2N^\theta-4)}$ and $\tilde{p}^{N-1}_{x,y}(1) := 1 - \tilde{p}^{N-1}_{x,y}(0) - \tilde{p}^{N-1}_{x,y}(2)$, we get

$$\begin{cases} p^N_{x,y,z}(0) = p^N_{x,y}(0) - p^N_z(1)\tilde{p}^{N-1}_{x,y}(0), \text{ if } (x,y,z) \in \mathcal{S}^N, \\ p^N_{x,y,z}(1) = p^N_{x,y}(1) + p^N_z(1)\left[\tilde{p}^{N-1}_{x,y}(1) - \tilde{p}^{N-1}_{x,y}(0)\right], \text{ if } (x,y,z) \in \mathcal{S}^N, \\ p^N_{x,y,z}(2) = p^N_{x,y}(2) + p^N_z(1)\left[\tilde{p}^{N-1}_{x,y}(2) - \tilde{p}^{N-1}_{x,y}(1)\right], \text{ if } (x,y,z) \in \mathcal{S}^N, \\ p^N_{x,y,z}(3) = p^N_z(1)\tilde{p}^{N-1}_{x,y}(2), \text{ if } (x,y,z) \in \mathcal{S}^N, \\ p^N_{0,y,z}(m) = p^N_{y,z}(m-1)\mathbb{1}_{\{m\neq 0\}}, \text{ if } (y,z) \in \mathcal{BT}^N \text{ and } m \in \{0,1,2,3\}, \\ p^N_{x,y,N}(m) = p^N_{x,y}(m)\mathbb{1}_{\{m\neq 3\}}, \text{ if } (x,y) \in \mathcal{BT}^N \text{ and } m \in \{0,1,2,3\}, \end{cases} \tag{4.19}$$

where $p^N_{x,y}(0)$ and $p^N_{y,z}(1)$ are as in (4.10). Like for the case $k = 2$, we observe that the absorption probabilities can be written in a factorized form, where, for $m \in \{0,1,2\}$, $\tilde{p}^{N-1}_{x,y}(m) = p^{N-1}_{x,y}(m)$ if ,and only if, $\theta = 0$.

As we had for one and two particles, the first four expressions in the system (4.19) are not valid for $x, y, z \in \{0, N\}$, if $\theta \neq 0$, since they do not match with the correct boundary conditions. Again, this is a result of the distortion on the Laplacian operator caused by θ, which is only eliminated when $\theta = 0$. Nevertheless, we can still write explicit expressions for $p^N_{x,y,z}(0)$, $p^N_{x,y,z}(1)$, $p^N_{x,y,z}(2)$ and $p^N_{x,y,z}(3)$ that include the boundary conditions, and, therefore, valid for any $x, y, z \in \{0, \ldots, N\}$ with $x < y < z$ (see Appendix B).

For $\theta = 0$, as we did for the case $k = 2$, we can get the same probabilities using a different argument that is based on, instead of starting with the ansatz (4.18), decomposing the absorption probabilities for three dual particles $p^N_{x,y,z}$ as

$$p^N_{x,y,z}(m) = A^m_{x,y} B^m_z + C^m_{x,y}, \tag{4.20}$$

where $W^m_{x,y} := W^m(x,y)$ and $W^m : \{0,\ldots,N\}^2 \to \mathbb{R}$, for $W \in \{A, C\}$, and $B^m_z := B^m(z)$ and $B^m : \{0,\ldots,N\} \to \mathbb{R}$. If $\theta = 0$, we have "continuity" of the coefficients of $p^N_{x,y,z}(m)$ until the boundary. Therefore, the expression for $p^N_{x,y,z}(m)$ in (4.20) is valid for every $x, y, z \in \{0, \ldots, N\}$. Also, the system

in (4.17) is reduced to four different types of equations:

$$\begin{cases} \Delta^{3D}_{N,full} p^N_{x,y,z}(m) = 0, \quad x = 1, \dots, N-5,\ y = x+2, \dots, N-3 \text{ and } z = y+2, \dots, N-1, \\ (\Delta^{2D}_{N,ref})_{x,y} p^N_{x,y,z}(m) + (\Delta^{1D}_N)_z p^N_{x,y,z}(m) = 0, \quad x = 1, \dots, N-4, y = x+1 \text{ and } z = x+3, \dots, N-1, \\ (\Delta^{2D}_{N,ref})_{y,z} p^N_{x,y,z}(m) + (\Delta^{1D}_N)_z p^N_{x,y,z}(m) = 0, \quad x = 1, \dots, N-4, y = x+2, \dots, N-2 \text{ and } z = y+1, \\ \Delta^{3D}_{N,ref} p^N_{x,y,z}(m) = 0, \quad x = 1, \dots, N-3, \end{cases}$$

with the same boundary conditions.

Using the boundary condition for $z = N$ and defining $p^N_{x,y}(3) := 0$, we get that $C^m_{x,y} = -A^m_{x,y} B^m_N + p^N_{x,y}(m)$. Therefore, $p^N_{x,y,z}(m) = A^m_{x,y}[B^m_z - B^m_N] + p^N_{x,y}(m)$. By a similar argument as the one for the case $k = 2$, we can only do this first step on the case $\theta = 0$. Using the other boundary condition, for $x = 0$, we obtain

$$A^m_{0,y}[B^m_z - B^m_N] + p^N_{0,y}(m) = \begin{cases} p^N_{y,z}(m-1), \text{ if } m = 1, 2, 3, \\ 0, \text{ if } m = 0. \end{cases}$$

If $m = 1, 2, 3$, then, $A^m_{0,y} \neq 0$ (since $p^N_{y,z}(m-1) \neq p^N_y(m-1) = p^N_{0,y}(m)$, if $z \neq N$) and

$$B^m_z - B^m_N = \frac{p^N_{y,z}(m-1) - p^N_{0,y}(m)}{A^m_0} = \frac{p^N_{y,z}(m-1) - p^N_y(m-1)}{A^m_0}.$$

Using the recurrence formula in (4.16) and that $p^N_z(1) \neq 0$, if $z \neq N$, we get that

$$B^m_z - B^m_N = \frac{p^N_z(1)[p^N_y(m-2) - p^N_y(m-1)]}{A^m_{0,y}} \iff \underbrace{\frac{B^m_z - B^m_N}{p^N_z(1)}}_{\text{function of z}} = \underbrace{\frac{p^{N-1}_y(m-2) - p^{N-1}_y(m-1)}{A^m_{0,y}}}_{\text{function of y}}$$

$$\implies \begin{cases} B^m_z - B^m_N = C p^N_z(1), \\ A^m_{0,y} = \frac{1}{C}[p^{N-1}_y(m-2) - p^{N-1}_y(m-1)], \end{cases}$$

where the last implication follows from a separation of variables argument, i.e., if we have an equality between two functions, one of the y variable and the other of the z variable, that is true for every $y \in \{1, \dots, N-3\}$ and $z \in \{y+2, \dots, N-1\}$, then, those functions must be constant. In other words, there must exist some non-zero constant $C \in \mathbb{R}$ satisfying the identities above. Then,

$$p^N_{x,y,z}(m) = \begin{cases} C A^m_{x,y} p^N_z(1) + p^N_{x,y}(m), \text{ if } m = 1, 2, 3, \\ A^0_{x,y}[B^0_z - B^0_N] + p^N_{x,y}(0), \text{ if } m = 0. \end{cases}$$

We are now missing $A^m_{x,y}$ for $m = 1, 2, 3$ and, as we did for the 2 particles case, for $m = 0$, we can use total law to easily get a formula for $p^N_{x,y,z}(0)$ avoiding harder computations. Then, for $m = 1, 2, 3$:

- For every $x \in \{1, \dots, N-5\}$, $y \in \{x+2, \dots, N-3\}$ and $z \in \{y+2, \dots, N-1\}$,

$$\Delta^{3D}_{N,full} p^N_{x,y,z}(m) = C[\Delta^{2D}_{N,full} A^m_{x,y}] \underbrace{p^N_z(1)}_{\neq 0} + CA^m_{x,y} \underbrace{[\Delta^{1D}_N p^N_z(1)]}_{=0} + \underbrace{\Delta^{2D}_{N,full} p^N_{x,y}(m)}_{=0}.$$

  Therefore, $\Delta^{2D}_{N,full} A^m_{x,y} = 0$.

- For every $x \in \{1, \dots, N-4\}$, and $z \in \{x+3, \dots, N-1\}$, if $y = x+1$, then

$$[(\Delta^{2D}_{N,ref}) + (\Delta^{1D}_N)_z] p^N_{x,y,z}(m) = C[\Delta^{2D}_{N,ref} A^m_{x,y}] \underbrace{p^N_z(1)}_{\neq 0} + CA^m_{x,y} \underbrace{(\Delta^{1D}_N p^N_z(1))}_{=0} + \underbrace{\Delta^{2D}_{N,ref} p^N_{x,y}(m)}_{=0}.$$

  Therefore, $\Delta^{2D}_{N,ref} A^m_{x,y} = 0$.

- For every $x \in \{1, \dots, N-4\}$ and $y \in \{x+2, \dots, N-2\}$, if $z = y+1$, then

$$[(\Delta^{1D}_N)_x + (\Delta^{2D}_{N,ref})_{y,z}] p^N_{x,y,z}(m) = 0 \Leftrightarrow A^m_{x,y-1} = \left[A^m_{x,y} \frac{p^N_y(1)}{p^N_{y+1}(1)} - (\Delta^{1D}_N)_x A^m_{x,y} \right] + \frac{p^N_{x,y+1}(m) - p^N_{x,y}(m)}{C p^N_{y+1}(1)}.$$

  If $x \in \{1, \dots, N-5\}$ and $y \in \{x+2, \dots, N-3\}$, then, using that $\Delta^{2D}_{N,full} A^m_{x,y} = 0$, i.e., $(\Delta^{1D}_N)_x A^m_{x,y} = -(\Delta^{1D}_N)_y A^m_{x,y}$, we can simplify the previous identity to get

$$CA^m_{x,y+1} = CA^m_{x,y} \frac{p^N_{y+2}(1)}{p^N_{y+1}(1)} + \frac{p^N_{x,y}(m) - p^N_{x,y+1}(m)}{p^N_{y+1}(1)}.$$

- For all $x \in \{1, \dots, N-3\}$, if $y = x+1$ and $z = x+2$, then

$$\Delta^{3D}_{N,ref} p^N_{x,y,z}(m) = 0 \;\; \Leftrightarrow CA^m_{x,x+1} = CA^m_{x-1,x+1} \frac{p^N_{x+2}(1)}{p^N_{x+1}(1)} + \frac{p^N_{x-1,x+1}(m) - p^N_{x,x+1}(m)}{p^N_{x+1}(1)}.$$

Therefore, $E^m_{x,y} := CA^m_{x,y}$ is solution of the following system of equations

$$\begin{cases} \Delta^{2D}_{N,full} E^m_{x,y} = 0, \quad x \in \{1, \dots, N-5\} \text{ and } y \in \{x+2, \dots, N-3\} & (4.21) \\ \Delta^{2D}_{N,ref} E^m_{x,y} = 0, \quad x \in \{1, \dots, N-4\} \text{ and } y = x+1 & (4.22) \\ E^m_{x,y+1} = E^m_{x,y} \dfrac{p^N_{y+2}(1)}{p^N_{y+1}(1)} + \dfrac{p^N_{x,y}(m) - p^N_{x,y+1}(m)}{p^N_{y+1}(1)}, \quad x \in \{1, \dots, N-5\} \text{ and } y \in \{x+2, \dots, N-3\} & (4.23) \\ E^m_{x,x+1} = E^m_{x-1,x+1} \dfrac{p^N_{x+2}(1)}{p^N_{x+1}(1)} + \dfrac{p^N_{x-1,x+1}(m) - p^N_{x,x+1}(m)}{p^N_{x+1}(1)}, \quad x \in \{1, \dots, N-3\} \text{ and } y = x+1 & (4.24) \\ E^m_{0,y} = p^{N-1}_y(m-2) - p^{N-1}_y(m-1), \quad y \in \{0, \dots, N-1\} & (4.25) \end{cases}$$

Also, combining (4.22) and (4.24) we get that, for every $x \in \{1, \dots, N-4\}$, if $y = x+1$, then

$$\begin{aligned} E^m_{x,x+2} = 2E^m_{x,x+1} - E^m_{x,x-1} &= \frac{2p^N_{x+2}(1) - p^N_{x+1}(1)}{p^N_{x+2}(1)} E^m_{x,x+1} + \frac{p^N_{x,x+1}(m) - p^N_{x-1,x+1}(m)}{p^N_{x+1}(1)} \\ &= E^m_{x,x+1} \frac{p^N_{x+3}(1)}{p^N_{x+2}(1)} + \frac{p^N_{x,x+1}(m) - p^N_{x,x+2}(m)}{p^N_{x+2}(1)}, \end{aligned} \tag{4.26}$$

where the last equality comes from the fact that $p_x^N(1)$ a discrete harmonic function. Therefore, (4.23) is also true for $y = x + 1$.

Remark that every solution of the last system of equations is solution of the problem

$$\begin{cases} E^m_{x,y+1} = E^m_{x,y}\frac{p^N_{y+2}(1)}{p^N_{y+1}(1)} + \frac{p^N_{x,y}(m) - p^N_{x,y+1}(m)}{p^N_{y+1}(1)}, & x \in \{1,\dots,N-5\} \text{ and } y \in \{x+1,\dots,N-3\}, \\ E^m_{x,x+1} = E^m_{x-1,x+1}\frac{p^N_{x+2}(1)}{p^N_{x+1}(1)} + \frac{p^N_{x-1,x+1}(m) - p^N_{x,x+1}(m)}{p^N_{x+1}(1)}, & x \in \{1,\dots,N-3\} \text{ and } y = x+1, \\ E^m_{0,y} = p_y^{N-1}(m-2) - p_y^{N-1}(m-1), & y \in \{0,\dots,N-1\}. \end{cases} \tag{4.27}$$

Therefore, if we guarantee existence and uniqueness of the solution for (4.27), then, the original problem has to have at most one solution.

The system in (4.27) allows us to completely determine $CA^m_{x,y}$, for every $x \in \{1,\dots,N-3\}$ and $y \in \{x+1,\dots,N-2\}$. We will give now a geometric way to interpret this last statement. Remark that, if $y = N-1$, by the exclusion rule, $z = N$. Therefore, $p^N_{x,N-1,N}(m)$ is completely determined. So, we only care about finding $p^N_{x,y,z}(m)$ for $y \leq N-2$. This implies that $CA^m_{x,y}$ can be thought of as a function CA^m defined in $\mathcal{T}^{N-2}$, where $\mathcal{T}^{N-2} = \{(x,y) \in (\Lambda_{N-2})^2 \mid x < y\}$, with boundary values in $\mathcal{BT}^{N-2} := \{(0,y) \mid y \in \{1,\dots,N-2\}\}$. The next picture represents geometrically the sets $\mathcal{T}^{N-2}$ and $\mathcal{BT}^{N-2}$.

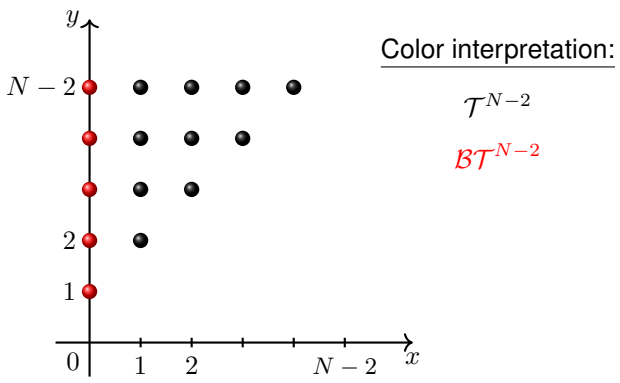

Figure 4.5: Geometric representation of $\mathcal{T}^{N-2}$ and $\mathcal{BT}^{N-2}$.

How can we interpret the system of equations solved by $CA^m_{x,y}$ using its domain? Starting from a point $(x_0, y_0) \in \{1,\dots,N-5\} \times \{x_0+2,\dots,N-2\}$, equation (4.23) allow us to recursively go down on the line $\{(x_0, z) \mid z \in \{x_0+1,\dots,N-2\}\}$ until we reach the point (x_0, x_0+2). Then, equation (4.26) allows us to go from (x_0, x_0+2) to the point (x_0, x_0+1). Now we can use equation (4.24) to change to the closest left vertical line, i.e., to $\{(x_0-1, z) \mid z \in \{x_0,\dots,N-2\}\}$. Repeating the same argument, we can reach the point (x_0-1, x_0) and then, using again equation (4.24) we can move to the new closest left vertical line. Iterating this process a finite number of movements, we will reach the boundary $\mathcal{BT}^{N-2}$ where we know the exact value of CA^m. If we start from any point of the form (x_0, x_0+1) where $x_0 \in \{1,\dots,N-4\}$, we can use equation (4.24) to get to the point $(x,y) = (x_0-1, x_0+1)$. Then, since now $x_0 \in \{1,\dots,N-5\}$ and $y = x_0+1$, we have reached a point that is in the previous situation, and therefore, we can repeat the same argument to reach one point on the line $x = 0$, where CA^m is well

defined. If we start from $(N-3, N-2)$, we can use equation (4.24) to move to $(N-4, N-2)$, from here use equation (4.26) to go to $(N-4, N-2)$ and repeat the same order of application of these two equations until we reach the point $(0, 2)$ where CA^m is completely determined. Finally, the cases starting from $(N-4, N-2)$ or $(N-4, N-3)$ are contained in the last case described above. See Figure 4.6 for an example of the previous arguments. Therefore, there is uniqueness of the solution for the previous system of discrete equations with one boundary condition (which implies that $C = 1$, by the restriction of the boundary condition). One possible way of finding $CA^m_{x,y}$, for every $(x, y) \in \mathcal{T}^{N-2}$, is going through the reverse path, i.e., starting from a point that can be computed directly from the boundary condition (such as $(1, 2)$), use it to compute all the others in the same vertical line, and, starting from the left most vertical column, move in the triangle, by going up and right to get all the values of CA^m in the triangle $\mathcal{T}^{N-2}$. Alternatively, if, for each $m \in \{1, 2, 3\}$, we find a function $f^m_{x,y}$ that solves the same system of equations, then, by uniqueness of solution, $A^m_{x,y} = f^m_{x,y}$.

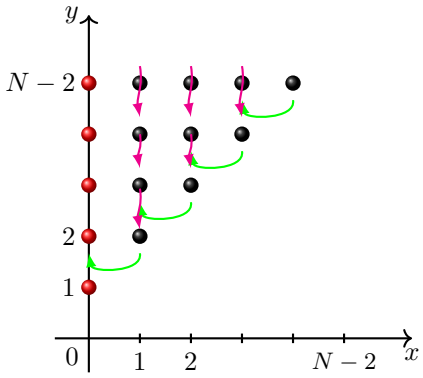

Figure 4.6: Example of how to use the system of equations to walk on the domain of CA^m and compute $CA^m_{x,y}$. The arrows in green represent the use of the equation (4.24) and the ones in magenta represent an application of the equation (4.23).

Then, $CA^m_{0,y} = p_y^{N-1}(m-2) - p_y^{N-1}(m-1) = p_{0,y}^{N-1}(m-1) - p_{0,y}^{N-1}(m)$. Since by the previous argument, we have uniqueness of the solution to the system of equations (4.27), if we show that $p_{x,y}^{N-1}(m-1) - p_{x,y}^{N-1}(m)$ is also solution to the same system, then $CA^m_{x,y}$ has to coincide with $p_{x,y}^{N-1}(m-1) - p_{x,y}^{N-1}(m)$, for every $(x, y) \in \mathcal{T}^{N-2} \cup \mathcal{BT}^{N-2}$.

Since, for every $k, j \in \mathbb{N}$ and $y, x \in \Lambda_N$,

$$p_y^{N+j}(1)p_x^{N+k}(1) = p_{y-j+k}^{N+k}(1)p_{x+j-k}^{N+j}(1), \text{ if } y-j+k, x+j-k \in \Lambda_N, \tag{4.28}$$

we have that:

- If $m = 3$, using (4.16) and (4.28), we get, for every $x \in \{1, \ldots, N-5\}$ and $y \in \{x+1, \ldots, N-3\}$,

$$p_{x,y}^{N-1}(2)p_{y+2}^{N}(1) = p_{y+2}^{N}(1)p_y^{N-1}(1)p_x^{N-2}(1) = p_{y+1}^{N-1}(1)p_x^{N-2}(1)p_{y+1}^{N}(1) = p_{x,y+1}^{N-1}(2)p_{y+1}^{N}(1),$$

  and, for every $x \in \{1, \ldots, N-3\}$ and $y = x+1$,

$$p_{x-1,x+1}^{N-1}(2)p_{x+2}^{N}(1) = p_{x+2}^{N}(1)p_{x+1}^{N-1}(1)p_{x-1}^{N-2}(1) = p_{x+1}^{N}(1)p_{x+1}^{N-1}(1)p_x^{N-2}(1) = p_{x,x+1}^{N-1}(2)p_{x+1}^{N}(1).$$

Then, $p_{x,y}^{N-1}(2) - p_{x,y}^{N-1}(3)$ [6] solves (4.24) and (4.23).

- If $m = 2$, for every $x \in \{1, \dots, N-5\}$ and $y \in \{x+1, \dots, N-3\}$, we get

$$
\begin{aligned}
&[p_{x,y}^{N-1}(1) - p_{x,y}^{N-1}(2)]p_{y+2}^{N}(1) + p_{x,y}^{N}(2) = [p_{y+2}^{N}(1) + p_{y}^{N}(1)]p_{x}^{N-1}(1) + p_{y}^{N-1}(1)p_{y+2}^{N}(1)[1 - 3p_{x}^{N-2}(1)] \\
&= 2p_{x}^{N-1}(1)p_{y+1}^{N}(1) + p_{y+1}^{N}(1)p_{y+1}^{N-1}(1)[1 - 3p_{x}^{N-2}(1)] = [p_{x,y+1}^{N-1}(1) - p_{x,y+1}^{N-1}(2)]p_{y+1}^{N}(1) + p_{x,y+1}^{N}(2),
\end{aligned}
$$

  where, on the first and third equality, we used (4.16) and that, for every $N \in \mathbb{N}$ and $x \in \Lambda_N$, $p_x^N(0) = 1 - p_x^N(1)$ and, on the second equality, we used (4.28) and that, for every $N \in \mathbb{N}$ and $x \in \Lambda_N$, $\Delta_N^{1D} p_x^N(1) = 0$. Also, for every $x \in \{1, \dots, N-3\}$ and $y = x+1$, we get that

$$
\begin{aligned}
&[p_{x,x+1}^{N-1}(1) - p_{x,x+1}^{N-1}(2)]p_{x+1}^{N}(1) + p_{x,x+1}^{N}(2) \\
&= p_{x+1}^{N-1}(1)p_{x+1}^{N}(1) + 2p_{x}^{N-1}(1)p_{x+1}^{N}(1) + -3p_{x+1}^{N}(1)p_{x+1}^{N-1}(1)p_{x}^{N-2}(1) \\
&= p_{x-1}^{N-1}(1)p_{x+1}^{N}(1) + p_{x+1}^{N-1}(1)p_{x+1}^{N}(1) - p_{x}^{N-1}(1)[p_{x}^{N}(1) - 2p_{x+1}^{N}(1)] - 3p_{x+1}^{N}(1)p_{x+1}^{N-1}(1)p_{x}^{N-2}(1) \\
&= p_{x+1}^{N}(1)p_{x-1}^{N-1}(1) + 2p_{x}^{N-1}(1)p_{x+2}^{N}(1) - 3p_{x+2}^{N}(1)p_{x+1}^{N-1}(1)p_{x-1}^{N-2}(1) \\
&= [p_{x-1,x+1}^{N-1}(1) - p_{x-1,x+1}^{N-1}(2)]p_{x+2}^{N}(1) + p_{x-1,x+1}^{N}(2),
\end{aligned}
$$

  where, on the first and fourth equality, we used (4.16) and that, for every $N \in \mathbb{N}$ and $x \in \Lambda_N$, $p_x^N(0) = 1 - p_x^N(1)$, on the second equality, we used that $p_{x-1}^{N-1}(1)p_{x+1}^{N}(1) = p_x^{N-1}(1)p_x^N(1)$, and, finally, on the third equality, we used (4.28) and that, for every $N \in \mathbb{N}$ and $x \in \Lambda_N$, $\Delta_N^{1D} p_x^N(1) = 0$. Therefore, $p_{x,y}^{N-1}(1) - p_{x,y}^{N-1}(2)$ solves (4.24) and (4.23).

- If $m = 1$ or $m = 0$, performing similar computations, we verify that $p_{x,y}^{N-1}(m-1) - p_{x,y}^{N-1}(m)$ solves (4.24) and (4.23).

Then, $p_{x,y}^{N-1}(m-1) - p_{x,y}^{N-1}(m)$ is the unique solution of the system (4.27). Also, we already knew that, independently from m,

$$
\begin{cases}
\Delta_{N,full}^{2D} p_{x,y}^{N-1}(m) = \Delta_{N-1,full}^{2D} p_{x,y}^{N-1}(m) = 0, \quad x \in \{1, \dots, N-5\} \text{ and } y \in \{x+2, \dots, N-3\} \\
\Delta_{N,ref}^{2D} p_{x,y}^{N-1}(m) = \Delta_{N-1,ref}^{2D} p_{x,y}^{N-1}(m) = 0, \quad x \in \{1, \dots, N-4\} \text{ and } y = x+1
\end{cases},
$$

which means that the solution we found for the system (4.27) is also a solution of the original problem, meaning that, $E_{x,y}^m = CA_{x,y}^m = p_{x,y}^{N-1}(m-1) - p_{x,y}^{N-1}(m)$. Then, if $m = 1, 2, 3$,

$$
p_{x,y,z}^{N}(m) = p_{x,y}^{N}(m) + p_z^N(1)[p_{x,y}^{N-1}(m-1) - p_{x,y}^{N-1}(m)]. \tag{4.29}
$$

Finally, using total law of probability,

$$
\begin{aligned}
p_{x,y,z}^{N}(0) &= 1 - p_{x,y}^{N}(1) - [p_{x,y}^{N-1}(0) - p_{x,y}^{N-1}(1)]p_z^N(1) - p_{x,y}^{N}(2) - [p_{x,y}^{N-1}(1) - p_{x,y}^{N-1}(2)]p_z^N(1) - p_{x,y}^{N-1}(2)p_z^N(1) \\
&= p_{x,y}^{N}(0) - p_z^N(1)p_{x,y}^{N-1}(0).
\end{aligned} \tag{4.30}
$$

[6] Recall that $p_{x,y}^{N-1}(3)$ is defined as zero.

Therefore, the recursive formula (4.29) is also true for $m = 0$ (where we use the convention $p^{N-1}_{x,y}(-1) = p^{N-1}_{x,y}(3) = p^N_{x,y}(3) = 0$).

Remark 4.1.3. *We expect that the argument presented above for $\theta = 0$ can be extended for any number of dual particles still keeping the recursive formula, i.e., for any $k \in \mathbb{N}$, we expect to have*

$$p^N_{x_1,\dots,x_{k+1}}(m) = p^N_{x_1,\dots,x_k}(m) + p^N_{x_{k+1}}(1)[p^{N-1}_{x_1,\dots,x_k}(m-1) - p^{N-1}_{x_1,\dots,x_k}(m)]. \tag{4.31}$$

Remark that we proved above this formula for $k = 2$ and $k = 3$.

Remark 4.1.4. *Even though, we did not need the following information to compute the absorption probabilities, observe that, because of our restriction of the parameters γ, ϵ, δ and β, there are some symmetries on the expressions of the absorption probabilities, i.e., for any $x, y, z \in \Lambda_N$ with $x < y < z$,*

$$p^N_x(1) = p^N_{N-x}(0),\ p^N_{x,y}(2) = p^N_{N-y,N-x}(0),\ p^N_{x,y}(1) = p^N_{N-y,N-x}(1),$$
$$p^N_{x,y,z}(3) = p^N_{N-z,N-y,N-x}(0),\ p^N_{x,y,z}(2) = p^N_{N-z,N-y,N-x}(1).$$

4.2 Stationary density and correlations via absorption probability

The case $\alpha = 1$ has a special highlight in the literature because is the only value of α for which is known, using MPA, a closed explicit formula for the k-points stationary correlation function, for $k \in \mathbb{N}$. Unfortunately, up to our knowledge, this method is not available neither for SEP(α), with $\alpha \in \mathbb{N}_2$, nor SIP(α), with $\alpha \in \mathbb{R}^+$. In this section, we will obtain explicit expressions for the stationary density profile and the 2-points and 3-points stationary correlation functions avoiding using any results from MPA known for SEP(1).

Notation: For what comes in the next subsections, some more notation will be required. Let $\nabla^{+,N}$ (resp. $\nabla^{-,N}$, denote the discrete right (resp. left) gradient, which is define, for every $f \in \mathcal{F}(\{0,\dots,N\})$ and $x \in \Lambda_N$, as

$$\nabla^{+,N} f(x) = N[f(x+1) - f(x)] \quad \text{(resp. } \nabla^{-,N} f(x) = N[f(x-1) - f(x)]). \tag{4.32}$$

We will denote by $\nabla^{+,N}_x$, $\nabla^{+,N}_y$ and $\nabla^{+,N}_z$ (resp. $\nabla^{-,N}_x$, $\nabla^{-,N}_y$ and $\nabla^{-,N}_z$) the operator defined above acting now on the first, second and third argument, respectively, of a function with at least three arguments.

Let $\eta \in \Omega_N$ and $k \in \mathbb{N}$ with $k \geq 2$. We will represent by $\{\eta_t\}_{t\geq 0}$ the continuous time Markov chain describing the dynamics of our models, where η_0 will represent the initial configuration of the process. Let μ_{ss} be the unique stationary measure of each of the studied processes and define the discrete stationary density profile ρ^N_{ss} as:

$$\rho^N_{ss}(x) = \mathbb{E}_{\mu_{ss}}[\eta(x)]. \tag{4.33}$$

Let us also define the k-points stationary correlation function φ_{ss}^N:

$$\varphi_{ss}^N(x_1,\dots,x_k) = \mathbb{E}_{\mu_{ss}}[\bar{\eta}(x_1)\dots\bar{\eta}(x_k)], \tag{4.34}$$

where $\bar{\eta}(x) = \eta(x) - \mathbb{E}_{\mu_{ss}}[\eta(x)]$ and $x_j \in \Lambda_N$ and $x_1 \leq \dots \leq x_k$, for any $j \in \{1,\dots,k\}$, and $k \in \mathbb{N}_2$. We are now interested in finding explicit expressions for each one of these functions. To do that, we will use an approach based on consequences of duality.

Remark 4.2.1. *For any permutation $\sigma : \Lambda_N \to \Lambda_N$, $\varphi_{ss}^N(x_1,\dots,x_k) = \varphi_{ss}^N(x_{\sigma_1},\dots,x_{\sigma_k})$. Then, to completely characterize φ_{ss}^N, it is enough to find its value for points $(x_1,\dots,x_k)$ that satisfy $x_1 \leq \dots \leq x_k$.*

4.2.1 Applications to SEP(1)

Fix $k \in \mathbb{N}$. Observe that, as we remarked on the proof of existence and uniqueness of invariant measure for SIP(α), since μ_{ss} is the unique stationary measure of an irreducible continuous time Markov chain and $f : \Omega_N \to \mathbb{R}$ given, for every $\eta \in \Omega_N$, by $f(\eta) = \bar{\eta}(x_1)\dots\bar{\eta}(x_k)$, is a bounded function (it is uniformly bounded by 2^K, because $\eta(x_j) \leq 1$, for every $j \in \{1,\dots,k\}$), then, for every initial configuration $\eta \in \Omega_N$,

$$\mathbb{E}_{\mu_{ss}}[\bar{\eta}(x_1)\dots\bar{\eta}(x_k)] = \lim_{t\to\infty} \mathbb{E}_\eta[\bar{\eta}_t(x_1)\dots\bar{\eta}_t(x_k)], \tag{4.35}$$

where $\mathbb{E}_\eta$ represents the expectation taken with respect to the probability measure P^η in the cádlag[7] space $\mathcal{D}([0,\infty),\Omega_N)$ for which $\eta_0 = \eta$ P^η-a.s. (recall the general definition of a Markov process in terms of families of probability measures - see, for example, [15]). Remark that, if $\hat{\eta} = \delta_{x_1} + \dots + \delta_{x_k} \in \Omega_N^{Ex,dual}$, with $x_1 < \dots < x_k$, then, from Theorem 3.4.1, since $\hat{\eta}(0) = \hat{\eta}(N) = 0$ (recall that $x_j \in \Lambda_N$, for every $j \in \{1,\dots,k\}$), we can simplify the expression of the duality function to

$$D^{SEP(1)}(\eta,\hat{\eta}) = \underbrace{\rho_0^{\hat{\eta}(0)}}_{=1}\left[\prod_{x=1}^{N-1}\mathbb{1}_{\{\eta(x)\geq\hat{\eta}(x)\}}\right]\underbrace{\rho_N^{\hat{\eta}(N)}}_{=1} = \prod_{i=1}^{k}\mathbb{1}_{\{\eta(x_i)\geq 1\}} = \prod_{i=1}^{k}\eta(x_i) \tag{4.36}$$

where the last equality comes from the fact that $\eta(x_j) \in \{0,1\}$, for all $j \in \{1,\dots,N\}$. Then, combining equations (4.36) and (4.35) and the duality property (3.1), we get that, for $x_1 < \dots < x_k$,

$$\begin{aligned}
\mathbb{E}_{\mu_{ss}}[\eta(x_1)\dots\eta(x_k)] &= \lim_{t\to\infty}\mathbb{E}_\eta[D^{SEP(1)}(\eta_t,\hat{\eta})] = \lim_{t\to\infty}\mathbb{E}_{\hat{\eta}}[D^{SEP(1)}(\eta,\hat{\eta}_t)] \\
&= \mathbb{E}_{\hat{\eta}}[\rho_0^{\hat{\eta}_\infty(0)}\rho_N^{\hat{\eta}_\infty(N)}] \qquad (4.37)\\
&= \sum_{m=0}^{k}\rho_0^m\rho_N^{k-m}\underbrace{\mathbb{P}_\eta(\hat{\eta}_\infty(0)=m,\hat{\eta}_\infty(N)=k-m)}_{=p^N_{x_1,\dots,x_k}(m)}, \qquad (4.38)
\end{aligned}$$

where the equality in (4.37) is obtained by remarking that the dual process $\{\hat{\eta}_t\}_{t\geq 0}$ describes a one dimensional (on a line segment) random walk of k dual particles with absorbing boundary points, and, therefore, as time goes to infinity, $\hat{\eta}_t$ will reach a stationary configuration with a certain number m of particles absorbed at the left reservoir and the remaining $k-m$ absorbed at the right reservoir.

[7] From the French "continue à droite, limite à gauche".

Hence, from duality, we can also use the absorption probabilities with one dual particle to obtain an explicit expression for (4.33) as:

$$\rho_{ss}^N(x) = \rho_0 p_x^N(1) + \rho_N p_x^N(0). \tag{4.39}$$

The last expression is obtain by using equation (4.38) with $k = 1$ and, therefore, $m \in \{0, 1\}$.

By similar arguments, the k-points stationary correlation function can be obtained once we have computed the absorption probabilities $p_{x_1,\dots,x_k}^N(m)$, with $m \in \{0, \dots, k\}$:

$$\begin{aligned}
&\varphi_{ss}^N(x_1, \dots, x_k) = \mathbb{E}_{\mu_{ss}}[\eta(x_1) \dots \eta(x_k)] - \rho_{ss}^N(x_1)\mathbb{E}_{\mu_{ss}}[\eta(x_2) \dots \eta(x_k)] - \\
&- \sum_{j_1=2}^{k} \rho_{ss}^N(x_{j_1})\mathbb{E}_{\mu_{ss}}[\eta(x_1) \dots \eta(x_{j_1-1})\eta(x_{j_1+1}) \dots \eta(x_k)] + \rho_{ss}^N(x_1)\rho_{ss}^N(x_2)\mathbb{E}_{\mu_{ss}}[\eta(x_{j_4}) \dots \eta(x_k)] + \\
&+ \sum_{j_2=3}^{k} \rho_{ss}^N(x_1)\rho_{ss}^N(x_{j_2})\mathbb{E}_{\mu_{ss}}[\eta(x_2) \dots \eta(x_{j_2-1})\eta(x_{j_2+1}) \dots \eta(x_k)] + \dots + \\
&+ \sum_{\substack{j_1,j_2=2 \\ j_1<j_2}}^{k} \rho_{ss}^N(x_{j_1})\rho_{ss}^N(x_{j_2})\mathbb{E}_{\mu_{ss}}[\eta(x_1) \dots \eta(x_{j_1-1})\eta(x_{j_1+1}) \dots \eta(x_{j_2-1})\eta(x_{j_2+1}) \dots \eta(x_k)] + \dots + \\
&+ (-1)^{k-1}k\rho_{ss}^N(x_1) \dots \rho_{ss}^N(x_k) + (-1)^k\rho_{ss}^N(x_1) \dots \rho_{ss}^N(x_k)
\end{aligned}$$

Rearranging the terms, denoting $(\mathbf{x}) = (x_1, \dots, x_k)$, we get

$$\varphi_{ss}^N(\mathbf{x}) = \mathbb{E}_{\mu_{ss}}\left[\prod_{l=1}^{k}\eta(x_l)\right] + \sum_{m=1}^{k-1}(-1)^m \sum_{\substack{j_1,\dots,j_m=1 \\ j_1<\dots<j_m}}^{k}\left[\prod_{l=1}^{m}\rho_{ss}^N(x_{j_l})\right]\mathbb{E}_{\mu_{ss}}\left[\prod_{\substack{l=1 \\ l\neq j_1,\dots,j_m}}^{k}\eta(x_l)\right] + (-1)^k\prod_{l=1}^{k}\rho_{ss}^N(x_l). \tag{4.40}$$

Using (4.40), for $k = 2$, we obtain the 2-points stationary correlation function:

$$\varphi_{ss}^N(x, y) = \mathbb{E}_{\mu_{ss}}[\eta(x)\eta(y)] - \rho_{ss}^N(x)\rho_{ss}^N(y), \tag{4.41}$$

and, for $k = 3$, the 3-points stationary correlation function is given by

$$\varphi_{ss}^N(x, y, z) = \mathbb{E}_{\mu_{ss}}[\eta(x)\eta(y)\eta(z)] - \rho_{ss}^N(x)\varphi_{ss}^N(y, z) - \rho_{ss}^N(y)\varphi_{ss}^N(x, z) - \rho_{ss}^N(z)\varphi_{ss}^N(x, y) - \rho_{ss}^N(x)\rho_{ss}^N(y)\rho_{ss}^N(z). \tag{4.42}$$

Thereby, using the results of Section 4.1, it is easy to derive explicit expressions for the discrete stationary density profile and the 2-points and 3-points stationary correlation functions.

Discrete stationary density profile:

Substituting in (4.39) the results obtained in (4.5), we get

$$\rho_{ss}^N(x) \quad = \quad \frac{(\rho_0 + \rho_N)(N^\theta - 1) + \rho_0(N - x) + \rho_N x}{N + 2N^\theta - 2}. \tag{4.43}$$

Then, if, for any $x \in \Lambda_N$, we assume that $\frac{x}{N} \to u \in [0,1]$, as N goes to infinity (here we are going from the microspace to the macrospace), the limit in N of (4.43) is given by

$$\lim_{N\to\infty} \rho_{ss}^N(x) = \bar{\rho}(u) := \begin{cases} \frac{\rho_0+\rho_1}{2}, & \text{if } \theta > 1 \\ \frac{\rho_0+\rho_1+\rho_0(1-u)+\rho_1 u}{3}, & \text{if } \theta = 1 \\ \rho_0(1-u)+\rho_1 u, & \text{if } \theta < 1 \end{cases},$$

where $\rho_1 := \rho_N$ and ρ_0 are as in Table 2.1.

We remark that, independently from the value of θ, the limit $\lim_{N\to\infty} \rho_{ss}^N(x)$ is a linear function of u but the coefficients depend on the value of θ and

$$\lim_{N\to\infty} \max_{x\in\Lambda_N} \left|\rho_{ss}^N(x) - \bar{\rho}(\tfrac{x}{N})\right| = 0. \tag{4.44}$$

2-points stationary correlation function:

Remark that $\varphi_{ss}^N(x,y)$ is defined for every $x, y \in \Lambda_N$, i.e., its domain is a discretized square. Since $\eta(x) = [\eta(x)]^2$, then $\varphi_{ss}^N(x,x) = \rho_{ss}(x)(1-\rho_{ss}(x))$ which is completely determined using the expression of $\rho_{ss}(x)$ given in (4.43). Also, by Remark 4.2.1, to find the value of $\varphi_{ss}^N(x,y)$, for $y \neq x$, we only need to focus now on the points (x,y) on the upper triangle $\mathcal{T}^N$.

Combining equation (4.41), the identity in (4.38) and the results obtained in the case $k=2$, namely, equation (4.10), a simple, but long, computation gives us, for $(x,y) \in \mathcal{T}^N$,

$$\begin{aligned} \varphi_{ss}^N(x,y) &= \rho_0^2 p_{x,y}^N(2) + \rho_0\rho_N p_{x,y}^N(1) + \rho_N^2 p_{x,y}^N(0) - [\rho_0 p_x^N(1) + \rho_N p_x^N(0)][\rho_0 p_y^N(1) + \rho_N p_y^N(0)] \\ &= -\frac{(\rho_N-\rho_0)^2}{N+2N^\theta-3} p_x^N(0) p_y^N(1). \end{aligned} \tag{4.45}$$

Observe that, for any $(x,y) \in \mathcal{T}_N$, since $p_x^N(0), p_y^N(1) > 0$, if $\rho_N \neq \rho_0$, then $\varphi_{ss}^N(x,y)$ takes negative values. This is due to the repelling of the particles caused by the exclusion rule. Also, for any possible value of $\theta \in \mathbb{R}$, if $x < y$, $\lim_{N\to\infty} \varphi_{ss}^N(x,y) = 0$, meaning that the 2-points stationary correlation function for SEP(1) decays to zero when we pass to the macroscopic space. The question now is, what is the order of its decay? The correct answer is that it has linear decay, i.e., of order N.

For any $x, y \in \Lambda_N$ with $x < y$, assuming that $\frac{x}{N} \to u$ and $\frac{y}{N} \to v$ as $N \to \infty$, then, the limit $\lim_{N\to\infty} N\varphi_{ss}^N(x,y)$ now depends on the value of θ, which means that, depending on the strength of the reservoirs, we will obtain different limit functions of parameters u and v, which are non-identically zero if $\theta \leq 1$.

Case 1: $\theta < 1$

$$\begin{aligned} \lim_{N\to\infty} N\varphi_{ss}^N(x,y) &= \lim_{N\to\infty} -\frac{N^3(\rho_N-\rho_0)^2}{(N+2N^\theta-3)(N+2N^\theta-2)^2} \left[\frac{x}{N} + \frac{N^\theta-1}{N}\right]\left[\frac{N+N^\theta-1}{N} - \frac{y}{N}\right] \\ &= -(\rho_1-\rho_0)^2 G^{2,Dir}(u,v), \end{aligned} \tag{4.46}$$

where ρ_1 and ρ_0 are the same as for the limit of the stationary density profile and $G^{2,Dir}(u,v) := u(1-v)$

is the Green function of the 2-dimensional Laplacian on the upper triangle

$$T = \{(u, v) \in [0, 1]^2 \mid u \leq v\}, \tag{4.47}$$

which is reflected on the diagonal $u = v$, and with homogeneous Dirichlet boundary conditions, i.e., $G^{2,Dir}$ is solution of the initial value problem

$$\begin{cases} \Delta^{2D} G^{2,Dir}(u, v) = \delta_{u=v}, \text{ if } (u, v) \in \text{int } T, & \textit{(2D Laplace equation)} \\ G^{2,Dir}(0, v) = G^{2,Dir}(u, 1) = 0, \text{ if } u, v \in [0, 1], & \textit{(homogeneous Dirichlet b.c.)} \end{cases} \tag{4.48}$$

with int T=$\{(u, v) \in (0, 1)^2 \mid u \leq v\}$ and Δ^{2D}=$\begin{cases} \frac{d^2}{du^2} + \frac{d^2}{dv^2}, \text{ in (int } T) \setminus DT \\ \frac{\partial}{\partial u} - \frac{\partial}{\partial v}, \quad \text{in } DT \end{cases}$, where DT=$\{(u, u) \mid u \in (0, 1)\}$.

Case 2: $\theta = 1$

$$\begin{aligned} \lim_{N\to\infty} N\varphi^N_{ss}(x, y) &= \lim_{N\to\infty} -\frac{N^3(\rho_N - \rho_0)^2}{3(N-1)(N+2N-2)^2} \left[\frac{x}{N} + \frac{N-1}{N}\right] \left[\frac{2N-1}{N} - \frac{y}{N}\right] \\ &= -\frac{(\rho_1 - \rho_0)^2}{9} G^{2,Rob}(u, v), \end{aligned}$$

where $G^{2,Rob}(u, v) := \dfrac{(u + 1)(2 - v)}{3}$ is the Green function of the 2-dimensional Laplacian (as above) on the upper triangle T, defined in (4.47), which is reflected on the line $u = v$ and with homogeneous Robin boundary conditions, i.e., $G^{2,Rob}$ is solution of the initial value problem

$$\begin{cases} \Delta^{2D} G^{2,Rob}(u, v) = \delta_{u=v}, \text{ if } (u, v) \in \text{int } T, & \textit{(2D Laplace equation)} \\ \frac{\partial}{\partial u} G^{2,Rob}(0, v) = G^{2,Rob}(0, v), \text{ if } v \in [0, 1], & \textit{(homogeneous Robin b. c.)} \\ \frac{\partial}{\partial v} G^{2,Rob}(u, 1) = -G^{2,Rob}(u, 1) \text{ if } u \in [0, 1], & \textit{(homogeneous Robin b. c.)} \end{cases}$$

where int T and Δ^{2D} are defined as above. In this regime, the limit $\lim_{N\to\infty} N\varphi^N_{ss}(x, y)$ is again a multiple of the solution of the 2-dimensional Laplace equation on T but with the homogeneous Dirichlet boundary conditions replaced by homogeneous Robin boundary conditions.

Case 3: $\theta > 1$

$$\lim_{N\to\infty} N\varphi^N_{ss}(x, y) = \lim_{N\to\infty} -\frac{N^3(\rho_N - \rho_0)^2}{(N + 2N^\theta - 3)(N + 2N^\theta - 2)^2} \left[\frac{N^\theta - 1 + x}{N}\right] \left[\frac{N + N^\theta - 1 - y}{N}\right] = 0.$$

This means that, for slow boundary ($\theta > 1$), the 2-points stationary correlation function has an order of decay higher than a linear function of N. In fact, the correct order to see a non-trivial limit in the case of very slow boundary, i.e., with $\theta > 1$, is N^θ. Then, if instead of assuming that $\frac{x}{N} \to u$ and $\frac{y}{N} \to v$ as

$N \to \infty$, we assume that $\frac{x}{N^\theta} \to u$ and $\frac{y}{N^\theta} \to v$ as $N \to \infty$, we obtain

$$\begin{aligned}\lim_{N\to\infty} N^\theta \varphi^N_{ss}(x,y) &= \lim_{N\to\infty} -\frac{N^{3\theta}(\rho_N-\rho_0)^2}{(N+2N^\theta-3)(N+2N^\theta-2)^2}\left[\frac{x}{N^\theta}+\frac{N^\theta-1}{N^\theta}\right]\left[\frac{N+N^\theta-1}{N^\theta}-\frac{y}{N^\theta}\right] \\ &= -\frac{(\rho_N-\rho_0)^2}{8}(u+1)(1-v).\end{aligned}$$

A summary of these results can be found in [7].

Before going to the next section where we look at the 3-points stationary correlation function, let us remark that, by using the forward Kolmogorov's equation, one can also find a system of equations for φ^N_{ss}, and, passing to the limit in N, recover the solutions above - see Appendix C.

3-points stationary correlation function:

If $x, y \in \Lambda_N$ with $x < y$, then $\varphi^N_{ss}(x,x,x) = \rho^N_{ss}(x) - 3[\rho^N_{ss}(x)]^2 + 2[\rho^N_{ss}(x)]^3$, $\varphi^N_{ss}(x,y,y) = \varphi^N_{ss}(x,y)[1-2\rho^N_{ss}(y)]$ and $\varphi^N_{ss}(x,x,y) = \varphi^N_{ss}(x,y)[1-2\rho^N_{ss}(x)]$, that are completely determined replacing the stationary density profile ρ^N_{ss} by the expression given in (4.43) and the 2-points stationary correlation function φ^N_{ss} by the expression given in (4.45).

Combining equation (4.42) and the identity in (4.38), we have, for $x, y, z \in \Lambda^3_N$ with $x < y < z$,

$$\begin{aligned}\varphi^N_{ss}(x,y,z) &= \rho_0^3 p^N_{x,y,z}(3) + \rho_0^2\rho_N p^N_{x,y,z}(2) + \rho_0\rho_N^2 p^N_{x,y,z}(1) + \rho_N^3 p^N_{x,y,z}(0) \\ &+ \frac{(\rho_N-\rho_0)^2}{N+2N^\theta-3}\Big\{\rho_0 p^N_z(1)[p^N_x(1)p^N_y(0) + 2p^N_x(0)p^N_y(1)] + \rho_N p^N_x(0)[p^N_z(0)p^N_y(1) + 2p^N_y(0)p^N_z(1)]\Big\} \\ &- [\rho_0 p^N_x(1) + \rho_N p^N_x(0)][\rho_0 p^N_y(1) + \rho_N p^N_y(0)][\rho_0 p^N_z(1) + \rho_N p^N_z(0)]. \qquad (4.49)\end{aligned}$$

Replacing in (4.49) the results obtained in (4.10) and (4.19), performing long (but simple) computations, we get an explicit formula for the 3-points stationary correlation function that is written in terms of the one particle absorption probabilities as we had for the 2-points stationary correlation function:

$$\varphi^N_{ss}(x,y,z) = 2\frac{(\rho_N-\rho_0)}{(N+2N^\theta-4)}[\varphi^N_{ss}(x,y)p^N_z(1) - p^N_x(0)\varphi^N_{ss}(y,z)]. \qquad (4.50)$$

Observe that, for any $x, z \in \Lambda_N$, $p^N_x(0), p^N_z(1) > 0$, then, the sign of $\varphi^N_{ss}(x,y,z)$ will depend on y and the difference between ρ_N and ρ_0.

Also, for any $\theta \in \mathbb{R}$, $\lim_{N\to\infty} N\varphi^N_{ss}(x,y,z) = 0$, meaning that, at the macroscopic level, the 3-points stationary correlation function decays to zero faster than a linear function of N, and, therefore, faster than the 2-points stationary correlation function. We expect that, as k (the total number of dual particles) increases, the corresponding k-points stationary correlation function converges faster to zero the higher the value of k. In the current case, what is then the correct order of decay? As we will see below, the order of decay is quadratic, i.e., N^2, if $\theta \leq 1$, and $N^{2\theta}$, if $\theta > 1$. For any $x, y, z \in \Lambda_N$ with $x < y < z$, assuming that $\frac{x}{N} \to u \in [0,1]$, $\frac{y}{N} \to v \in [0,1]$ and $\frac{z}{N} \to w \in [0,1]$ as $N \to \infty$, then, the limit $\lim_{N\to\infty} N^2\varphi^N_{ss}(x,y,z)$, similarly to what to have for the 2-points stationary correlation function, depends on the value of θ.

Case 1: $\theta < 1$

$$\lim_{N\to\infty} N^2\varphi_{ss}^N(x,y,z) = \lim_{N\to\infty} \frac{-2N^5(\rho_N-\rho_0)^3\left(N^{\theta-1}-N^{-1}+\frac{x}{N}\right)\left(1-\frac{2y}{N}\right)\left(N^{\theta-1}-N^{-1}+1-\frac{z}{N}\right)}{(N+2N^\theta-3)(N+2N^\theta-4)(N+2N^\theta-2)^3}$$
$$= -2(\rho_1-\rho_0)^3 G^{3,Dir}(u,v,w),$$

where $\rho_1 := \rho_N$ and ρ_0 are as in Table 2.1 and $G^{3,Dir}(u,v,w) = u(1-2v)(1-w)$ is the Green function of the 3-dimensional Laplacian on the simplex

$$S = \{(u,v,w) \in [0,1]^3 \mid u \le v \le w\}, \tag{4.51}$$

which is reflected on the planes S_1, S_2 and on the line S_3, where $S_1 = \{(u,v,w) \in S \mid u = v\}$, $S_2 = \{(u,v,w) \in S \mid v = w\}$ and $S_3 = \{(u,v,w) \in S \mid u = v = w\}$, and with homogeneous Dirichlet boundary conditions, i.e., $G^{3,Dir}$ is solution of the initial value problem

$$\begin{cases} \Delta^{3D} G^{3,Dir}(u,v,w) = (1-w)\delta_{u=v} + u\delta_{w=v}, \text{ if } (u,v,w) \in \text{int } S, & \textit{(3D Laplace equation)} \\ G^{3,Dir}(0,v,w) = G^{3,Dir}(u,v,1) = 0, \text{ if } (u,v),(v,w) \in Q, & \textit{(homogeneous Dirichlet b.c.)} \end{cases}$$

where int $S = \{(u,v,w) \in (0,1)^3 \mid u \le v \le w\}$,

$$\Delta^{3D} = \begin{cases} \frac{d^2}{du^2} + \frac{d^2}{dv^2} + \frac{d^2}{dw^2}, \text{ in (int } S) \setminus (\text{ int } S_1 \cup \text{ int } S_2), \\ \frac{d^2}{du^2} + \frac{d}{dv} - \frac{d}{dw}, \text{ in (int } S_1) \setminus (\text{ int } S_3), \\ \frac{d^2}{dw^2} + \frac{d}{du} - \frac{d}{dv}, \text{ in (int } S_2) \setminus (\text{ int } S_3), \\ \frac{d}{du} - \frac{d}{dw}, \text{ in int } S_3, \end{cases}$$

and int $S_j = (\text{int } S) \cap S_j$, for $j \in \{1,2,3\}$.

Case 2: $\theta = 1$

$$\lim_{N\to\infty} N^2\varphi_{ss}^N(x,y,z) = \lim_{N\to\infty} \frac{-2N^5(\rho_N-\rho_0)^3\left(1-N^{-1}+\frac{x}{N}\right)\left(1-\frac{2y}{N}\right)\left(1-N^{-1}+1-\frac{z}{N}\right)}{(3N-3)(3N-4)(3N+-2)^3}$$
$$= -\frac{2(\rho_1-\rho_0)^3}{9} G^{3,Rob}(u,v,w),$$

where $G^{3,Rob}(u,v,w) = \frac{(1+u)(1-2v)(2-w)}{27}$ is the Green function of the 3-dimensional Laplacian (as above) on the simplex S, defined in (4.51), which is reflected on the same planes and line and with homogeneous Robin boundary conditions, i.e., $G^{3,Rob}$ is solution of the initial value problem

$$\begin{cases} \Delta^{3D} G^{3,Dir}(u,v,w) = (1-w)\delta_{u=v} + u\delta_{w=v}, \text{ if } (u,v,w) \in \text{int } S, & \textit{(3D Laplace equation)} \\ \frac{\partial}{\partial u} G^{3,Rob}(0,v,w) = G^{3,Rob}(0,v,w), \text{ if } (v,w) \in T, & \textit{(homogeneous Robin b.c.)} \\ \frac{\partial}{\partial w} G^{3,Rob}(u,v,1) = -G^{3,Rob}(u,v,1) \text{ if } (u,v) \in T, & \textit{(homogeneous Robin b.c.)} \end{cases}$$

Case 3: $\theta > 1$

$$\lim_{N\to\infty} N^2\varphi^N_{ss}(x,y,z) = \lim_{N\to\infty} \frac{-2N^5(\rho_N-\rho_0)^3\left(N^{\theta-1}-N^{-1}+\frac{x}{N}\right)\left(1-\frac{2y}{N}\right)\left(N^{\theta-1}-N^{-1}+1-\frac{z}{N}\right)}{(N+2N^\theta-3)(N+2N^\theta-4)(N+2N^\theta-2)^3} = 0.$$

As we had for the 2-points correlation function, this means that, for slow boundary, the 3-points stationary correlation function has an order of decay higher than a quadratic function of N. In fact, the correct order to see a non-trivial limit in the case of slow boundary is $N^{2\theta}$.

Then, if instead of assuming that $\frac{x}{N}\to u$, $\frac{y}{N}\to v$ and $\frac{z}{N}\to w$ as $N\to\infty$, we assume that $\frac{x}{N^\theta}\to u$, $\frac{y}{N^\theta}\to v$ and $\frac{z}{N^\theta}\to w$ as $N\to\infty$, we obtain

$$\lim_{N\to\infty} N^{2\theta}\varphi^N_{ss}(x,y,z) = -\frac{(\rho_1-\rho_0)^3}{2}\frac{(1+u)v(1-w)}{8}.$$

Remark 4.2.2. *Like we remarked for the 2-points stationary correlation function, by using the forward Kolmogorov's equation, one can also find a system of equations for the 3-points stationary correlation function and, passing to the limit in N, recover the solutions above - see Appendix C.*

We could now continue with this same type of computations to find the absorption probabilities for four, five, and so on, particles and, using them, find explicit expressions for the 4-th, 5-th, and so on, stationary correlation functions. The issue is that the systems of equations that we will have to solve, for fixed $k\in\mathbb{N}$, will have $2^{k+1}-1$ different types of equations, which, as k increases, grows very fast. How to get this last number? If we have k particles to place on the system (here, we will be considering N at least $2(k+1)$ so we can have the configuration with all particles separated by at least one empty site and no particles at sites 1 and $N-1$). Then, to place the first particle, it can either go to the point 1 of somewhere in $\{2,\dots,N-1-k\}$; the second can be placed after the first as nearest neighbor or not; the third can be after the second as nearest neighbor or not, and so on; for the k-th particle, it has four possibilities that generate different equations: either it is nearest neighbor to the particle $k-1$ and is not in site $N-1$, or it is nearest neighbor to the particle $k-1$ and is in site $N-1$, or it is not nearest neighbor to the particle $k-1$ and is in site $N-1$, or even, it is not nearest neighbor to the particle $k-1$ and is not in site $N-1$. Until now, we have counted $2\times\underbrace{2\times\cdots\times 2}_{k-2 \text{ times}}\times 4$ different types of equations. Since, because of our choice of N, we cannot have all particles to be close to each other and have the first at 1 and the last at $N-1$, then, to the 2^{k+1} possibilities counted above, we have to extracted this case that is not allowed. We then obtain that the number of possible different equations is $2^{k+1}-1$. As k increases, the number of equations that we would need to solve to find the absorption probabilities with k dual particles grows exponentially, making our computations longer and harder. Because of this, we will stop here (with 3 dual particles) the computations for SEP(1). Next, we will apply the same ideas for the general SEP(α).

4.3 SEP(α) with $\alpha \geq 2$ - Absorption Probabilities

4.3.1 Case k = 1

With just one dual particle, SEP(1) and SEP(α) describe the same model, with the jump rates on the bulk that we had for SEP(1) rescaled by α (which does not affect the computations of the absorption probabilities). We get exactly the same results as in Section 4.1.1.

4.3.2 Case k = 2

Recall that fixing $k = 2$ means that we now have only two dual particles and, as before, $m \in \{0, 1, 2\}$. Let us keep the same notation we used in SEP(1) for the absorption probabilities $p^N_{x,y}(m)$. Since in SEP(α), with $\alpha \geq 2$, we allow more than one particle per site, we are also interested in computing $p^N_{x,x}(m)$. Here $p^N_{x,x}(m)$ denotes the probability that, starting from a dual configuration with 2 particles at site x (in a pile), m particles being absorbed at the left reservoir.

As we did for SEP(1), conditioning on the first jump, we get a system of ten equations with two boundary conditions that can be compacted in

$$\begin{cases} \mathfrak{U}^\theta_N p^N_{x,y}(m) = 0, \text{ for } x = 1, \dots, N-2 \text{ and } y = x+1, \dots, N-1, \\ p^N_{0,y}(m) = p^N_y(m-1)\mathbb{1}_{\{m\neq 0\}}, \text{ if } y = 0, \dots, N, \\ p^N_{x,N}(m) = p^N_x(m)\mathbb{1}_{\{m\neq 2\}}, \text{ if } x = 0, \dots, N. \end{cases} \tag{4.52}$$

where the operator $\mathfrak{U}^\theta_N : \mathcal{F}(\mathcal{BCT}^N) \to \mathcal{F}(\mathcal{CT}^N)$, with

$$\mathcal{BCT}^N := \{(x, y) \in \{0, \dots, N\}^2 \mid x \leq y\} \quad \text{and} \quad \mathcal{CT}^N := \{(x, y) \in \Lambda^2_N \mid x \leq y\}, \tag{4.53}$$

is defined as, for every function $f \in \mathcal{F}(\mathcal{BCT}^N)$ and for every $(x, y) \in \mathcal{CT}^N$,

$$\mathfrak{U}^\theta_N f(x, y) = a_x[f(x-1, y) - f(x, y)] + b_y[f(x, y+1) - f(x, y)] + d_{x,y}[f(x+1, y) + f(x, y-1) - 2f(x, y)], \tag{4.54}$$

where $a_x = \frac{N^2}{N^\theta}\mathbb{1}_{x=1} + N^2\mathbb{1}_{x\neq 1}$, $b_y = \frac{N^2}{N^\theta}\mathbb{1}_{y=N-1} + N^2\mathbb{1}_{y\neq N-1}$ and $d_{x,y} = -\frac{N^2}{\alpha}\mathbb{1}_{y=x+1} + N^2\mathbb{1}_{y\neq x,x+1}$. For every $x, y \in \{2, \dots, N-2\}$, if $x < y$ and $|x - y| \geq 2$, $\mathfrak{U}^\theta_N = \Delta^{2D}_{N,full}$, and if $y = x$, $\mathfrak{U}^\theta_N = \Delta^{2D}_{N,ref}$, where $\Delta^{2D}_{N,full}$ and $\Delta^{2D}_{N,ref}$ have the exact same meaning as in SEP(1). Also, if $x \in \{2, \dots, N-2\}$ and $y = x + 1$,

$$\mathfrak{U}^\theta_N p^N_{x,x+1}(m) = \Delta^{2D}_{N,full} p^N_{x,x+1}(m) + \frac{1}{\alpha - 1}\Delta^{2D}_{N,ref} p^N_{x,x+1}(m).$$

This shows that if $\alpha \geq 2$, the operator that we obtain for $x \in \{2, \dots, N-2\}$ and $y = x + 1$ is no longer the 2-dimensional reflected Laplacian as we had for SEP(1). Instead, over the line $\{(x, x+1) \mid x \in \Lambda_N\}$, we will observe a super position of the operators that act above and over this line.

As we did for SEP(1), can we find $p^N_{x,y}(m)$ by starting with an ansatz? The natural ansatz to take is, for every $(x,y) \in \mathcal{CT}^N$,

$$p^N_{x,y}(m) = \begin{cases} A^m x + B^m y + C^m xy + D^m, \text{ if } |x-y| \geq 2, \\ \tilde{A}_m x + \tilde{B}_m y + \tilde{C}_m xy + \tilde{D}_m, \text{ if } y = x+1, \\ \overline{A}_m x + \overline{B}_m x^2 + \overline{C}_m, \text{ if } y = x, \end{cases} \tag{4.55}$$

with $A^m, B^m, C^m, D^m, \overline{A}_m, \overline{B}_m, \overline{C}_m, \overline{D}_m, \tilde{A}_m, \tilde{B}_m, \tilde{C}_m, \tilde{D}_m \in \mathbb{R}$ to be determined.

If $\theta = 0$, using the ansatz in equation (4.55) to solve the system in (4.52), performing long (but, again, simple) computations, for every $x, y \in \{0, \dots, N\}$ with $x \leq y$, we obtain

$$\begin{array}{ll} \underline{\text{If } x < y:} & \underline{\text{If } x = y:} \\ \begin{cases} p^N_{x,y}(0) = \frac{-1+\alpha y}{-1+\alpha N} p^N_x(0) \\ p^N_{x,y}(1) = \frac{(\alpha N+1)x+(\alpha N-1)y-2\alpha xy}{N(-1+\alpha N)} \\ p^N_{x,y}(2) = \frac{-1+\alpha(N-x)}{-1+\alpha N} p^N_y(1) \end{cases} & \begin{cases} p^N_{x,x}(0) = \frac{x(-1+\alpha x)}{N(-1+\alpha N)} + \frac{1}{2N(-1+\alpha N)}, \\ p^N_{x,x}(1) = \frac{(\alpha N+1)x+(\alpha N-1)x-2\alpha x^2}{N(-1+\alpha N)} - \frac{1}{N(-1+\alpha N)}, \\ p^N_{x,x}(2) = \frac{(-1+\alpha N-\alpha x)(N-x)}{N(-1+\alpha N)} + \frac{1}{2N(-1+\alpha N)}. \end{cases} \end{array} \tag{4.56}$$

The previous results were checked using Mathematica (see Appendix A). Therefore, in this case, $\tilde{A}_m = A^m$, $\tilde{B}_m = B^m$, $\tilde{C}_m = C^m$ and $\tilde{D}_m = D^m$. We could not obtain the explicit expressions for the absorption probabilities in the case $\theta \neq 0$, since taking different coefficients throughout the line $y = x+1$ and $\theta \neq 0$, we are lead to a system with a solution not of this form (see Appendix A). Also, we observe that on the diagonal line $y = x$, the expression of $p^N_{x,x}(m)$ differs from the expression of $p^N_{x,y}(m)$ inside the triangle, i.e., $(x,y) \in \mathcal{T}^N$ (with $\mathcal{T}^N$ as SEP(1)), by a constant that depend on α, N and m.

4.4 Stationary density function and correlations via absorption probabilities

4.4.1 Applications to SEP(α) with $\alpha \geq 2$

As we did for SEP(1), we want now to compute the stationary density profile and the 2-points stationary correlation function (defined in Section 4.2), and find their limit as N goes to infinity.

If $\hat{\eta} = \delta_{x_1} + \cdots + \delta_{x_k} \in \Omega_N^{Ex,dual}$ with $x_1 < \cdots < x_k$, from Theorem 3.4.1 and recalling that, since $x_j \in \Lambda_N$ for every $j \in \{1, \dots, k\}$, then $\hat{\eta}(0) = \hat{\eta}(N) = 0$, we can simplify the duality function to

$$\begin{aligned} D^{SEP(\alpha)}(\eta, \hat{\eta}) &= \underbrace{\left(\frac{\rho_0}{\alpha}\right)^{\hat{\eta}(0)}}_{=1} \left[\prod_{x=1}^{N-1} \frac{\eta(x)!(\alpha - \hat{\eta}(x))!}{(\eta(x) - \hat{\eta}(x))!\alpha!} \mathbb{1}_{\{\eta(x) \geq \hat{\eta}(x)\}}\right] \underbrace{\left(\frac{\rho_N}{\alpha}\right)^{\hat{\eta}(N)}}_{=1} \\ &= \underbrace{\left[\prod_{\substack{x=1 \\ x \neq x_1, \dots, x_k}}^{N-1} \mathbb{1}_{\{\eta(x) \geq 0\}}\right]}_{=1} \left[\prod_{i=1}^{k} \frac{(\eta(x_i))!}{(\eta(x_i)-1)!} \frac{(\alpha-1)!}{\alpha!} \mathbb{1}_{\{\eta(x_i) \geq 1\}}\right] = \frac{1}{\alpha^k} \prod_{i=1}^{k} \eta(x_i) \end{aligned} \tag{4.57}$$

where the last equality comes from the fact that, for every $i \in \{1, \dots, k\}$, $\eta(x_i)\mathbb{1}_{\{\eta(x_i)\geq 1\}} = \eta(x_i)$. Here ρ_0 and ρ_N are the left and right density of the reservoirs of the initial process, respectively. Then, we can simply repeat the argument we did for SEP(1) in Section 4.2.1 to get that, for $x_1 < \cdots < x_k$,

$$\mathbb{E}_{\mu_{ss}}[\eta(x_1)\dots\eta(x_k)] = \sum_{m=0}^{k} \rho_0^m \rho_N^{k-m} \underbrace{\mathbb{P}_\eta(\hat{\eta}_\infty(0) = m, \hat{\eta}_\infty(N) = k-m)}_{=p^N_{x_1,\dots,x_k}(m)}. \tag{4.58}$$

Now, if we allow only two of the x_j in $\hat{\eta}$ to be equal (we consider $x_1 = x_2$ without loss of generality), the expression of the duality function becomes

$$\begin{aligned} D^{SEP(\alpha)}(\eta, \hat{\eta}) &= \left[\prod_{i=3}^{k} \frac{\eta(x_i)!}{(\eta(x_i)-1)!}\frac{(\alpha-1)!}{\alpha!}\mathbb{1}_{\{\eta(x_i)\geq 1\}}\right] \frac{\eta(x_1)!}{(\eta(x_1)-2)!}\frac{(\alpha-2)!}{\alpha!}\mathbb{1}_{\{\eta(x_i)\geq 2\}} \\ &= \frac{1}{\alpha^{k-2}}\left[\prod_{i=3}^{k}\eta(x_i)\right]\frac{\eta(x_1)[\eta(x_1)-1]}{\alpha(\alpha-1)} = \frac{1}{\alpha^{k-1}(\alpha-1)}\left[\prod_{i=3}^{k}\eta(x_i)\right]\eta(x_1)[\eta(x_1)-1], \end{aligned} \tag{4.59}$$

if $k \geq 3$, and $D^{SEP(\alpha)}(\eta, \hat{\eta}) = \frac{1}{\alpha(\alpha-1)}\eta(x_1)[\eta(x_1)-1]$, if $k = 2$. Therefore, if $x_1 = x_2 < \cdots < x_k$,

$$\mathbb{E}_{\mu_{ss}}[\eta(x_1)\dots\eta(x_k)] - \mathbb{E}_{\mu_{ss}}[\eta(x_2)\dots\eta(x_k)] = \frac{\alpha-1}{\alpha}\sum_{m=0}^{k}\rho_0^m\rho_N^{k-m}\underbrace{\mathbb{P}_\eta(\hat{\eta}_\infty(0) = m, \hat{\eta}_\infty(N) = k-m)}_{=p^N_{x_1,\dots,x_k}(m)}.$$

Again, using duality, explicit formulas for the discrete stationary density profile and the 2-points stationary correlation function for SEP(α) can be obtained once we have computed the absorption probabilities $p^N_{x_1,x_2}(m)$, with $m \in \{0,1,2\}$. For $k \geq 3$, to obtain all the quantities needed to compute the k-points stationary correlation function, we would need now to simplify the expression of the duality function considering more cases besides having only two particles at the same site.

Discrete stationary density profile:

Since the absorption probabilities for one dual particle are the same for SEP(1) and SEP(α), by equation (4.58), the stationary density profile for SEP(α) is given as in (4.43) (here ρ_0 and ρ_N are the densities of the left and right reservoir for SEP(α)). If we had chosen also for SEP(α), $\epsilon + \gamma = \beta + \delta = 1$, then, the density profile would be given by $\alpha\rho^N_{ss}$, where ρ^N_{ss} represent the density profile of SEP(1).

2-points stationary correlation function:

As we did for SEP(1), combining equation (4.41), the identity in (4.58) with $k = 2$ (and $m \in \{0,1,2\}$) and the results obtained in (4.56), if $\theta = 0$, then, for every $(x, y) \in \mathcal{CT}^N$,

$$\begin{aligned} \varphi^N_{ss}(x,y) &= \rho_0^2 p^N_{x,y}(2) + \rho_0\rho_N p^N_{x,y}(1) + \rho_N^2 p^N_{x,y}(0) - [\rho_0 p^N_x(1) + \rho_N p^N_x(0)][\rho_0 p^N_y(1) + \rho_N p^N_y(0)] \\ &\quad + [\rho_0 p^N_x(1) + \rho_N p^N_x(0) - \frac{\rho_0^2}{\alpha}p^N_{x,x}(2) - \frac{\rho_0\rho_N}{\alpha}p^N_{x,x}(1) - \frac{\rho_N^2}{\alpha}p^N_{x,x}(0)]\mathbb{1}_{y=x} \end{aligned} \tag{4.60}$$

Simplifying the previous expression, we get that

$$\varphi_{ss}^N(x,y) = -\frac{(\rho_0-\rho_N)^2}{-1+\alpha N}p_x^N(0)p_y^N(1) + \left\{\frac{(\rho_0-\rho_N)^2}{2N(-1+\alpha N)} + \rho_{ss}^N(x) - \frac{\rho_0^2}{\alpha}p_{x,x}^N(2) - \frac{\rho_0\rho_N}{\alpha}p_{x,x}^N(1) - \frac{\rho_N^2}{\alpha}p_{x,x}^N(0)\right\}\mathbb{1}_{y=x}. \tag{4.61}$$

Observe that, for any $x,y \in \Lambda_N$, since $p_x^N(0), p_y^N(1) > 0$, if $\rho_N \neq \rho_0$, then, for $x \neq y$, $\varphi_{ss}^N(x,y) < 0$. Remark that, for $y = x$, since $-\frac{(\rho_0-\rho_N)^2}{-1+\alpha N}p_x^N(0)p_y^N(1) + \frac{(\rho_0-\rho_N)^2}{2N(-1+\alpha N)} < 0$, then $\varphi_{ss}^N(x,x) < \rho_{ss}^N(x)$. In fact, if $N \geq 2$ (which is clearly the case), the only real solutions of the polynomial $N - 2Nx + 2x^2$ are $\frac{N\pm\sqrt{N(N-2)}}{2}$, and since, $\frac{N-\sqrt{N(N-2)}}{2} < 1$ and $\frac{N+\sqrt{N(N-2)}}{2} > N-1$, the result follows. Also, as we had for SEP(α), if $x < y$, $\lim_{N\to\infty}\varphi_{ss}^N(x,y) = 0$, meaning that the 2-points stationary correlation function for SEP(α) decays to zero when we pass to the macroscopic space. Like we had for SEP(1), it has decay of order N. Also, $\varphi_{ss}^N(x,x) - \frac{\rho_{ss}^N(x)[\alpha-\rho_{ss}^N(x)]}{\alpha}$ decays to zero as N goes to infinity, with order of decay equal to N. For any $(x,y) \in \mathcal{CT}^N$, assuming that $\frac{x}{N} \to u$ and $\frac{y}{N} \to v$ as $N \to \infty$, then

$$\lim_{N\to\infty} N\left[\varphi_{ss}^N(x,y) - \rho_{ss}^N(x)\frac{\alpha-\rho_{ss}^N(x)}{\alpha}\mathbb{1}_{y=x}\right] = -\left(1-\frac{1}{\alpha}\mathbb{1}_{v=u}\right)\frac{(\rho_0-\rho_1)^2}{\alpha}G^{2,Dir}(u,v),$$

where $G^{2,Dir}$ is given as in (4.48). Observe that this case and the previous ($\alpha = 1$) differ from a multiplicative constant $1/\alpha$. This means that, macroscopically, changing the parameter α, rescales the limit function of $N\varphi_{ss}^N(x,y)$ obtain in (4.46) by a factor of $1/\alpha$.

In the next section, we will apply the same strategies for SIP(α).

4.5 SIP(α) - Absorption Probabilities

4.5.1 Case k = 1

If $k = 1$, then SEP(α) and SIP(α) describe the exact same model, therefore, for every $x \in \Lambda_N$, the absorption probabilities $p_x^N(m)$, with $m \in \{0,1\}$, are the same as in Section 4.1.1.

4.5.2 Case k = 2

Let $m \in \{0,1,2\}$ and let us keep the same notation and conventions for the absorption probabilities $p_{x,y}^N(m)$. Like in SEP(α), in SIP(α), we allow more than one particle per site, and therefore, it makes sense to consider $p_{x,y}^N(m)$ with $x = y$ with the same meaning as in SEP(α), with $\alpha \geq 2$. As we did for SEP(α), conditioning on the first jump, we get a system of 10 equations for $p_{x,y}^N(m)$ with two boundary conditions that can be compacted in:

$$\begin{cases} \mathfrak{W}_N^\theta p_{x,y}^N(m) = 0, \text{ for } x = 1,\dots,N-2 \text{ and } y = x+1,\dots,N-1 \\ p_{0,y}^N(m) = p_y^N(m-1)\mathbb{1}_{\{m\neq 0\}}, \text{ if } y = 0,\dots,N \\ p_{x,N}^N(m) = p_x^N(m)\mathbb{1}_{\{m\neq 2\}}, \text{ if } x = 0,\dots,N \end{cases}, \tag{4.62}$$

where the operator $\mathfrak{W}_N^\theta : \mathcal{F}(\mathcal{BCT}^N) \to \mathcal{F}(\mathcal{CT}^N)$ (see (4.53)) is defined, for every function $f \in \mathcal{F}(\mathcal{BCT}^N)$ and for every $(x,y) \in \mathcal{CT}^N$, by

$$\mathfrak{W}_N^\theta f(x,y) = a_x[f(x-1,y) - f(x,y)] + b_y[f(x,y+1) - f(x,y)] + e_{x,y}[f(x+1,y) + f(x,y-1) - 2f(x,y)],$$

where $a_x = \frac{N^2}{N^\theta}\mathbb{1}_{x=1} + N^2\mathbb{1}_{x\neq 1}$, $b_y = \frac{N^2}{N^\theta}\mathbb{1}_{y=N-1} + N^2\mathbb{1}_{y\neq N-1}$ and $e_{x,y} = \frac{N^2}{\alpha}\mathbb{1}_{y=x+1} + N^2\mathbb{1}_{y\neq x,x+1}$. For every $x,y \in \{2,\dots,N-2\}$, if $x<y$ and $|x-y|\geq 2$, $\mathfrak{W}_N^\theta = \Delta_{N,full}^{2D}$, and if $y=x$, $\mathfrak{W}_N^\theta = \Delta_{N,ref}^{2D}$. Also, if $x \in \{2,\dots,N-2\}$ and $y = x+1$,

$$\mathfrak{W}_N^\theta p_{x,x+1}^N(m) = \Delta_{N,full}^{2D} p_{x,x+1}^N(m) + \frac{1}{\alpha+1}\Delta_{N,ref}^{2D} p_{x,x+1}^N(m).$$

This shows that, like in SEP(α), over the line $\{(x,x+1) \mid x \in \Lambda_N\}$, we observe a super position of the operators that act above and over this line. Then, the natural ansatz to take is the same as in (4.55).

If $\theta = 0$, using that ansatz to solve the system in (4.62), performing, once again, long computations, if $x,y \in \{0,\dots,N\}$ with $x \leq y$,

If $x<y$:

$$\begin{cases} p_{x,y}^N(0) = \frac{x(1+\alpha y)}{N(1+\alpha N)} = \frac{1+\alpha y}{1+\alpha N} p_x^N(0), \\ p_{x,y}^N(1) = \frac{(\alpha N-1)x + (1+\alpha N)y - 2\alpha xy}{N(1+\alpha N)}, \\ p_{x,y}^N(2) = \frac{(1+\alpha[N-x])(N-y)}{N(1+\alpha N)} = \frac{1+\alpha[N-x]}{1+\alpha N} p_y^N(1). \end{cases}$$

If $x=y$:

$$\begin{cases} p_{x,x}^N(0) = \frac{x(1+\alpha x)}{N(1+\alpha N)} - \frac{1}{2N(1+\alpha N)}, \\ p_{x,x}^N(1) = \frac{(\alpha N-1)x + (1+\alpha N)x - 2\alpha x^2}{N(1+\alpha N)} + \frac{1}{N(1+\alpha N)}, \\ p_{x,x}^N(2) = \frac{(1+\alpha[N-x])(N-x)}{N(1+\alpha N)} - \frac{1}{2N(1+\alpha N)}. \end{cases} \tag{4.63}$$

Like in SEP(α), this means that on the diagonal $y=x$, the expression of $p_{x,y}^N(m)$ changes from the one inside the triangle, i.e., for $(x,y) \in \mathcal{T}^N$, by a constant that depends on α, N and m. Also, in this case, $\tilde{A}_m = A^m$, $\tilde{B}_m = B^m$, $\tilde{C}_m = C^m$ and $\tilde{D}_m = D^m$. We could not obtain the explicit expressions for the absorption probabilities for $\theta \neq 0$, by the same reasons presented in SEP(α).

4.6 Stationary density function and correlations via absorption probabilities

4.6.1 Applications to SIP(α)

As we did for SEP(α), we want now to compute the stationary density profile and the 2-points stationary correlation function for SIP(α) by using the absorption probabilities obtained above. Fix $k \in \mathbb{N}$. Note that, if $\hat{\eta} = \delta_{x_1} + \cdots + \delta_{x_k}$, where $x_1 < \cdots < x_k$, then, from Theorem 3.5.1, and, since $_j \in \Lambda_N$, for every $j \in \{1,\dots,k\}$, $\hat{\eta}(0) = \hat{\eta}(N) = 0$, we can simplify the expression of the duality function to

$$D^{SIP}(\eta,\hat{\eta}) = \underbrace{\left(\frac{\rho_0}{\alpha}\right)^{\hat{\eta}(0)}}_{=1} \left[\prod_{i=1}^k \frac{(\eta(x_i))!}{(\eta(x_i)-1)!}\frac{\Gamma(\alpha)}{\Gamma(\alpha+1)}\mathbb{1}_{\{\eta(x_i)\geq 1\}}\right] \underbrace{\left(\frac{\rho_N}{\alpha}\right)^{\hat{\eta}(N)}}_{=1} = \frac{1}{\alpha^k}\prod_{i=1}^k \eta(x_i), \tag{4.64}$$

where the last equality comes from the fact that, for every $i \in \{1, \dots, k\}$, $\eta(x_i)\mathbb{1}_{\{\eta(x_i)\geq 1\}} = \eta(x_i)$ and the property of the gamma function given by $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Here ρ_0 and ρ_N represent the density of the left and right reservoirs of the initial process, respectively. Then, repeating the argument we did for SEP(1) in Section 4.2.1, using equation (4.64) and the duality property (3.1), we get that

$$\mathbb{E}_{\mu_{ss}}[\eta(x_1)\dots\eta(x_k)] = \sum_{m=0}^{k} \rho_0^m \rho_N^{k-m} \underbrace{\mathbb{P}_\eta(\hat{\eta}_\infty(0) = m, \hat{\eta}_\infty(N) = k-m)}_{=p^N_{x_1,\dots,x_k}(m)}. \tag{4.65}$$

Again, if now we allow $x_1 = x_2$ but all the other x_j different, we obtain, analogous to what we had for SEP(α), that $D^{SIP(\alpha)}(\eta, \hat{\eta}) = \frac{1}{\alpha(\alpha+1)}\eta(x_1)[\eta(x_1)-1]$, if $k=2$, and, therefore, for SIP(α) we have that

$$\mathbb{E}_{\mu_{ss}}[\eta(x_1)\dots\eta(x_k)] + \mathbb{E}_{\mu_{ss}}[\eta(x_2)\dots\eta(x_k)] = \frac{\alpha+1}{\alpha}\sum_{m=0}^{k} \rho_0^m \rho_N^{k-m} \underbrace{\mathbb{P}_\eta(\hat{\eta}_\infty(0) = m, \hat{\eta}_\infty(N) = k-m)}_{=p^N_{x_1,\dots,x_k}(m)}.$$

Thus, we can obtain the stationary density function and the 2-points stationary correlation function once we have computed the absorption probabilities $p^N_{x_1,\dots,x_k}(m)$, with $m \in \{0,\dots,k\}$ and $k \leq 2$.

Discrete stationary density profile:

The SEP(α) and SIP(α), with only one dual particle, describe the same model - equal density profile.

2-points stationary correlation function:

Again, combining (4.41), (4.65) and the results in (4.63), for every $(x,y) \in \mathcal{CT}^N$,

$$\begin{aligned}\varphi^N_{ss}(x,y) = {}& \rho_0^2 p^N_{x,y}(2) + \rho_0\rho_N p^N_{x,y}(1) + \rho_N^2 p^N_{x,y}(0) - [\rho_0 p^N_x(1) + \rho_N p^N_x(0)][\rho_0 p^N_y(1) + \rho_N p^N_y(0)] \\ & - [\rho_0 p^N_x(1) + \rho_N p^N_x(0) - \frac{\rho_0^2}{\alpha}p^N_{x,x}(2) - \frac{\rho_0\rho_N}{\alpha}p^N_{x,x}(1) - \frac{\rho_N^2}{\alpha}p^N_{x,x}(0)]\mathbb{1}_{y=x}\end{aligned} \tag{4.66}$$

Simplifying the previous expression, we get, if $\theta = 0$, that

$$\begin{aligned}\varphi^N_{ss}(x,y) = {}& \frac{(\rho_0-\rho_N)^2}{-1+\alpha N} p^N_x(0)p^N_y(1) \\ & + \left\{-\frac{(\rho_0-\rho_N)^2}{2N(-1+\alpha N)} - \rho^N_{ss}(x) + \frac{\rho_0^2}{\alpha}p^N_{x,x}(2) + \frac{\rho_0\rho_N}{\alpha}p^N_{x,x}(1) + \frac{\rho_N^2}{\alpha}p^N_{x,x}(0)\right\}\mathbb{1}_{y=x}.\end{aligned} \tag{4.67}$$

Observe that, for any $x, y \in \Lambda_N$, since $p^N_x(0), p^N_y(1) > 0$, if $\rho_N \neq \rho_0$, then, for $x \neq y$, $\varphi^N_{ss}(x,y) > 0$. Remark that, for $y = x$, by the same argument we used for SEP(α), we also have that $\varphi^N_{ss}(x,x) + \rho^N_{ss}(x) > 0$, if $N \geq 2$. For SIP(α), we still have that $\lim_{N\to\infty}\left[\varphi^N_{ss}(x,y) + \frac{\rho^N_{ss}(x)[\alpha+\rho^N_{ss}(x)]}{\alpha}\mathbb{1}_{y=x}\right] = 0$, meaning that the 2-points stationary correlation function for SEP(α) decays to zero when we pass to the macroscopic space. The order of decay of order is still N. Finally, for any $x, y \in \Lambda_N$, assuming that $\frac{x}{N} \to u$ and $\frac{y}{N} \to v$ as $N \to \infty$, then $\lim_{N\to\infty} N\left[\varphi^N_{ss}(x,y) + \rho^N_{ss}(x)\frac{\alpha+\rho^N_{ss}(x)}{\alpha}\mathbb{1}_{y=x}\right] = \left(1 + \frac{1}{\alpha}\mathbb{1}_{v=u}\right)\frac{(\rho_0-\rho_N)^2}{\alpha}G^{2,Dir}(u,v)$, where $G^{2,Dir}$ has the same interpretation as in SEP(1) and SEP(α). Observe that the difference from the analogous result obtained for SEP(α) is the change of sign from a minus to a plus.

Chapter 5

Conclusions

5.1 Achievements and results discussion

We can think of this work as a constructive description of some of the results one can obtain when duality is available. This thesis is organized in three main chapters, Chapters 2, 3 and 4, where the first two contain the ingredients needed for applications presented in Chapter 4.

The main achievements of the present work are: the presentation of the relationship, through duality, between all the continuous time Markov processes with reservoirs attached to the boundary referred throughout the dissertation, with their corresponding versions replacing the reservoirs dynamics by absorbing boundary points; obtaining explicit expressions for the stationary density profile and 2-points (also for SEP(1), 3-points) stationary correlation functions, with a method based on finding absorption probabilities of the dual processes that are solution of some discrete equations with boundary condition; and finding the macroscopic limit functions of these same functions relating them with Green functions that are solution of an initial value problem with different boundary conditions depending on the strength of the interaction of the reservoirs with the system, i.e., on value of the parameter θ.

In Chapter 2, the main result is the existence and uniqueness of invariant measures for both processes taken in consideration when we are in equilibrium and out of equilibrium. Here we concluded that, when the density of the left and right reservoirs are equal, not only exists a unique invariant measure, but that it is also reversible and of homogeneous product form. For different densities of the left and right reservoirs, using a classical theorem from continuous times Markov chains, we also guaranteed existence and uniqueness of the invariant measure for SEP(α) and the proof, based on duality, for existence and uniqueness of the invariant measure for SIP(α) was presented on the following chapter.

In Chapter 3, we concluded that both models introduced in Chapter 2 have a dual process that conserves the bulk dynamics exchanging the reservoir's dynamics to absorbing boundary points. To show these duality relations, we used not only the very important observation that we can generate duality functions by applying symmetries of the Markov generator to a known duality function, but also powerful tools from Lie algebra's theory, namely, representations of the Lie algebras $su(2)$ and $su(1,1)$. We concluded that once a reversible measure is available for a given continuous time Markov process

with a countable state space, self-duality comes for free using the cheap self-duality function and that non-trivial duality functions can be generated from this one. We also concluded that duality can be used to prove the non-trivial result about existence and uniqueness of stationary measure for the model SIP(α) that has a countable (and not finite) state space.

At last, in Chapter 4, we could obtain explicit expressions for the absorption probabilities with $k = 1, 2, 3$ dual particles for SEP(1) with $\theta \in \mathbb{R}$ and with $k = 1, 2$ dual particles for SEP(α) and SIP(α) with $\theta = 0$. Let us now compare the results:

- The next table compares the absorption probabilities $p^N_{x,y}(m)$, with $x, y \in \Lambda_N$ and $x < y$, taking $\theta = 0$, for SEP(1), SEP(α) and SIP(α).

Values for m	SEP(1)	SEP(α)	SIP(α)
$m = 0$	$p^N_x(0) - p^N_y(1)p^{N-1}_x(0)$	$p^N_x(0) - p^N_y(1)\frac{x}{N-\alpha^{-1}}$	$p^N_x(0) - p^N_y(1)\frac{x}{N+\alpha^{-1}}$
$m = 1$	$\frac{(N+1)x+(N-1)y-2xy}{N(N-1)}$	$\frac{(\alpha N+1)x+(\alpha N-1)y-2\alpha xy}{N(-1+\alpha N)}$	$\frac{(\alpha N-1)x+(1+\alpha N)y-2\alpha xy}{N(1+\alpha N)}$
$m = 2$	$p^{N-1}_x(1)p^N_y(1)$	$\frac{N-\alpha^{-1}-x}{N-\alpha^{-1}}p^N_y(1)$	$\frac{N+\alpha^{-1}-x}{N+\alpha^{-1}}p^N_y(1)$

Table 5.1: Absorption probabilities $p^N_{x,y}(m)$, with $x, y \in \Lambda_N$ where $x < y$ and $\theta = 0$: SEP(1) versus SEP(α) versus SIP(α).

We observe that, for $m \in \{0, 1, 2\}$, $p^N_{x,y}(m)$, for all models, can be factorized, for every $x, y \in \Lambda_N$ with $x < y$, as $p^N_{x,y}(m) = p^N_x(m) + p^N_y(1)F_x(m)$, with

$$F_x(m) = \begin{cases} p^{N-1}_x(m) - p^{N-1}_x(m-1), \text{ for SEP(1)} \\ p^{N-\alpha^{-1}}_x(m) - p^{N-\alpha^{-1}}_x(m-1), \text{ for SEP}(\alpha) \\ p^{N+\alpha^{-1}}_x(m) - p^{N+\alpha^{-1}}_x(m-1), \text{ for SIP}(\alpha) \end{cases}, \text{ where } p^{N+r}_x(m) := \begin{cases} \frac{x}{N+r}, \text{ if } m = 0, \\ \frac{N+r-x}{N+r}, \text{ if } m = 1, \\ 0, \text{ if } m = -1, 2, \end{cases} \tag{5.1}$$

for every $r \in \mathbb{R}$. So, if $\alpha^{-1} \in \mathbb{N}$, then $p^{N+\alpha^{-1}}_x(1)$ (resp. $p^{N+\alpha^{-1}}_x(0)$) represents the probability that, starting from one dual particle at site x on a lattice of size $N+\alpha^{-1}$ (the size increased), one particle is absorbed at 0 (resp. N). On the other hand, if $\alpha^{-1} \notin \mathbb{N}$, then $p^{N\pm\alpha^{-1}}_x(1)$ and $p^{N\pm\alpha^{-1}}_x(0)$ are no longer absorption probabilities since to see them as that we would need to consider a lattice of non-integer size (the size N decreases by α^{-1}, for SEP(α), and increases by α^{-1}, for SIP(α)) that, by our construction of the microscopic space from the macroscopic space, is not allowed.

Let us now compare the absorption probabilities $p^N_{x,y}(m)$, with $x, y \in \Lambda_N$ and $y = x$, taking $\theta = 0$, for SEP(α) and SIP(α) (recall that it does not make sense to talk about $p^N_{x,x}(m)$ for SEP(1), since only one particle is allowed per site) - see the Table 5.2.

Clearly, for SEP(α) and SIP(α), for $m \in \{0, 1, 2\}$, $p^N_{x,x}(m)$ can be factorized, for every $x \in \Lambda_N$ as $p^N_{x,x}(m) = p^N_x(m) + p^N_x(1)F_x(m) + H(m)$, where $F_x(m)$ is as in (5.1) and $H(m)$ is a real number

Values for m	SEP(α)	SIP(α)
$m=0$	$p_x^N(0)-p_x^N(1)\frac{x}{N-\alpha^{-1}}+\frac{\alpha^{-1}}{2N(N-\alpha^{-1})}$	$p_x^N(0)-p_x^N(1)\frac{x}{N+\alpha^{-1}}-\frac{\alpha^{-1}}{2N(N+\alpha^{-1})}$
$m=1$	$\frac{(\alpha N+1)x+(\alpha N-1)x-2\alpha x^2}{N(-1+\alpha N)}-\frac{\alpha^{-1}}{N(N-\alpha^{-1})}$	$\frac{(\alpha N-1)x+(1+\alpha N)x-2\alpha x^2}{N(1+\alpha N)}+\frac{\alpha^{-1}}{N(N+\alpha^{-1})}$
$m=2$	$\frac{N-\alpha^{-1}-x}{N-\alpha^{-1}}p_x^N(1)+\frac{\alpha^{-1}}{2N(N-\alpha^{-1})}$	$\frac{N+\alpha^{-1}-x}{N+\alpha^{-1}}p_x^N(1)-\frac{\alpha^{-1}}{2N(N+\alpha^{-1})}$

Table 5.2: Absorption probabilities $p_{x,x}^N(m)$, with $x\in\Lambda_N$ and $\theta=0$: SEP(α) versus SIP(α).

that depends on m, but not on x, and is defined, for $m\in\{0,1,2\}$, as

$$H(m):=\begin{cases}\frac{(-1)^m\alpha^{-1}}{2N(N-\alpha^{-1})}[\mathbb{1}_{m\neq 0}+\mathbb{1}_{m\neq 2}], \text{ for SEP}(\alpha),\\ \frac{(-1)^{m+1}\alpha^{-1}}{N(N+\alpha^{-1})}[\mathbb{1}_{m\neq 0}+\mathbb{1}_{m\neq 2}], \text{ for SIP}(\alpha).\end{cases} \tag{5.2}$$

Summarizing, for SEP(α) and SIP(α), if $\theta=0$, for $m\in\{0,1,2\}$, $p_{x,y}^N(m)$ can be factorized, for every $x,y\in\Lambda_N$ with $x\leq y$, as

$$p_{x,y}^N(m)=p_x^N(m)+p_y^N(1)F_x(m)+H(m)\mathbb{1}_{y=x}, \tag{5.3}$$

where $F_x(m)$ and $H(m)$ are the ones defined above.

- As we saw in Chapter 4, the stationary density profiles of SEP(α) and SIP(α) are exactly the same, since with only one particle, these processes describe the same dynamics. Let us then compare the results for the 2-points stationary correlation function for SEP(α) and SIP(α) for the choice $\theta=0$ - see Table 5.3.

SEP(α)	SIP(α)
$-\alpha\frac{(\rho_0-\rho_N)^2}{N-\alpha^{-1}}p_x^N(0)p_y^N(1)+f_1(x)\mathbb{1}_{y=x}$	$\alpha\frac{(\rho_0-\rho_N)^2}{N+\alpha^{-1}}p_x^N(0)p_y^N(1)+f_2(x)\mathbb{1}_{y=x}$

Table 5.3: 2-points stationary correlation function, $\varphi_{ss}^N(x,y)$, with $x,y\neq 0$, $x,y\neq N$ and $\theta=0$: SEP(α) versus SIP(α).

Above, $f_1,f_2:\Lambda_N\to\mathbb{R}$ are given by

$$f_j(x)=(-1)^j\left[\frac{(\rho_0-\rho_N)^2}{2N(-1+\alpha N)}-\rho_{ss}^N(x)+\frac{\rho_0^2}{\alpha}p_{x,x}^N(2)+\frac{\rho_0\rho_N}{\alpha}p_{x,x}^N(1)+\frac{\rho_N^2}{\alpha}p_{x,x}^N(0)\right],$$

with $j\in\{1,2\}$. Therefore, for SEP(α) and SIP(α), $\varphi_{ss}^N(x,y)$, with $x,y\in\Lambda_N$, factorizes as

$$\begin{aligned}&\varphi_{ss}^N(x,y)+C_\alpha\rho_{ss}^N(x)\frac{\alpha+C_\alpha\rho_{ss}^N(x)}{\alpha}\mathbb{1}_{y=x}\\&=\frac{(\rho_0-\rho_N)^2}{\alpha}\left\{NH(1)p_x^N(0)p_y^N(1)+\left(H(0)+\frac{C_\alpha}{\alpha}\left[NH(1)p_x^N(0)p_y^N(1)+H(0)\right]\right)\mathbb{1}_{y=x}\right\}\end{aligned} \tag{5.4}$$

where $H(0)$ and $H(1)$ are the same as in (5.2) and $C_\alpha=-1$, for SEP(α), and $C_\alpha=1$ for SIP(α). Also, for SEP(α), we have $\varphi_{ss}^N(x,y)<0$, while for SIP(α), $\varphi_{ss}^N(x,y)>0$, for all $x,y\in\Lambda_N$ with $y\neq x$.

Finally, in Chapter 4, we concluded that duality, for our models, can be used to obtain explicit formulas for the stationary density profile and for k-points stationary centered (and also non-centered) correlation function provided that there is a closed formula for the absorption probabilities starting from a configuration on the dual process initialized with k particles. By the methods presented in this chapter, we concluded that solving systems of equations with some boundary conditions, with patience (due to the extensive computations that are involved), we can obtain the same results known for SEP(1) with $\theta = 0$ and with general θ that were obtained using MPA. More, not only this method works for SEP(1), but for SEP(α) and SIP(α) where MPA is not available, and even, we believe, can be applied for other interacting particle systems with boundaries where duality is available with such a duality function that can be related to moments of the initial process.

5.2 Future Work

We left, as future work, proving the general recursive formula for the absorption probabilities for $k \geq 4$ dual particles for SEP(1) using our method. Once this is achieved, using (4.40), one can easily obtain the general formula for the k-points stationary correlation function for SEP(1) that we expect to match with the one obtained using MPA. Also, we would like to obtain the solution for the system of equations that the absorption probabilities for 2 dual particles with $\theta \in \mathbb{R}$ for SEP(α) and SIP(α) solve and from here obtain explicit expressions for the 2-points stationary correlation function for SEP(α) and SIP(α) and study its corresponding limiting function, meaning, find its order of decay to zero. We expect this approach to leave to a hard path. An alternative approach could be, as is presented in Appendix (C), apply Kolmogorov's equation to obtain the equation satisfied by the 2-points stationary correlation function for these models and try to find the solution of that. We expect that the operators that will appear will be related to the ones obtained for the absorption probabilities $p^N_{x,y}(m)$ with an extra error function taking values only on the diagonal $y = x$. From here, we would also like to see if the stationary centered correlation functions for higher order (3-points, 4-points, and so on) can also be written in an analogous factorized form as in (5.4) using absorption probabilities of just one dual particle or even recursively like it is known for SEP(1).

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Appendix A

Links for the Mathematica's code

Since the code used in Mathematica to verify some of this thesis computations is too extensive, we leave here the links for the Mathematica files that are free for observation:

- Absorption probabilities for SEP(1), with $\theta \in \mathbb{R}$:

  `https://www.wolframcloud.com/obj/b0f3b160-65cb-4c51-81af-9133b7566771`

- Absorption probabilities for SEP(α) with $\theta = 0$:

  `https://www.wolframcloud.com/obj/8ad986db-b7c6-4ea1-9678-df592edb049b`

- Absorption probabilities for SIP(α) with $\theta = 0$:

  `https://www.wolframcloud.com/obj/12adc15a-7b9c-4662-bb19-535255ea24ad`

- Stationary density, 2-nd and 3-rd point stationary centered correlation functions for SEP(1), with $\theta \in \mathbb{R}$:

  `https://www.wolframcloud.com/obj/0eacec5b-2cdc-4769-9746-501f80cddc4a`

- Stationary density and 2-nd point stationary centered correlation function for SEP(α) with $\theta = 0$:

  `https://www.wolframcloud.com/obj/0b141ad0-413f-486e-9f4b-925174a9d466`

- Stationary density and 2-nd point stationary centered correlation function for SIP(α) with $\theta = 0$:

  `https://www.wolframcloud.com/obj/e3d1eff2-bbdd-4b88-97be-fc10ea727b13`

Appendix B

Formulas for absorption probabilities including the boundary

B.1 SEP(1)

B.1.1 Case k = 1

For every $x \in \{0, \dots, N\}$,

$$p_x^N(1) = \frac{N^\theta - 1 + N - x}{N + 2N^\theta - 2} + \frac{N^\theta - 1}{N + 2N^\theta - 2}\mathbb{1}_{\{x=0\}} - \frac{N^\theta - 1}{N + 2N^\theta - 2}\mathbb{1}_{\{x=N\}},$$
$$p_x^N(0) = \frac{N^\theta - 1 + x}{N + 2N^\theta - 2} - \frac{N^\theta - 1}{N + 2N^\theta - 2}\mathbb{1}_{\{x=0\}} + \frac{N^\theta - 1}{N + 2N^\theta - 2}\mathbb{1}_{\{x=N\}}.$$

B.1.2 Case k = 2

For every $x, y \in \{0, \dots, N\}$ with $x < y$,

$$\begin{cases} p_{x,y}^N(0) = \frac{(N^\theta-1+x)(N^\theta-2+y)}{[N+2N^\theta-2][N+2N^\theta-3]} - \frac{(N^\theta-1)(N^\theta-2+y)}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{x=0\}} + \\ \qquad\qquad \frac{(N^\theta-1)(N^\theta-1+x)}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{y=N\}} - \frac{(N^\theta-1)^2}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{x=0,y=N\}} \\ p_{x,y}^N(1) = \frac{(N+1)x+(N-1)y-2xy+2(N^\theta-1)(N^\theta+N-1)}{[N+2N^\theta-2][N+2N^\theta-3]} + \frac{(N^\theta-1)(2y-N-1)}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{x=0\}} + \\ \qquad\qquad \frac{(N^\theta-1)(N-1-2x)}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{y=N\}} + \frac{2(N^\theta-1)^2}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{x=0,y=N\}} \\ p_{x,y}^N(2) = \frac{(N^\theta-2+N-x)(N^\theta-1+N-y)}{[N+2N^\theta-2][N+2N^\theta-3]} - \frac{(N^\theta-1)(N^\theta-2+N-x)}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{y=N\}} + \\ \qquad\qquad \frac{(N^\theta-1)(N^\theta-1+N-y)}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{x=0\}} - \frac{(N^\theta-1)^2}{[N+2N^\theta-2][N+2N^\theta-3]}\mathbb{1}_{\{x=0,y=N\}} \end{cases}.$$

B.1.3 Case k = 3

For every $x, y, z \in \{0, \dots, N\}$ with $x < y < z$,

$$
\begin{cases}
p^N_{x,y,z}(0) = \frac{(N^\theta-1+x)(N^\theta-2+y)(N^\theta-3+z)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]} - \frac{(N^\theta-1)(N^\theta-2+y)(N^\theta-3+z)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0\}} + \\
\qquad \frac{(N^\theta-1)(N^\theta-1+x)(N^\theta-2+y)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{z=N\}} - \frac{(N^\theta-1)^2(N^\theta-2+y)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0,z=N\}} \\
\\
p^N_{x,y,z}(1) = \frac{(N^\theta-2)(1+2N+N^\theta)x+(N^{2\theta}+2NN^\theta-2N^\theta-3N+3)y+(N^\theta-1)(N^\theta+2N-2)z+(5+N-N^\theta)xy+(2+N-N^\theta)xz}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]} + \\
\qquad \frac{(-1+N-N^\theta)yz-3xyz+3(N^\theta-2)(N^\theta-1)(N^\theta+N-1)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]} + \frac{(N^\theta-1)^2[3y+N^\theta-N-2]}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0,z=N\}} \\
\qquad\qquad \frac{(N^\theta-1)[(N^\theta-N-5)y+(N^\theta-N-2)z+3yz-(N^\theta-2)(N^\theta+2N+1)]}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0\}} + \\
\qquad\qquad \frac{(N^\theta-1)[(-N^\theta+2+N)x+(-N^\theta+N-1)y+3xy+(N^\theta-1)(N^\theta+2N-2)]}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{z=N\}} \\
\\
p^N_{x,y,z}(2) = \frac{(N-N^\theta+2)(N^\theta+N-1)x+(-N^{2\theta}+2N^\theta+N^2-N-3)y+(N^\theta+N-2)(-N^\theta+N-1)z-(1+2N+N^\theta)xy}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]} + \\
\frac{(2-2N-N^\theta)xz+(5-2N-N^\theta)yz+3xyz+3(N^\theta-1)(N^\theta+N-2)(N^\theta+N-1)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]} + \frac{(N^\theta-1)^2[-3y+N^\theta+2N-2]}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0,z=N\}} + \\
\qquad\qquad \frac{(N^\theta-1)[(N^\theta+2N+1)y+(N^\theta+2N-2)z-3yz+(N^\theta+N-1)(N^\theta-N-2)]}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0\}} + \\
\qquad\qquad \frac{(N^\theta-1)[(-N^\theta-2N+2)x+(-N^\theta-2N+5)y-3xy-(N^\theta-N+1)(N^\theta+N-2)]}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{z=N\}} \\
\\
p^N_{x,y,z}(3) = \frac{(N^\theta+N-1-z)(N^\theta+N-2-y)(N^\theta+N-3-x)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]} - \frac{(N^\theta-1)(N^\theta-2+N-y)(N^\theta-3+N-x)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{z=N\}} + \\
\qquad \frac{(N^\theta-1)(N^\theta-1+N-z)(N^\theta-2+N-y)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0\}} - \frac{(N^\theta-1)^2(N^\theta-2+N-y)}{[N+2N^\theta-2][N+2N^\theta-3][N+2N^\theta-4]}\mathbb{1}_{\{x=0,z=N\}}
\end{cases}
$$

Appendix C

Correlations via Kolmogorov's equation

C.1 2-points stationary correlation for SEP(1)

Let us keep the notation φ_{ss}^N to represent the 2-points stationary correlation for SEP(1). We want to find a system of equations for φ_{ss}^N using Kolmogorov's equation. To simplify notation, we will write here η_x to represent $\eta(x)$. So, by the forward Kolmogorov's equation, we have that

$$0 = \frac{d}{dt}\varphi_{ss}^N(x,y) = \mathbb{E}_{\mu_{ss}}[\mathcal{L}_1^{Ex}\eta_x\eta_y] - \rho_{ss}^N(x)\mathbb{E}_{\mu_{ss}}[\mathcal{L}_1^{Ex}\eta_y] - \rho_{ss}^N(y)\mathbb{E}_{\mu_{ss}}[\mathcal{L}_1^{Ex}\eta_x]. \tag{C.1}$$

Since

$$\mathcal{L}_1^{Ex}\eta_x = \begin{cases} \frac{1}{N^\theta}\left[\epsilon - \eta_1\right] + (\nabla^{+,N}\eta_1), \text{ if } x = 1 \\ \Delta_N^{1D}\eta_x, \text{ if } x \neq 1, N-1 \\ \frac{1}{N^\theta}\left[\delta - \eta_{N-1}\right] + (\nabla^{-,N}\eta_{N-1}), \text{ if } x = N-1 \end{cases}$$

and

$$\mathcal{L}_1^{Ex}\eta_x\eta_y = \begin{cases} \frac{1}{N^\theta}\left[\epsilon - \eta_1\right]\eta_y + (\nabla^{+,N}\eta_1)\eta_y + \eta_1(\Delta_N^{1D}\eta_y), \text{ if } x = 1, y = x+2, \ldots, N-2 \\ \frac{1}{N^\theta}\left[\epsilon - \eta_1\right]\eta_2 + (\nabla^{+,N}\eta_2)\eta_1, \text{ if } x = 1, y = 2 \\ \frac{1}{N^\theta}\left[\epsilon - \eta_1\right]\eta_{N-1} + (\nabla^{+,N}\eta_1)\eta_{N-1} + \eta_1(\nabla^{-,N}\eta_{N-1}), \text{ if } x = 1, y = N-1 \\ \eta_y(\Delta_N^{1D}\eta_x) + \eta_x(\Delta_N^{1D}\eta_y), \text{ if } x \in \{2, \ldots, N-3\}, y \in \{x+2, \ldots, N-1\} \\ \eta_{x+1}(\nabla^{-,N}\eta_x) + \eta_x(\nabla^{+,N}\eta_{x+1}), \text{ if } x \in \{2, \ldots, N-2\}, y = x+1 \\ \frac{1}{N^\theta}\left[\delta - \eta_{N-1}\right]\eta_x + (\nabla^{-,N}\eta_{N-1})\eta_x + (\Delta_N^{1D}\eta_x)\eta_{N-1}, \text{ if } x \in \{2, \ldots, N-3\}, y = N-1 \\ \frac{1}{N^\theta}\left[\delta - \eta_{N-1}\right]\eta_{N-2} + (\nabla^{-,N}\eta_{N-2})\eta_{N-1}, \text{ if } x = N-2, y = N-1 \end{cases},$$

substituting in (C.1) and performing long computations, we get

$$\mathfrak{O}_N^\theta \varphi_{ss}^N(x,y) + g_N(x,y) = 0, \tag{C.2}$$

where $\mathfrak{O}_N^\theta$ is the same as in (4.8) and $g_N(x,y) = -(\nabla^{+,N}\rho_{ss}^N(x))^2\delta_{y=x+1}$. The previous result can also be found in [8] for a correlation function that is time-dependent (hence, starting not necessarily from the stationary measure but from any probability measure μ_N defined on Ω_N^{Ex}).

If we take the limit as N goes to infinity, the equation (C.2) is converted to an initial value problem: since, as N goes to infinity, $\nabla^{+,N} \to \frac{d}{dx}$ and $\rho_{ss}^N \to \bar{\rho}$, and

$$\frac{d}{du}\bar{\rho}(u) = \begin{cases} \rho_1 - \rho_0, \text{ if } \theta < 1 \\ \frac{\rho_1-\rho_0}{3} \text{ if } \theta = 1 \\ 0, \text{ if } \theta > 1 \end{cases},$$

then, denoting by φ_{ss} the limit in N of φ_{ss}^N,

- if $\theta < 1$, φ_{ss} is solution to $\begin{cases} -\Delta^{2D}\varphi_{ss}(u,v) = (\rho_1-\rho_0)^2\delta_{u=v}, \text{ if } (u,v) \in \text{int } Q \\ \varphi_{ss}(0,v) = \varphi_{ss}(u,1) = 0, \text{ if } (u,v) \in Q \end{cases}$.
- If $\theta = 1$, φ_{ss} is solution to $\begin{cases} -\Delta^{2D}\varphi_{ss}(u,v) = \left(\frac{\rho_1-\rho_0}{3}\right)^2\delta_{u=v}, \text{ if } (u,v) \in \text{int } Q \\ \frac{\partial}{\partial u}\varphi_{ss}(0,v) = \varphi_{ss}(0,v), \text{ if } v \in [0,1] \\ \frac{\partial}{\partial v}\varphi_{ss}(u,1) = -\varphi_{ss}(u,1) \text{ if } u \in [0,1] \end{cases}$.

Remark C.1.1. *We observe that the previous computations can be done by taking the expectation with respect to any probability measure μ^N, and, in that case, we obtain the system of equations for the 2-points time-dependent centered correlation function, $\varphi_t^N(x,y)$.*

C.2 3-points stationary correlation for SEP(1)

Let us keep the notation φ_{ss}^N to represent the 3-points stationary correlation for SEP(1). Performing similar (but longer) computations like the ones for the 2-points stationary centered correlation function, we obtain

$$\mathcal{R}_N^\theta \varphi_{ss}^N(x,y,z) + f_N(x,y,z) = 0, \tag{C.3}$$

where $\mathcal{R}_N^\theta$ is the same as in (4.17) and

$$f_N(x,y,z) = -\left[\nabla^{+,N}\rho_{ss}^N(x)\right]\left[(\nabla^{+,N})_x\varphi_{ss}^N(x,z)\right]\delta_{y=x+1} - \left[\nabla^{+,N}\rho_{ss}^N(y)\right]\left[(\nabla^{+,N})_y\varphi_{ss}^N(x,y)\right]\delta_{z=y+1}. \tag{C.4}$$

If we take the limit as N goes to infinity, the equation is converted to an initial value problem: since, as N goes to infinity, $\nabla^{+,N} \to \frac{d}{dx}$, $(\nabla^{+,N})_x \to \frac{\partial}{\partial x}$ (the same substituting x by y) and $\rho_{ss}^N \to \bar{\rho}$, then, denoting by φ_{ss} the limit in N of φ_{ss}^N, we have:

- If $\theta < 1$, $\begin{cases} -\Delta^{3D}\varphi_{ss}(u,v,w) = (\rho_N - \rho_0)^3[(1-w)\delta_{u=v} + u\delta_{w=v}], \text{ if } (u,v,w) \in \text{int } S \\ \varphi_{ss}(0,v,w) = \varphi_{ss}(u,v,1) = 0, \text{ if } (u,v),(v,w) \in Q \end{cases}$.

- If $\theta = 1$, $\begin{cases} -\Delta^{3D}\varphi_{ss}(u,v,w)(u,v,w) = (\rho_N - \rho_0)^3[(1-w)\delta_{u=v} + u\delta_{w=v}], \text{ if } (u,v,w) \in \text{int } S \\ \frac{\partial}{\partial u}\varphi_{ss}(0,v,w) = \varphi_{ss}(0,v,w), \text{ if } (v,w) \in T \\ \frac{\partial}{\partial w}\varphi_{ss}(u,v,1) = -\varphi_{ss}(u,v,1) \text{ if } (u,v) \in T \end{cases}$.

Remark C.2.1. *We observe that the previous results can be easily extended to obtain the system of equations for the 3-points time-dependent centered correlation function, $\varphi_t^N(x,y,z)$, by taking the expectation in the definition of $\varphi_{ss}^N(x,y,z)$ with respect to any probability measure μ^N instead of the invariant measure.*

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