TÉCNICO LISBOA

Thom Polynomials for Degeneracy Loci of 2-forms and Maps to an Almost Symplectic Manifold

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Abstract

Let M be a smooth $2m$-manifold. In some cases, the degeneracy loci of a geometric structure on M give rise to homology classes in $H_*(M)$. The computation of their Poincaré duals (usually referred to as Thom polynomials) is an active area of research. In this thesis we compute the Thom polynomials of degeneracy loci in two settings.

Chapter 2 gives some theoretical background and necessary results needed in the rest of the thesis.

In chapter 3, we study the degeneracy loci of sections of the bundle $\Lambda^2 T^*M \to M$ or, in other words, of 2-forms over M. There will be a Thom polynomial P_{R_k} for each integer $k \in \{0, ..., m\}$. To compute P_{R_k}, we define it first as a certain cohomological obstruction to the existence of sections with rank everywhere greater than $2k$. We compute those obstructions classes and then show that they are indeed the Poincaré duals of the degeneracy loci.

In chapter 4, we consider a smooth map $i : M \to N$ between a $2m$-manifold M and an almost symplectic $2n$-manifold N with $m \leq n$. We study the degeneracy loci of sections of the bundle $Hom(TM, i^*TN) \to M$. In this setting, the degeneracy loci may not give rise to homology classes, but the definition of Thom polynomials as obstructions classes remains valid and that is the one we will use. The procedure to compute the Thom polynomials in this case will be the same as the one used in chapter 3.

Keywords: Degeneracy Loci, Characteristic Classes, Classifying Spaces, Thom Polynomials

Resumo

Seja M uma $2m$-variedade diferenciável. Por vezes, os loci de degenerescência de uma estrutura geométrica em M dão origem a classes de homologia em $H_*(M)$. O cálculo dos seus duais de Poincaré (usualmente denominados por polinómios de Thom) é uma área ativa de investigação. Nesta tese, calculamos os polinómios de Thom de loci de degenerescência em dois casos.

No capítulo 2 introduzimos preliminares teóricos e conceitos necessários no resto da tese.

No capítulo 3, estudamos os loci de degenerescência de secções do fibrado $\Lambda^2 T^*M \to M$ ou, por outras palavras, de 2-formas sobre M. Haverá um polinómio de Thom P_{R_k} para cada inteiro $k \in \{0, ..., m\}$. Para calcular P_{R_k}, definimo-lo primeiro como uma certa obstrução cohomológica à existência de secções com rank em todo o lado maior do que $2k$. Calculamos essas obstruções e por fim mostramos que são os duais de Poincaré dos loci de degenerescência.

No capítulo 4, consideramos uma aplicação diferenciável $i : M \to N$ entre uma $2m$-variedade M e uma $2n$-variedade quase-simplética N, com $m \leq n$. Estudamos neste caso os loci de degenerescência do fibrado $Hom(TM, i^*TN) \to M$. Neste contexto, os loci poderão não dar origem a classes de homologia, mas a definição de polinómios de Thom como classes de obstrução permanece válida e é essa que usamos. O procedimento para calcular os polinómios de Thom é o mesmo que o usado no capítulo 3.

Palavras-Chave: Loci de Degenerescência, Classes Características, Espaços Classificantes, Polinómios de Thom

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Chapter 1

Introduction

A central problem in singularity theory and enumerative geometry is to study cohomological properties of singularities and degeneracy loci of geometric structures. Perhaps the simplest examples are the Chern classes. The i-th Chern class $c_i(E)$ of an n-vector bundle E can be interpreted as the Poincaré dual of the locus of points where $n-i+1$ generic sections of E become linearly dependent. Another typical (and more complicated) example concerns the degeneracy loci of a vector bundle map: let E and F be complex vector bundles of ranks m and n, respectively, over a compact manifold M. A vector bundle map $E \to F$ is the same as a section s of the bundle

$$\begin{array}{c} Hom(E,F) \\ \big\downarrow \big\uparrow s \\ M \end{array}$$

Given a positive integer k, the **degeneracy locus** of rank k of s is the set of points $x \in M$ where the rank of $s(x)$ is at most k. Let us denote this degeneracy locus by $F_k \subset M$. For a generic section, F_k gives rise to an homology class $[F_k] \in H_*(M;\mathbb{Z}_2)$, independent of the chosen section. In [Tho57], R. Thom observed that there exists a universal polynomial $P_{F_k}(x_1,...,x_m,y_1,...,y_n)$ such that, when evaluated on the Chern classes $c_i(E)$ and $c_i(F)$, yields the Poincaré dual of $[F_k]$. Later, I. Porteous in [Por71] found a formula for P_{F_k}, based on Giambelli's formula for Schubert classes (see equation (10) in page 146 of [Ful97]). For this reason, the formula for P_{F_k} is usually called the Giambelli-Thom-Porteous formula:

$$P_{F_k}(c(E),c(F)) = det((c(F)/c(E))_{n-i+j})_{m\times m}. \tag{1.1}$$

Here, $(c(F)/c(E))_i$ is the i-th coefficient of the formal quotient of the total classes $c(F)$ and $c(E)$. Note that if $P_{F_k}(c(E),c(F)) \neq 0$, then there cannot exist a section s with rank everywhere greater than k. The class $P_{F_k}(c(E),c(F))$ is thus a **cohomological obstruction** to the existence of such sections.

One can replace $Hom(E,F)$ by another vector bundle and compute the Poincaré duals of degeneracy loci of other geometric structures. In many cases, there exist universal polynomials, like P_{F_k}, which evaluate to the Poincaré duals of the degeneracy loci. Such polynomials are usually called **Thom Polynomials**, as was R. Thom who initiated their study in the case of singularities of smooth maps. The problem of computing Thom polynomials is, in general, hard and not many examples are known.

In this thesis, we compute the Thom polynomials of two different types of degeneracy loci. In chapter 3, we consider the bundle of 2-forms $\Lambda^2 T^*M \to M$ over a compact $2m$-manifold M and compute the Thom polynomials of the following degeneracy loci:

$$F_k = \{x \in M \mid rank(s(x)) \leq 2k\},$$

where $s : M \to \Lambda^2 T^*M$ is a generic section. We conclude the following:

Theorem 1.1. *Let $w_i(M) \in H^i(M)$ denote the i-th Stiefel Whitney class of TM. The Poincaré dual of $[F_k] \in H_*(M)$ is given by*

$$P_{F_k}(w_1(M), ..., w_{2m}(M)) = det(w_{2(m-k)-2i+j})_{2(m-k)\times 2(m-k)}. \tag{1.2}$$

These Thom polynomials have already been computed for the bundle of complex 2-forms (see for instance Theorem 3.1 of [FNR05]). They are given by formula (1.2) but with Chern classes instead of Stiefel-Whitney classes. M. Kazarian, in the beginning of page 4 of [Kaz06], observes that it is a general principle that one may obtain Thom polynomials for real singularities from the Thom polynomials of complex singularities by substituting Chern classes in the formula by Stiefel-Whitney classes and reducing the coefficients to $\mathbb{Z}_2$. This however needs to be checked in each case and we conclude that it is indeed true for degeneracy loci of 2-forms. Observe that, as with the Giambelli-Thom-Porteous classes, the class in (1.2) is a cohomological obstruction to the existence of sections of $\Lambda^2 T^*M$ whose rank is everywhere greater than $2k$.

In chapter 4, we consider a more complicated problem with degeneracy loci that may not give rise to homology classes. In spite of that, we can interpret Thom polynomials as certain cohomological obstructions and compute them for these degeneracy loci as well. Let M be a $2m$-manifold, N a $2n$-manifold with $2m \leq 2n$ and $i : M \to N$ a smooth map. Endow N with an almost symplectic form ω (meaning a non-degenerate 2-form not necessarily closed). The differential di can be seen as a section of the bundle $Hom(TM, i^*TN) \to M$. And, given $x \in M$, the rank of $(di)_x$ can be any integer $l \in \{0, ..., 2m\}$ and the rank of $(di)_x^*\omega_x$ can be any integer $2k$ for $k \in \{0, ..., \lfloor l/2 \rfloor\}$. One can thus try to obtain cohomological obstructions to the existence of sections s homotopic to di whose ranks satisfy $rank(s(x)) > l$ and $rank(s(x)^*\omega_x) > 2k$.

Theorem 1.2. *Let $w_i(M)$ denote the i-th Stiefel Whitney class of TM and $c_i(N)$ denote the i-th Chern class of TN. Pick integers $l \in \{0, ..., 2m\}$ and $k \in \{0, ..., \lfloor l/2 \rfloor\}$. Then, if the class $\mathfrak{o}_{l,k} \in H^*(M)$ below is different from zero, there cannot exist a section $s : M \to Hom(TM, i^*TN)$, homotopic to di, such that*

$rank(s(x)) > l$ or $rank(s(x)^*\omega_x) > 2k$ for every $x \in M$.

$$\mathfrak{o}_{l,k} = det \begin{pmatrix} \{(i^*c(M)/w(M))_{2n-l-i+j}\}_{i,j=1}^{i=2m-l,j=2(m-k)} \\ \\ \{w_{l-2k-2i+j}(M)\}_{i=1,j=1-2m+l}^{i,j=l-2k} \end{pmatrix} \tag{1.3}$$

Note that the elements of the upper submatrix in (1.3) follow the pattern in the Giambelli-Thom-Porteous classes (1.1) and the elements of lower submatrix follow the pattern in (1.2). The obstructions $\mathfrak{o}_{l,k}$ are thus a mix of 1) obstructions to the existence of sections whose rank is everywhere greater than l with 2) obstructions to the existence of sections s such that $rank(s(x)^*\omega_x) > 2k$ for all $x \in M$. What is perhaps surprising is that $\mathfrak{o}_{l,k}$ is not just the product of the obstructions coming from (1.1) with the ones coming from (1.2).

Structure of the Thesis

We start with a chapter on some preliminary concepts which will be needed later. We will provide an introduction to fibre bundles, classifying spaces and characteristic classes. Formulas (1.1), (1.2) and (1.3) are deeply related to what are known as Schur polynomials, so a quick introduction to those will also be given. Chapter 2 ends with some necessary results about homotopy pushouts and locally trivial stratifications.

In chapter 3, we treat the case of degeneracy loci of 2-forms and compute the classes in (1.2). M. Kazarian introduced a method to obtain Thom Polynomials for loci arising from group actions (details in [Kaz06]). In [FR04], L. M. Fehér and R. Rimányi continue Kazarian's work giving an interpretation of Thom polynomials as cohomological obstructions and describing a method to compute them, introduced by the second author and called the method of restricting equations. This interpretation of Thom polynomials and the method of restricting equations are described in section 3.3 and are used in section 3.5 to compute the classes (1.2). In section 3.5. we also show that the Thom polynomials are indeed the Poincaré duals of the degeneracy loci.

In chapter 4, we treat the case of degeneracy loci of smooth maps to an almost symplectic manifold and compute the classes in (1.3). To do so, we use the same interpretation of Thom polynomials as cohomological obstructions and the method of restricting equations. The classes (1.3) are computed in section 4.5.

In chapters 3 and 4, we will only work with cohomology with coefficients in $\mathbb{Z}_2$. Thus, any time the coefficient ring is not mentioned, one should assume it is $\mathbb{Z}_2$.

Chapter 2

Preliminaries

2.1 Fibre Bundles

2.1.1 First Definitions

Let us start by defining fibre bundles.

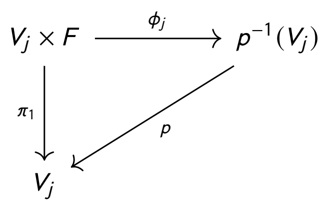
Definition 2.1. Let E and B be topological spaces, $p : E \to B$ a continuous map, G a topological group and F a space endowed with an effective [1] left action of G (or G-action). The tuple $\mathcal{E} = (E, B, p, F, G)$ is called a **fibre bundle** with typical fibre F and structure group G if the following condition is satisfied:

- B has an open cover indexed by a set J $\{V_j\}_{j \in J}$ such that for each $j \in J$, there is an homeomorphism

$$\phi_j : V_j \times F \to p^{-1}(V_j)$$

  that

  - makes the following diagram commute:



    with π_1 denoting the projection onto the first factor;

  - Fixing $b \in V_j$, the map $\phi_{j,b} : F \to p^{-1}(b)$ defined by $\phi_{j,b}(x) = \phi_j(b, x)$ is such that for each $i, j \in J$ and $b \in V_i \cap V_j$, the homeomorphism $\phi_{j,b}^{-1}\phi_{i,b} : F \to F$ coincides with the action of an element of G on F.
  - Moreover, the map $g_{ij} : V_i \cap V_j \to G$ given by $g_{ij}(b) = \phi_{j,b}^{-1}\phi_{i,b}$ is continuous.

Remark 2.2 (Terminology)**.**

[1] A continuous action of a topological group G on a space F is a continuous homomorphism $G \to Aut(F)$. The action is said to be effective if this homomorphism is injective. This means that every element of the group is uniquely determined by its action on F.

- The spaces E and B are named the total and base spaces, respectively. The map $p : E \to B$ is called the projection map, the functions ϕ_j are the coordinate functions and g_{ij} the transitions functions. The fibres $p^{-1}(b)$ for each $b \in B$ will be denoted by F_b.
- When the typical fibre F and structure group G are obvious from context or do not matter, the notation $p : E \to B$ will be used to denote the fibre bundle $\mathcal{E} = (E, B, p, F, G)$.
- A open cover $\{U_j\}_{j\in J}$ of B, along with maps ϕ_j as in the definition, is called a trivializing cover B.

Definition 2.3. Let $p : E \to B$ define a fibre bundle. A **section** of the bundle is a map $s : B \to E$ such that $p \circ s = id_B$.

Definition 2.4. Consider the action of $GL(n;\mathbb{R})$ on $\mathbb{R}^n$ given by the usual matrix product on the left. A fibre bundle with structure group $GL(n;\mathbb{R})$ and typical fibre $\mathbb{R}^n$ with that action is called a **vector bundle** of rank n (or an n-vector bundle).

Example 2.5. Let M be an m-dimensional. It is easy to prove that the tangent bundle $TM \to M$ is an m-vector bundle.

Example 2.6. Consider $Gr_k(\mathbb{R}^n)$ the grassmannian of k-planes in $\mathbb{R}^n$ and define

$$\gamma^n(\mathbb{R}^k) = \{(P, v) \in Gr_n(\mathbb{R}^k) \times \mathbb{R}^n \mid v \in P\}$$

The projection onto the first factor $\gamma^n(\mathbb{R}^k) \to Gr_n(\mathbb{R}^k)$ can be proved to define an n-vector bundle. It is called the tautological bundle over $Gr_n(\mathbb{R}^k)$.

Having defined fibre bundles, one now wishes to study what it means for two to be isomorphic.

Definition 2.7. Let (E, B, p, F, G) and (E', B', p', F, G) be two fibre bundles with the same fibre and group. By a **bundle map** between them, one means a map $h : E \to E'$ that satisfies two conditions:

1. h maps each fibre $F_b \subset E$, $b \in B$ homeomorphically into some fibre $F_{b'} \subset E'$ with $b' \in B'$.

   This in particular implies the existence of a map $f : B \to B'$ that makes the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

2. Let $\{(V_j, \phi_j)\}_{j\in J}$ and $\{(V'_j, \psi_j)\}_{j\in J'}$ be any trivializing covers of B and B', respectively, and, given $b \in V_i \cap f^{-1}(V'_j)$, let $h_b : F_b \to F_{f(b)}$ be the restriction of h to the fibre over b. Then, the composition $\tilde{g}_{ij}(b) = \psi^{-1}_{j,f(b)} h_b \phi_{i,b}$ coincides with the action of an element of G and the so defined map $\tilde{g}_{ij} : V_i \cap f^{-1}(V'_j) \to G$ is continuous.

Remark 2.8.

- It is easy to see that if a map satisfies condition 2 for two covers $\{(V_j, \phi_j)\}_{j\in J}$ and $\{(V'_j, \psi_j)\}_{j\in J'}$, then it satisfies the condition for any other trivializing covers of B and B'.

- It is readily checked that the identity $E \to E$ satisfies these conditions and the composition of bundle maps is a bundle map, so the set of bundles with fixed fibre and group defines, in this way, a category.

Definition 2.9. Let $\mathcal{E}$ and $\mathcal{E}'$ be two bundles with the same typical fibre, structure group and base space. One says that $\mathcal{E}$ and $\mathcal{E}'$ are **isomorphic bundles** if there is a bundle map between them that induces the identity map on the base space.

A fibre bundle with base B, total space E and fibre F can also be denoted by $F \hookrightarrow E \to B$.

Proposition 2.10. Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibre bundle, fix points $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$ and let F_0 be the fibre over x_0. There is a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(F_0, x_0) \to \pi_i(E, x_0) \xrightarrow{p_*} \pi_i(B, b_0) \to \pi_{i-1}(F_0, x_0) \to \cdots$$

Proof. See Theorem 17.1 of [Ste51]. □

2.1.2 Construction of Bundles

It turns out that, given a base B and a fibre F with an effective G-action, the transition functions determine completely the structure of a bundle (up to isomorphism).

Theorem 2.11. *Let G a topological group, F a space endowed with an effective left G-action, $\{U_j\}_{j\in J}$ an open cover of a space B and $\{g_{ij}\}$ a family of maps $g_{ij} : U_i \cap U_j \to G$ that satisfy the relation*

$$g_{ij}g_{jk} = g_{ik} \qquad \forall i, j, k \in J. \tag{2.1}$$

Then, there exists a fibre bundle $\mathcal{E}$ with base space B, fibre F, structure group G and transition functions $\{g_{ij}\}$ for the cover $\{U_j\}_{j\in J}$. This bundle is unique up to isomorphism.

Proof. See Theorem 3.2 in [Ste51]. □

Let us now briefly discuss three methods one uses to obtain new bundles out of old ones.

Definition 2.12 (Pullback)**.** Let $\mathcal{E} = (E, B, p, F, G)$ be a fibre bundle and $f : B' \to B$ a map from some space B' to the base space. The **pull-back of** $\mathcal{E}$ by f is the bundle $f^*\mathcal{E}$ with base space B', total space given by

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = p(e)\},$$

and projection map $p' : f^*E \to B'$ defined by $p'(b', e) = b'$.

Let $\pi_2 : f^*E \to E$ denote the projection onto the second factor. The picture of the pullback one should have in mind is the following commutative square:

$$\begin{array}{ccc} f^*E & \xrightarrow{\pi_2} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

If $\{(U_j, \phi_j\}_{j\in J}$ is a trivializing cover of B, then the cover $\{f^{-1}(U_j)\}_{j\in J}$ with the functions

$$\psi_j : f^{-1}(U_j) \times F \to p'^{-1}(f^{-1}(U_j))$$
$$(b', f) \mapsto (b', \phi_j(f(b), f))$$

is a trivializing cover of B'. Moreover, if g_{ij} are the transition functions of $\mathcal{E}$ relative to $\{U_j\}_{j\in J}$, one can check that those of $f^*\mathcal{E}$ relative to $\{f^{-1}(U_j)\}_{j\in J}$ are given by

$$g'_{ij} : f^{-1}(U_i \cap U_j) \to G$$
$$b' \mapsto g_{ij}(f(b'))$$

There is a simple and useful property of pullbacks concerning sections:

Proposition 2.13. Let $p : E \to B$ be a fibre bundle and $g : B' \to B$ some map. Then, there is a section of the pullback bundle if and only if there is a map $f : B' \to E$ making the diagram

$$\begin{array}{ccc} & & E \\ & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{g} & B \end{array}$$

commute.

Proof. If there is a section $s : B' \to g^*E$, then $f = \pi_2 \circ s$ makes the diagram commute. On the other hand, if there is such a map f, then define $s(b') = (g(b'), f(b'))$. It is obviously continuous, the image is contained in g^*E and it makes the diagram commute. □

Definition 2.14 (Cartesian Products)**.** Now consider two bundles $\mathcal{E}_1 = (E_1, B_1, p_1, F_1, G_1)$ and $\mathcal{E}_2 = (E_2, B_2, p_2, F_2, G_2)$. The **product bundle** $\mathcal{E}_1 \times \mathcal{E}_2$ has total space $E_1 \times E_2$, base space $B_1 \times B_2$ and projection map $p = p_1 \times p_2$. Moreover, given $\{U_j\}_{j\in J_1}$ a trivializing cover of B_1 and $\{V_j\}_{j\in J_2}$ a trivializing cover of B_2, then $\{U_j \times V_i\}$ is a trivializing cover of $B_1 \times B_2$. The coordinate and the transition functions are also the products of the ones of $\mathcal{E}_1$ and $\mathcal{E}_2$. The typical fibre and structure group are $F_1 \times F_2$ and $G_1 \times G_2$, respectively.

Definition 2.15. (Whitney Sum) Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two bundles with the same base space and consider the diagonal embedding of B:

$$d : B \to B \times B$$
$$b \mapsto (b, b)$$

The **Whitney sum of** $\mathcal{E}_1$ **and** $\mathcal{E}_2$ is the bundle $\mathcal{E}_1 \oplus \mathcal{E}_2 = d^*(\mathcal{E}_1 \times \mathcal{E}_2)$.

2.1.3 Principal and Associated Bundles

Let G be any topological group. G acts on itself by left multiplication and this action is effective. When both the fibre and structure group are G with this action, the bundle is called a **principal** G**-bundle**.

Definition 2.16. Two bundles $\mathcal{E}_1 = (E_1, B, p_1, F_1, G)$ and $\mathcal{E}_2 = (E_2, B, p_2, F_2, G)$, with the same base space and structure group and different fibres, are said to be **associated** if there is a trivializing cover of both such that the transition functions of $\mathcal{E}_1$ subordinate to that cover are the same as those of $\mathcal{E}_2$.

Example 2.17. The bundle $\Lambda^k(T^*M) \to M$ of k-forms on M is associated to $TM \to M$.

Definition 2.18. Let E be a right G-space and F a left G-space. The **balanced product** of E and F, denoted by $E \times_G F$, is the quotient of $E \times F$ by the equivalence relation $(e, f) \sim (e \cdot g, g^{-1} \cdot f)$ for all $e \in E, f \in F$ and $g \in G$.

Proposition 2.19. Let $\mathcal{E} = (E, B, p, G, G)$ be a principal G-bundle and F a space with an effective [2] left G-action. Then,

$$q : E \times_G F \to B$$
$$[e, f] \mapsto p(e)$$

defines a fibre bundle with fibre space F, denoted by $\mathcal{E}[F]$. Moreover, a trivializing cover of $\mathcal{E}$ also trivializes $\mathcal{E}[F]$ and the transition functions of the latter are the ones of $\mathcal{E}$.

Proof. Let $\{U_j, \phi_j\}$ be a trivializing cover of B for $\mathcal{E}$. Then $\phi'_j : U_j \times F \to q^{-1}(U_j)$, defined as $\phi'_j(b, f) = [\phi_j(b, 1), f]$, is a coordinate function (with inverse $\phi'^{-1}_j([e, f]) = (p(e), \phi_j^{-1}(e) \cdot f))$. The transition functions are

$$g'_{ij}(b) \cdot f = {\phi'_{j,b}}^{-1} \phi'_{i,b}(f) = \phi_{j,b}^{-1} \phi_{i,b}^{-1}(1) \cdot f = g_{ij}(b) \cdot f$$

All these maps are continuous so the statement follows. □

A few properties of the balanced product should be mentioned:

Proposition 2.20. Consider H a subgroup of a topological group G, X a right G-space, Y a left and right G-space and Z a left G-space. Then,

[2] If the action is not effective this becomes a G/H bundle with $H = \{g \in G \mid g \cdot e = e \;\; \forall e \in E\}$. The transition functions are in this case the composition of the transition functions of $\mathcal{E}$ with the quotient $G \to G/H$.

1. $X \times_G * \cong X/G$, where $*$ denotes the singleton set;

2. $X \times_G G \cong X$, where G is regarded as a left G-space with the action of left multiplication;

3. $(X \times_G Y) \times_G Z \cong X \times_G (Y \times_G Z)$. Here $X \times_G Y$ is given the action of G coming from the right action of G on Y and the same goes for the left action on $Y \times_G Z$;

4. $X \times_G G \times_H Y \cong X \times_H Y$, where G is regarded as a right H-space with the action given by multiplication on the right;

5. $X \times_G (G/H) \cong X/H$.

6. If $p : E \to B$ is a principal G-bundle, F has a right G-action and $f : X \to B$ is some map, then

$$f^*(E \times_G F) \cong f^*E \times_G F.$$

Here the right action of G on f^*E is just $(x, e) \cdot g = (x, e \cdot g)$, for all $(x, e) \in f^*E, g \in G$.

Proof. In all cases, the homeomorphisms are fairly obvious. One only needs to pay attention to continuity, which follows from the fact that if $f : Y \to Z$ is an open G-equivariant map, then $g : X \times_G Y \to X \times_G Z$ defined as $g([x, y]) = [x, f(y)]$ is also open. Details for points 1.-5. can be found in [Bre72]. The isomorphism of point 6. is just

$$\begin{aligned} F : f^*(E \times_G F) &\to f^*E \times_G F \\ (x, [e, f]) &\mapsto [(x, e), f] \end{aligned}$$

□

Let us now study some properties related to restricting the structure group.

Definition 2.21. Let $H \subset G$ be a subgroup of a topological group G and $\mathcal{E} = (E, B, p, F, G)$ a fibre bundle. One says that $\mathcal{E}$ admits a **reduction of structure group** to H if there exists a local trivialization of $\mathcal{E}$ whose transition functions all have values in H.

Example 2.22. Let M be an m-manifold and consider the tangent bundle $TM \to M$, which has structure group $GL(m; \mathbb{R})$. It is easy to see that TM admits a reduction of structure group to $O(m)$ if and only if M admits a riemannian metric. It is a basic fact that every manifold admits a metric, so actually every tangent bundle admits such a reduction. The same goes for all associated bundles like the bundles of forms on M.

Example 2.23. Similarly, if a $2m$-dimensional manifold M is endowed with an **almost symplectic form** (a non-degenerate 2-form) then one can show that TM admits a reduction of structure group to the symplectic group $Sp(2m)$.

2.2 Classifying Spaces

2.2.1 Classification Problem

By Theorem 2.11, the converse of Proposition 2.19 also holds: given a fibre bundle $\mathcal{E}$ with fibre space F, one can construct its principal associated bundle η. Furthermore, one has

Proposition 2.24. Let $\mathcal{E} = (E, B, p, F, G)$ be a fibre bundle and η its associated principal bundle. Then,

$$\eta[F] \cong \mathcal{E}$$

Proof. This is a direct consequence of Proposition 2.19 and the uniqueness in Theorem 2.11. □

As a consequence,

Corollary 2.25. Two fibre bundles $\mathcal{E}_i = (E_i, B, p_i, F, G)$, $i = 1, 2$ are isomorphic if and only if their associated principal bundles are isomorphic.

By this corollary, the classification of bundles is reduced to classifying principal bundles. So fix a topological group G and consider the contravariant functor $\mathcal{P}_G : Top \to Set$ such that

$$\mathcal{P}_G(B) = \{\text{Isomorphism classes of principal } G\text{-bundles over } B\}$$
$$\mathcal{P}_G(f) = f^*$$

A first result is that, for paracompact base spaces, this functor descends to the homotopy category $hTop$:

Theorem 2.26. *Let $p : E \to B$ be a principal G-bundle and $f, g : B' \to B$. If B is paracompact, then*

$$f \simeq g \implies f^*\mathcal{E} \cong g^*\mathcal{E}$$

Proof. Theorem 4.9.9 of [Hus94] proves the result for numerable fibre bundles (see Definition 4.9.2 of [Hus94]). It is a standard result that fibre bundles over paracompact spaces are numerable. □

2.2.2 Classifying Spaces

Now, the question is whether this induced functor $\mathcal{P}_G : hTop \to Set$ is representable. Let $p : E \to B$ be a principal G-bundle with B paracompact. Then, by the previous theorem, there is a well-defined natural transformation

$$T_{\mathcal{E}} : [X, B] \to \mathcal{P}_G(X)$$

Definition 2.27. A principal bundle $\mathcal{E}$ is called **universal** if $T_{\mathcal{E}}$ is bijective for every space X. The base space of such a bundle is called the **classifying space** of G and is denoted by BG. The total space is denoted by EG.

In other words, every principal bundle η over X is the pullback of $EG \to BG$ by a unique map from X to BG. This map is called the classifying map of η.

By Yoneda's Lemma, if a universal bundle exists, it is unique up to isomorphism and the classifying space is unique up to homotopy equivalence. Using this, one can give a criterion for a principal bundle to be universal:

Proposition 2.28. If $p : E \to B$ is a principal G-bundle, B is paracompact and E is contractible, then the bundle is isomorphic to $EG \to BG$ and $B \simeq BG$.

Proof. See [Dol63], Theorem 7.5. □

Remark 2.29.

- If F is a space endowed with an effective, right G-action, denote by $Fib_F(X)$ the set of isomorphism classes of fibre bundles over X with fibre F and structure group G. Proposition 2.24 implies there is a bijection

  $$\mathcal{P}_G(X) \to Fib_F(X)$$
  $$\eta \mapsto \eta[F]$$

  So, the classifying space BG also classifies bundles with fibre F. Moreover, by point 6 of Proposition 2.20, the natural transformation

  $$[X, BG] \to Fib_F(X)$$

  maps each $f : X \to BG$ to f^*.

- The space $EG \times_G F$ will also be denoted by F_{hG}.

Example 2.30. Let $Gr_n(\mathbb{R}^\infty)$ be the set of n-planes in $\mathbb{R}^\infty$ and

$$\gamma^n = \{(P, v) \in Gr_n(\mathbb{R}^\infty) \times \mathbb{R}^n \mid v \in P\}.$$

The projection $\gamma^n \to Gr_n(\mathbb{R}^\infty)$ defines an n-vector bundle. In Lemma 5.3 of [MS74], the authors show that γ^n classifies n-vector bundles. Therefore, $BGL(n;\mathbb{R}) \simeq Gr_n(\mathbb{R}^\infty)$. If M is an m-manifold, the classifying map of the tangent bundle is denoted by $\tau_M : M \to BGL(m;\mathbb{R})$.

2.2.3 Milnor Construction

The universal bundle of a given topological group G can be constructed in the following way, introduced by John Milnor.

Definition 2.31. The join of two spaces X and Y is the quotient space

$$X \star Y := X \times I \times Y/\sim$$

where $\sim$ is the equivalence relation generated by $(x, 0, y) \sim (x', 0, y)$ and $(x, 1, y) \sim (x, 1, y')$ for all $x, x' \in X$ and $y, y' \in Y$.

In the following, let us use the notation $X^{\star n} = X \star \cdots \star X$, where X appears n times.

Theorem 2.32. *Let G be a topological group. Define*

$$EG = colim_n G^{\star n}$$

where $G^{\star n} \subset G^{\star n+1}$ through the inclusions $\sum_{i=1}^n t_i g_i \mapsto \sum_{i=1}^n t_i g_i + 0g$. Define also an action of G on EG by

$$\left(\sum_{i=1}^n t_i g_i\right) \cdot g = \sum_{i=1}^n t_i g_i g$$

Finally, write $BG = EG/G$, the orbit space of the action. Then, the projection

$$p : EG \to BG$$

defines a universal principal G-bundle.

Proof. See sections 4.11 and 4.12 of [Hus94]. □

A nice property of this construction is that it is functorial.

Proposition 2.33. Let $\phi : G \to H$ be a continuous homomorphism. Then, ϕ induces a diagram between universal bundles

$$\begin{array}{ccc} EG & \xrightarrow{E\phi} & EH \\ \downarrow & & \downarrow \\ BG & \xrightarrow{B\phi} & BH \end{array}$$

such that, if $\psi : H \to K$ is another such homomorphism, then $E(\psi \circ \phi) = E\psi \circ E\phi$ and $B(\psi \circ \phi) = B\psi \circ B\phi$.

Proof. $E\phi$ is defined as $E\phi(\sum_{i=1}^n t_i g_i) = \sum_{i=1}^n t_i \phi(g_i)$. One can check that this map is continuous and that is equivariant with respect to the actions of G and H. Thus, it induces a map on the quotients $BG \to BH$. The composition property is also clear from the definition of $E\phi$. □

Example 2.34. As $O(n)$ is a maximal compact subgroup of $GL(n;\mathbb{R})$, it follows that the inclusion $O(n) \overset{i}{\hookrightarrow} GL(n;\mathbb{R})$ is an homotopy equivalence (Theorem 2 of [Mos49]). The maps i, Ei and Bi induce maps between the homotopy exact sequences of the bundles $O(n) \hookrightarrow EO(n) \to BO(n)$ and $GL(n;\mathbb{R}) \hookrightarrow EGL(n;\mathbb{R}) \to BGL(n;\mathbb{R})$. Since $O(n) \simeq GL(n;\mathbb{R})$ and $EO(n) \simeq EGL(n;\mathbb{R})$, it follows that $BO(n) \simeq GL(n;\mathbb{R})$.

2.2.4 Properties of classifying spaces

We end the section about classifying spaces on a quick listing of properties that will be needed in the main chapters. Let $H \subset G$ be topological groups. The action of G on EG restricts to an action of H.

Proposition 2.35.

1. The inclusion $BH \hookrightarrow EG/H$ is an homotopy equivalence.
2. If F is a space with a left H-action, then the inclusion $EH \times_H F \hookrightarrow EG \times_H F$ is an homotopy equivalence.

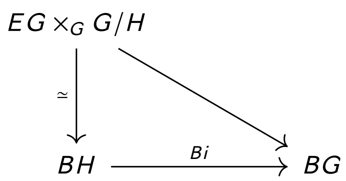
Proof.

1. One can show that $EG \to EG/H$ is a principal H-bundle. Since EG is contractible, it follows from Proposition 2.28 that $BH \simeq EG/H$.
2. The inclusion in question and $BH \hookrightarrow EG/H$ induce maps between the homotopy exact sequences of the bundles $F \hookrightarrow EH \times_H F \to BH$ and $F \hookrightarrow EG \times_H F \to EG/H$. Using the the previous point and the 5-lemma, the result follows.

□

The orbit space G/H has a natural G-action given by left multiplication by elements of G. Then, one may consider the associated bundle $EG \times_G G/H \to BG$.

Proposition 2.36. If $i : H \hookrightarrow G$ denotes the inclusion map, then the following diagram commutes:



Proof. By point 5 of Proposition 2.20,

$$EG \times_G G/H \simeq EG/H$$

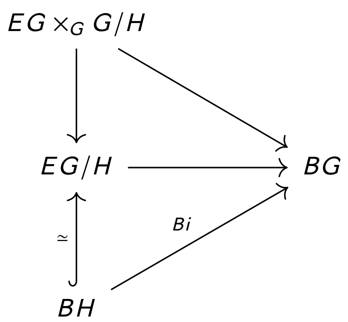
The composition

$$\begin{aligned} EG \times_G G/H &\to EG/H \to BG \\ [x, gH] &\mapsto x \cdot g \quad \mapsto xg = x \end{aligned}$$

is just the projection $EG \times_G G/H \to BG$. On the other hand, using the Milnor construction, one sees that the composition

$$\begin{array}{ccccc} BH & \hookrightarrow & EG/H & \to & BG \\ \sum_{i=1}^{n} t_i h_i & \mapsto & \sum_{i=1}^{n} t_i h_i & \mapsto & \sum_{i=1}^{n} t_i h_i \end{array}$$

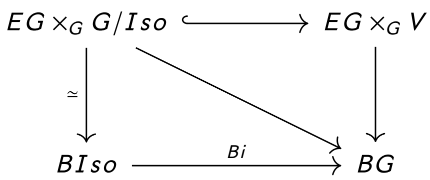
is just Bi. This translates into the commuting diagram



from which the result follows. □

There is a particularly important special case of last proposition:

Corollary 2.37. Let G be a topological group and V be a vector space endowed with a left G-action. Fix an element $p \in V$ and denote by $Iso \subset G$ the isotropy group of p. Then $O(p)$ the orbit of p is isomorphic to G/Iso and the following diagram commutes

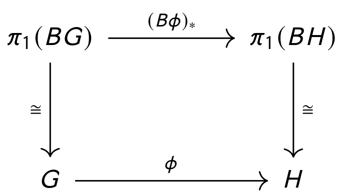


Proposition 2.38. Let G and H be topological groups. Then,

$$B(G \times H) \simeq BG \times BH$$

Proof. This follows just from the fact that $EG \times EH \to BH \times BG$ is a principal $G \times H$ bundle and $EG \times EH$ is contractible. □

Proposition 2.39. If G is a discrete topological group, then $\pi_1(BG) \cong G$. Moreover, if H is another discrete topological group and $\phi : G \to H$ is a continuous homomorphism, then the following diagram commutes:



Proof. Since G is discrete, the local triviality condition of the bundle $EG \to BG$ translates into the triviality condition of coverings. Hence, it is a covering. The action of G on EG makes it a subgroup of the Deck transformation group. Actually, given f any automorphism of the covering and $x \in EG$, there is some $g \in G$ such that $x \cdot g = f(x)$ so, by the unique lifting property, g, as an automorphism, is equal to f. Thus, G is, in fact, the Deck transformation group. Let us fix a point $x_0 \in BG$ and a point $\tilde{x}_0 \in EG$ over x_0 and write $\pi_1(BG)$ for $\pi_1(BG; x_0)$. EG is contractible so Proposition 1.39 of [Hat02] implies that there is an isomorphism $\pi_1(BG) \to G$ sending each loop $[\gamma] \in \pi_1(BG)$ to the element g that takes $\tilde{x}_0 \in EG$ to $\tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lifting of γ that begins in $\tilde{x}_0$. This proves the first statement.

The lifting of $B\phi \circ \gamma$ that begins in $E\phi(\tilde{x}_0)$ is just $E\phi\circ$. The commutativity of the square will follow from the equality $E\phi(\tilde{x}_0) \cdot \phi(g) = E\phi \circ \tilde{\gamma}(1)$. To prove this equality, write $\tilde{x}_0 = \sum_{i=1}^{n} t_i g_i$. Then,

$$E\phi \circ \tilde{\gamma}(1) = E\phi(\tilde{x}_0 \circ g) = E\phi\left(\sum_{i=1}^{n} t_i g_i g\right) = \sum_{i=1}^{n} t_i \phi(g_i)\phi(g) = E\phi(\tilde{x}_0) \cdot \phi(g)$$

□

2.3 Characteristic Classes

In this section, we restrict our attention to vector bundles. Characteristic classes are cohomology classes associated to vector bundles, invariant by isomorphisms. We will define and present some properties of three types of characteristic classes: Stiefel-Whitney classes, Euler classes and Chern classes.

2.3.1 Stiefel-Whitney Classes

Let $\mathcal{E}$ be a vector bundle with base space B. The Stiefel-Whitney classes of $\mathcal{E}$ are the only cohomology classes

$$w_i(\mathcal{E}) \in H^i(B; \mathbb{Z}_2)$$

that satisfy the following four axioms.

Axioms for Stiefel-Whitney Classes:

1. $w_0(\mathcal{E}) = 1$ and $w_i(\mathcal{E}) = 0$ for $i > n$ if $\mathcal{E}$ has rank n.

2. If η is a vector bundle with base B' and there exists a bundle map between $\mathcal{E}$ and η that induces $f : B \to B'$, then

$$w_i(\mathcal{E}) = f^* w_i(\eta)$$

3. If $\mathcal{E}$ and η are vector bundles with the same base space, then

$$w_i(\mathcal{E} \oplus \eta) = \sum_{j=0}^{i} w_j(\mathcal{E}) w_{i-j}(\eta)$$

4. For the line bundle $\gamma^1(\mathbb{R}) \to \mathbb{P}^1$, $w_1(\gamma^1(\mathbb{R})) \neq 0$.

Theorem 2.40. *For each vector bundle $\mathcal{E}$ there exists a unique set of cohomology classes that satisfy these four axioms.*

Proof. See chapter 8 of [MS74]. □

For simplicity of notation, define the total Stiefel-Whitney class of $\mathcal{E}$ by

$$w(\mathcal{E}) = 1 + w_1(\mathcal{E}) + \ldots + w_n(\mathcal{E}) \in H^*(B; \mathbb{Z}_2)$$

With this notation, the third axiom translates to the equality $w(\mathcal{E} \oplus \eta) = w(\mathcal{E})w(\eta)$.

Proposition 2.41. Let $\mathcal{E}$ and η denote vector bundles over the same base B and τ the product vector bundle over B (of some rank n). Then,

- If $\mathcal{E}$ and η are isomorphic, then $w(\mathcal{E}) = w(\eta)$.
- $w(\tau)_i = 0$ for $i > 0$ and $w(\mathcal{E} \oplus \tau) = w(\mathcal{E})$.
- $w(\mathcal{E} \times \eta) = w(\mathcal{E}) \times w(\eta)$, where the $\times$ on the right denotes the cross-product of cohomology classes.

Proof. The first two items are simple applications of the axioms. For the third and details on the other, see chapter 4 of [MS74]. □

If one computes $w_i(\gamma^n)$ the Stiefel-Whitney classes of the universal bundle of n-vector bundles, then, by axiom 2, the classes of any vector bundle $p : E \to B$ are given by $f^* w_i(\gamma^n)$, where $f : B \to Gr_n(\mathbb{R}^\infty)$ is its classifying map. It is important then to understand the cohomology of $Gr_n(\mathbb{R}^\infty)$.

Proposition 2.42. Let $\mathbb{P}^\infty = Gr_1(\mathbb{R}^\infty)$ be the infinite projective space. Then,

$$H^*(\mathbb{P}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[t]$$

where $t = w_1(\gamma^1)$.

Proof. See Lemma 4.3 of [MS74]. □

The cohomology ring of $(\mathbb{P}^\infty)^n$ is then given by $\mathbb{Z}_2[t_1, ..., t_n]$ where t_i is the generator of the cohomology of the i-th factor in $(\mathbb{P}^\infty)^n$.

Proposition 2.43. Let w_i denote the i-th Stiefel-Whitney class of $\gamma^n \to Gr_n(\mathbb{R}^\infty)$ and $j : (\mathbb{P}^\infty)^n \to Gr_n(\mathbb{R}^\infty)$ the canonical inclusion. Then the map

$$H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) \xrightarrow{j^*} H^*((\mathbb{P}^\infty)^n; \mathbb{Z}_2) = \mathbb{Z}_2[t_1, ..., t_n]$$

is injective. Furthermore, it sends w_i to $e_i(t_1, ..., t_n)$ the i-th elementary symmetric polynomial in the variables $t_1, ..., t_n$ [3].

Proof. $(\gamma^1)^n \to (\mathbb{P}^\infty)^n$ is an n-vector bundle and $w((\gamma^1)^n) = w(\gamma^1)^n = \prod_{i=1}^n (1 + t_i)$. Moreover, j is covered by the bundle map

$$h : (\gamma^1)^n \to \gamma^n$$
$$(x_i, v_i) \mapsto (j(x_1, ..., x_n), v_1 + ... + v_n)$$

so $j^* w = \prod_{i=1}^n (1 + t_i)$ and note that the term of degree i in the product is e_i. This implies that the classes w_i do not satisfy any polynomial relations. The injectivity of j^* is then proved if one shows that they generate the cohomology of $Gr_n(\mathbb{R}^\infty)$. See Theorem 7.1 of [MS74] for the rest of the proof. These fact that w_i do not satisfy any polynomial relations and generate the cohomology of $Gr_n(\mathbb{R}^\infty)$ also prove the next theorem. □

The variables $t_1, ..., t_n$ are called the Stiefel-Whitney roots.

Theorem 2.44. *The cohomology ring of $Gr_n(\mathbb{R}^\infty)$ is given by*

$$H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, ..., w_n].$$

Remark 2.45. Let $\mathbb{Z}_2[t_1, ..., t_n]^{S_n}$ denote the algebra of symmetric polynomials in the Stiefel-Whitney roots. Proposition 2.43 implies that

$$H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}_2) \cong \mathbb{Z}_2[t_1, ..., t_n]^{S_n}.$$

2.3.2 Euler Class

For orientable vector bundles, there is an important characteristic class called the **Euler class**.

Definition 2.46. Let $p : E \to B$ be an n-vector bundle. For any $b \in B$, denote by F_b the fibre over b. The bundle is said to be **orientable** if for each F_b there is a choice of generator $u_b \in H^n(F_b, F_b \backslash \{0\}; \mathbb{Z})$ satisfying the following local compatibility condition: For each point of the base, there exists a neighborhood

[3] See Definition 2.73

$V \subset B$ and $u \in H^n(p^{-1}(V), p^{-1}(V)\backslash\{0\}; \mathbb{Z})$ such that for every $b \in V$,

$$\begin{aligned} H^n(p^{-1}(V), p^{-1}(V)\backslash\{0\}; \mathbb{Z}) &\to H^n(F_b, F_b\backslash\{0\}; \mathbb{Z}) \\ u &\mapsto u_b \end{aligned}$$

An **orientation** is a choice of generator for each fibre.

Theorem 2.47 (Thom Isomorphism Theorem)**.** *Let $p : E \to B$ be an oriented n-vector bundle. Then, $H^i(E, E\backslash 0; \mathbb{Z}) = 0$ for $0 < i < n$ and there exists a unique class*

$$u \in H^n(E, E\backslash 0; \mathbb{Z})$$

such that for each $b \in B$,

$$\begin{aligned} H^n(E, E\backslash 0; \mathbb{Z}) &\to H^n(F_b, F_b\backslash\{0\}; \mathbb{Z}) \\ u &\mapsto u_b \end{aligned}$$

where u_b is the chosen generator of F_b. Moreover, the map

$$\begin{aligned} H^i(E; \mathbb{Z}) &\to H^{i+n}(E, E\backslash 0; \mathbb{Z}) \\ a &\mapsto a \cup u \end{aligned}$$

is an isomorphism.

Proof. See chapter 10 of [MS74]. □

Remark 2.48. The class u is usually denoted by u^E, the Thom class of E. Observe that the Thom class is functorial, meaning that if $h : E \to E'$ is a bundle map, then

$$h^*(u^{E'}) = u^E.$$

This follows from the uniqueness properties of the Thom class and the fact that h is an isomorphism on each fibre.

Definition 2.49. The **Euler class** of an oriented n-vector bundle $\mathcal{E}$ is the cohomology class

$$e(\mathcal{E}) \in H^n(B; \mathbb{Z})$$

the image of $u^E \in H^n(E, E\backslash 0; \mathbb{Z})$ by the composition

$$H^n(E, E\backslash 0; \mathbb{Z}) \to H^n(E; \mathbb{Z}) \xrightarrow{(p^*)^{-1}} H^n(B; \mathbb{Z})$$

Proposition 2.50. Let $\mathcal{E}$ and η be oriented vector bundles over B and $\mathcal{E}'$ an oriented vector bundle over some base B'. Then,

- If there is an orientation preserving[4] bundle map from $\mathcal{E}$ to $\mathcal{E}'$ that induces a map $f : B \to B'$, then

$$e(\mathcal{E}) = f^* e(\mathcal{E}').$$

- $e(\mathcal{E} \oplus \eta) = e(\mathcal{E})e(\eta)$ and $e(\mathcal{E} \times \eta) = e(\mathcal{E}) \times e(\eta)$

- If $\mathcal{E}$ has a nowhere zero section, then $e(\mathcal{E}) = 0$.

Proof. These are Property 9.2, Property 9.6 and Property 9.7 of [MS74], respectively. □

Remark 2.51. Note that the last property gives an interpretation of the Euler class as an obstruction to the existence of non vanishing sections of a vector bundle.

The projection $\mathbb{Z} \to \mathbb{Z}_2$ induces a restriction of coefficients $H^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z}_2)$.

Proposition 2.52. Under this restriction, $e(\mathcal{E}) \mapsto w_n(\mathcal{E})$.

Proof. See Property 9.5 of [MS74]. □

Let $p : E \to B$ define an oriented n-vector bundle, denote by $p_0 : E_0 \to B$ the restriction of p to E_0 and let e denote the Euler class of the bundle. e appears in a long exact sequence of cohomology groups:

Theorem 2.53 (Gysin Sequence)**.** *One can associate to $p : E \to B$ a long exact sequence of the form*

$$\cdots \to H^i(B;\mathbb{Z}) \xrightarrow{\cup e} H^{i+n}(B;\mathbb{Z}) \xrightarrow{p_0^*} H^{i+n}(E_0;\mathbb{Z}) \to H^{i+1}(B;\mathbb{Z}) \to \cdots$$

Where $\cup e$ is the map that sends $x \in H^i(B;\mathbb{Z})$ to $x \cup e$.

Proof. See Theorem 12.2 in [MS74]. □

2.3.3 Chern Classes

Definition 2.54. A complex vector bundle of complex rank n is a fibre bundle with fibre $\mathbb{C}^n$ and structure group $Gl(n;\mathbb{C})$ with the action of $A \in Gl(n;\mathbb{C})$ on $v \in \mathbb{C}^n$ given by the matrix product Av.

Example 2.55. Similar to the real case, one has $Gr_n(\mathbb{C}^k)$ the grassmannian of complex n-planes in $\mathbb{C}^k$,

$$\gamma^n(\mathbb{C}^k) = \{(P, v) \in Gr_n(\mathbb{C}^k) \times \mathbb{C}^k \mid v \in P\}$$

and $\gamma^n(\mathbb{C}^k) \to Gr_n(\mathbb{C}^k)$ defines a complex n-vector bundle.

Example 2.56. Moreover, $Gr_n(\mathbb{C}^\infty)$ is the grassmannian of complex n-planes in $\mathbb{C}^\infty$, $\gamma^n(\mathbb{C}^\infty)$ is defined analogously and $\gamma^n(\mathbb{C}^\infty) \to Gr_n(\mathbb{C}^\infty)$ classifies complex n-vector bundles. Hence, $BGL(n;\mathbb{C}) \simeq Gr_n(\mathbb{C}^\infty)$. Furthermore, $U(n)$ is a maximal compact subgroup of $GL(n;\mathbb{C})$, so $U(n) \simeq GL(n;\mathbb{C})$ and, as with Example 2.34, one has $BO(n) \simeq BGL(n;\mathbb{C})$.

[4] If E_1 and E_2 are orientend n-vector bundles, a bundle map $F : E_1 \to E_2$ is said to be **orientation preserving** if its restriction to each fibre $(F_1)_b$ sends the chosen generator of $H^n((F_1)_b, (F_1)_b\backslash\{0\})$ to the chosen generator of $H^n((F_2)_b, (F_2)_b\backslash\{0\})$.

With the usual identification $\mathbb{C}^n = \mathbb{R}^{2n}$ ($x + iy = (x, y)$ for $x, y \in \mathbb{R}^n$), one can see $Gl(n;\mathbb{C})$ as a subgroup of $Gl(2n;\mathbb{R})$ under the identification

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \qquad A, B \in Gl(n;\mathbb{R})$$

Now let $\mathcal{E}$ be a complex n-vector bundle. Under these identifications, $\mathcal{E}$ is also a real $2n$-vector bundle. Denote it by $\mathcal{E}_{\mathbb{R}}$.

Lemma 2.57. If $\mathcal{E}$ is a complex vector bundle, $\mathcal{E}_{\mathbb{R}}$ is orientable and has a canonical choice for orientation.

Proof. See Lemma 14.1 of [MS74]. □

Just as real vector bundles have associated characteristic classes satisfying certain axioms, complex vector bundles have also characteristic classes, now with coefficients in $\mathbb{Z}$. So let $\mathcal{E}$ be a complex vector bundle with complex dimension n and base space B. The **Chern Classes** of $\mathcal{E}$ are cohomology classes

$$c_i(\mathcal{E}) \in H^{2i}(B;\mathbb{Z})$$

that satisfy the following four axioms:

Axioms for Chern Classes:

1. $c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ for $i > n$.
2. If η is a complex vector bundle with base B' and there exists a bundle map between $\mathcal{E}$ and η that induces $f : B \to B'$, then

$$c_i(\mathcal{E}) = f^* c_i(\eta).$$

3. Let $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + ... + c_n(\mathcal{E}) \in H^*(B;\mathbb{Z})$ and let η be a complex vector bundle also with base B. Then,

$$c(\mathcal{E} \oplus \eta) = c(\mathcal{E})c(\eta).$$

4. For the complex line bundle $\gamma^1(\mathbb{C}) \to \mathbb{CP}^1$, $c_1(\gamma^1(\mathbb{C}))$ is a generator of $H^2(\mathbb{CP}^1;\mathbb{Z})$.

Theorem 2.58. *For each complex vector bundle $\mathcal{E}$ there exists a canonical choice of cohomology classes that satisfy these four axioms.*

Proof. One can construct the Chern classes in several different ways. In chapter 14 of [MS74] for instance, these are constructed using the Euler class associated to the canonical orientation. See Remark 2.63 for another possibility. □

Proposition 2.59. Let $\mathcal{E}$ be a complex vector bundle of complex dimension n, with base B. Then,

1. $c_n(\mathcal{E}) = e(\mathcal{E})$.
2. The restriction of coefficients $H^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z}_2)$ sends $c(\mathcal{E})$ to $w(\mathcal{E}_{\mathbb{R}})$.

If η is another complex vector bundle over B and τ the product complex vector bundle over B of some dimension. Then,

3. If $\mathcal{E}$ and η are isomorphic, then $c(\mathcal{E}) = c(\eta)$.
4. $c(\mathcal{E} \oplus \tau) = c(\mathcal{E})$.
5. $c(\mathcal{E} \times \eta) = c(\mathcal{E}) \times c(\eta)$.

Proof. Point 1. follows immediately from the construction in [MS74]. By Proposition 2.52 and point 1., point 2. follows for $c_n(\mathcal{E})$. For the lower classes, one uses induction on the complex dimension of the bundle and their definition in chapter 14 of [MS74]. Point 3. is obvious from axiom 2 and point 4. is Lemma 14.3 in [MS74]. To prove point 5., one can use the same argument as in the proof of Lemma 14.8 in [MS74]. □

Similarly to the real case, one has

Proposition 2.60. Let $\mathbb{CP}^\infty = Gr_1(\mathbb{C}^\infty)$ be the infinite complex projective space. Then,

$$H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[x]$$

where $x = c_1(\gamma^1(\mathbb{C}^\infty))$.

Proof. Follows from Theorem 14.5 in [MS74]. □

Theorem 2.61. *Let c_i denote the i-the Chern class of $\gamma^n(\mathbb{C}^\infty) \to Gr_n(\mathbb{C}^\infty)$. The cohomology ring of $Gr_n(\mathbb{C}^\infty)$ is given by*

$$H^*(Gr_n(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1, ..., c_n]$$

Proof. See Theorem 14.5 of [MS74]. □

Proposition 2.62. Let $j : \mathbb{CP}^\infty \to Gr_n(\mathbb{C}^\infty)$ be the canonical inclusion. Then the map

$$H^*(Gr_n(\mathbb{C}^\infty);\mathbb{Z}) \xrightarrow{j^*} H^*((\mathbb{CP}^\infty)^n;\mathbb{Z}) = \mathbb{Z}[x_1, ..., x_n]$$

is injective. Furthermore, it sends c_i to $e_i(x_1, ..., x_n)$, the i-th elementary symmetric polynomial in the variables $x_1, ..., x_n$.

Proof. The proof of the second statement follows as in the case of Stiefel-Whitney classes. The injectivity part follows from this fact and the previous theorem. □

The variables $x_1, ..., x_n$ are called the chern roots.

Remark 2.63. To construct the Chern classes, one could also prove first that $H^*(BU(n))$ is a polynomial ring in n variables $c_1, ..., c_n$ with $c_i \in H^{2i}(BU(n))$ as in Theorem 5.5 in [MT91]. Then, define $c_i(\gamma^n(\mathbb{C}^\infty)) = c_i$ and use the universality of this bundle to define the chern classes of any bundle $\mathcal{E}$ as $c_i(\mathcal{E}) = f^*c_i$ where f is its classifying map.

Remark 2.64. Let $\mathbb{Z}[x_1, ..., x_n]^{S_n}$ denote the algebra of symmetric polynomials in the Chern roots. Proposition 2.62 implies that

$$H^*(Gr_n(\mathbb{C}^\infty)) \cong \mathbb{Z}[x_1, ..., x_n]^{S_n}.$$

We now know the cohomology of $BU(n) \simeq Gr_n(\mathbb{C}^\infty)$ and $BO(n) \simeq Gr_n(\mathbb{R}^\infty)$. Note that $O(n) \subset U(n)$ and, under the identification $GL(n;\mathbb{C}) \subset GL(2n;\mathbb{R})$, one has $U(n) = O(2n) \cap GL(n;\mathbb{C})$, so $U(n) \subset O(2n)$. The maps induced by these inclusions are computed in the following propositions:

Proposition 2.65. Consider $H^*(BO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_1, ..., w_n]$ and $H^*(BU(n);\mathbb{Z}_2) = \mathbb{Z}_2[c_1, ..., c_n]$, where c_i denotes the reduction of the i-th Chern class to $\mathbb{Z}_2$ coefficients. The inclusion $O(n) \subset U(n)$ induces the map in cohomology

$$H^*(BU(n);\mathbb{Z}_2) \to H^*(BO(n);\mathbb{Z}_2)$$
$$c_i \mapsto w_i^2$$

Proof. See Theorem 5.11 (1) of [MT91]. □

Proposition 2.66. The inclusion $U(n) \subset O(2n)$ induces

$$H^*(BO(2n);\mathbb{Z}_2) \to H^*(BU(n);\mathbb{Z}_2)$$
$$w_{2i} \mapsto c_i$$
$$w_{2i-1} \mapsto 0$$

Proof. See Theorem 3.5.11 (2) of [MT91]. □

One can also ask whether the inclusions $O(n) \times O(m) \hookrightarrow O(n+m)$ and $U(n) \times U(m) \hookrightarrow U(n+m)$ yield easy relations between characteristic classes. And indeed,

Proposition 2.67. These inclusions induce in cohomology the following maps:

$$H^*(BO(n+m)) \to H^*(BO(m) \times BO(n))$$
$$w_i \mapsto \sum_{j+k=i} w_j \times w_k$$

$$H^*(BU(n+m)) \to H^*(BU(m) \times BU(n))$$
$$c_i \mapsto \sum_{j+k=i} c_j \times c_k$$

Proof. See Theorems 3.5.11 (3) and 5.8 (3) of [MT91]. □

2.4 Schur Polynomials

It turns out that the characteristic classes that we will compute in the next chapter can be written as **Schur polynomials** in the Stiefel-Whitney roots t_i.

Definition 2.68. A **partition** of length n (or an n-partition) of a non-negative integer k is a tuple $\lambda = (\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1 + ... + \lambda_n = k$. If $\lambda = (\lambda_1, ..., \lambda_n)$ and $\delta = (\delta_1, ..., \delta_n)$ are two partitions of the same length, then $\lambda + \delta = (\lambda_1 + \delta_1, ..., \lambda_n + \delta_n)$. If λ is a partition of length $n \leq m$, its associated partition of length m is $\tilde{\lambda} = (\lambda_1, ..., \lambda_n, 0, ...0)$.

A partition $\lambda = (\lambda_1, ..., \lambda_n)$ can be represented through a Ferrers diagram. This is a diagram of dots with n rows and λ_i dots on the i-th row. For example, the Ferrers diagram for the partition $\lambda = (4, 3, 1)$ is

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{matrix}$$

Definition 2.69. The conjugate of a partition λ is the partition λ' obtained from λ by transposing its Ferrers diagram.

Definition 2.70. Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a tuple of non-negative integers. The alternant $a_\alpha(x_1, ..., x_n)$ is the polynomial

$$a_\alpha(x_1, ..., x_n) = det \begin{pmatrix} x_1^{\alpha_1} & \cdots & x_1^{\alpha_n} \\ \vdots & \ddots & \vdots \\ x_n^{\alpha_1} & \cdots & x_n^{\alpha_n} \end{pmatrix}$$

Definition 2.71 (Schur Polynomial)**.** Let n be a non-negative integer and δ be the partition $(n-1, n-2, ..., 1, 0)$. For a partition λ of length $\leq n$, the **Schur polynomial** s_λ of λ in n variables is the polynomial

$$s_\lambda = \frac{a_{\tilde{\lambda}+\delta}}{a_\delta}$$

where $\tilde{\lambda}$ is the partition of length n associated to λ.

Remark 2.72. s_λ is a symmetric polynomial. It is symmetric because it is a quotient of alternants, which are alternating polynomials. Theorem 2.74 gives another possible definition for s_λ, one for which it is clear that s_λ is a polynomial with coefficients in $\mathbb{Z}$.

Definition 2.73. The i-th elementary symmetric polynomial in n variables is the polynomial

$$e_i(x_1, ..., x_n) = \sum_{\substack{j_1+\cdots+j_n=i \\ j_1,...,j_n\leq 1}} x_1^{j_1} \cdots x_n^{j_n}$$

The polynomials e_i with negative i or $i > n$ are defined to be 0.

Theorem 2.74 (Second Jacobi-Trudi Formula)**.** *Let λ be a partition of length $\leq n$ and λ' its conjugate. Then,*

$$s_\lambda(x_1, ..., x_n) = det(e_{\lambda'_i+j-i})_{i,j=1}^n = det\begin{pmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \cdots & e_{\lambda'_1+n-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \cdots & e_{\lambda'_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_n-n+1} & e_{\lambda'_n-n+2} & \cdots & e_{\lambda'_n} \end{pmatrix}$$

Proof. See formula 3.5 in Chapter I.3 of [Mac99]. □

An important special case is the Schur polynomial of $\delta = (n-1, n-2, ..., 1, 0)$.

Proposition 2.75.

$$s_\delta(x_1, ..., x_n) = det\begin{pmatrix} e_{n-1} & e_n & \cdots & e_{2n-2} \\ e_{n-3} & e_{n-2} & \cdots & e_{2n-4} \\ \vdots & \vdots & \ddots & \vdots \\ e_{-n+1} & e_{-n+2} & \cdots & 1 \end{pmatrix} = \prod_{1\leq i<j\leq n}(x_i + x_j) \qquad (2.2)$$

Proof. A proof can be found in [gri]. □

2.5 Homotopy Pushouts

Another construction that will be useful in the subsequent chapters is the notion of a homotopy pushout. We will consider certain decompositions of spaces (called stratifications) and homotopy pushouts describe the way in which the pieces are glued into the whole space.

Definition 2.76. Consider maps $X \xleftarrow{f} Z \xrightarrow{g} Y$. The **double mapping cylinder** of f and g is the quotient space

$$M(f,g) = \frac{X \sqcup Z \times I \sqcup Y}{(z,0) \sim f(z) \quad (z,1) \sim g(z)}$$

There are canonical inclusions

$$i_X : X \to M(f,g), \qquad i_Y : Y \to M(f,g)$$

and a canonical homotopy $h : Z \times I \to M(f,g)$ between $i_X \circ f$ and $i_Y \circ g$ making the following square homotopy commutatitve:

$$\begin{array}{ccc} Z & \xrightarrow{\quad g \quad} & Y \\ {\scriptstyle f}\big\downarrow & \overset{h}{\Rightarrow} & \big\downarrow{\scriptstyle i_Y} \\ X & \xhookrightarrow{\quad i_X \quad} & M(f,g) \end{array}$$

This square is called the **standard homotopy pushout** of f and g.

Given any other homotopy commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\quad g \quad} & Y \\ {\scriptstyle f}\big\downarrow & \overset{H}{\Longrightarrow} & \big\downarrow{\scriptstyle k} \\ X & \xhookrightarrow{\quad h \quad} & W \end{array}$$

one can construct a map

$$\begin{aligned} \theta_H : M(f,g) &\to W \\ x &\mapsto h(x) \\ y &\mapsto k(y) \\ (z,t) &\mapsto H(z,t) \end{aligned}$$

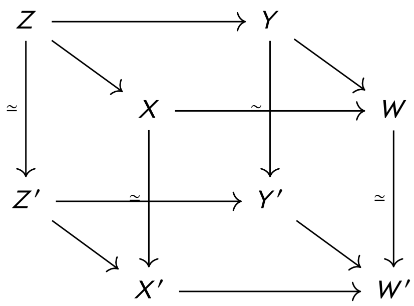
Definition 2.77. A homotopy commutative square

$$\begin{array}{ccc} Z & \xrightarrow{\ g\ } & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ X & \xrightarrow{\ h\ } & W \end{array}$$

is said to be a **homotopy pushout** is there exists a homotopy $H : Z \times I \to W$ between $h \circ f$ and $k \circ g$ such that θ_H is a homotopy equivalence.

Homotopy pushouts are invariant under homotopies:

Proposition 2.78. If the diagram



homotopy commutes and the vertical maps are equivalences, then the top square is a homotopy pushout if and only if the bottom square is a homotopy pushout.

Proof. See Proposition 6.3.2 of [Ark11]. □

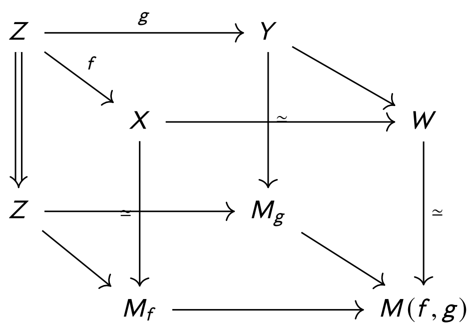
Let us present some properties of homotopy pushouts that will be needed later.

Theorem 2.79. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \longrightarrow & W \end{array}$$

be a homotopy pushout, f an n-equivalence and g an m-equivalence. Then, k is an $(n+m)$-equivalence.

Proof. One can assume that $W = M(f,g)$. With this, it is clear that the following diagram homotopy commutes:



Then the result is a direct application of the homotopy excision theorem (Theorem 4.23 in [Hat02]) on the bottom square. Indeed, the bottom square is a pushout and the connectedness of the maps on this square is the same as the one of the maps on top. □

Theorem 2.80. *Let the following strictly commutative square*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & W \end{array}$$

be a pushout. If f (or g) is a cofibration, then the square is a homotopy pushout.

Proof. See Proposition 6.2.6 of [Ark11]. □

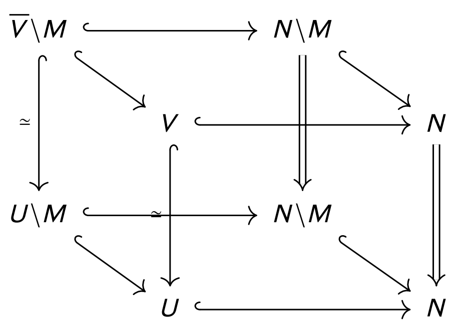
Example 2.81. Let N be a manifold, $M \subset N$ a closed submanifold and $U \subset N$ an open tubular neighborhood of M. Then,

$$\begin{array}{ccc} U \backslash M & \hookrightarrow & N \backslash M \\ \downarrow & & \downarrow \\ U & \hookrightarrow & N \end{array}$$

is a pushout. If $V \subset \overline{V} \subset U$ is a smaller tubular neighbourhood of M,

$$\begin{array}{ccc} \overline{V} \backslash M & \overset{i}{\hookrightarrow} & N \backslash M \\ \downarrow & & \downarrow \\ \overline{V} & \hookrightarrow & N \end{array}$$

is also a pushout and, furthermore, $i : \overline{V} \backslash M \to N \backslash M$ defines an NDR-pair. Indeed, $\overline{V} \backslash M$ is closed and a deformation retract of an open neighbourhood $U \backslash M$. Hence, i is a cofibration and thus, this last square is a homotopy pushout. Using the following diagram, Proposition 2.78 implies that the first square is also a homotopy pushout.



Proposition 2.82. A homotopy pushout

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow{h} & W \end{array}$$

induces a long exact sequence in cohomology

$$\cdots \to H^*(W) \xrightarrow{(h^*,k^*)} H^*(X) \oplus H^*(Y) \xrightarrow{f^*-g^*} H^*(Z) \to H^{*+1}(W) \to \cdots$$

Proof. The square can be assumed to be a standard homotopy pushout. Then, $W = U \cup V$, where U is an open neighborhood of X that deformation retracts onto X, V is an open neighborhood of Y that deformation retracts onto Y and $U \cap V \cong I \times Z$. Then, the sequence of the statement is the Mayer-Vietoris cohomology sequence associated to this decomposition of W. □

2.6 Locally Trivial Stratifications

Many times, a manifold is best understood by partitioning it into submanifolds in a controlled way. One way to obtain such a partition is through **locally trivial stratifications**.

Definition 2.83. Let M be a manifold, $K = \{0, ..., n\} \subset \mathbb{N}$, for some integer n, and $\{F_k\}_{k\in K}$ a family of closed subsets of M totally ordered by inclusion.

$$F_0 \subset F_1 \subset F_n = M$$

This family is said to be a finite stratification if, for each $k \in K$, the space $R_k = F_k \backslash F_{k-1}$ is an embedded submanifold of M. The submanifolds R_k are called the **strata** of the filtration.

The stratification is said to satisfy the **frontier condition** if the strata R_k satisfy the following property:

$$R_j \cap \overline{R_k} \neq \emptyset \implies R_j \subset \overline{R_k}$$

Definition 2.84. Let M be a manifold with a stratification $\{F_k\}_{k\in K}$, N a manifold with a stratification $\{G_k\}_{k\in K}$. A **diffeomorphism of stratifications** is a diffeomorphism $f : M \to N$ such that $f(F_k) = G_k$.

Definition 2.85. Let M be a manifold and $\{F_k\}_{k\in K}$ be a finite stratification. The stratification is said to be **locally trivial** if for each $k \in K$ and $x \in R_k = F_k \backslash F_{k-1}$, there is an open neighborhood $V \subset M$ of x, a stratified manifold U and a diffeomorphism of stratifications

$$\phi : V \to (V \cap R_k) \times U$$

Here, if $\{G_{k'}\}_{k'\in K}$ is the stratification of U, then $\{(V \cap R_k) \times G_{k'}\}_{k'\in K}$ is the stratification of $(V \cap R_k) \times U$.

We will use two ways of constructing stratifications out of old ones. The first one, introduces a stratification on the total space of a bundle from a stratification on the typical fibre. The second, introduces a stratification on the domain M of a map $s : M \to N$ out of a stratification $\{F_k\}$ on N, if s is transversal to the stratification $\{F_k\}$.

Proposition 2.86. Let $p : E \to B$ define a fibre bundle with fibre F and suppose F has a locally trivial stratification $\{F_k\}_{k\in K}$ that is preserved by the action of the structure group. Then, E has a locally trivial Stratification given by $\{\bigcup_{x\in B}(F_k)_x\}_{k\in K}$.

Proof. For each $k \in K$, let $R_k = F_k \backslash F_{k-1}$, $G_k = \bigcup_{x\in B}(F_k)_x$ and $S_k = G_k \backslash G_{k-1}$. Fix $k \in K$ and $x \in S_k$. There is an open neighborhood $W \subset B$ of x and a diffeomorphism $\psi_1 : p^{-1}(W) \to W \times F$ (since the action of the structure group is stratification preserving, ψ_1 is also stratification preserving). Let $\psi_1(x) = (b, f)$. Then, there exists an open neighborhood $\tilde{V} \subset F$ of f, a stratified manifold U and a stratification preserving diffeomorphism $\psi_2 : \tilde{V} \to (\tilde{V} \cap R_k) \times U$. Then, defining $V = \psi_1^{-1}(W \times \tilde{V})$, the following composition is the desired stratification preserving diffeomorphism $\phi : V \to (V \cap S_k) \times U$:

$$V \xrightarrow{\psi_1} W \times \tilde{V} \xrightarrow{id_W \times \psi_2} \underbrace{W \times (\tilde{V} \cap R_k)}_{=\psi_1(V \cap S_k)} \times U \xrightarrow{\psi_1^{-1} \times id_U} (V \cap S_k) \times U$$

□

Proposition 2.87. Let $s : M \to N$ be a map of manifolds and $\{F_k\} \subset N$ a locally trivial stratification of N. If s is transversal to each strata $F_k \backslash F_{k-1}$, then the family $\{s^{-1}(F_k)\}$ is a locally trivial stratification of M.

Proof. Let m be the dimension of M. Given k, let us use the notation $R_k = F_k \backslash F_{k-1}$ and $S_k = s^{-1}(F_k \backslash F_{k-1})$. Since s is transversal to R_k, it follows that S_k is an embedded submanifold of M, so $\{s^{-1}(F_k)\}$ forms a stratification of M. Let us now show that it is locally trivial.
Fix $k \in K$ and $x \in S_k$. There exists $V' \subset N$ an open neighborhood of $s(x)$, U' a stratified manifold and a stratification preserving diffeomorphism $\psi : V' \to (V' \cap R_k) \times U'$. Let d be the dimension of U' and $\pi_2 : (V' \cap R_k) \times U' \to U'$ be the canonical projection. Then, as s is transversal to R_k, it follows that the following composition, denoted by ϕ_2, is a submersion at x:

$$s^{-1}(V') \xrightarrow{s} V' \xrightarrow{\psi} (V' \cap R_k) \times U' \xrightarrow{\pi_2} U'$$

Therefore, there exists $V \subset s^{-1}(V')$ an open neighborhood of x, $U \subset U'$ an open neighborhood of $\phi_2(x)$ and local coordinates $\alpha_V : V \to \mathbb{R}^m$ and $\beta_U : U \to \mathbb{R}^d$ such that $\beta_U \circ \phi_2 \circ \alpha_V^{-1}$ is the projection given by

$$(x_1, ..., x_m) \mapsto (x_{m-d+1}, ..., x_m).$$

In the coordinates given by α_V, points in $V \cap S_k$ are written as $(x_1, ..., x_{m-d}, 0, ..., 0)$. Let $\phi_1 : V \to V \cap S_k$ be the map written in the coordinates given by α_V as

$$(x_1, ..., x_m) \mapsto (x_1, ..., x_{m-d}).$$

Then, the following map is a diffeomorphism of stratifications:

$$\begin{aligned} \phi : V &\to (V \cap S_k) \times U \\ x &\mapsto (\phi_1(x), \phi_2(x)) \end{aligned}$$

□

Locally trivial stratifications satisfying the frontier condition admit triangulations:

Theorem 2.88. *Given $\{F_k\}$ a locally trivial stratification of a manifold M, such that $\{F_k\}$ satisfies the frontier condition, there is a triangulation of M with each F_k a subcomplex.*

Proof. It is easy to prove that every locally trivial stratification satisfying the frontier condition is a Whitney stratification (see the beginning of section 5 in page 480 of [Mat12] for the definition of Whitney stratification).

In page 491 of [Mat12], after Definition 8.2, the author shows that Whitney stratifications are Thom-Mather stratifications (see Definition 8.1 of [Mat12] for the definition). The result then follows from the fact that Thom-Mather stratifications admit triangulations (see Proposition 5 of [Gor78]). □

For stratifications $\{F_k\}$ where $dim(F_k) - dim(F_{k-1}) \geq 2$, Theorem 2.88 in particular implies that F_k has a well defined homology class $[F_k] \in H^*(M)$. The next theorem (applied with $K = F_k$ and $L = \bigcup_{l<k} F_l$) gives a description of the Poincaré dual of $[F_k]$.

Theorem 2.89. *Suppose M is a compact orientable m-manifold and $L \subset K \subset M$ are compact subsets such that $K \backslash L$ is an orientable submanifold of dimension k and L is a union of submanifolds of dimensions $\leq k - 2$. Suppose further that there exists a triangulation of K with L a subcomplex. Then, K has a well defined homology class $[K] \in H_k(M)$ and its Poincaré dual is the unique class in $H^{m-k}(M)$ whose restriction to $H^{m-k}(M \backslash L)$ is the Thom class of the normal bundle of $K \backslash L$ in $M \backslash L$.*

Proof. Let V be an open neighborhood of L in K that deformation retracts onto L. The sum of k-simplices of the triangulation of K generates $H_k(K, V) \cong H_k(K, L)$.
Since L is a union of manifolds of dimensions $\leq k - 2$, the map $H_k(K) \to H_k(K, L)$ is an isomorphism, so the sum of k-simplices is a generator of $H_k(K)$ - it is the fundamental class. The image of this fundamental class in $H_k(M)$ is $[K]$. Now let $j : M \backslash L \to M$ be the inclusion map and let U be a tubular neighborhood of $K \backslash L$ in $M \backslash L$.

$$\begin{array}{ccccc}
H^{m-k}(M) & \xrightarrow{\quad j^* \quad} & H^{m-k}(M \backslash L) & & \\
{\scriptstyle D}\uparrow & & \uparrow & & \\
H_k(M) & & H^{m-k}(M \backslash L, M \backslash K) & \xrightarrow{\cong} & H^{m-k}(U, U \backslash (K \backslash L)) \\
\uparrow & & {\scriptstyle D}\uparrow & & \\
H_k(K) & \xrightarrow{\qquad\quad} & H_k(K, L) & &
\end{array}$$

Here, both instances of D denote duality maps, the one on the right being a relative version of duality proved in Theorem 6.2.17 of [Spa66]. Since the codimension of L in M is greater than $m - k + 1$, the top map is an isomorphism. Lastly, the fundamental class in $H_k(K, L)$ is mapped to the Thom class in $H^{m-k}(U, U \backslash (K \backslash L))$ because D is an isomorphism. Therefore, the dual of $[K]$ is the unique class in $H^{m-k}(M)$ that restricts to the (image of the) Thom class in $H^{m-k}(M \backslash L)$. □

Chapter 3

Degeneracy Loci of 2-forms

3.1 Introduction

Let M be a compact $2m$-manifold and consider the vector bundle $\Lambda^2 T^*M \to M$ of 2-forms over M. Given a generic section s of $\Lambda^2 T^*M$ and an integer $k \in \{0, ..., m\}$, the degeneracy locus of points $x \in M$ where $rank(s(x)) \leq 2k$ gives rise to an homology class. The purpose of this chapter is to compute the Poincaré dual of this homology class, following the methods of M. Kazarian in [Kaz06] and of L. M. Fehér and R. Rimányi in [FR04].

The chapter starts by studying the typical fibre $\Lambda^2(\mathbb{R}^{2m})^*$ of $\Lambda^2 T^*M$ and the properties of the following spaces:

$$R_k = \{\omega \in \Lambda^2(\mathbb{R}^{2m})^* \ \mid \ \text{rank}(\omega) = 2k\} \subset \Lambda^2(\mathbb{R}^{2m})^*.$$

In section 2.3, we define the Thom polynomials as cohomological obstructions. Then, in section 2.5, we compute those cohomological obstructions and show that they are indeed the Poincaré duals of the homology classes of the degeneracy loci. We finish the chapter by computing the Poincaré dual of such a class in a specific example.

3.2 The Homogeneous Spaces R_k and their Normal Bundles

Consider the vector space $\Lambda^2(\mathbb{R}^{2m})^*$ endowed with the action of $GL(2m;\mathbb{R})$ given by

$$\begin{aligned} GL(2m;\mathbb{R}) \times \Lambda^2(\mathbb{R}^{2m})^* &\to \Lambda^2(\mathbb{R}^{2m})^* \\ (A, \omega) &\mapsto A^*\omega. \end{aligned} \tag{3.1}$$

And define $R_k = \{\omega \in \Lambda^2(\mathbb{R}^{2m})^* \ \mid \ \mathrm{rank}(\omega) = 2k\}$.

Proposition 3.1. The sets $\{R_k\}_{k=0,\dots,m}$ are the orbits of action (3.1).

Proof. Given $\omega \in R_k$, denote by $O(\omega)$ the orbit of ω. For any $A \in GL(2m;\mathbb{R})$, $rank(A^*\omega) = rank(\omega)$, so $O(\omega) \subset R_k$. The other inclusion follows from the general fact that, for any 2-form $\sigma \in R_k$, there exists a basis $\{e_i\}_{i=1,\dots,2m}$ of $\mathbb{R}^{2m}$ such that σ is represented by

$$J = \begin{pmatrix} J_{2k} & 0 \\ 0 & 0_{2(m-k)} \end{pmatrix} = \left(\begin{array}{cc|c} 0 & \mathcal{I}_{2k} & \\ -\mathcal{I}_{2k} & 0 & 0 \\ \hline 0 & & 0_{2(m-k)} \end{array}\right). \tag{3.2}$$

Let $\{e_i\}_{i=1,\dots,2m}$ be such a basis for ω. Given any other $\omega' \in R_k$, let $\{f_i\}_{i=1,\dots,2m}$ be a basis of $\mathbb{R}^{2m}$ such that ω' is represented by J. Define $A \in GL(2m;\mathbb{R})$ by $A(e_i) = f_i$. Then, $\omega' = A^*\omega$ and $R_k \subset O(\omega)$. □

Proposition 3.2. $\overline{R_k} = \bigcup_{j\leq k} R_j = \{\omega \in \Lambda^2(\mathbb{R}^{2m})^* \mid \mathrm{rank}(\omega) \leq 2k\}$

Proof. Pick a basis $\{e_i\}_{i=1,\dots,2m}$ of $\mathbb{R}^{2m}$. A form ω, represented in this basis by a matrix J, has rank $\leq 2k$ if and only if all $(2k+1)\times(2k+1)$ minors of J are zero. Hence the set $\{\omega \in \Lambda^2(\mathbb{R}^{2m})^* \mid \mathrm{rank}(\omega) \leq 2k\}$ is closed. This set obviously contains R_k so it also contains $\overline{R_k}$.

Now, given some form $\sigma \in \{\omega \in \Lambda^2(\mathbb{R}^{2m})^* \mid \mathrm{rank}(\omega) \leq 2k\}$ of rank $2k' \leq 2k$, pick a basis $\{e_i\}_{i=1,\dots,2m}$ such that σ is represented by

$$\begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $M_1 \in \mathcal{M}_{2k'\times 2k'}(\mathbb{R})$ is skew-symmetric and non-singular. Consider a sequence $\sigma_n \in R_k$ given, in the basis $\{e_i\}_{i=1,\dots,2m}$, by

$$\begin{pmatrix} M_1 & 0 & 0 \\ 0 & \frac{1}{n}M_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where $M_2 \in \mathcal{M}_{2(k-k')\times 2(k-k')}(\mathbb{R})$ is also skew-symmetric and non-singular. As $\sigma_n \to \sigma$, we have $\sigma \in \overline{R_k}$. □

Note that both $\overline{R_k}$ and $\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_k}$ are invariant under the action (3.1). Proposition 3.1 implies that each R_k is an immersed submanifold of $\Lambda^2(\mathbb{R}^{2m})^*$ and Proposition 3.2 says that the closures $\overline{R_k}$ form a stratification satisfying the frontier condition.

$$\overline{R_0} \subset \overline{R_1} \subset \cdots \subset \overline{R_m} = \Lambda^2(\mathbb{R}^{2m})^*$$

The next theorem improves these results.

Theorem 3.3. *Let*

$$d_k = 2(m-k). \tag{3.3}$$

R_k is an embedded submanifold of $\Lambda^2(\mathbb{R}^{2m})^*$ of codimension $\frac{1}{2}(d_k^2 - d_k)$. Furthermore, the stratification $\{\overline{R_k}\}_{k=0,\dots,m}$ is locally trivial.

Proof. Fix $k \in \{0,\dots,m\}$ and $\omega \in R_k$. Pick a basis $\{e_i\}_{i=1,\dots,2m}$ for $\mathbb{R}^{2m}$ such that ω is represented by J, as in (3.2). Take a neighborhood $U \subset \Lambda^2(\mathbb{R}^{2m})^*$ of ω such that every form $\sigma \in U$ is written in $\{e_i\}_{i=1,\dots,2m}$ as

$$G = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}$$

with $A \in \mathcal{M}_{2k\times 2k}(\mathbb{R})$ skew-symmetric and non-singular, $B \in \mathcal{M}_{2k\times 2(m-k)}(\mathbb{R})$ and $C \in \mathcal{M}_{2(m-k)\times 2(m-k)}(\mathbb{R})$ skew-symmetric. Multiplying G on the right by $\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}$ gives

$$\begin{pmatrix} A & 0 \\ -B^T & B^TA^{-1}B + C \end{pmatrix}$$

The rank of this matrix is given by $2(k+k')$ with k' equal to the rank of $B^TA^{-1}B+C$. Note that $B^TA^{-1}B+C$ is a $2(m-k)\times 2(m-k)$ skew-symmetric matrix, so it represents a 2-form in $\mathbb{R}^{2(m-k)}$, in the basis $\{e_i\}_{i=2k+1,\dots,2m}$. Consider the stratification of $\Lambda^2(\mathbb{R}^{2(m-k)})^*$ given by $\{\overline{R'_k}\}_{k=0,\dots,m-k}$ with

$$R'_k = \{\omega \in \Lambda^2\mathbb{R}^{2(m-k)} \mid rank(\omega) = 2k\}.$$

Then, the map f defined by

$$(\Lambda^2(\mathbb{R}^{2k})^* \cap GL(2k;\mathbb{R})) \times \mathcal{M}_{2k\times 2(m-k)}(\mathbb{R}) \times \Lambda^2(\mathbb{R}^{2(m-k)})^* \xrightarrow{f} U \subset \Lambda^2(\mathbb{R}^{2m})^*$$
$$(A,B,X) \mapsto \begin{pmatrix} A & B \\ -B^T & X + B^TA^{-1}B \end{pmatrix}$$

is a diffeomorphism of stratifications, as it satisfies the property: X has rank $2k'$ if and only if $f(A,B,X)$ has rank $2(k+k')$.

This proves that $\{\overline{R_k}\}$ is a locally trivial stratification and R_k is embedded. Its dimension is $\frac{1}{2}2k(2k-1) + (2k)2(m-k) + \frac{1}{2}2(m-k)(2(m-k)-1)$. Hence,

$$codim(R_k) = \frac{1}{2}2m(2m-1) - dim(R_k) = \frac{1}{2}(d_k^2 - d_k)$$

□

Fix an element $\omega \in R_k$ and denote by $Iso(k)$ the isotropy group of the action (3.1) at ω. Consider the bijective smooth immersion ϕ given by

$$\begin{aligned} GL(2m;\mathbb{R})/Iso(k) &\xrightarrow{\phi} R_k \\ A &\mapsto A\cdot\omega \end{aligned}$$

The map ϕ makes R_k an immersed submanifold of $\Lambda^2(\mathbb{R}^{2m})^*$ but, since R_k is embedded, the smooth structure induced by ϕ must be the same as the one defined in the proof of Theorem 3.3. To better understand $R_k \cong GL(2m;\mathbb{R})/Iso(k)$, let us compute $Iso(k)$:

Theorem 3.4. *$Iso(k) \cong (Sp(2k;\mathbb{R})\times GL(2(m-k);\mathbb{R})) \ltimes \mathcal{M}_{2(m-k)\times 2k}(\mathbb{R})$, where $Sp(2k)\times GL(2(m-k);\mathbb{R})$ acts on $\mathcal{M}_{2(m-k)\times 2k}$ in the natural way.*

Proof. Pick a basis $\{e_i\}_{i=1,\ldots,2m}$ such that ω is represented by J, as in (3.2). Represent also the elements of $GL(2m;\mathbb{R})$ by matrices using the basis $\{e_i\}_{i=1,\ldots,2m}$. Then $Iso(k)$ is composed of non-singular $2m\times 2m$ matrices A such that $A^TJA = J$. Decompose A in blocks as

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_1 is $2k\times 2k$ and the dimensions of the other blocks are determined by those of A_1. Then,

$$A^TJA = J \Leftrightarrow \begin{pmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{pmatrix}\begin{pmatrix} J_{2k} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} J_{2k} & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} A_1^TJ_{2k}A_1 & A_1^TJ_{2k}A_2 \\ A_2^TJ_{2k}A_1 & A_2^TJ_{2k}A_2 \end{pmatrix} = \begin{pmatrix} J_{2k} & 0 \\ 0 & 0 \end{pmatrix}.$$

This equality implies $A_1 \in Sp(2k;\mathbb{R})$ and $A_2 = 0$. Since A must be non-singular, $A_4 \in GL(2(m-k);\mathbb{R})$. One can thus form a short exact sequence

$$0 \to \mathcal{M}_{2(m-k)\times 2k}(\mathbb{R}) \xrightarrow{f} Iso(k) \xrightarrow{g} Sp(2k;\mathbb{R})\times GL(2(m-k);\mathbb{R}) \to 0 \tag{3.4}$$

with

$$f(A_3) = \begin{pmatrix} I & 0 \\ A_3 & I \end{pmatrix} \quad \text{and} \quad g\begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix} = (A_1, A_4).$$

Moreover, the inclusion

$$\begin{aligned} Sp(2k;\mathbb{R})\times GL(2(m-k);\mathbb{R}) &\hookrightarrow Iso(k) \\ (A_1, A_4) &\mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} \end{aligned}$$

provides a right inverse to g, so (3.4) splits and $Iso(k)$ is the semidirect product stated in the theorem. □

Now let g be an $O(2m)$-invariant metric on $\Lambda^2(\mathbb{R}^{2m})^*$ and let $NR_k \to R_k$ be the normal bundle of R_k with respect to g. It is a $codim(R_k)$-vector bundle. $NR_k \backslash R_k \to R_k$ is the bundle, with fibre homotopy equivalent to $S^{codim(R_k)-1}$, obtained by removing the zero section from NR_k. Using the riemannian exponential map, one can see NR_k as a tubular neighborhood U of R_k inside the open submanifold $\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1}$. Under the identification $NR_k \cong U$, $\pi : NR_k \to R_k$ can be seen as a retraction of U onto R_k. In the following, we will often use the notation NR_k to mean both the normal bundle and the tubular neighborhood and also denote by π both the bundle projection and the retraction.

Proposition 3.5. $\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{k-1}}$ is the homotopy pushout of $R_k \xleftarrow{\pi} NR_k \backslash R_k \hookrightarrow \Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k}$.

Proof. Denote by $U = NR_k$ the tubular neighborhood of R_k and consider the pushout

$$\begin{array}{ccc} U\backslash R_k & \hookrightarrow & \Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k} \\ \downarrow & & \downarrow \\ U & \hookrightarrow & \Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{k-1}} \end{array}$$

By Example 2.81, this is also a homotopy pushout. Hence, using the equivalences

$$\begin{array}{ccc} U\backslash R_k & \xleftarrow{\cong} & NR_k\backslash R_k \\ \downarrow & & \downarrow \pi \\ U & \xleftarrow{\simeq} & R_k \end{array}$$

and Proposition 2.78, the square

$$\begin{array}{ccc} NR_k\backslash R_k & \hookrightarrow & \Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k} \\ \pi \downarrow & & \downarrow \\ R_k & \hookrightarrow & \Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{k-1}} \end{array} \tag{3.5}$$

is a homotopy pushout. □

Remark 3.6. $GL(2m, \mathbb{R})$ and $Iso(k)$ are semisimple Lie groups[1] so they deformation retract to their maximal compact subgroups (Theorem 2 in [Mos49]). Let us denote by G^c the maximal compact subgroup of a given Lie group G. So $GL(2m;\mathbb{R})^c = O(2m)$ and, since $Sp(2k) \cap O(2k) \cong U(k)$, one has $Iso(k)^c = Iso(k) \cap O(2m) \cong U(k) \times O(2(m-k))$.
Define also $R_k^c := O(2m)/Iso(k)^c$. Using the homotopy equivalences $GL(2m;\mathbb{R}) \simeq O(2m)$ and $Iso(k) \simeq Iso(k)^c$, the homotopy long exact sequence of the bundles $Iso(k) \hookrightarrow GL(2m;\mathbb{R}) \to R_k$ and $Iso(k)^c \hookrightarrow O(2m) \to R_k^c$ and the 5-lemma, one checks that $R_k \simeq R_k^c$.

[1] A semimsimple Lie group is a Lie group whose Lie algebra is semisimple. See section 3.1 of [Hum72] for the definition of semisimple Lie algebra. $GL(n;\mathbb{R})$ and $Sp(2n;\mathbb{R})$ are semisimple Lie groups and a finite product of semisimple Lie groups is semisimple.

Recall that $\mathcal{I}so(k)$ is the isotropy group of a fixed $\omega \in R_k$. Consider the following action of $O(2m)$ on $T_\omega\Lambda^2(\mathbb{R}^{2m})^*$:

$$A \cdot v = (dA)_\omega v \qquad \forall A \in O(2m), \quad v \in T_\omega\Lambda^2(\mathbb{R}^{2m})^* \tag{3.6}$$

where $(dA)_\omega : T_\omega\Lambda^2(\mathbb{R}^{2m})^* \longrightarrow T_{A^*\omega}\Lambda^2(\mathbb{R}^{2m})^*$ is the differential of the map

$$\Lambda^2(\mathbb{R}^{2m})^* \xrightarrow{A} \Lambda^2(\mathbb{R}^{2m})^*$$
$$\sigma \mapsto A^*\sigma$$

Note that $(dA)_\omega$ sends vectors in $T_\omega R_k$ to vectors in $T_{A^*\omega}R_k$ so, by invariance of the metric g, $(dA)_\omega$ also sends vectors in $(T_\omega R_k)^\perp$ to vectors in $(T_{A^*\omega}R_k)^\perp$. In particular, if $A \in \mathcal{I}so(k)^c$ then $(dA)_\omega$ sends vectors in $(T_\omega R_k)^\perp$ to vectors in $(T_\omega R_k)^\perp$ so if we restrict to elements $A \in \mathcal{I}so(k)^c$ and $v \in (T_\omega R_k)^\perp$, then formula (3.6) yields an action of $\mathcal{I}so(k)^c$ on $(T_\omega R_k)^\perp$.

Proposition 3.7. Let $V_k := (T_\omega R_k)^\perp$. Then one has the following diagram:

$$\begin{array}{ccccc} O(2m)\times_{\mathcal{I}so(k)^c} V_k & \xrightarrow{\cong} & NR_k|_{R_k^c} & \longrightarrow & NR_k \\ & \searrow & \downarrow & \lrcorner & \downarrow \pi \\ & & R_k^c & \overset{\simeq}{\hookrightarrow} & R_k \end{array}$$

Proof. The inclusion $R_k^c \hookrightarrow R_k$ is an equivalence by Remark 3.6. Consider the following map:

$$O(2m)\times_{\mathcal{I}so(k)^c} V_k \xrightarrow{f} NR_k|_{R_k^c}$$
$$[A, v] \mapsto (dA)_\omega v$$

For $A \in O(2m)$, $(dA)_\omega$ restricts to an isomorphism between V_k and $(T_{A^*\omega}R_k)^\perp$ so f is well defined and the restriction of f to each fibre is an isomorphism. It follows that f is a bundle isomorphism. □

To finish this section, let us compute V_k. Recall that we defined $d_k = 2(m-k)$.

Theorem 3.8. *Under the identification of $\mathcal{I}so(k)^c$ with $U(k)\times O(d_k)$, the representation V_k of $\mathcal{I}so(k)^c$ is isomorphic to $\Lambda^2(\mathbb{R}^{d_k})^*$ endowed with the action of $U(n)\times O(d_k)$ given by*

$$(A, B)\cdot\sigma = B^*\sigma \qquad \forall (A, B) \in U(k)\times O(d_k), \quad \sigma \in \Lambda^2\mathbb{R}^{d_k}.$$

Proof. Pick a basis $\{e_i\}_{i=1,\dots,2m}$ of $\mathbb{R}^{2m}$ such that ω is represented by J, as in (3.2).

$$J = \begin{pmatrix} J_{2k} & 0 \\ 0 & 0 \end{pmatrix}$$

Given $V \in \mathfrak{gl}(2m)$,

$$\frac{d}{dt}\bigg|_{t=0}(exp(tV)\cdot\omega) = \frac{d}{dt}\bigg|_{t=0}(exp(tV)^T Jexp(tV)) = V^T J + JV$$

Write $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$ with V_1 a $2k \times 2k$ matrix.

$$V^T J + JV = \begin{pmatrix} V_1^T J_{2k} + J_{2k}V_1 & J_{2k}V_2 \\ V_2^T J_{2k} & 0 \end{pmatrix}.$$

Since J_{2k} is non-singular, $V_1^T J_{2k} + J_{2k}V_1$ spans all the $2k \times 2k$ skew-symmetric matrices and $J_{2k}V_2$ spans all $2k \times 2(m-k)$ matrices. Therefore,

$$T_\omega R_k = a(\mathfrak{gl}(2m))_\omega = \left\{ \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \,\middle|\, A^T = -A \right\}.$$

And so,

$$V_k \cong \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \,\middle|\, C^T = -C \right\} \cong \Lambda^2(\mathbb{R}^{d_k})^*.$$

The isotropy action of $A \in \mathrm{Iso}(k)^c$ on $M \in T_\omega\Lambda^2(\mathbb{R}^{2m})^*$ is given by $A \cdot M = A^T MA$ and this yields the action on $\Lambda^2\mathbb{R}^{d_k}$ stated in the theorem:

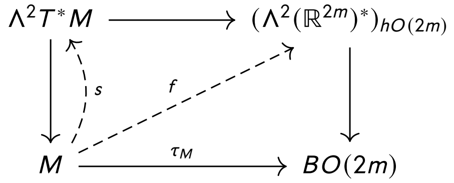
$$\begin{pmatrix} A_1^T & 0 \\ 0 & A_4^T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A_4^T C A_4 \end{pmatrix}.$$

□

Remark 3.9. Observe that A_4 is the only term that acts on the elements of V_k so V_k is reduced to an $O(d_k)$-representation.

3.3 Cohomology of Degeneracy Loci

The bundle $\Lambda^2 T^*M \to M$ is associated to TM so its structure group is $O(2m)$. By Theorem 2.13, sections of $\Lambda^2 T^*M$ are in one-to-one correspondence with lifts f of τ_M (the classifying map of TM) to the total space of the universal bundle with fibre $\Lambda^2(\mathbb{R}^{2m})^*$.



(3.7)

There is an induced stratification of $(\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)}$, given by $\{(\overline{R_k})_{hO(2m)}\}_{k=0,\dots,m}$. Moreover, by definition of f, the image of s is contained in $\Lambda^2 T^*M \setminus \overline{R_k}$ if and only if the image of f is contained in $(\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)} \setminus (\overline{R_k})_{hO(2m)} = (\Lambda^2(\mathbb{R}^{2m})^* \setminus \overline{R_k})_{hO(2m)}$.

This section is devoted to defining cohomological obstructions to lifting f to $(\Lambda^2(\mathbb{R}^{2m})^* \setminus \overline{R_k})_{hO(2m)}$ and finding a method to compute them. Starting with some map f as in (3.7), the existence of a map g homotopic to f that avoids $(\overline{R_k})_{hO(2m)}$ is expressed in the commutativity (up to homotopy) of the following diagram:

$$\begin{array}{ccc} & & (\Lambda^2(\mathbb{R}^{2m})^* \setminus \overline{R_k})_{hO(2m)} \\ & \overset{g}{\nearrow} & \big\downarrow \iota \\ & & (\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)} \\ & \overset{f}{\nearrow} & \big\downarrow \\ M & \longrightarrow & BO(2m) \end{array} \tag{3.8}$$

If g exists, then one has in cohomology $f^* = g^* \circ \iota^*$ so the kernel of ι^* is contained in the kernel of f^*. Thus, for g to exist, f^* needs to satisfy the equations

$$f^*(x) = 0 \qquad \forall x \in ker(\iota^*). \tag{3.9}$$

The generators of $ker(\iota^*)$ can therefore be regarded as cohomology classes which obstruct the existence of a lifting of f. These will be referred to as the **obstruction classes**.

Up to degree $codim(R_k)$ in cohomology, there are no obstructions to the existence of such a lifting g.

Proposition 3.10. ι is a $(codim(R_k) - 1)$-equivalence. In particular, for degrees $< codim(R_k)$, ι^* is injective.

Proof. Given $n \leq k$, consider the following square:

$$\begin{array}{ccc} (NR_n \setminus R_n)_{hO(2m)} & \overset{i_n}{\hookrightarrow} & (\Lambda^2(\mathbb{R}^{2m})^* \setminus \overline{R_n})_{hO(2m)} \\ \pi_n \big\downarrow & & \big\downarrow j_n \\ (R_n)_{hO(2m)} & \overset{l_n}{\hookrightarrow} & (\Lambda^2(\mathbb{R}^{2m})^* \setminus \overline{R_{n-1}})_{hO(2m)} \end{array} \tag{3.10}$$

All the spaces are well defined as the fibres are invariant by the $O(2m)$-action (the tubular neighborhood $NR_n \setminus R_n$ is invariant because the chosen metric g is invariant so the exponential map is equivariant). Using the arguments of Proposition 3.5 and Example 2.81, one can see that (3.10) is a homotopy pushout.

$(NR_n \backslash R_n)_{hO(2m)} \xrightarrow{\pi_n} (R_n)_{hO(2m)}$ is a bundle with fibre homotopy equivalent to $S^{codim(R_n)-1}$, so by the exact sequence of this bundle, π_n is a $(codim(R_n) - 1)$-equivalence. Then, Theorem 2.79 implies that j_n is also a $(codim(R_n) - 1)$-equivalence.

Now, note that $d_k = 2(m-k) \leq d_n$ for $n \leq k$, so $codim(R_k) = \frac{1}{2}d_k(d_k - 1) \leq codim(R_n)$ for $n \leq k$. Note also that $\iota = j_0 \circ j_1 \circ \cdots \circ j_k$. It follows that $j_0, ..., j_k$ are all $(codim(R_k) - 1)$-equivalences and hence, so is ι. Thus, in cohomology, ι^* is an isomorphism up to degree $codim(R_k) - 2$ and injective in degree $codim(R_k) - 1$. □

Remark 3.11. Proposition 3.10 could also be proved using transversality. Indeed, let $n < codim(R_k)$ and take some map $f : S^n \to \Lambda^2(\mathbb{R}^{2m})^*$. Theorem 3.2.5 in [Hir76] implies that there is a smooth map g homotopic to f which is transversal to all submanifodls $R_0, ..., R_k$. By definition of transversality, it follows that the image of g does not intersect any of these sets, therefore it is contained in $\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k}$. Thus, the inclusion $\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k} \hookrightarrow \Lambda^2(\mathbb{R}^{2m})^*$ is a $(codim(R_k) - 1)$-equivalence. Using the exact sequences of the bundles $\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k} \hookrightarrow (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k})_{hO(2m)} \to BO(2m)$ and $\Lambda^2(\mathbb{R}^{2m})^* \hookrightarrow (\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)} \to BO(2m)$ and the 5-lemma, one concludes that ι is also a $(codim(R_k) - 1)$-equivalence.

In degree $codim(R_k)$, however, obstructions appear and if the Euler classes [2] $e_n \in H^{codim(R_n)}((R_n)_{hO(2m)})$ of the normal bundles $(NR_n)_{hO(2m)} \to (R_n)_{hO(2m)}$ are not zero-divisors for $n \geq k$, then the obstructions can be computed using the following maps ψ_n.

$$\begin{array}{ccc}
(R_n)_{hO(2m)} & \xhookrightarrow{I_n} & (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{n-1}})_{hO(2m)} \\
& \searrow^{\psi_n} & \Big\downarrow \\
& & (\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)}
\end{array}$$

Theorem 3.12. *Suppose that for every $n \geq k$, $e_n \in H^{codim(R_n)}((R_n)_{hO(2m)})$ is not a zero-divisor. Then, in $H^{codim(R_k)}((\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)})$,*

$$ker(\iota^*) = \bigcap_{n=k+1}^{m} ker(\psi_n^*)$$

Moreover, in degree $codim(R_k)$, $dim_{\mathbb{Z}_2}(ker(\iota^*)) = 1$, and therefore $ker(\iota^*)$ is generated by a single non-zero class which will be denoted by $\mathfrak{o}_k$.

Proof. Given $n \geq k$, consider the homotopy pushout:

$$\begin{array}{ccc}
(NR_n \backslash R_n)_{hO(2m)} & \xhookrightarrow{i_n} & (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_n})_{hO(2m)} \\
\Big\downarrow \pi_n & & \Big\downarrow j_n \\
(R_n)_{hO(2m)} & \xhookrightarrow{I_n} & (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{n-1}})_{hO(2m)}
\end{array}$$

[2] Since the cohomology coefficients are $\mathbb{Z}_2$, by Euler class one means the top Stiefel Whitney class. It is however easier just saying "Euler class" and, due to Proposition 2.52, this terminology should cause no confusion.

Passing to cohomology, Proposition 2.82 then gives the Mayer-Vietoris sequence

$$\cdots \to H^*((\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_{n-1}})_{hO(2m)}) \xrightarrow{(I_n^*, j_n^*)} H^*((R_n)_{hO(2m)}) \oplus H^*((\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_n})_{hO(2m)}) \to$$
$$\xrightarrow{\pi_n^* - i_n^*} H^*((NR_n\backslash R_n)_{hO(2m)}) \to \cdots \tag{3.11}$$

Consider also the Gysin Sequence

$$\cdots \to H^{*-codim(R_n)}((R_n)_{hO(2m)}) \xrightarrow{\cup e_n} H^*((R_n)_{hO(2m)}) \xrightarrow{\pi_n^*} H^*((NR_n\backslash R_n)_{hO(2m)}) \to$$
$$\to H^{*-codim(R_n)+1}((R_n)_{hO(2m)}) \to \cdots \tag{3.12}$$

where $\cup e_n$ denotes the map given by cup product with e_n. Since e_n is not a zero divisor, the map

$$\cup e_n : H^{*-codim(R_n)}((R_n)_{hO(2m)}) \to H^*((R_n)_{hO(2m)})$$

is injective for $* \geq codim(R_n)$. Then, exactness of (3.12) implies that π_n^* is surjective for $* \geq codim(R_n)$. This turns the Mayer-Vietoris sequence (3.11) into a short exact sequence for each degree $* \geq codim(R_n)$:

$$0 \to H^*((\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_{n-1}})_{hO(2m)}) \xrightarrow{(I_n^*, j_n^*)} H^*((R_n)_{hO(2m)}) \oplus H^*((\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_n})_{hO(2m)}) \to$$
$$\xrightarrow{\pi_n^* - i_n^*} H^*((NR_n\backslash R_n)_{hO(2m)}) \to 0 \tag{3.13}$$

In particular, the pair (I_n^*, j_n^*) is injective for $* \geq codim(R_n)$. As $codim(R_k) \geq codim(R_n)$ for all $n \geq k$, the pair (I_n^*, j_n^*) is injective in degree $codim(R_k)$.

Starting with $n = k+1$, it follows that, in degree $codim(R_k)$, one has $ker(\iota^*) = ker(I_{k+1} \circ \iota^*) \cap ker(j_{k+1} \circ \iota^*)$. Note that $\iota \circ I_{k+1} = \psi_{k+1}$, so $ker(\iota^*) = ker(\psi_{k+1}^*) \cap ker(j_{k+1}^* \circ \iota^*)$.

$$\begin{array}{ccc}
 & & (\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_{k+2}})_{hO(2m)} \\
 & & \downarrow{\scriptstyle j_{k+2}} \\
(R_{k+2})_{hO(2m)} & \xrightarrow{I_{k+2}} & (\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_{k+1}})_{hO(2m)} \\
 & & \downarrow{\scriptstyle j_{k+1}} \\
(R_{k+1})_{hO(2m)} & \xrightarrow{I_{k+1}} & (\Lambda^2(\mathbb{R}^{2m})^*\backslash\overline{R_k})_{hO(2m)} \\
 & {\scriptstyle \psi_{k+1}} \searrow & \downarrow{\scriptstyle \iota} \\
 & & (\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)}
\end{array}$$

The result now follows from applying the same reasoning to $ker(j^*_{k+1} \circ \iota^*)$ and then to the maps that follow. For instance, in the next step, one has $ker(j^*_{k+1} \circ \iota^*) = ker(l^*_{k+2} \circ j^*_{k+1} \circ \iota^*) \cap ker(j^*_{k+2} \circ j^*_{k+1} \circ \iota^*)$ and $\iota \circ j_{k+1} \circ l_{k+2} = \psi_{k+2}$. Hence, $ker(\iota^*) = ker(\psi^*_{k+1}) \cap ker(\psi^*_{k+2}) \cap ker(j^*_{k+2} \circ j^*_{k+1} \circ \iota^*)$. In the last step, one has $ker(\iota^*) = \bigcap_{n=k+1}^{m-1} ker(\psi^*_n) \cap ker(j^*_{m-1} \circ \cdots \circ j^*_{k+1} \circ \iota^*)$, but $(\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{m-1}})_{hO(2m)} = (R_m)_{hO(2m)}$ and $j^*_{m-1} \circ \cdots \circ j^*_{k+1} \circ \iota^* = \psi^*_m$.

To prove that $dim_{\mathbb{Z}_2}(ker(\iota^*)) = 1$ (in cohomology degree $codim(R_k)$), we may apply Lemma 3.13 below to short exact sequence (3.13) with $n = k$ to show that $dim_{\mathbb{Z}_2}(ker(j^*_k)) = dim_{\mathbb{Z}_2}(ker(\pi^*_k))$.

From Gysin sequence (3.12) also with $n = k$, it follows that $ker(\pi^*_k) = Im(\cup e_k) = \langle e_k \rangle$ in cohomology of dimension $codim(R_k)$, hence $dim_{\mathbb{Z}_2}(ker(j^*_k)) = 1$. To conclude, note that $\iota = j_0 \circ j_1 \circ \cdots \circ j_k$ and j^*_n are isomorphisms in degree $codim(R_k)$ for $n < k$ (this was observed in the last paragraph of the proof of Proposition 3.10). Therefore, $dim_{\mathbb{Z}_2}(ker(\iota^*)) = dim_{\mathbb{Z}_2}(ker(j^*_k)) = 1$ □

Lemma 3.13. If $0 \longrightarrow A \xrightarrow{(l,j)} B \oplus C \xrightarrow{\pi - i} D \longrightarrow 0$ is a short exact sequence of vector spaces over a field K, then $dim_K(ker(j)) = dim_K(ker(\pi))$.

Proof. A short exact sequence as the one in the statement yields a pullback of the form

$$\begin{array}{ccc} A & \xrightarrow{l} & B \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{i} & D \end{array} \tag{3.14}$$

By the universal property of the pullback there is a map $\alpha : ker(\pi) \longrightarrow A$ such that $l \circ \alpha$ is the inclusion $ker(\pi) \hookrightarrow B$ and $j \circ \alpha = 0$.

$$\begin{array}{ccccc} ker(\pi) & & \overset{\hookrightarrow}{\longrightarrow} & & B \\ & \overset{\exists\alpha}{\searrow} & & & \\ & & A & \xrightarrow{l} & B \\ {\scriptstyle 0}\searrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi} \\ & & C & \xrightarrow{i} & D \end{array}$$

Using again the universal property of the pullback (3.14), one can check that the left square of the next diagram is also a pullback.

$$\begin{array}{ccccc} ker(\pi) & \xrightarrow{\alpha} & A & \xrightarrow{l} & B \\ \downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi} \\ 0 & \longrightarrow & C & \xrightarrow{i} & D \end{array}$$

The universal property of the pullback for the left square is the universal property for the kernel of j. Therefore, $ker(\pi) \cong ker(j)$. □

Theorem 3.12 transforms the problem of computing $ker(\iota^*)$ into one of solving the equations $\psi_n^*(x) = 0$ for all $n > k$. In [FR04], the authors also use the equations $\psi_n^*(x) = 0$ to compute the obstruction classes (the generators of $ker(\iota^*)$) but in a more general context. The authors refer to these equations as the **restriction equations** and refer to the obstruction classes as **Thom polynomials**. In the context considered in this chapter, the restriction equations are not very hard to solve and the solution is given in Theorem 3.17. However, before solving the equations, one must first check that we are indeed in the conditions of Theorem 3.12, meaning that the Euler classes are not zero-divisors. That is the goal of next section.

3.4 The Euler classes

The Euler class of $(NR_k)_{hO(2m)} \to (R_k)_{hO(2m)}$ can be easily computed with an appropriate description of the normal bundle:

Lemma 3.14. There is a bundle morphism

$$\begin{array}{ccc} EIso(k)^c \times_{Iso(k)^c} V_k & \longrightarrow & (NR_k)_{hO(2m)} \\ \downarrow & & \downarrow \\ BIso(k)^c & \xrightarrow{\simeq} & (R_k)_{hO(2m)} \end{array}$$

Proof. By Proposition 3.7, $R_k^c \simeq R_k$. Using the 5-lemma with the long exact sequences of the bundles $R_k^c \hookrightarrow (R_k^c)_{hO(2m)} \to BO(2m)$ and $R_k \hookrightarrow (R_k)_{hO(2m)} \to BO(2m)$, one can show that $(R_k^c)_{hO(2m)} \simeq (R_k)_{hO(2m)}$. Therefore, one has the following bundle morphism:

$$\begin{array}{ccc} \left(NR_k|_{(R_k^c)}\right)_{hO(2m)} & \longrightarrow & (NR_k)_{hO(2m)} \\ \downarrow & \lrcorner & \downarrow \\ (R_k^c)_{hO(2m)} & \overset{\simeq}{\hookrightarrow} & (R_k)_{hO(2m)} \end{array}$$

On the other hand,

$$\left(NR_k|_{(R_k^c)}\right)_{hO(2m)} \simeq EO(2m) \times_{O(2m)} \left(O(2m) \times_{Iso(k)^c} V_k\right) \simeq EO(2m) \times_{Iso(k)^c} V_k \simeq$$
$$\simeq EIso(k)^c \times_{Iso(k)^c} V_k$$

The first equivalence follows from Proposition 3.7, the second from point 4 of Proposition 2.20 and the third from Proposition 2.35. Also from Proposition 2.35 and point 5 of Proposition 2.20 it follows that

$$(R_k^c)_{hO(2m)} \simeq EO(2m) \times_{O(2m)} O(2m)/Iso(k)^c \simeq BIso(k)^c$$

□

Recall that $d_k = 2(m-k)$. By Proposition 2.38 and Remark 3.6, $BIso(k)^c \simeq BU(k) \times BO(d_k)$ so, from Theorems 2.44 and 2.61, it follows that

$$H^*((R_k)_{hO(2m)}) \cong H^*(BIso(k)^c) \cong \mathbb{Z}_2[c_1, ..., c_k, w_1, ..., w_{d_k}]. \tag{3.15}$$

Using this identification, one can write a formula for e_k.

Theorem 3.15. *The Euler class of $(NR_k)_{hO(2m)} \to (R_k)_{hO(2m)}$ is the Schur polynomial of the partition $\delta = (d_k - 1, d_k - 2, ..., 1, 0)$ in the Stiefel-Whitney roots $t_1, ..., t_{d_k}$ or, equivalently,*

$$e_k = det \begin{pmatrix} w_{d_k-1} & w_{d_k} & \cdots & w_{2d_k-2} \\ w_{d_k-3} & w_{d_k-2} & \cdots & w_{2d_k-4} \\ \vdots & \vdots & \ddots & \vdots \\ w_{-d_k+1} & w_{-d_k+2} & \cdots & 1 \end{pmatrix}.$$

Proof. By the pullback formula for the Euler class and Lemma 3.14, the goal is to compute the Euler class of $EIso(k)^c \times_{Iso(k)^c} V_k \to BIso(k)^c$. Take π to be the projection

$$BIso(k)^c \cong BU(k) \times BO(d_k) \to BO(d_k)$$

and consider $\Lambda^2(\mathbb{R}^{d_k})^*$ endowed with the action of $O(d_k)$ given by

$$A \cdot \omega = A^* \sigma \qquad \forall A \in O(d_k), \sigma \in \Lambda^2(\mathbb{R}^{d_k})^*.$$

Remark 3.9 implies the existence of the next bundle map

$$\begin{array}{ccc} EIso(k)^c \times_{Iso(k)^c} V_k & \longrightarrow & EO(d_k) \times_{O(d_k)} \Lambda^2(\mathbb{R}^{d_k})^* \\ \downarrow & & \downarrow \\ BIso(k)^c & \xrightarrow{\pi} & BO(d_k) \end{array}$$

So one may compute the Euler class of $EO(d_k) \times_{O(d_k)} \Lambda^2(\mathbb{R}^{d_k})^* \to BO(d_k)$ and pull it back with π^*. Now consider the inclusion of the diagonal matrices $(\mathbb{Z}_2)^{d_k} \overset{j}{\hookrightarrow} O(d_k)$. The restriction of the action on $\Lambda^2(\mathbb{R}^{d_k})^*$ to this subgroup yields another square of bundles:

$$\begin{array}{ccc} (E\mathbb{Z}_2)^{d_k} \times_{(\mathbb{Z}_2)^{d_k}} \Lambda^2(\mathbb{R}^{d_k})^* & \longrightarrow & EO(d_k) \times_{O(d_k)} \Lambda^2(\mathbb{R}^{d_k})^* \\ \downarrow & & \downarrow \\ (B\mathbb{Z}_2)^{d_k} & \xhookrightarrow{j} & BO(d_k) \end{array}$$

Observe that, by Proposition 2.43, j^* is injective on cohomology and, with the identifications $H^*(BO(d_k)) = \mathbb{Z}_2[w_1, ..., w_{d_k}]$ and $H^*((B\mathbb{Z}_2)^{d_k}) = \mathbb{Z}_2[t_1, ..., t_{d_k}]$, j^* sends w_i to the i-th elementary symmetric polynomial in the variables $t_1, ..., t_{d_k}$. It is therefore sufficient to compute the Euler class of $(E\mathbb{Z}_2)^{d_k} \times_{(\mathbb{Z}_2)^{d_k}} \Lambda^2(\mathbb{R}^{d_k})^* \to$

$(B\mathbb{Z}_2)^{d_k}$ and write it in terms of the elementary symmetric polynomials.

$\Lambda^2(\mathbb{R}^{d_k})^*$ has a basis given by $\{v_{ij}\}_{i<j}$ where v_{ij} is the skew-symmetric $d_k \times d_k$ matrix with zeros everywhere except in positions (i, j) and (j, i) where it has a 1 and a -1, respectively. Given an element $A \in (\mathbb{Z}_2)^{d_k}$, the action of A on v_{ij} is given by

$$A \cdot v_{ij} = A v_{ij} A^T.$$

Hence, depending on A, the action yields either v_{ij} or $-v_{ij}$. Given $s \in \{1, ..., d_k\}$, the generator of the s-th factor of $(\mathbb{Z}_2)^{d_k}$ is the diagonal matrix $A_s = diag(1, ..., -1, ..., 1)$ with 1's along the diagonal, except in the s-th position, where it has a -1. One has, in fact,

$$A_s \cdot v_{ij} = \begin{cases} -v_{ij}, & \text{if } s = i \text{ or } j \\ v_{ij}, & \text{otherwise} \end{cases}$$

Therefore, for each (i, j) with $i < j$, the subspace spanned by v_{ij} is a one dimensional subrepresentation that is acted on by $\mathbb{Z}_2 \times \mathbb{Z}_2$, the i-th and j-th factors of $(\mathbb{Z}_2)^{d_k}$. Moreover, the map

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{+} \mathbb{Z}_2$$
$$(a, b) \mapsto a + b$$

reduces this action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $Span(v_{ij})$ to the (only) non-trivial action of $\mathbb{Z}_2$ on $Span(v_{ij}) \cong \mathbb{R}$. Considering $\mathbb{R}$ endowed with the non-trivial action of $\mathbb{Z}_2$, the $+$ map is covered by a map of bundles:

$$\begin{array}{ccc} (E\mathbb{Z}_2)^2 \times_{\mathbb{Z}_2 \times \mathbb{Z}_2} Span(v_{ij}) & \longrightarrow & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathbb{R} \\ \downarrow & & \downarrow \\ (B\mathbb{Z}_2)^2 & \xrightarrow{+} & B\mathbb{Z}_2 \end{array}$$

Furthermore, it can easily be checked that $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} \mathbb{R} \cong \gamma^1$. Therefore, with $H^*(B\mathbb{Z}_2) = \mathbb{Z}_2[t]$, the Euler class of the bundle on the left is $+^*(t)$.

Since $\mathbb{Z}_2$ is discrete, Proposition 2.39 implies that $\pi_1(B\mathbb{Z}_2) = \mathbb{Z}_2$ and the map $\pi_1((B\mathbb{Z}_2)^2) \to \pi_1(B\mathbb{Z}_2)$ induced by $+$ is just $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{+} \mathbb{Z}_2$. Since $\pi_1(B\mathbb{Z}_2) = H_1(B\mathbb{Z}_2)$, $+^*$ is the dual of $+$ and it follows that $+^*(t) = t_1 + t_2$, where $H^*((B\mathbb{Z}_2)^2) = \mathbb{Z}_2[t_1, t_2]$.

Therefore, the Euler class of the v_{ij} summand is $t_i + t_j$. Since $V_k = \bigoplus_{i<j} Span(v_{ij})$, the bundle $(E\mathbb{Z}_2)^{d_k} \times_{(\mathbb{Z}_2)^{d_k}} V_k$ decomposes as a Whitney sum of line bundles $(E\mathbb{Z}_2)^{d_k} \times_{(\mathbb{Z}_2)^{d_k}} Span(v_{ij})$ for $i < j$ and its Euler class is thus the product of the Euler classes of the summands:

$$e_k = \prod_{\substack{i,j=1 \\ i<j}}^{d_k} (t_i + t_j)$$

It follows from Proposition 2.75 that such a product is equal to $s_\delta(t_1, ..., t_{d_k})$ and so the result follows from substituting the i-th elementary symmetric polynomial with w_i in the determinantal formula (2.2). □

Since the cohomology ring in (3.15) is a polynomial ring and the Euler class e_k is clearly non-zero, it follows that e_k is not a zero divisor and therefore the hypothesis of Theorem 3.12 is satisfied.

3.5 Computing the Obstructions

Recall that, by Theorem 3.12, to compute the obstruction class $\mathfrak{o}_k$, one must solve the restriction equations $\psi_n^*(x) = 0$ for all $n > k$. To do so, we first need the following lemma:

Lemma 3.16. Recall that $d_k = 2(m-k)$. One has isomorphisms

$$H^*((R_k)_{hO(2m)}) \cong \mathbb{Z}_2[c_1, ..., c_k, w_1, ..., w_{d_k}]$$
$$H^*((\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)}) \cong \mathbb{Z}_2[w_1, ..., w_{2m}].$$

Under these identifications, the maps ψ_k^* are given by

$$\begin{aligned} \psi_k^* : \mathbb{Z}_2[w_1, ..., w_{2m}] &\to \mathbb{Z}_2[c_1, ..., c_k, w_1, ..., w_{d_k}] \\ w &\mapsto cw \end{aligned} \tag{3.16}$$

where c and w denote the total characteristic classes.

Proof. The first isomorphism is the one in (3.15). The second one comes from the fact that the fibres of the bundle $(\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)} \to BO(2m)$ are vector spaces, hence contractible, so the total space is homotopy equivalent to $BO(2m)$ by the long exact sequence of the bundle. Thus, one has

$$H^*((\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)}) \cong H^*(BO(2m)) \cong \mathbb{Z}_2[w_1, ..., w_{2m}],$$

the second isomorphism coming from Theorem 2.44.

To prove the last claim, observe that the following square commutes, by Corollary 2.37.

$$\begin{array}{ccc} (R_k)_{hO(2m)} & \xhookrightarrow{\psi_k} & (\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)} \\ \downarrow \simeq & & \downarrow \simeq \\ BIso(k)^c & \hookrightarrow & BO(2m) \end{array}$$

Denote also by $\psi_k : BIso(k)^c \hookrightarrow BO(2m)$ the bottom map and note that ψ_k factors through the composition of inclusions:

$$\begin{array}{ccc} BIso(k)^c \cong BU(k) \times BO(d_k) & \xrightarrow{\quad \psi_k \quad} & BO(2m) \\ & {\scriptstyle \alpha_k} \searrow \quad \nearrow {\scriptstyle \beta_k} & \\ & BO(2k) \times BO(d_k) & \end{array} \tag{3.17}$$

Writing $H^*(BO(2k) \times BO(d_k)) = \mathbb{Z}_2[v_1, ..., v_{2k}, w_1, ..., w_{d_k}]$, $v = 1 + v_1 + v_2 + ...$ and $w = 1 + w_1 + w_2 + ...$, Proposition 2.67 implies that $\beta_k^*(w) = vw$ and Proposition 2.66 implies that $\alpha_k^*(v) = c$, so composing β_k^* and α_k^*, the result follows. □

Finally, we are ready to compute the obstructions.

Theorem 3.17. *For each $0 < k \leq m$, the kernel of ι^* in cohomology of degree $codim(R_k)$ is generated by*

$$\mathfrak{o}_k = s_\delta(t_1, ..., t_{d_k}) = det \begin{pmatrix} w_{d_k-1} & w_{d_k} & \cdots & w_{2d_k-2} \\ w_{d_k-3} & w_{d_k-2} & \cdots & w_{2d_k-4} \\ \vdots & \vdots & \ddots & \vdots \\ w_{-d_k+1} & w_{-d_k+2} & \cdots & 1 \end{pmatrix}$$

where $w_i = 0$ for $i > 2m$ or $i < 0$ and $d_k = 2(m-k)$.

This looks exactly like the formula for the Euler class. The only difference is that in the Euler class, the elements w_i for $d_k < i \leq 2m$ are zero, while those in the obstruction class are not.

Proof. By Theorem 3.12, one only needs to check that $\psi_n^*(\mathfrak{o}_k) = 0$ for all $k+1 \leq n \leq 2m$. For such an $n > k$, by (3.16), one has

$$\psi_n^*(\mathfrak{o}_k) = det \begin{pmatrix} (cw)_{d_k-1} & (cw)_{d_k} & \cdots & (cw)_{2d_k-2} \\ (cw)_{d_k-3} & (cw)_{d_k-2} & \cdots & (cw)_{2d_k-4} \\ \vdots & \vdots & \ddots & \vdots \\ (cw)_{-d_k+1} & (cw)_{-d_k+2} & \cdots & 1 \end{pmatrix}. \tag{3.18}$$

Note that $\psi_n^*(\mathfrak{o}_k) \in H^{codim(R_k)}((R_n)_{hO(2m)})$, so for all instances of w_i in (3.18), one has $w_i = 0$ for $i > d_n = 2(m-n)$. Denote the matrix in (3.18) by M. The element of M in position (i, j) is of the form $(cw)_{d_k-2i+j}$. By the product formula,

$$(cw)_{d_k-2i+j} = \sum_{s=0}^{(d_k-2i+j)/2} c_s w_{d_k-2i+j-2s} = w_{d_k-2i+j} + c_1 w_{d_k-2i+j-2} + ...$$

Thus, M can be written as a product of matrices:

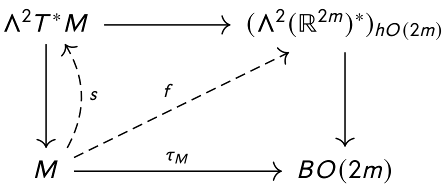
$$M = \begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_{d_k} \\ 0 & 1 & c_1 & \cdots & c_{d_k-1} \\ 0 & 0 & 1 & \cdots & c_{d_k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} w_{d_k-1} & w_{d_k} & \cdots & w_{2d_k-2} \\ w_{d_k-3} & w_{d_k-2} & \cdots & w_{2d_k-4} \\ \vdots & \vdots & \ddots & \vdots \\ w_{-d_k+1} & w_{-d_k+2} & \cdots & 1 \end{pmatrix} \tag{3.19}$$

The first matrix of the product (3.19) has determinant equal to 1 so

$$det(M) = det \begin{pmatrix} w_{d_k-1} & w_{d_k} & \cdots & w_{2d_k-2} \\ w_{d_k-3} & w_{d_k-2} & \cdots & w_{2d_k-4} \\ \vdots & \vdots & \ddots & \vdots \\ w_{-d_k+1} & w_{-d_k+2} & \cdots & 1 \end{pmatrix} \tag{3.20}$$

But $w_i = 0$ for $i > d_n$ and $n > k \implies d_n < d_k$, therefore $w_i = 0$ for $i \geq d_k - 1$. Hence, the first row of the matrix in (3.20) is composed of only zeroes and so has zero determinant. It then follows that $\psi_n^*(\mathfrak{o}_k) = det(M) = 0$. □

Recall diagram (3.7):



Here s was any section, f was the map to the universal bundle induced by s and τ_M was the classifying map of TM. We saw in (3.9) that $f^*(\mathfrak{o}_k) = 0$ was a necessary condition for the existence of a lift of f to $(\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k})_{hO(2m)}$, i.e, a map homotopic to f that avoids $\overline{R_k}$. Under the equivalence $(\Lambda^2(\mathbb{R}^{2m})^*)_{hO(2m)} \simeq BO(2m)$, condition $f^*(\mathfrak{o}_k) = 0$ is given by $\tau_M^*(\mathfrak{o}_k) = 0$. But, by the properties of the Stiefel-Whitney classes, $\tau_M^*(w_i) = w_i(M)$ where the latter is the Stiefel-Whitney class of TM. Therefore, $\tau_M^*(\mathfrak{o}_k) = 0$ translates into

$$det \begin{pmatrix} w_{d_k-1}(M) & w_{d_k}(M) & \cdots & w_{2d_k-2}(M) \\ w_{d_k-3}(M) & w_{d_k-2}(M) & \cdots & w_{2d_k-4}(M) \\ \vdots & \vdots & \ddots & \vdots \\ w_{-d_k+1}(M) & w_{-d_k+2}(M) & \cdots & 1 \end{pmatrix} = 0 \tag{3.21}$$

A trivial first observation about condition (3.21) is that, as one would expect, if the manifold admits a non-degenerate 2-form - or equivalently, an almost complex structure - then the classes are automatically zero. This can be seen using the fact that, on such manifolds, $w_i = 0$ for odd i[3]. Odd columns of the matrix in (3.21) have only odd classes so certainly there will be columns of zeroes and so the determinant will be zero.

Consider the bundle $\Lambda^2 T^*M \to M$ and let $R_k = \{\omega_x \in \Omega^2(T_xM) \mid x \in M, rank(\omega_x) = 2k\}$. Hopefully, this abuse of notation will not cause confusion. Equation (3.21) gives an obstruction to the existence of a section avoiding $\overline{R_k}$ and it is valid for any manifold M. But when M is compact, the classes $\tau_M^* \mathfrak{o}_k$ gain another interpretation. They are actually the Poincaré duals of the degeneracy loci $\overline{R_k}$. Let us now show that.

Lemma 3.18. R_k is an embedded submanifold and the family $\{\overline{R_k}\}$ is a locally trivial stratification. Moreover, given a section $s : M \to \Lambda^2 T^*M$ transversal to the sets R_k, the spaces $(R_k)_M := s^{-1}(R_k) \subset M$ are also embedded submanifolds and $\{(\overline{R_k})_M\}$ is also a locally trivial stratification.

Proof. Local coordinates for R_k come from a trivializing cover of $\Lambda^2 T^*M$ and local coordinates for the fibre $R_k \subset \Lambda^2(\mathbb{R}^{2m})^*$. By Theorem 3.3, the fibre $R_k \subset \Lambda^2(\mathbb{R}^{2m})^*$ is an embedded submanifold of $\Lambda^2(\mathbb{R}^{2m})^*$, so $R_k \subset \Lambda^2 T^*M$ is an embedded submanifold of $\Lambda^2 T^*M$. $\{\overline{R_k}\}$ is a locally trivial stratification by Proposition 2.86. By Theorem 1.3.3 of [Hir76], $(R_k)_M$ is an embedded submanifold of M and by Proposition 2.87, $\{(\overline{R_k})_M\}$ forms a locally trivial stratification of M. ◻

By Theorem 3.2.5 of [Hir76], generically, a section s is transversal to the spaces R_k.

Theorem 3.19. *The sets $\overline{(R_k)}_M$ give rise to homology classes $[\overline{(R_k)}_M] \in H_{2m-codim(R_k)}(M)$. Moreover, letting $D : H_*(M) \to H^{2m-*}(M)$ denote the Poincaré duality map, one has*

$$D([\overline{(R_k)}_M]) = \tau_M^* \mathfrak{o}_k \tag{3.22}$$

Proof. By Lemma 3.18, Theorem 2.88 and the fact that $codim(R_{k-1}) - codim(R_k) \geq 2$, it follows that $\overline{(R_k)}_M$ give rise to homology classes. To prove (3.22), we will show that both $D([\overline{(R_k)}_M])$ and $\tau_M^* \mathfrak{o}_k$ are the restriction of the Thom class of $N(R_k)_M \to M$ to $H^{codim(R_k)}(M)$.

To avoid cluttering the proof, let us reduce the notation $X_{hO(2m)}$ to just X_h and write $*$ for the degree $codim(R_k)$ in cohomology. Let us denote by $u^{N_h} \in H^*((NR_k)_h, (NR_k \backslash R_k)_h)$ the Thom class of $(NR_k \backslash R_k)_h \to (R_k)_h$ and denote by $T_k \in H^*(BO(2m))$ the image of u^{N_h} by the composition

$$H^*((NR_k)_h, (NR_k \backslash R_k)_h) \cong H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1})_h, (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_k)_h) \to$$
$$\to H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1})_h) \cong H^*((\Lambda^2(\mathbb{R}^{2m})^*_h)) \cong H^*(BO(2m)).$$

We begin by showing that $\tau_M^* \mathfrak{o}_k$ is the restriction of the Thom class of $N(R_k)_M \to M$ to $H^*(M)$. First note that $\mathfrak{o}_k \in H^*(BO(2m))$ is equal to T_k. Indeed, recall that $\mathfrak{o}_k$ is the generator of $ker(\iota^*) \cap H^{codim(R_k)}(BO(2m))$,

[3] This is a consequence of the fact that an almost complex structure gives TM a structure of complex vector bundle. Then, point 2 of Proposition 2.59 implies that $w_i = 0$ for odd i.

which is one-dimensional by Theorem 3.12. Recall also the inclusions $j_k : (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_k)_h \hookrightarrow (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1})_h$. One has $j_k^*(T_k) = 0$ by exactness of the following sequence

$$H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1})_h, (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_k)_h) \to H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1})_h) \xrightarrow{j_k^*} H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_k)_h), \qquad (3.23)$$
$$u^{N_h} \mapsto T_k$$

Since $\iota = j_0 \circ \cdots \circ j_k$, one has $\iota^*(T_k) = 0$. This means that either $T_k = 0$ or $T_k = \mathfrak{o}_k$. By exactness of (3.23) and the fact that u^{N_h} generates $H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_{k-1})_h, (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R}_k)_h)$, it suffices to show that j_k^* is not injective to see that $T_k \neq 0$. Indeed, if j_k^* were injective then ι^* would too be injective because $\iota = j_0 \circ \cdots \circ j_k$ and the maps j_n^* for $n < k$ are injective in degree $codim(R_k)$. However, ι^* is not injective.

Take $h : \Lambda^2 T^* M \to (\Lambda^2(\mathbb{R}^{2m})^*)_h$ a bundle map over τ_M and denote by u^N the Thom class of $NR_k \to R_k$. The restriction of h to the tubular neighborhood NR_k yields a bundle map between NR_k and $(NR_k)_h$. Functoriality of the Thom class implies that $h^*(u^{N_h}) = u^N$. This, together with the commutativity of (3.24) implies that $h^*\mathfrak{o}_k$ is the restriction of u^N to $H^*(\Lambda^2 T^* M \backslash \overline{R_{k-1}})$.

$$\begin{array}{ccccc} u^{N_h} & H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{k-1}})_h, (\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_k})_h) & \to & H^*((\Lambda^2(\mathbb{R}^{2m})^* \backslash \overline{R_{k-1}})_h) & \mathfrak{o}_k \\ \downarrow & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} & \downarrow \\ u^N & H^*(\Lambda^2 T^* M \backslash \overline{R_{k-1}}, \Lambda^2 T^* M \backslash \overline{R_k}) & \longrightarrow & H^*(\Lambda^2 T^* M \backslash \overline{R_{k-1}}) & h^*\mathfrak{o}_k \end{array} \qquad (3.24)$$

Now, denoting by u^{N_M} the Thom class of $N(R_k)_M \to M$, one applies the same reasoning using the section s to map $h^*\mathfrak{o}_k$ to the restriction of u^{N_M} to $H^*(M \backslash \overline{R_{k-1}})$. Since s is transversal to every stratum, there is a bundle map

$$\begin{array}{ccc} N(R_k)_M & \xrightarrow{s|_{N(R_k)_M}} & NR_k \\ \downarrow & & \downarrow \\ (R_k)_M & \xrightarrow{s|_{(R_k)_M}} & R_k \end{array}$$

Hence $s^*(u^N) = u^{N_M}$. Then, a commuting diagram as (3.24) shows that $s^*h^*\mathfrak{o}_k$ is the restriction of u^{N_M} to $H^*(M \backslash \overline{R_{k-1}})$.

$$\begin{array}{ccccc} u^N & H^*(\Lambda^2 T^* M \backslash \overline{R_{k-1}}, \Lambda^2 T^* M \backslash \overline{R_k}) & \to & H^*(\Lambda^2 T^* M \backslash \overline{R_{k-1}}) & h^*\mathfrak{o}_k \\ \downarrow & \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle s^*} & \downarrow \\ u^{N_M} & H^*(M \backslash \overline{(R_{k-1})_M}, M \backslash \overline{(R_k)_M}) & \longrightarrow & H^*(M \backslash \overline{(R_{k-1})_M}) & s^*h^*\mathfrak{o}_k \end{array} \qquad (3.25)$$

Because $codim(R_{k-1}) > codim(R_k)+1$, the restriction map $H^*(M) \to H^*(M \backslash \overline{R_{k-1}})$ is an isomorphism. And, under the identification $H^*(M) \cong H^*(M \backslash \overline{R_{k-1}})$, $s^*h^*\mathfrak{o}_k$ translates to $\tau_M^*\mathfrak{o}_k$. Finally, use Theorem 2.89 with $K = (\overline{R}_k)_M$ and $L = (\overline{R}_{k-1})_M$ to conclude that $D([\overline{(R_k)_M}])$ is also the restriction of u^{N_M} to $H^*(M \backslash \overline{R_{k-1}}) \cong H^*(M)$. $\square$

3.6 An Example

Computing the determinant in (3.21) yields in general intricate equations relating the w_i's. However, for low cohomology degrees, the formulas turn out to be relatively simple. For instance, for $\mathfrak{o}_{m-1}$ the obstruction in degree 1, one has $\mathfrak{o}_{m-1} = w_1$. Note that $\tau_M^* \mathfrak{o}_1 = w_1(M)$ is zero if and only if M is orientable [4]. The next obstruction in higher degree is $\tau_M^* \mathfrak{o}_{m-2} \in H^6(M; \mathbb{Z}_2)$,

$$\mathfrak{o}_{m-2} = det \begin{pmatrix} w_3 & w_4 & w_5 & w_6 \\ w_1 & w_2 & w_3 & w_4 \\ 0 & 1 & w_1 & w_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = w_3^2 + w_1 w_5 + w_1 w_2 w_3 + w_1^2 w_4.$$

If the manifold M is orientable, $w_1(M) = 0$, thus there is no obstruction in degree 1 and $\tau_M^* \mathfrak{o}_{m-2} = w_3(M)^2$. The next proposition proves that $M = Gr_3^+(\mathbb{R}^7)$, the grassmannian of oriented 3-planes in $\mathbb{R}^7$, is an orientable 12-manifold with $w_3(M)^2 \neq 0$. Hence, every 2-form on this manifold cannot have rank greater than $2(m-2) = 8$ everywhere.

Proposition 3.20. $M = Gr_3^+(\mathbb{R}^7)$ is an orientable 12-manifold with $w_3(M)^2 \neq 0$.

Proof. The fact that $M = Gr_3^+(\mathbb{R}^7)$ is an orientable 12-manifold is an easy check. The proof of the identity $w_3(M)^2 \neq 0$ can be subdivided into two main steps:

1. $w_3(M)^2 \neq 0$ iff $w_3(Gr_3(\mathbb{R}^7))^2$ is not a multiple of $w_1(\gamma^3(\mathbb{R}^7))$:

   This follows from a version of the Gysin sequence for double coverings and $\mathbb{Z}_2$ coefficients, found in Corollary 12.3 of [MS74]. The sequence applied to the covering $Gr_3^+(\mathbb{R}^7) \xrightarrow{\pi} Gr_3(\mathbb{R}^7)$ takes the form

   $$\cdots \to H^{*-1}(Gr_3(\mathbb{R}^7)) \xrightarrow{\cup w_1(\gamma^3(\mathbb{R}^7))} H^*(Gr_3(\mathbb{R}^7)) \xrightarrow{\pi^*} H^*(M) \to H^*(Gr_3(\mathbb{R}^7)) \to \cdots$$

   As $ker(\pi^*) = Im(\cup w_1(\gamma^3(\mathbb{R}^7))) = \langle w_1(\gamma^3(\mathbb{R}^7)) \rangle$, given $x \in H^*(Gr_3(\mathbb{R}^7))$, $\pi^*(x) = 0$ iff x is a multiple of $w_1(\gamma^3(\mathbb{R}^7))$. Now, note that there is a bundle map

   $$\begin{array}{ccc} TGr_3^+(\mathbb{R}^7) & \longrightarrow & TGr_3(\mathbb{R}^7) \\ \downarrow & & \downarrow \\ Gr_3^+(\mathbb{R}^7) & \xrightarrow{\pi} & Gr_3(\mathbb{R}^7) \end{array}$$

   implying that $\pi^* w_3(Gr_3(\mathbb{R}^7)) = w_3(M)$.

2. $w_3(Gr_3(\mathbb{R}^7))^2$ is not a multiple of $w_1(\gamma^3(\mathbb{R}^7))$:

   Denote by w_i the i-th Stiefel-Whitney class of $\gamma^3 \to BO(3)$. To prove 2., we will show that

   $$w_3(Gr_3(\mathbb{R}^7)) = w_3(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))^3.$$

[4] A proof of this fact can be found in Theorem 12.1 of [Hus94].

This is sufficient because the inclusion $Gr_3(\mathbb{R}^7) \hookrightarrow BO(3)$ induces an isomorphism in cohomology of degree 6 and w_3 is not a multiple of w_1 and thus $w_3(\gamma^3(\mathbb{R}^7))$ is not a multiple of $w_1(\gamma^3(\mathbb{R}^7))$. Hence, $w_3(Gr_3(\mathbb{R}^7))$ is also not a multiple of $w_1(\gamma^3(\mathbb{R}^7))$.

In the end of page 4 and in page 5 of [Alb], the author shows that $w_1(Gr_3(\mathbb{R}^7) = w_1(\gamma^3(\mathbb{R}^7))$ and $w_2(Gr_3(\mathbb{R}^7)) = w_2(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))^2$. Moreover, by Theorem 5.12 of [MT91], one has

$$
\begin{aligned}
Sq^1(w_2) &= w_3 + w_1 w_2 \\
Sq^1(w_1) &= w_1^2
\end{aligned}
$$

where Sq^1 denotes the first Steenrod square[5] and $H^*(BO(3), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, w_3]$. By naturality of the Steenrod squares, it follows that

$$
\begin{aligned}
Sq^1(w_2(Gr_3(\mathbb{R}^7))) &= w_3(Gr_3(\mathbb{R}^7)) + w_1(Gr_3(\mathbb{R}^7))w_2(Gr_3(\mathbb{R}^7)), \\
Sq^1(w_2(\gamma^3(\mathbb{R}^7))) &= w_3(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))w_2(\gamma^3(\mathbb{R}^7)), \\
Sq^1(w_1(\gamma^3(\mathbb{R}^7))) &= w_1(\gamma^3(\mathbb{R}^7))^2.
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
w_3(Gr_3(\mathbb{R}^7)) &= Sq^1(w_2(Gr_3(\mathbb{R}^7))) - w_1(Gr_3(\mathbb{R}^7))w_2(Gr_3(\mathbb{R}^7)) \\
&= Sq^1(w_2(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))^2) - w_1(\gamma^3(\mathbb{R}^7))(w_2(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))^2) \\
&= Sq^1(w_2(\gamma^3(\mathbb{R}^7))) + Sq^1(w_1(\gamma^3(\mathbb{R}^7))^2) - w_1(\gamma^3(\mathbb{R}^7))w_2(\gamma^3(\mathbb{R}^7)) - w_1(\gamma^3(\mathbb{R}^7))^3 \\
&= w_3(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))w_2(\gamma^3(\mathbb{R}^7)) - w_1(\gamma^3(\mathbb{R}^7))w_2(\gamma^3(\mathbb{R}^7)) - w_1(\gamma^3(\mathbb{R}^7))^3 \\
&= w_3(\gamma^3(\mathbb{R}^7)) + w_1(\gamma^3(\mathbb{R}^7))^3. \qquad (\mathbb{Z}_2 \text{ coefficients})
\end{aligned}
$$

The first to last equality follows from the fact that $Sq^1(w_1(\gamma^3(\mathbb{R}^7))^2) = 0$.

□

[5] See chapter 4.L. of [Hat02] for a definition of Steenrod squares and basic properties.

Chapter 4

Thom Polynomials of Smooth Maps to an Almost Symplectic Manifold

4.1 Introduction

In the last chapter, we computed the Poincaré duals of homology classes $[(\overline{R_k})_M] \in H_*(M)$ given by degeneracy loci of 2-forms. To compute these Poincaré dual classes, we first defined and computed, for each $k \in \{0, ...m\}$, certain cohomological obstructions $\tau_M^* \mathfrak{o}_k$, whose non-triviality obstructed the existence of sections which have everywhere rank greater than $2k$. Then, we proved that each class $\tau_M^* \mathfrak{o}_k$ was in fact the Poincaré dual of $[(\overline{R_k})_M]$. Although the cohomological obstruction $\tau_M^* \mathfrak{o}_k$ turned out to be equal to the Poincaré dual of a degeneracy locus, the definition of $\tau_M^* \mathfrak{o}_k$ was independent of the existence of $[(\overline{R_k})_M]$. In this chapter, we will consider the following problem: let M be a $2m$-manifold, N a $2n$-manifold with $2m \leq 2n$ and $i : M \to N$ a smooth map. Endow N with an almost symplectic form ω (meaning a non-degenerate 2-form not necessarily closed). Take the bundle $Hom(TM, i^*TN) \to M$[1] and consider the following sets, for $l \in \{0, ..., 2m\}$ and $k \in \{0, ..., \lfloor l/2 \rfloor\}$:

$$S_{l,k} = \{\phi : T_x M \to T_{i(x)} N \mid rank(\phi) = l, rank(\phi^* \omega) = 2k\}$$

These sets may not give rise to homology classes, but one can nonetheless define and compute, analogously to chapter 3, cohomological obstructions to the existence of sections of $Hom(TM, i^*TN)$ that avoid the sets $S_{l,k}$. That is the goal of the present chapter. To do so, we will follow the same methods as in the previous chapter, starting by studying the typical fibre $Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ of the bundle in question.

[1] $Hom(TM, i^*TN) \to M$ is the pullback by $(id_M, i) : M \to M \times N$ of the bundle $Hom(TM, TN) \to M \times N$. This bundle in turn is the one associated to $TM \times TN$ with fibre $Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ and action of $GL(2n; \mathbb{R}) \times GL(2m; \mathbb{R})$ given by $(A, B) \cdot \phi = A\phi B^{-1}$ for all $(A, B) \in GL(2n; \mathbb{R}) \times GL(2m; \mathbb{R})$ and $\phi \in Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$.

4.2 The Homogeneous Spaces $S_{l,k}$ and their Normal Bundles

Let ω be a non-degenerate 2-form on $\mathbb{R}^{2n}$ and consider the action of $Sp(2n)\times GL(2m;\mathbb{R})$ on $Hom(\mathbb{R}^{2m},\mathbb{R}^{2n})$ given by

$$(A,B)\cdot\phi = A\circ\phi\circ B^{-1}, \tag{4.1}$$

for $(A,B)\in Sp(2n)\times GL(2m;\mathbb{R})$ and $\phi\in Hom(\mathbb{R}^{2m},\mathbb{R}^{2n})$.

Proposition 4.1. The spaces $S_{l,k}=\{\phi:\mathbb{R}^{2m}\to\mathbb{R}^{2n}\mid rank(\phi)=l, rank(\phi^*\omega)=2k\}$ are the orbits of this action.

Proof. Take $\phi\in S_{l,k}$ and denote the orbit of ϕ by O_ϕ. Given $(A,B)\in Sp(2n)\times GL(2m;\mathbb{R})$, since A and B are invertible, the action preserves the rank of ϕ and since $A\in Sp(2n)$, $(A\phi B^{-1})^*\omega=(B^{-1})^*\phi^*\omega$, so the rank of the pullback form is also preserved. Hence, $O_\phi\subset S_{l,k}$.

On the other hand, given $\psi\in S_{l,k}$, there is some $A\in Sp(2n)$ such that $A(Im(\phi))=Im(\psi)$. This is proved in Lemma 4.2 below. Since $Im(A\phi)=Im(\psi)$, there exists some change of basis matrix B such that $A\phi B^{-1}=\psi$. Thus, $S_{l,k}\subset O_\phi$. □

Given a subspace $P\subset\mathbb{R}^{2n}$, the symplectic complement of P is the vector space

$$P^\omega=\{u\in\mathbb{R}^{2n}\mid\omega(u,v)=0\ \forall v\in P\}.$$

Lemma 4.2. Let $\phi,\psi\in S_{l,k}$.

1. The condition $rank(\phi^*\omega)=2k$ is equivalent to $dim(Im(\phi)\cap Im(\phi)^\omega)=l-2k$.

2. Since $rank(\phi^*\omega)=rank(\psi^*\omega)$, there exists $A\in Sp(2n)$ such that $A(Im(\phi))=Im(\psi)$.

Proof.

1. Let $W\subset\mathbb{R}^{2m}$ be a complement to $ker(\phi)$ in the kernel of $\phi^*\omega$, denoted by $rad(\phi^*\omega)$, so

   $$rad(\phi^*\omega)=W\oplus ker(\phi).$$

   Note that $\phi|_W$ is injective and $Im(\phi)\cap Im(\phi)^\omega=\phi(W)$, so $dim(W)=dim(Im(\phi)\cap Im(\phi)^\omega)$. Since $rank(\phi)=l$, it follows that $dim(ker(\phi))=2m-l$ and so one has

   $$dim(Im(\phi)\cap Im(\phi)^\omega)=dim(W)=dim(rad(\phi^*\omega))-(2m-l)$$

   But $dim(rad(\phi^*\omega))=2m-rank(\phi^*\omega)$. Thus, one has

   $$dim(Im(\phi)\cap Im(\phi)^\omega)=2m-rank(\phi^*\omega)-(2m-l)=l-rank(\phi^*\omega).$$

2. By Point 1., there exist a basis $\{u_1, ..., u_{2k}, v_1, ..., v_{l-2k}\}$ for $Im(\phi)$ such that $\{v_1, ..., v_{l-2k}\}$ is a basis for $Im(\phi) \cap Im(\phi)^\omega$ and suppose that ω restricted to $Span(\{u_1, ..., u_{2k}\})$ is represented in $\{u_1, ..., u_{2k}\}$ by the matrix J_{2k} in 3.2. One can extend this basis to $\{u_1, ..., u_{2k}, v_1, ..., v_{2(n-k)}\}$ a basis of $\mathbb{R}^{2n}$ such that $\omega(u_i, v_j) = 0$ and ω restricted to $Span(\{v_1, ..., v_{2(n-k)}\})$ is represented in $\{v_1, ..., v_{2(n-k)}\}$ by a matrix $J_{2(n-k)}$ obtained from J_{2k} by replacing k with $n - k$. In the same way, one can construct a basis $\{u'_1, ..., u'_{2k}, v'_1, ..., v'_{2(n-k)}\}$ of $\mathbb{R}^{2n}$ such that $\{u'_1, ..., u'_{2k}, v'_1, ..., v'_{l-2k}\}$ is a basis for $Im(\psi)$, $\{v'_1, ..., v'_{l-2k}\}$ is a basis for $Im(\psi) \cap Im(\psi)^\omega$, ω restricted to $Span(\{u'_1, ..., u'_{2k}\})$ is represented by J_{2k}, $\omega(u'_i, v'_j) = 0$ and ω restricted to $Span(\{v'_1, ..., v'_{l-2k}\})$ is represented by $J_{2(n-k)}$. Then, the linear isomorphism A defined by $A(u_i) = u'_i$ and $A(v_i) = v'_i$ can be checked to be in $Sp(2n)$ and satisfy $A(Im(\phi)) = Im(\psi)$.

□

Remark 4.3. Not all pairs (l, k) satisfy $S_{l,k} \neq \emptyset$. In fact, for $l \in \{0, ..., 2m\}$ and $k \in \{0, ..., \lfloor l/2 \rfloor\}$, $S_{l,k} \neq \emptyset$ if and only if $n - l + k \geq 0$. If $S_{l,k} \neq \emptyset$, then take $\phi \in S_{l,k}$. By definition, $dim(Im(\phi)) = l$ and, as $rank(\phi^*\omega) = 2k$, it follows that $dim(Im(\phi) \cap Im(\phi)^\omega) = l - 2k$. Since ω is non-degenerate, $dim(Im(\phi)^\omega) = 2n - l$. Hence,

$$Im(\phi) \cap Im(\phi)^\omega \subset Im(\phi)^\omega \implies l - 2k \leq 2n - l \Leftrightarrow n - l + k \geq 0$$

On the other hand, if $n - l + k \geq 0$, then take a basis $\{f_1, ..., f_{2n}\}$ of $\mathbb{R}^{2n}$ such that

$$\omega(f_i, f_j) = \begin{cases} 1 & \text{for } j = i + n, \\ -1 & \text{for } i = j + n, \\ 0 & \text{otherwise.} \end{cases}$$

Pick also a basis $\{e_1, ..., e_{2m}\}$ of $\mathbb{R}^{2m}$ and define ϕ by

$$\phi(e_i) = \begin{cases} f_i & \text{for } i \leq l - k, \\ f_{i+n-k} & \text{for } l - k + 1 \leq i \leq l, \\ 0 & \text{for } i > l. \end{cases}$$

$rank(\phi) = l$ and $rank(\phi^*\omega) \geq 2k$ since $\omega(\phi(e_i), \phi(e_j)) = 1$ for $i = l - 2k + 1, ..., l - k$ and $j = i + k$. Because $n - l + k \geq 0$, $\omega(\phi(e_i), \phi(e_j)) = 0$ for all $i, j \leq l - k$ so $rank(\phi^*\omega) \leq 2k$. It follows that $\phi \in S_{l,k}$, so $S_{l,k} \neq \emptyset$.

Let us define a relation $\geq$ between pairs (l, k) and (l', k'):

$$(l, k) \geq (l', k') \Leftrightarrow l \geq l' \text{ and } k \geq k'. \tag{4.2}$$

Proposition 4.4. The closure of $S_{l,k}$ is given by:

$$\overline{S}_{l,k} = \bigcup_{(l,k)\geq(l',k')} S_{l',k'}.$$

Remark 4.11 will show that $\geq$ is not a total order. This implies that, in contrast with the family $\{\overline{R_k}\}$ of chapter 3, the sets $\overline{S_{l,k}}$ are not contained in each other in succession.

Proof. Pick bases for $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$. Given $\phi : \mathbb{R}^{2m} \to \mathbb{R}^{2n}$, denote by M the matrix representing ϕ in the chosen bases. Then, $rank(\phi) \leq l$ iff all $l+1 \times l+1$ minors of M are zero. In the same way, $rank(\phi^*\omega) \leq 2k$ iff all $2k+1 \times 2k+1$ minors the matrix representing $\phi^*\omega$ are zero. Hence, $\bigcup_{(l,k)\geq(l',k')} S_{l',k'}$ is closed. It also contains $S_{l,k}$, so $\overline{S}_{l,k} \subset \bigcup_{(l,k)\geq(l',k')} S_{l',k'}$. To prove the other inclusion, take $(l',k') \leq (l,k)$ and consider two cases:

1. $l' - 2k' \leq l - 2k$:

   Pick a basis $\{f_i\}_{i=1,\ldots,2n}$ of $\mathbb{R}^{2n}$ such that ω is represented by

$$J = \left(\begin{array}{c|ccc} J_{2k'} & & 0_{2k'\times 2(n-k')} & \\ \hline & 0 & 0 & I_{l-2k} \\ 0_{2(n-k')\times 2k'} & 0 & J_{2(n-l+k)+2(k-k')} & 0 \\ & -I_{l-2k} & 0 & 0 \end{array}\right)$$

   where each J_{2p} is the matrix J_{2k} in (3.2) with k replaced by p. Pick also some basis $\{e_i\}_{i=1,\ldots,2m}$ for $\mathbb{R}^{2m}$ and consider the homomorphism $\psi \in S_{l',k'}$ represented in $\{e_i\}$ and $\{f_i\}$ by

$$\begin{pmatrix} I_{l'} & 0 \\ 0 & 0_{(2n-l')\times(2m-l')} \end{pmatrix}.$$

   Consider the sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ where each ψ_j is represented by

$$\begin{pmatrix} I_{l'} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{j}I_{(l-2k)-(l'-2k')} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j}I_{k-k'} & 0 & 0 \\ 0 & 0 & 0 & 0_{(n-l+k)\times(k-k')} & 0 \\ 0 & 0 & 0 & \frac{1}{j}I_{k-k'} & 0 \\ 0 & 0 & 0 & 0 & 0_{(n-k)\times(2m-l)} \end{pmatrix}.$$

   One can check that $rank(\psi_j^*\omega) = 2k$ and $rank(\psi_j) = l$. Since $\psi_j \to \psi$, one has $\psi \in \overline{S_{l,k}}$.

2. $l' - 2k' > l - 2k$:

Now pick a basis $\{f_i\}_{i=1,...,2n}$ of $\mathbb{R}^{2n}$ such that ω is represented by

$$J = \left(\begin{array}{c|ccc} J_{2k'} & & 0_{2k'\times 2(n-k')} & \\ \hline & 0 & 0 & I_{l'-2k'} \\ 0_{2(n-k')\times 2k'} & 0 & J_{2(n-l'+k')} & 0 \\ & -I_{l'-2k'} & 0 & 0 \end{array}\right)$$

Pick also some basis $\{e_i\}_{i=1,...,2m}$ for $\mathbb{R}^{2m}$ and consider the homomorphism $\psi \in S_{l',k'}$ represented in $\{e_i\}$ and $\{f_i\}$ by

$$\begin{pmatrix} I_{l'} & 0 \\ 0 & 0_{(2n-l')\times(2m-l')} \end{pmatrix}.$$

Let us suppose, to simplify computations, that $l - l'$ is even. The other case is similar. Consider the sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ where each ψ_j is represented by

$$\left(\begin{array}{cccccc|c} I_{2k'} & 0 & 0 & 0 & 0 & 0 & \\ 0 & I_{l-2k} & 0 & 0 & 0 & 0 & \\ 0 & 0 & I_{k-k'-\frac{l-l'}{2}} & 0 & 0 & 0 & \\ 0 & 0 & 0 & I_{k-k'-\frac{l-l'}{2}} & 0 & 0 & \\ 0 & 0 & 0 & 0 & \frac{1}{j}I_{\frac{l-l'}{2}} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0_{n-\frac{l+l'}{2}+k'} & \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{j}I_{\frac{l-l'}{2}} & \\ 0 & 0 & 0 & 0 & 0 & 0_{n-\frac{l+l'}{2}+k'} & \\ 0 & 0 & 0 & \frac{1}{j}I_{k-k'-\frac{l-l'}{2}} & 0 & 0 & \\ \hline & & & & & & 0_{(\frac{l+l'}{2}-k-k')\times(2m-l)} \end{array}\right).$$

One can check that $rank(\psi_j^*\omega) = 2k$ and $rank(\psi_j) = l$. Since $\psi_j \to \psi$, one has $\psi \in \overline{S_{l,k}}$.

In either case, given any other $\phi \in S_{l',k'}$, there exists a pair $(A, B) \in Sp(2n) \times GL(2m; \mathbb{R})$ such that $(A, B)\cdot\psi = \phi$. Thus, $(A, B)\cdot\psi_j$ is a sequence in $S_{l,k}$ converging to ϕ and so $\phi \in \overline{S_{l,k}}$. □

Lemma 4.5. Given integers $l \in \{0, ..., 2m\}$ and $k \in \{0, ..., \lfloor l/2 \rfloor\}$ and an l-plane $P \subset \mathbb{R}^{2n}$ such that $dim(P \cap P^\omega) = l - 2k$, let $\{f_1, ..., f_{2n}\}$ be a basis of $\mathbb{R}^{2n}$ such that

- $\{f_1, ..., f_{l-2k}\}$ is a basis for $P \cap P^\omega$,
- $\{f_1, ..., f_l\}$ is a basis for P, such that ω restricted to $Span(f_{l-2k+1}, ..., f_l)$ is represented by J_{2k} in (3.2),
- $\{f_{l+1}, ..., f_{2n-l+2k}\}$ is a basis for a complement of $P \cap P^\omega$ in P^ω such that ω restricted to this complement is represented by $J_{2(n-l+k)}$ as in (3.2),

- $\{f_{2n-l+2k+1}, \ldots, f_{2n}\}$ is a basis for a complement of $P + P^\omega$ in $\mathbb{R}^{2n}$.

If $A \in Sp(2n)$ satisfies $A(P) = P$, then it is represented in the basis $\{f_i\}$ by

$$A = \begin{pmatrix} B_1 & B_2 & C_1 & C_2 \\ 0 & B_4 & 0 & C_4 \\ 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & (B_1^T)^{-1} \end{pmatrix}, \tag{4.3}$$

where $B_1 \in GL(l-2k;\mathbb{R})$, $B_2 \in \mathcal{M}_{l-2k\times 2k}(\mathbb{R})$, $B_4 \in Sp(2k)$, $D_1 \in Sp(2(n-l+k))$, $D_2 \in \mathcal{M}_{(2(n-l+k))\times l-2k}(\mathbb{R})$ and C_1, C_4 and C_2 satisfy the equations:

$$C_1 = B_1 D_2^T J_{2(n-l+k)} D_1 \tag{4.4}$$

$$C_4 = J_{2k}(B_4^T)^{-1} B_2^T (B_1^T)^{-1} \tag{4.5}$$

$$B_1^{-1} C_2 - C_2^T (B_1^T)^{-1} = C_4^T J_{2k} C_4 + D_2^T J_{2(n-l+k)} D_2. \tag{4.6}$$

Note that C_2 is completely determined by (4.6) and the choice of a $l-2k \times l-2k$ symmetric matrix.

Proof. In a basis like $\{f_i\}$, the $2n \times 2n$ matrix J representing ω is of the form

$$J = \begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix} \tag{4.7}$$

where

$$G_1 = \begin{pmatrix} 0_{l-2k} & 0 \\ 0 & J_{2k} \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & I_{l-2k} \\ 0_{2k\times 2(n-l+k)} & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} J_{2(n-l+k)} & 0 \\ 0 & 0_{l-2k} \end{pmatrix},$$

A transformation $A \in Sp(2n)$ satisfying $A(P) = P$ is represented in $\{f_i\}$ as a matrix A of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

where $A_1 \in \mathcal{M}_{l\times l}(\mathbb{R})$ and $A_3 \in \mathcal{M}_{2n-l\times 2n-l}(\mathbb{R})$. Since $A \in Sp(2n)$, A satisfies the equation $A^T J A = J$. Unravelling this equation, one gets

$$\begin{pmatrix} A_1^T & 0 \\ A_2^T & A_3^T \end{pmatrix}\begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A_1^T G_1 & A_1^T G_2 \\ A_2^T G_1 - A_3^T G_2^T & A_2^T G_2 + A_3^T G_3 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A_1^T G_1 A_1 & A_1^T G_1 A_2 + A_1^T G_2 A_3 \\ -(A_1^T G_1 A_2 + A_1^T G_2 A_3)^T & A_2^T G_1 A_2 - A_3^T G_2^T A_2 + A_2^T G_2 A_3 + A_3^T G_3 A_3 \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix}.$$

So we get three independent equations:

1. $A_1^T G_1 A_1 = G_1$
2. $A_1^T G_1 A_2 + A_1^T G_2 A_3 = G_2$
3. $A_2^T G_1 A_2 - A_3^T G_2^T A_2 + A_2^T G_2 A_3 + A_3^T G_3 A_3 = G_3$.

To solve equation (1), let us write A_1 in blocks:

$$A_1 = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where $B_1 \in \mathcal{M}_{l-2k\times l-2k}(\mathbb{R})$ and $B_4 \in \mathcal{M}_{2k\times 2k}(\mathbb{R})$. Then,

$$A_1^T G_1 A_1 = G_1 \Leftrightarrow \begin{pmatrix} B_1^T & B_3^T \\ B_2^T & B_4^T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix} \Leftrightarrow \begin{pmatrix} B_3^T J_{2k} B_3 & B_3^T J_{2k} B_4 \\ B_4^T J_{2k} B_3 & B_4^T J_{2k} B_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix} \quad (4.8)$$

Equation (4.8) implies that $B_4^T J_{2k} B_4 = J_{2k}$ so $B_4 \in Sp(2k)$; $B_3^T J_{2k} B_4 = 0$ and so $B_3 = 0$ since both J_{2k} and B_4 are non-singular. The other two equations resulting from (4.8) do not give more restrictions. Therefore, A_1 is given by

$$A_1 = \begin{pmatrix} B_1 & B_2 \\ 0 & B_4 \end{pmatrix}$$

with $B_4 \in Sp(2k)$, $B_2 \in \mathcal{M}_{(l-2k)\times 2k}(\mathbb{R})$ and because A_1 must be non-singular, $B_1 \in GL(l-2k;\mathbb{R})$. To solve equations (2) and (3), let us write

$$A_2 = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

where $C_1 \in \mathcal{M}_{(l-2k)\times 2(n-l+k)}(\mathbb{R})$, $C_4 \in \mathcal{M}_{2k\times(l-2k)}(\mathbb{R})$, $D_1 \in \mathcal{M}_{2(n-l+k)\times 2(n-l+k)}(\mathbb{R})$, and $D_4 \in \mathcal{M}_{(l-2k)\times(l-2k)}(\mathbb{R})$. Solving (2), one has

$$\begin{aligned}
& A_1^T G_1 A_2 + A_1^T G_2 A_3 = G_2 \\
\Leftrightarrow & \begin{pmatrix} B_1^T & 0 \\ B_2^T & B_4^T \end{pmatrix}\left(\begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} + \begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\right) = \begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix} \\
\Leftrightarrow & \begin{pmatrix} 0 & 0 \\ B_4^T J_{2k} C_3 & B_4^T J_{2k} C_4 \end{pmatrix} + \begin{pmatrix} B_1^T D_3 & B_1^T D_4 \\ B_2^T D_3 & B_2^T D_4 \end{pmatrix} = \begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix} \\
\Leftrightarrow & \begin{pmatrix} B_1^T D_3 & B_1^T D_4 \\ B_4^T J_{2k} C_3 + B_2^T D_3 & B_4^T J_{2k} C_4 + B_2^T D_4 \end{pmatrix} = \begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

which results in the following equations:

$$D_4 = (B_1^T)^{-1}, \quad D_3 = C_3 = 0,$$
$$C_4 = J_{2k}(B_4^T)^{-1}B_2^T(B_1^T)^{-1} \qquad (J_{2k}^{-1} = -J_{2k}). \tag{4.9}$$

Solving equation (3) in turn yields

$$\begin{aligned}
&A_2^T G_1 A_2 - A_3^T G_2^T A_2 + A_2^T G_2 A_3 + A_3^T G_3 A_3 = G_3 \\
\Leftrightarrow &\begin{pmatrix} 0 & 0 \\ 0 & C_4^T J_{2k} C_4 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ D_4^T C_1 & D_4^T C_2 \end{pmatrix} + \begin{pmatrix} 0 & C_1^T D_4 \\ 0 & C_2^T D_4 \end{pmatrix} + \begin{pmatrix} D_1^T J_{2(n-l+k)} D_1 & D_1^T J_{2(n-l+k)} D_2 \\ D_2^T J_{2(n-l+k)} D_1 & D_2^T J_{2(n-l+k)} D_2 \end{pmatrix} = \begin{pmatrix} J_{2(n-l+k)} & 0 \\ 0 & 0 \end{pmatrix} \\
\Leftrightarrow &\begin{pmatrix} D_1^T J_{2(n-l+k)} D_1 & C_1^T D_4 + D_1^T J_{2(n-l+k)} D_2 \\ D_2^T J_{2(n-l+k)} D_1 - D_4^T C_1 & C_4^T J_{2k} C_4 - D_4^T C_2 + C_2^T D_4 + D_2^T J_{2(n-l+k)} D_2 \end{pmatrix} = \begin{pmatrix} J_{2(n-l+k)} & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

which results in the equations

$$D_1^T J_{2(n-l+k)} D_1 = J_{2(n-l+k)} \tag{4.10}$$
$$C_1 = B_1 D_2^T J_{2(n-l+k)} D_1 \tag{4.11}$$
$$B_1^{-1} C_2 - C_2^T (B_1^T)^{-1} = C_4^T J_{2k} C_4 + D_2^T J_{2(n-l+k)} D_2. \tag{4.12}$$

Equation (4.10) implies that $D_1 \in Sp(2(n-l+k))$; (4.12) implies that C_2 is fully determined by C_4,D_2,B_1 and a $l-2k \times l-2k$ symmetric matrix. Putting it all together, one gets the matrix (4.3). □

Now fix an element $\phi \in S_{l,k}$ and denote by $Iso(l,k)$ the isotropy group of ϕ.

Theorem 4.6.

1. *$Iso(l,k) \cong H_{l,k} \ltimes N_{l,k}$, where*

$$H_{l,k} = GL(l-2k;\mathbb{R}) \times Sp(2k) \times Sp(2(n-l+k)) \times GL(2m-l;\mathbb{R})$$
$$N_{l,k} = \mathcal{M}_{(l-2k)\times 2k}(\mathbb{R}) \times \mathcal{M}_{(2n-2l+2k)\times(l-2k)}(\mathbb{R}) \times Sym(l-2k;\mathbb{R}) \times \mathcal{M}_{(2m-l)\times l}(\mathbb{R})$$

and $(B_1, B_4, D_1, F_3) \in H_{l,k}$ acts on $(B_2, D_2, S, F_2) \in N_{l,k}$ by

$$(B_1, B_4, D_1, F_3) \cdot (B_2, D_2, S, F_2) = \left(B_1 B_2 B_4^{-1}, D_1 D_2 B_1^T, B_1 S B_1^T, F_3 F_2 \begin{pmatrix} B_1^{-1} & 0 \\ 0 & B_4^{-1} \end{pmatrix} \right).$$

2. *$S_{l,k}$ is an immersed submanifold of $Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ of codimension $\frac{1}{2}((l-2k)^2 - (l-2k)) + (2m - l)(2n-l)$.*

Proof.

1. Pick a basis $\{e_1, ..., e_{2m}\}$ for $\mathbb{R}^{2m}$ such that $\{\phi(e_1), ..., \phi(e_{l-2k})\}$ is a basis for $Im(\phi) \cap Im(\phi)^\omega$ and $\{\phi(e_1), ..., \phi(e_l)\}$ is a basis for $Im(\phi)$. Take also a basis $\{f_1, ..., f_{2n}\}$ for $\mathbb{R}^{2n}$ such that

   - $f_i = \phi(e_i)$ for $1 \le i \le l$;
   - $\{f_{l+1}, ..., f_{2n-l+2k}\}$ forms a basis for a complement of $Im(\phi) \cap Im(\phi)^\omega$ in $Im(\phi)^\omega$;
   - $\{f_{2n-l+2k+1}, ..., f_{2n}\}$ forms a basis for a complement of $Im(\phi) + Im(\phi)^\omega$ in $\mathbb{R}^{2n}$.

   Observe that $\{f_i\}_{i=1,...,2n}$ is a basis of the form considered in Lemma 4.5. In the bases $\{e_1, ..., e_{2m}\}$ and $\{f_1, ..., f_{2n}\}$, ϕ is represented by

$$\begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}.$$

   Furthermore, given a pair $(A, B) \in Iso(l, k)$, A must fix $Im(\phi)$ so A must be of the form in (4.3). Write also

$$B = \left(\begin{array}{cc|c} E_1 & E_2 & \multirow{2}{*}{F_1} \\ E_3 & E_4 & \\ \hline \multicolumn{2}{c|}{F_2} & F_3 \end{array}\right),$$

   where $E_1 \in \mathcal{M}_{l-2k \times l-2k}(\mathbb{R})$, $E_4 \in \mathcal{M}_{2k \times 2k}(\mathbb{R})$ and $F_3 \in \mathcal{M}_{2m-l \times 2m-l}(\mathbb{R})$. Then, $(A, B) \in Iso(l, k)$ is equivalent to

$$\begin{aligned} &(A, B) \cdot \phi = \phi \Leftrightarrow A\phi B^{-1} = \phi \Leftrightarrow A\phi = \phi B \\ \Leftrightarrow &\begin{pmatrix} B_1 & B_2 & C_1 & C_2 \\ 0 & B_4 & 0 & C_4 \\ 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & (B_1^T)^{-1} \end{pmatrix} \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{array}{cc|c} E_1 & E_2 & \multirow{2}{*}{F_1} \\ E_3 & E_4 & \\ \hline \multicolumn{2}{c|}{F_2} & F_3 \end{array}\right) \\ \Leftrightarrow &\left(\begin{array}{cc|c} B_1 & B_2 & \multirow{2}{*}{0} \\ 0 & B_4 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) = \left(\begin{array}{cc|c} E_1 & E_2 & \multirow{2}{*}{F_1} \\ E_3 & E_4 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right). \end{aligned}$$

   It follows that $E_1 = B_1$, $E_2 = B_2$, $E_3 = 0$, $E_4 = B_4$ and $F_1 = 0$. Moreover, since B must be non-singular, F_3 must also be non-singular. Therefore,

$$B = \left(\begin{array}{cc|c} B_1 & B_2 & \multirow{2}{*}{$0_{l \times 2m-l}$} \\ 0 & B_4 & \\ \hline \multicolumn{2}{c|}{F_2} & F_3 \end{array}\right) \tag{4.13}$$

   with $F_2 \in \mathcal{M}_{2m-l \times l}(\mathbb{R})$ and $F_3 \in GL(2m-l; \mathbb{R})$. Thus, any element of $Iso(l, k)$ must be represented by a pair (A, B) with A as in (4.3) and B as in (4.13). On the other hand, given any pair (A, B) with A

as in (4.3) and B as in (4.13), it is easily checked that it belongs in $\mathcal{I}so(l,k)$.

One thus has a short exact sequence

$$0 \to N_{l,k} \overset{f}{\hookrightarrow} \mathcal{I}so(l,k) \overset{g}{\to} H_{l,k} \to 0 \tag{4.14}$$

where

$$f(B_2, D_2, S, F_2) = \left(\begin{pmatrix} I & B_2 & C_1 & C_2 \\ 0 & I & 0 & C_4 \\ 0 & 0 & I & D_2 \\ 0 & 0 & 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ F_2 & 0 \end{pmatrix} \right),$$

with C_1 determined by (4.4), C_4 determined by (4.5) and C_2 determined by (4.6) and the symmetric matrix S;

$$g(A, B) = (B_1, B_4, D_1, F_3),$$

with A as in (4.3) and B as in (4.13). Moreover, (4.14) splits with right inverse of g given by the inclusion $H_{l,k} \hookrightarrow \mathcal{I}so(l,k)$:

$$(B_1, B_4, D_1, F_3) \mapsto \left(\begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_4 & 0 & 0 \\ 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & (B_1^T)^{-1} \end{pmatrix}, \left(\begin{array}{cc|c} B_1 & 0 & \\ 0 & B_4 & 0 \\ \hline \multicolumn{2}{c|}{0} & F_3 \end{array}\right) \right). \tag{4.15}$$

One can easily check that the action of $H_{l,k}$ on $N_{l,k}$ determined by sequence (4.14) is the one stated in the theorem.

2. Since $S_{l,k}$ is an orbit of (4.1), it follows that $S_{l,k} \cong Sp(2n) \times GL(2m;\mathbb{R})/\mathcal{I}so(l,k)$ is an immersed submanifold of $Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ of dimension

$$\begin{aligned} dim(S_{l,k}) &= dim(Sp(2n)) + dim(GL(2m;\mathbb{R}) - dim(\mathcal{I}so(l,k)) \\ &= n(2n+1) + (2m)^2 - ((l-2k)^2 + k(2k+1) \\ &\quad + (n-l+k)(2(n-l+k)+1) + (2m-l)^2 + 2k(l-2k) \\ &\quad + (2n-2l+2k)(l-2k) + (2m-l)l + \frac{1}{2}(l-2k)(l-2k+1)) \\ &= 2m2n - \left(\frac{1}{2}((l-2k)^2 - (l-2k)) + (2m-l)(2n-l)\right) \\ &= dim\left(Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})\right) - codim(S_{l,k}) \end{aligned}$$

□

Remark 4.7. $Iso(l,k)^c \cong O(l-2k) \times U(k) \times U(n-l+k) \times O(2m-l)$. Under this identification, the inclusion $Iso(l,k)^c \hookrightarrow U(n) \times O(2m) = Sp(2n)^c \times GL(2m;\mathbb{R})^c$ is given by (4.15).

Since the sets $\overline{S_{l,k}}$ are not totally ordered by inclusion, care must be taken when constructing the homotopy pushouts. Consider the following order relation for pairs (l,k) and (l',k') such that $codim(S_{l,k}) \neq codim(S_{l',k'})$:

$$(l',k') \prec (l,k) \Leftrightarrow codim(S_{l',k'}) > codim(S_{l,k}). \tag{4.16}$$

There may, however, exist different pairs whose corresponding spaces have the same codimension. Extend $\prec$ for such pairs choosing some order for them. This makes $\prec$ a total order for the pairs (l,k) defining non-empty strata $S_{l,k}$. Denote by $\preceq$ the corresponding non-strict total order.

Proposition 4.8. The order $\prec$ refines the order $<$, defined in (4.2).

Proof. The spaces $S_{l,k}$ are semi-algebraic sets in the sense of Definition 2.1.4 in [BCR98]. Furthermore, Propositions 2.8.13 and 2.8.14 of [BCR98] imply that if $S_{l',k'} \subset \overline{S_{l,k}}$, then $codim(S_{l',k'}) > codim(S_{l,k})$. Thus, the result follows from Proposition 4.4. □

To simplify notation, write $X = Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$. Now fix a pair (l,k) and define $F_{l,k} \subset X$ to be the set

$$F_{l,k} = \bigcup_{(l',k') \prec (l,k)} S_{l',k'}.$$

Lemma 4.9. $F_{l,k}$ and $F_{l,k} \cup S_{l,k}$ are both closed subsets of X.

Proof. If $S_{l',k'} \subset F_{l,k}$ then, by Proposition 4.4, $\overline{S_{l',k'}} \subset F_{l,k}$. Hence, $F_{l,k}$ is closed. By the same reasoning, $F_{l,k} \cup S_{l,k} = F_{l,k} \cup \overline{S_{l,k}}$ so $F_{l,k} \cup S_{l,k}$ is also closed. □

Pick some $U(n) \times O(2m)$-invariant metric g on X. Let $\pi_k : NS_{l,k} \to S_{l,k}$ be the normal bundle of $S_{l,k}$ in $X \backslash F_{l,k}$ (with respect to g) and let $NS_{l,k} \backslash S_{l,k}$ be the normal bundle minus the zero section.

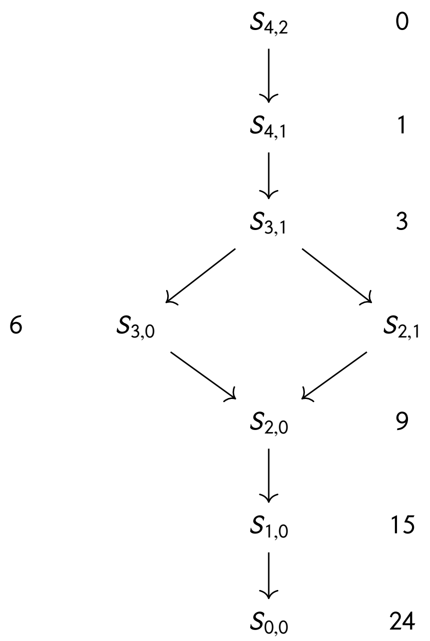
Proposition 4.10. For each pair (l,k), the square

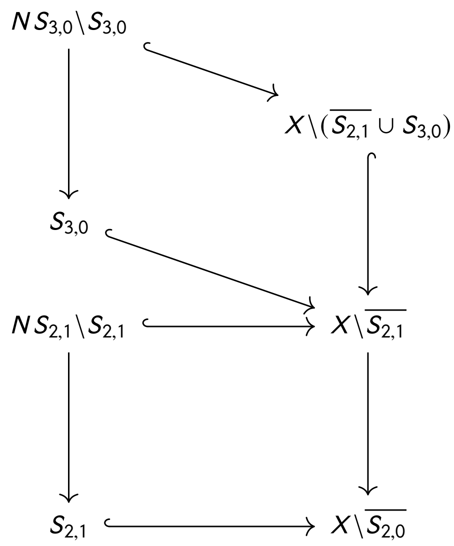
$$\begin{array}{ccc} NS_{l,k} \backslash S_{l,k} & \hookrightarrow & X \backslash (F_{l,k} \cup S_{l,k}) \\ \downarrow{\scriptstyle \pi_{l,k}} & & \downarrow \\ S_{l,k} & \hookrightarrow & X \backslash F_{l,k} \end{array}$$

is a homotopy pushout.

Proof. The proof is completely analogous to the proof of Proposition 3.5. One just substitutes in the proof R_k by $S_{l,k}$, NR_k by $NS_{l,k}$ $\Lambda^2(\mathbb{R}^{2m})^*$ by X, $\overline{R_k}$ by $F_{l,k} \cup S_{l,k}$ and $\overline{R_{k-1}}$ by $F_{l,k}$. □

Remark 4.11. It is not obvious which are the strata $S_{l',k'}$ such that $(l',k') \prec (l,k)$. One could expect, as was the case with the family $\{R_k\}$ in chapter 2, that $F_{l,k} \cup S_{l,k} = \overline{S_{l,k}}$, but this is not always true. To better understand why, it may be useful to look at some concrete cases. Let $m = 2$ and $n = 3$.





The image on the left shows the non-empty strata connected by arrows expressing the relation $\geq$, defined in 4.2. There is an arrow pointing from (l, k) to (l', k') if $(l, k) \geq (l', k')$ (not all arrows are displayed, only the ones going to directly below strata). The numbers on the side represent $codim(S_{l,k})$. Note that there is a bifurcation signalling that the spaces $\overline{S_{l,k}}$ are not contained in succession. As a consequence, $S_{2,1}$ has codimension higher than $S_{3,0}$ but it is not contained in its closure. The image on the right shows the homotopy pushouts of the spaces starring in the bifurcation. In this case, we first remove the stratum $S_{2,1}$ and then $S_{3,0}$. So $F_{2,1} \cup S_{2,1} = \overline{S_{2,1}}$ but $F_{3,0} \cup S_{3,0} = \overline{S_{2,1}} \cup S_{3,0} \neq \overline{S_{3,0}}$

In this example, due to its simplicity, if $l > l'$, then $codim(S_{l,k}) < codim(S_{l',k'})$, independently of k and k'. However, this does not happen in general, as one can see when $m = 4$ and $n = 5$. For instance, in this case, $codim(S_{4,2}) < codim(S_{5,0})$ and $codim(S_{5,0}) = codim(S_{4,1})$ so there are even two different strata with the same codimension.

Let $S^c_{l,k} = U(n) \times O(2m)/Iso(l, k)^c$ be the orbits of the action (4.1) restricted to the maximal compact subgroup of $Sp(2n) \times GL(2m; \mathbb{R})$. Consider the tangent action of $(A, B) \in U(n) \times O(2m)$ on $v \in T_\phi X$. For $(A, B) \in U(n) \times O(2m)$ and $v \in T_\phi S_{l,k}$, one has $(A, B) \cdot v \in T_{(A,B)\cdot\phi} S_{l,k}$. In particular, if $(A, B) \in Iso(l, k)^c$, then $(A, B) \cdot v \in T_\phi S_{l,k}$ for $v \in T_\phi S_{l,k}$. Since g is $U(n) \times O(2m)$-invariant, it follows that the normal space $(T_\phi S_{l,k})^\perp$ is also invariant by the tangent action. Therefore, the restriction of the tangent action to $Iso(l, k)^c$ induces an action of $Iso(l, k)^c$ on $(T_\phi S_{l,k})^\perp$, making $(T_\phi S_{l,k})^\perp$ an $Iso(l, k)^c$-representation, called the orthogonal $Iso(l, k)^c$-representation. The restriction $NS_{l,k}|_{S^c_{l,k}}$ of the normal bundle to $S^c_{l,k}$ can be described by the orthogonal $Iso(l, k)^c$-representation:

Theorem 4.12. *Let $V_{l,k} = (T_\phi S_{l,k})^\perp$. Then,*

1. *There is a diagram*

$$\begin{array}{ccccc} (U(n)\times O(2m))\times_{\mathcal{Iso}(l,k)^c} V_{l,k} & \xrightarrow{\cong} & NS_{l,k}|_{S^c_{l,k}} & \longrightarrow & NS_{l,k} \\ & \searrow & \downarrow & \lrcorner & \downarrow \\ & & S^c_{l,k} & \xhookrightarrow{\simeq} & S_{l,k} \end{array}$$

2. *Consider the vector space $\Lambda^2(\mathbb{R}^{l-2k})^* \times \mathcal{M}_{(2n-l)\times(2m-l)}(\mathbb{R})$ endowed with the action of $O(l-2k) \times U(k) \times U(n-l+k) \times O(2m-l)$ given by*

$$(A_1, A_2, A_3, B_1)\cdot(\sigma, M) = \left(A_1^*\sigma, \begin{pmatrix} A_3 & 0 \\ 0 & A_1 \end{pmatrix} M B_1^T\right) \tag{4.17}$$

for all $(A_1, A_2, A_3, B_1) \in O(l-2k) \times U(k) \times U(n-l+k) \times O(2m-l)$ and $(\sigma, M) \in \Lambda^2(\mathbb{R}^{l-2k})^* \times \mathcal{M}_{(2n-l)\times(2m-l)}(\mathbb{R})$.

Then, under the identification $\mathcal{Iso}(l,k)^c \cong O(l-2k) \times U(k) \times U(n-l+k) \times O(2m-l)$, one has $V_{l,k} \cong \Lambda^2(\mathbb{R}^{l-2k})^* \times \mathcal{M}_{(2n-l)\times(2m-l)}(\mathbb{R})$ as $\mathcal{Iso}(l,k)^c$-representations.

Proof.

1. The homotopy equivalence $S^c_{l,k} \hookrightarrow S_{l,k}$ comes from the fact that both inclusions $U(n)\times O(2m) \hookrightarrow Sp(2n)\times GL(2m;\mathbb{R})$ and $\mathcal{Iso}(l,k)^c \hookrightarrow \mathcal{Iso}(l,k)$ are homotopy equivalences. The 5-lemma applied to the exact sequences of the bundles $\mathcal{Iso}(l,k)^c \hookrightarrow U(n)\times O(2m) \to S^c_{l,k}$ and $\mathcal{Iso}(l,k) \hookrightarrow Sp(2n)\times GL(2m;\mathbb{R}) \to S_{l,k}$ yields the desired equivalence. Moreover, the map

$$\begin{aligned} (U(n)\times O(2m))\times_{\mathcal{Iso}(l,k)^c} V_{l,k} &\xrightarrow{f} NS_{l,k}|_{S^c_{l,k}} \\ [(A,B),v] &\mapsto (A,B)\cdot v \end{aligned}$$

is well defined and restricts to an isomorphism between the fibres since $(A,B) \in U(n)\times O(2m)$ is an isomorphism that maps $V_{l,k}$ onto $(T_{(A,B)\cdot\phi}S_{l,k})^\perp$. It follows that f is an isomorphism between bundles.

2. Let us firstly compute $T_\phi S_{l,k} = a(\mathfrak{sp}(2n)\times\mathfrak{gl}(2m))$. Given $(V_1, V_2) \in \mathfrak{sp}(2n)\times\mathfrak{gl}(2m)$,

$$\frac{d}{dt}\Big|_{t=0}(exp(tV_1), exp(tV_2))\cdot\phi = \frac{d}{dt}\Big|_{t=0} exp(tV_1)\phi exp(-tV_2) = V_1\phi - \phi V_2$$

By picking bases for $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$ as the ones in the beginning of the proof of Point 1 of Theorem 4.6, we can assume $\phi = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}$ and that ω is represented by a matrix J of the form in (4.7), but substituting $2m$ by l in every block dimension. Write

$$V_1 = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

with $X_1 \in \mathcal{M}_{l\times l}(\mathbb{R})$, $X_4 \in \mathcal{M}_{2n-l\times 2n-l}(\mathbb{R})$, $Y_1 \in \mathcal{M}_{l\times l}(\mathbb{R})$ and $Y_4 \in \mathcal{M}_{2m-l\times 2m-l}(\mathbb{R})$.

$$V_1\phi - \phi V_2 = \begin{pmatrix} X_1 & 0 \\ X_3 & 0 \end{pmatrix} - \begin{pmatrix} Y_1 & Y_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1 - Y_1 & -Y_2 \\ X_3 & 0 \end{pmatrix} \tag{4.18}$$

Since $\mathfrak{gl}(2m) = \mathcal{M}_{2m\times 2m}(\mathbb{R})$, it follows that $X_1 - Y_1$ spans all matrices in $\mathcal{M}_{l\times l}(\mathbb{R})$ and $-Y_2$ spans all matrices in $\mathcal{M}_{l\times 2m-l}(\mathbb{R})$. $V_1 \in \mathfrak{sp}(2n) = \{X \in \mathcal{M}_{2n\times 2n}(\mathbb{R}) \mid X^T J = -JX\}$ so the matrices X_i have some restrictions imposed on them. To obtain the restrictions imposed on X_3, let us expand the equation $V_1^T J = -JV_1$:

$$\begin{aligned} &\begin{pmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{pmatrix}\begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix} = -\begin{pmatrix} G_1 & G_2 \\ -G_2^T & G_3 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \\ \Leftrightarrow &\begin{pmatrix} X_1^T G_1 - X_3^T G_2^T & X_1^T G_2 + X_3^T G_3 \\ X_2^T G_1 - X_4^T G_2^T & X_2^T G_2 + X_4^T G_3 \end{pmatrix} = -\begin{pmatrix} G_1 X_1 + G_2 X_3 & G_1 X_2 + G_2 X_4 \\ -G_2^T X_1 + G_3 X_3 & -G_2^T X_2 + G_3 X_4 \end{pmatrix}. \end{aligned}$$

Thus, one gets three independent equations:

(a) $X_1^T G_1 - X_3^T G_2^T = -G_1 X_1 - G_2 X_3$;

(b) $X_1^T G_2 + X_3^T G_3 = -G_1 X_2 - G_2 X_4$;

(c) $X_2^T G_2 + X_4^T G_3 = G_2^T X_2 - G_3 X_4$.

To solve these equations, let us write further the matrices X_i in blocks:

$$X_1 = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad X_2 = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \quad X_3 = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad X_4 = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where $A_1 \in \mathcal{M}_{l-2k\times l-2k}(\mathbb{R})$, $B_1 \in \mathcal{M}_{l-2k\times 2(n-l+k)}(\mathbb{R})$, $C_1 \in \mathcal{M}_{2(n-l+k)\times l-2k}(\mathbb{R})$ and $D_1 \in \mathcal{M}_{2(n-l+k)\times 2(n-l+k)}(\mathbb{R})$ (the dimensions of the other blocks are determined by those of A_1, B_1, C_1 and D_1). Equation (2a) translates to

$$\begin{aligned} &\begin{pmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix} - \begin{pmatrix} C_1^T & C_3^T \\ C_2^T & C_4^T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I_{l-2k} & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} - \begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \\ \Leftrightarrow &\begin{pmatrix} 0 & A_3^T J_{2k} \\ 0 & A_4^T J_{2k} \end{pmatrix} - \begin{pmatrix} C_3^T & 0 \\ C_4^T & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ J_{2k} A_3 & J_{2k} A_4 \end{pmatrix} - \begin{pmatrix} C_3 & C_4 \\ 0 & 0 \end{pmatrix} \\ \Leftrightarrow &\begin{pmatrix} -C_3^T & A_3^T J_{2k} \\ -C_4^T & A_4^T J_{2k} \end{pmatrix} = \begin{pmatrix} -C_3 & -C_4 \\ -J_{2k} A_3 & -J_{2k} A_4 \end{pmatrix}, \end{aligned}$$

which implies that

$$C_3 = C_3^T, \quad C_4 = -A_3^T J_{2k}, \quad J_{2k} A_4 = -A_4^T J_{2k}.$$

Equation (2b) translates to

$$\begin{pmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{pmatrix}\begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_1^T & C_3^T \\ C_2^T & C_4^T \end{pmatrix}\begin{pmatrix} J_{2(n-l+k)} & 0 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & J_{2k} \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} - \begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 & A_1^T \\ 0 & A_2^T \end{pmatrix} + \begin{pmatrix} C_1^T J_{2(n-l+k)} & 0 \\ C_2^T J_{2(n-l+k)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ J_{2k} B_3 & J_{2k} B_4 \end{pmatrix} - \begin{pmatrix} D_3 & D_4 \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} C_1^T J_{2(n-l+k)} & A_1^T \\ C_2^T J_{2(n-l+k)} & A_2^T \end{pmatrix} = \begin{pmatrix} -D_3 & -D_4 \\ J_{2k} B_3 & J_{2k} B_4 \end{pmatrix},$$

which implies that

$$D_3 = C_1^T J_{2(n-l+k)}, \quad D_4 = -A_1^T, \quad C_2 = J_{2(n-l+k)} B_3^T J_{2k}, \quad B_4 = -J_{2k} A_2^T.$$

Equation (2c) translates to

$$\begin{pmatrix} B_1^T & B_3^T \\ B_2^T & B_4^T \end{pmatrix}\begin{pmatrix} 0 & I_{l-2k} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_1^T & D_3^T \\ D_2^T & D_4^T \end{pmatrix}\begin{pmatrix} J_{2(n-l+k)} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I_{l-2k} & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} - \begin{pmatrix} J_{2(n-l+k)} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0 & B_1^T \\ 0 & B_2^T \end{pmatrix} + \begin{pmatrix} D_1^T J_{2(n-l+k)} & 0 \\ D_2^T J_{2(n-l+k)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix} - \begin{pmatrix} J_{2(n-l+k)} D_1 & J_{2(n-l+k)} D_2 \\ 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} D_1^T J_{2(n-l+k)} & B_1^T \\ D_2^T J_{2(n-l+k)} & B_2^T \end{pmatrix} = \begin{pmatrix} -J_{2(n-l+k)} D_1 & -J_{2(n-l+k)} D_2 \\ B_1 & B_2 \end{pmatrix}$$

which implies that

$$D_1^T J_{2(n-l+k)} = -J_{2(n-l+k)} D_1, \quad D_2 = J_{2(n-l+k)} B_1^T, \quad B_2 = B_2^T.$$

Putting all restrictions together, V_1 must be of the form

$$V_1 = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & J_{2k} A_2^T \\ C_1 & J_{2(n-l+k)} B_3^T J_{2k} & D_1 & J_{2(n-l+k)} B_1^T \\ C_3 & -A_3^T J_{2k} & -C_1^T J_{2(n-l+k)} & -A_1^T \end{pmatrix}$$

where

$$J_{2k} A_4 = -A_4^T J_{2k}, \quad B_2^T = B_2, \quad C_3^T = C_3, \quad D_1^T J_{2(n-l+k)} = -J_{2(n-l+k)} D_1.$$

In particular,

$$X_3 = \begin{pmatrix} C_1 & -J_{2(n-l+k)} B_3^T J_{2k} \\ C_3 & A_3^T J_{2k} \end{pmatrix}.$$

The matrices C_1, B_3 and A_3 have no restrictions imposed on them, so, by (4.18), the tangent space at ϕ is

$$T_\phi S_{l,k} = a(\mathfrak{sp}(2n) \times \mathfrak{gl}(2m)) = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_1} & M_2 \\ \hline M_3 & M_4 & \\ M_5 & M_6 & 0_{2n-l \times 2m-l} \end{array} \right) \;\middle|\; M_5^T = M_5 \right\}.$$

with $M_1 \in \mathcal{M}_{l \times l}(\mathbb{R})$ and $M_5 \in \mathcal{M}_{l-2k \times l-2k}(\mathbb{R})$. Finally, taking a complement,

$$V_{l,k} = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & \\ Z_1 & 0 & Z_2 \end{array} \right) \;\middle|\; Z_1^T = -Z_1 \right\} = \Lambda^2 \mathbb{R}^{l-2k} \times \mathcal{M}_{2n-l \times 2m-l}(\mathbb{R}).$$

An element $(A, B) \in Iso(l,k)^c$ is written as

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & B_1 \end{pmatrix}$$

with $A_1 \in O(l-2k), A_2 \in U(k), A_3 \in U(n-l+k)$ and $B_1 \in O(2m-l)$ (see (4.15)). The action on $V_{l,k}$ is just

$$A \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & \\ Z_1 & 0 & Z_2 \end{array} \right) B^{-1} = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_1 \end{pmatrix} \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & \\ Z_1 & 0 & Z_2 \end{array} \right) \begin{pmatrix} A_1^T & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & B_1^T \end{pmatrix} =$$

$$= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \\ \hline 0 & 0 & \\ A_1 Z_1 A_1^T & 0 & \begin{pmatrix} A_3 & 0 \\ 0 & A_1 \end{pmatrix} Z_2 B_1^T \end{array} \right)$$

so $V_{l,k}$ is indeed isomorphic to $\Lambda^2 (\mathbb{R}^{l-2k})^* \times \mathcal{M}_{2n-l \times 2m-l}(\mathbb{R})$ with action given by (4.17).

□

Remark 4.13. Note that the $\Lambda^2 \mathbb{R}^{l-2k}$ term of $V_{l,k}$ is only acted on by $O(l-2k)$ and $\mathcal{M}_{(2n-l)\times(2m-l)}(\mathbb{R})$ is acted on by $O(l-2k) \times U(n-l+k) \times O(2m-l)$.

4.3 Cohomological Obstructions

Since N has an almost symplectic form, TN admits a reduction of structure group to $Sp(2n)$. Also, $U(n)$ is a maximal compact subgroup of $Sp(2n)$, so $BSp(2n) \simeq BU(n)$.

Consider the bundle $p : Hom(TM, TN) \to M \times N$, where

$$Hom(TM, TN) = \{(x, y, \phi) \mid x \in M, y \in N \text{ and } \phi : T_x M \to T_y N\}$$

and p is the projection on the first two coordinates. In other words, $Hom(TM, TN)$ is the bundle with fibre $Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$ associated to $TM \times TN \to M \times N$, so its classifying map is $\tau_M \times \tau_N$. The following diagram of bundles then implies that the classifying map of $Hom(TM, i^*TN)$ is $(\tau_M, \tau_N \circ i)$.

$$\begin{array}{ccc} Hom(TM, i^*TN) & \longrightarrow & Hom(TM, TN) \\ \downarrow & & \downarrow \\ M & \xrightarrow{(id_M, i)} & M \times N \end{array}$$

Let $X = Hom(\mathbb{R}^{2m}, \mathbb{R}^{2n})$. We wish to define cohomological obstructions to the existence of a lift g of a map f as the one given below.

$$\begin{array}{ccc} & & (X \backslash (F_{l,k} \cup S_{l,k}))_h \\ & \overset{g}{\nearrow} & \downarrow \iota \\ & \overset{f}{\nearrow} & X_h \\ & & \downarrow \\ M & \xrightarrow{(\tau_M, \tau_N \circ i)} & BU(n) \times BO(2m) \end{array} \tag{4.19}$$

Here, the notation $hU(n) \times O(2m)$ has been reduced to h.

As was the case with (3.8) in chapter 3, the obstructions come from the kernel of ι^*, in the sense that, for g to exist, f must satisfy equations like (3.9):

$$f^*(x) = 0 \qquad \forall x \in ker(\iota^*).$$

The goal is thus to find generators for $ker(\iota^*)$, which we will call the **obstruction classes**.

Proposition 4.14. ι is a $(codim(S_{l,k}) - 1)$-equivalence. In particular for degrees $< codim(S_{l,k})$, ι^* is injective.

Proof. The proof is similar to the proof of Proposition 3.10. One first observes that for each pair $(l', k') < (l, k)$ (see (4.16)) and for (l, k) itself, the following square is a homotopy pushout.

$$\begin{array}{ccc} (NS_{l',k'} \backslash S_{l',k'})_h & \hookrightarrow & (X \backslash (F_{l',k'} \cup S_{l',k'}))_h \\ \downarrow \pi_{l',k'} & & \downarrow j_{l',k'} \\ (S_{l',k'})_h & \hookrightarrow & (X \backslash F_{l',k'})_h \end{array}$$

The map $\pi_{l',k'}$ is a $(codim(S_{l',k'})-1)$-equivalence, so $j_{l',k'}$ is also a $(codim(S_{l',k'})-1)$-equivalence. Since $S_{l',k'} \subset F_{l,k} \cup S_{l,k}$, it follows that $j_{l',k'}$ is a $(codim(S_{l,k})-1)$-equivalence. Finally, one notes that ι is the composition of maps $j_{l',k'}$ for pairs (l',k') where $S_{l',k'} \subset F_{l,k} \cup S_{l,k}$ (for instance, in the first example of Remark 4.11, for $(l,k) = (3,0)$, $\iota = j_{3,0} \circ j_{2,1} \circ \cdots \circ j_{0,0}$). Hence, ι is a $(codim(S_{l,k})-1)$-equivalence. □

Now denote the Euler class of $(NS_{l,k})_h \to (S_{l,k})_h$ by $e_{l,k}$ and denote by $\psi_{l,k}$ the inclusions

$$
\begin{array}{ccc}
(S_{l,k})_h & \xhookrightarrow{I_{l,k}} & (X\backslash F_{l,k})_h \\
& \searrow^{\psi_{l,k}} & \downarrow \\
& & X_h
\end{array}
\tag{4.20}
$$

Theorem 4.15. *If for every pair (l',k') such that $(l,k) \preceq (l',k')$ (see (4.16)) one has that $e_{l',k'}$ is not a zero-divisor, then in $H^{codim(S_{l,k})}(X_h)$,*

$$
ker(\iota^*) = \bigcap_{(l,k) \prec (l',k')} ker(\psi^*_{l',k'})
\tag{4.21}
$$

Moreover, in degree $codim(S_{l,k})$, $dim_{\mathbb{Z}_2}(ker(\iota^*)) = 1$, so $ker(\iota^*)$ is generated by a single non-zero class $\mathfrak{v}_{l,k}$.

Proof. The proof follows exactly as in Theorem 3.12. For each pair (l',k') such that $(l,k) \preceq (l',k')$, the homotopy pushout

$$
\begin{array}{ccc}
(NS_{l',k'}\backslash S_{l',k'})_h & \xhookrightarrow{i_{l',k'}} & (X\backslash(F_{l',k'} \cup S_{l',k'}))_h \\
\downarrow \pi_{l',k'} & & \downarrow j_{l',k'} \\
(S_{l',k'})_h & \xhookrightarrow{I_{l',k'}} & (X\backslash F_{l',k'})_h
\end{array}
$$

yields a long exact sequence of cohomology:

$$
\begin{aligned}
\cdots \to H^*((X\backslash F_{l',k'})_h)) \xrightarrow{(I^*_{l',k'}, j^*_{l',k'})} H^*((S_{l,k})_h) \oplus H^*((X\backslash(F_{l',k'} \cup S_{l',k'}))_h) \to \\
\xrightarrow{\pi^*_{l',k'} - i^*_{l',k'}} H^*((NS_{l',k'}\backslash S_{l',k'})_h) \to \cdots
\end{aligned}
\tag{4.22}
$$

The Gysin Sequence

$$
\begin{aligned}
\cdots \to H^{*-codim(S_{l',k'})}((S_{l',k'})_h) \xrightarrow{\cup e_{l',k'}} H^*((S_{l',k'})_h) \xrightarrow{\pi^*_{l',k'}} H^*((NS_{l',k'}\backslash S_{l',k'})_h) \to \\
\to H^{*-codim(S_{l',k'})+1}((S_{l',k'})_h) \to \cdots
\end{aligned}
\tag{4.23}
$$

together with the fact that $e_{l',k'}$ is not a zero divisor, implies that $\pi_{l',k'}$ is surjective in degrees $* \geq codim(S_{l',k'})$. Sequence (4.22), together with surjectivness of $\pi_{l',k'}$, implies that the pair $(I^*_{l',k'}, j^*_{l',k'})$ is injective in degrees $* \geq codim(S_{l',k'})$. In particular, $(I^*_{l',k'}, j^*_{l',k'})$ is injective in degree $codim(S_{l,k}) \geq$

$codim(S_{l',k'})$. Let us write (l_1, k_1) for the smallest pair (with respect to $\prec$) such that $(l,k) \prec (l_1,k_1)$. Then,

$$ker(\iota^*) = ker(l^*_{l_1,k_1} \circ \iota^*) \cap ker(j^*_{l_1,k_1} \circ \iota^*) = ker(\psi^*_{l_1,k_1}) \cap ker(j^*_{l_1,k_1} \circ \iota^*).$$

In the same way, if (l_2, k_2) is the minimum pair such that $(l_1,k_1) \prec (l_2,k_2)$, then

$$ker(j^*_{l_1,k_1} \circ \iota^*) = ker(\psi^*_{l_2,k_2}) \cap ker(j^*_{l_2,k_2} \circ j^*_{l_1,k_1} \circ \iota^*).$$

Equality (4.21) follows from continuing this reasoning and noting both that $(2m, m)$ is the maximum with respect to $\prec$ and $j^*_{2m,m-1} \circ \cdots \circ j^*_{l_1,k_1} \circ \iota^* = \psi_{2m,m}$.

To prove that $dim_{\mathbb{Z}_2}(ker(\iota^*)) = 1$, observe that (4.23) for $(l',k') = (l,k)$ implies that $ker(\pi_{l,k}) = Im(\langle e_{l,k}\rangle)$ so $dim_{\mathbb{Z}_2}(ker(\pi_{l,k})) = 1$. Lemma 3.13 then implies that $dim_{\mathbb{Z}_2}(ker(j_{l,k})) = dim_{\mathbb{Z}_2}(ker(\pi_{l,k})) = 1$ and ι^* is a composition of $j^*_{l,k}$ with $j^*_{l',k'}$ for $(l',k') \prec (l,k)$, which are isomorphisms in degree $codim(S_{l,k})$. □

Hence, the problem is reduced to solving the equations $\psi^*_{l',k'} x = 0$, which are called the restricting equations (see the discussion immediately before section 3.4). Before solving the restricting equations, let us show that indeed the Euler classes $e_{l,k}$ are not zero-divisors.

4.4 The Euler Classes

Lemma 4.16. There is a bundle morphism:

$$\begin{array}{ccc} EIso(l,k)^c \times_{Iso(l,k)^c} V_{l,k} & \longrightarrow & (NS_{l,k})_h \\ \downarrow & & \downarrow \\ BIso(l,k)^c & \xrightarrow{\simeq} & (S_{l,k})_h \end{array}$$

Proof. This now follows exactly the proof of Lemma 3.14. One first notes that $(S^c_{l,k})_h \simeq (S_{l,k})_h$ and restricts the normal bundle to $(S^c_{l,k})_h$. Then, one uses the equivalences

$$\left(NS_{l,k}|_{(S^c_{l,k})}\right)_h \simeq EU(n) \times EO(2m) \times_{U(n)\times O(2m)} \left(U(n) \times O(2m) \times_{Iso(l,k)^c} V_{l,k}\right) \simeq$$
$$\simeq EU(n) \times EO(2m) \times_{Iso(k)^c} V_{l,k} \simeq EIso(k)^c \times_{Iso(k)^c} V_k$$

$$(S^c_{l,k})_h \simeq EU(n) \times EO(2m) \times_{U(n)\times O(2m)} \left(U(n) \times O(2m)/Iso(l,k)^c\right) \simeq BIso(l,k)^c$$

□

Remark 4.17.

- Let us denote by w, c, d and v the total Stiefel-Whitney (or Chern reduced mod 2) classes in $H^*(BO(l-2k))$, $H^*(BU(k))$, $H^*(BU(n-l+k))$ and $H^*(BO(2m-l))$, respectively. For $i<0$ or $i>l-2k$, $w_i=0$ and the same goes for the other classes with the appropriate bounds.

- Denote also by t_i the Stiefel-Whitney roots corresponding to w, s_i the ones corresponding to v and u_j the Chern roots of d.

- Following Remark 4.7, $BIso(l,k)^c \simeq BO(l-2k)\times BU(k)\times BU(n-l+k)\times BO(2m-l)$. Thus, one has

$$H^*((S_{l,k})_h) \cong \mathbb{Z}_2[w_1,...,w_{l-2k},c_1,...,c_k,d_1,...d_{n-l+k},v_1,...,v_{2m-l}]$$

Theorem 4.18. *The Euler class of* $(NS_{l,k})_h$ *is the product*

$$e_{l,k} = det(w_{\delta_i-i+j})_{i,j=1}^{l-2k}\cdot det((wd/v)_{2n-l-i+j})_{i,j=1}^{2m-l}$$

where $\delta=(l-2k-1,l-2k-2,...,1)$ *and the total class* wd/v *is the one that satisfies* $v\cup(wd/v)=wd$.

Note that the first determinant in the product is the Schur polynomial in the variables $t_1,...,t_{l-2k}$ associated to the partition δ.

Proof. According to Remark 4.13, we may consider the projection of $Iso(l,k)^c$ onto $G_{l,k}=O(l-2k)\times U(n-l+k)\times O(2m-l)$, inducing

$$\begin{array}{ccc} EIso(l,k)^c\times_{Iso(l,k)^c}V_{l,k} & \longrightarrow & EG_{l,k}\times_{G_{l,k}}\Lambda^2(\mathbb{R}^{l-2k})^*\oplus\mathcal{M}_{2n-l\times 2m-l}(\mathbb{R}) \\ \downarrow & & \downarrow \\ BIso(k)^c & \xrightarrow{\pi} & BG_{l,k} \end{array}$$

and compute the Euler class of the bundle on the right. Note that, by the same remark, $\Lambda^2(\mathbb{R}^{l-2k})^*\oplus\mathcal{M}_{2n-l\times 2m-l}(\mathbb{R})$ is the direct sum of two subrepresentations and observe that the action on $\Lambda^2(\mathbb{R}^{l-2k})^*$ is the same as the action considered in Theorem 3.8 (substituting $2m$ by l). Thus, the Euler class of this factor is $s_\delta(t_1,...,t_{l-2k})$. Focusing on the other factor, observe that it factors itself into two subrepresentations. Indeed, let $(A,B,C)\in O(l-2k)\times U(n-l+k)\times O(2m-l)$ and $M=\begin{pmatrix}M_1\\M_2\end{pmatrix}\in\mathcal{M}_{2n-l\times 2m-l}(\mathbb{R})$ with M_1 a $2(n-l+k)\times 2m-l$ matrix and M_2 a $l-2k\times 2m-l$ matrix. The action on M is then given by

$$(A,B,C)\cdot M=\begin{pmatrix}B&0\\0&A\end{pmatrix}\begin{pmatrix}M_1\\M_2\end{pmatrix}C^T=\begin{pmatrix}BM_1C^T\\AM_2C^T\end{pmatrix}.$$

So $\mathcal{M}_{2n-l\times 2m-l}(\mathbb{R})=\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R})\oplus\mathcal{M}_{l-2k\times 2m-l}(\mathbb{R})$ as a representation. Let us study the classes coming from the factor $\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R})$. Take the inclusion $I_{l,k}=(S^1)^{n-l+k}\times(\mathbb{Z}_2)^{2m-l}\hookrightarrow G_{l,k}$ inducing the diagram

$$\begin{array}{ccc} EI_{l,k}\times_{I_{l,k}}\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R}) & \longrightarrow & EG_{l,k}\times_{G_{l,k}}\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R}) \\ \downarrow & & \downarrow \\ BI_{l,k} & \xrightarrow{\quad j \quad} & BG_{l,k} \end{array}$$

and observe that j^* is an injective map such that $j^*(d_i)$ and $j^*(v_i)$ are the i-th elementary symmetric polynomials in the variables u_i and s_i, respectively. As a representation of $I_{l,k}$, $\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R})$ breaks up into a direct sum of copies of $\mathbb{C}$:

$$\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R}) = \bigoplus_{i,j=1}^{\substack{i=n-l+k\\ j=2m-l}} \mathbb{C}_{i,j}$$

where $\mathbb{C}_{i,j}$ denotes $\mathbb{C}$ with $I_{l,k}$ acting through the projection onto the i-th S^1 and j-th $\mathbb{Z}_2$ factors by

$$(S^1\times\mathbb{Z}_2)\times\mathbb{C}\to\mathbb{C}$$
$$((e^{i\theta},a),z)\mapsto ae^{i\theta}z$$

Let us fix a copy $\mathbb{C}_{i,j}$. The composition $S^1\times\mathbb{Z}_2\overset{j}{\hookrightarrow} S^1\times S^1\overset{p}{\to} S^1$, where $p(z,w)=zw$, induces a bundle diagram

$$\begin{array}{ccc} ES^1\times E\mathbb{Z}_2\times_{S^1\times\mathbb{Z}_2}\mathbb{C} & \longrightarrow & ES^1\times_{S^1}\mathbb{C} \\ \downarrow & & \downarrow \\ BS^1\times B\mathbb{Z}_2 & \xrightarrow{\quad Bp\circ Bj \quad} & BS^1 \end{array}$$

where the action of S^1 on $\mathbb{C}$ considered on $ES^1\times_{S^1}\mathbb{C}$ is $(e^{i\theta},z)\mapsto e^{i\theta}z$. The cohomology of BS^1 is generated by $t=c_1(\gamma^1(\mathbb{C}))\in H^2(BS^1)$. It is easy to check that $ES^1\times_{S^1}\mathbb{C}$ is just the tautological bundle $\gamma^1(\mathbb{C})$ so the map $Bp\circ Bj$ pulls back t to the Euler class of the bundle $ES^1\times E\mathbb{Z}_2\times_{S^1\times S^1}\mathbb{C}$. We wish now to write explicitly the map induced in cohomology by $Bp\circ Bj$. Let us write

$$H^*(BS^1)=\mathbb{Z}_2[t]$$
$$H^*(BS^1\times BS^1)=\mathbb{Z}_2[x_1,x_2]$$
$$H^*(BS^1\times B\mathbb{Z}_2)=\mathbb{Z}_2[u,s].$$

By Proposition 2.65, the map induced by $Bj: BS^1\times\mathbb{Z}_2\to BS^1\times BS^1$ sends $x_1+x_2\mapsto u+s^2$. To understand Bp^*, consider the long exact sequences of the bundles $S^1\hookrightarrow ES^1\to BS^1$ and $S^1\times S^1\hookrightarrow ES^1\times ES^1\to BS^1\times BS^1$.

$$\cdots\to\pi_2(ES^1)\to\pi_2(BS^1)\to\pi_1(S^1)\to\pi_1(ES^1)\to\cdots$$
$$\cdots\to\pi_2(ES^1\times ES^1)\to\pi_2(BS^1\times BS^1)\to\pi_1(S^1\times S^1)\to\pi_1(ES^1\times ES^1)\to\cdots$$

From the naturality of the sequences and the fact that ES^1 is contractible, there is a commuting square:

$$\begin{array}{ccc} \pi_2(BS^1 \times BS^1) & \xrightarrow{\cong} & \pi_1(S^1 \times S^1) \\ \big\downarrow{\scriptstyle Bp_*} & & \big\downarrow{\scriptstyle p_*} \\ \pi_2(BS^1) & \xrightarrow{\cong} & \pi_1(S^1) \end{array}$$

Since $BS^1 = \mathbb{CP}^\infty$ is simply connected, the Hurewicz Theorem (Theorem 4.37 of [Hat02]) and naturality of the Hurewicz map h imply that

$$\begin{array}{ccc} \pi_2(BS^1 \times BS^1) & \xrightarrow[\cong]{h} & H_2(BS^1 \times BS^1; \mathbb{Z}) \\ \big\downarrow{\scriptstyle Bp_*} & & \big\downarrow{\scriptstyle Bp_*} \\ \pi_2(BS^1) & \xrightarrow[\cong]{h} & H_2(BS_1; \mathbb{Z}) \end{array}$$

It is easy to see that $p_* : \pi_1(S^1 \times S^1) \to \pi_1(S^1)$ is just $(a, b) \mapsto a + b$ ($\pi_1(S^1) = \mathbb{Z}$), so the same expression holds for Bp_*. Reducing the homology coefficients to $\mathbb{Z}_2$ and dualizing, one gets

$$Bp^* : H^2(BS^1) \to H^2(BS^1 \times BS^1)$$
$$t \mapsto x_1 + x_2$$

Hence $(Bp \circ B\iota)^*(t) = u + s^2$ and the Euler class of the $\mathcal{M}_{2(n-l+k)\times 2m-l}(\mathbb{R})$ factor is

$$\prod_{i,j=1}^{\substack{i=n-l+k \\ j=2m-l}} (u_i + s_j^2)$$

The class of the $\mathcal{M}_{l-2k\times 2m-l}(\mathbb{R})$ factor is obtained in a similar but easier way. One considers first the bundle diagram induced by the inclusion $(\mathbb{Z}_2)^{l-2k} \times (\mathbb{Z}_2)^{2m-l} \hookrightarrow G_{l,k}$, then decomposes the restricted representation into a direct sum

$$\bigoplus_{i,j=1}^{\substack{i=l-2k \\ j=2m-l}} \mathbb{R}_{ij}$$

of $(l-2k)(2m-l)$ copies of $\mathbb{R}$ where $R_{i,j}$ is acted on by $(\mathbb{Z}_2)^{l-2k} \times (\mathbb{Z}_2)^{2m-l}$ through the projections onto the i-th and $l - 2k + j$-th $\mathbb{Z}_2$ factors. The Euler class of $\mathbb{R}_{i,j}$ is obtained in a similar way and is $t_i + s_j$. Hence, the Euler class of the $\mathcal{M}_{l-2k\times 2m-l}(\mathbb{R})$ factor is

$$\prod_{i,j=1}^{\substack{i=l-2k \\ j=2m-l}} (t_i + s_j)$$

The Euler class of the factor $\mathcal{M}_{2n-l\times 2m-l}(\mathbb{R})$ will then be

$$\prod_{i,j=1}^{\substack{i=n-l+k \\ j=2m-l}} (u_i + s_j^2) \prod_{i,j=1}^{\substack{i=l-2k \\ j=2m-l}} (t_i + s_j) \tag{4.24}$$

It remains to show that $det((wd/v)_{2n-l-i+j})_{i,j=1}^{2m-l}$ is mapped to (4.24) when replacing w, c and v by their expansions in terms of the Stiefel-Whitney and Chern roots. A useful notation for w, d and v in terms of their roots is

$$w = \prod_{i=1}^{l-2k}(1+t_i t), \quad d = \prod_{i=1}^{n-l+k}(1+u_i t^2), \quad v = \prod_{j=1}^{2m-l}(1+s_j t),$$

where w_i and v_i are the coefficients of t^i and d_i is the coefficient of t^{2i}. Let

$$p(t) = \sum_{j=0}^{2n-l} p_j t^j = wd = \prod_{i=1}^{l-2k}(1+t_i t)\prod_{i=1}^{n-l+k}(1+u_i t^2),$$
$$q(t) = \sum_{j=0}^{2m-l} q_j t^j = v = \prod_{j=1}^{2m-l}(1+s_j t).$$

The resultant of $q(t)$ and $p(t)$, usually denoted by $Res(q(t), p(t))$ is the following determinant:

$$Res(q(t),p(t)) = det\begin{pmatrix} 1 & q_1 & \cdots & q_{2m-l} & 0 & \cdots & & 0 \\ 0 & 1 & q_1 & \cdots & q_{2m-l} & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & q_1 & \cdots & q_{2m-l} \\ 1 & p_1 & \cdots & p_{2n-l} & 0 & \cdots & & 0 \\ 0 & 1 & p_1 & \cdots & p_{2n-l} & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & p_1 & \cdots & p_{2n-l} \end{pmatrix}. \tag{4.25}$$

One can check using formula (4.25) that if $a(t)$, $b(t)$ and $c(t)$ are polynomials, then $Res(a(t)b(t), c(t)) = Res(a(t), c(t))Res(b(t), c(t))$. And the same happens for the other slot. Thus, $Res(q(t), p(t))$ can be computed by computing the resultants of each pair of factors in the products defining $p(t)$ and $q(t)$. For instance, given some i, j, one sees that

$$Res(1+s_j t, 1+t_i t) = det\begin{pmatrix} 1 & s_j \\ 1 & t_i \end{pmatrix} = t_i + s_j.$$

Doing the same for factors of the form $1 + u_i t^2$ and $1 + s_j t$, it follows that $Res(q(t), p(t))$ =(4.24). Now let $x(t) = \sum_{i=0}^{\infty} x_i t^i = 1/q(t)$. Then,

$$X = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_{2n+2m-2l-1} \\ 0 & 1 & x_1 & \cdots & x_{2n+2m-2l-2} \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & & \cdots & 1 \end{pmatrix},$$

is a matrix of determinant 1, so multiplying the matrix in (4.25) by X does not affect the determinant. Following section 2.4 (i) of [Arb+85], one can see that after multiplying by $det(X)$, one has

$$Res(q(t), p(t)) = det \begin{pmatrix} (p/q)_{2n-l} & (p/q)_{2n-l+1} & \cdots & (p/q)_{2n+2m-2l-1} \\ (p/q)_{2n-l-1} & (p/q)_{2n-l} & \cdots & \\ \vdots & & & \vdots \\ (p/q)_{2n-2m+1} & \cdots & & (p/q)_{2n-l} \end{pmatrix},$$

where $\sum_{i=0}^{\infty}(p/q)_i t^i = p(t)/q(t)$. Note that $(p/q)_i = (wd/v)_i$, so this determinant is $det((wd/v)_{2n-l-i+j})_{i,j=1}^{2m-l}$ after replacing w, d and v by their expressions in terms of Stiefel-Whitney and Chern roots. □

Once again, it is obvious that $e_{l,k}$ is not a zero-divisor so we can move on to computing the cohomological obstructions.

4.5 Computing the Obstructions

To solve the restricting equations, one must first write a suitable expression for the maps $\psi_{l,k}$ in (4.20).

Lemma 4.19. One has isomorphisms

$$H^*((S_{l,k})_h) \cong \mathbb{Z}_2[w_1, ..., w_{l-2k}, c_1, ..., c_k, d_1, ...d_{n-l+k}, v_1, ..., v_{2m-l}],$$
$$H^*(X_h) \cong \mathbb{Z}_2[w_1, ..., w_{2m}, c_1, ..., c_n].$$

Under these identifications, the maps $\psi_{l,k}^*$ are given by

$$\begin{aligned} \psi_{l,k}^* : \mathbb{Z}_2[w_1, ..., c_n] &\to \mathbb{Z}_2[w_1, ..., v_{2m-l}] \\ w &\mapsto wcv \\ c &\mapsto w^2cd \end{aligned} \tag{4.26}$$

Proof. The first isomorphism was obtained in Remark 4.17, the second comes from the fact that X is contractible, so $X_h \simeq BU(n) \times BO(2m)$. To prove (4.26), observe that there is a commuting diagram by Corollary 2.37:

$$\begin{array}{ccc} (S_{l,k})_h & \xhookrightarrow{\psi_{l,k}} & X_h \\ \downarrow \simeq & & \downarrow \simeq \\ BIso(l,k)^c & \xhookrightarrow{j} & BU(n) \times BO(2m) \end{array}$$

Thus, under the identifications $(S_{l,k})_h \simeq BIso(l,k)^c$, $X_h \simeq BU(n) \times BO(2m)$, $\psi_{l,k}$ is the inclusion $BIso(l,k)^c \overset{j}{\hookrightarrow} BU(n) \times BO(2m)$. Consider the identification

$$BIso(l,k)^c \cong BO(l-2k) \times BU(k) \times BU(n-l+k) \times BO(2m-l)$$

and denote by $\pi_1 : BU(n) \times BO(2m) \to BU(n)$ and $\pi_2 : B(n) \times BO(2m) \to BO(2m)$ the canonical projections. Then, by Remark 4.7, $\pi_1 \circ j$ decomposes as

$$\begin{array}{ccc} BO(l-2k) \times BU(k) \times BU(n-l+k) \times BO(2m-l) & \xhookrightarrow{\pi_1 \circ j} & BU(n) \\ \downarrow a_1 & & \uparrow b_1 \\ BU(l-2k) \times BU(k) \times BU(n-l+k) \times BO(2m-l) & \xrightarrow{p_1} & BU(l-2k) \times BU(k) \times BU(n-l+k) \end{array}$$

where p_1 is the projection on the first three factors. By Proposition 2.67, $b_1^*(c) = b\tilde{c}d$, where $b, \tilde{c}$ and d are the total Chern classes in $H^*(BU(l-2k))$, $H^*(BU(k))$ and $H^*(BU(n-l+k))$, respectively. $p_1^*(b\tilde{c}d) = b\tilde{c}d$ and, by Proposition 2.65, $a_1^*(b\tilde{c}d) = \tilde{w}^2\tilde{c}d$, where $\tilde{w}$ is the total Stiefel-Whitney class in $H^*(BO(l-2k))$. In the same way, $\pi_2 \circ j$ decomposes as

$$\begin{array}{ccc} BO(l-2k) \times BU(k) \times BU(n-l+k) \times BO(2m-l) & \xhookrightarrow{\pi_2 \circ j} & BO(2m) \\ \downarrow a_2 & & \uparrow b_2 \\ BO(l-2k) \times BO(2k) \times BU(n-l+k) \times BO(2m-l) & \xrightarrow{p_2} & BO(l-2k) \times BO(2k) \times BO(2m-l) \end{array} \tag{4.27}$$

$b_2^*(w) = \tilde{w}xv$, where x and v are the total Stiefel-Whitney classes in $H^*(BO(2k))$ and $H^*(BO(2m-l))$, respectively. $p_2^*(\tilde{w}xv) = \tilde{w}xv$ and, by Proposition 2.66, $a_2^*(\tilde{w}xv) = \tilde{w}\tilde{c}v$. □

Finally, we are ready to compute the obstructions.

Theorem 4.20. *Recall the inclusion ι defined in (4.19). For each pair (l, k), the kernel of ι^* in degree $codim(S_{l,k})$ is generated by*

$$\mathfrak{v}_{l,k} = det \begin{pmatrix} \{(c/w)_{2n-l-i+j}\}_{i,j=1}^{i=2m-l, j=2(m-k)} \\ \\ \{w_{l-2k-2i+j}\}_{i=1, j=1-2m+l}^{i,j=l-2k} \end{pmatrix} \tag{4.28}$$

Proof. According to Theorem 4.15, one only needs to check that $\psi_{l',k'}^*(\mathfrak{v}_{l,k}) = 0$ for all pairs (l', k') such that $(l, k) \prec (l', k')$. Moreover, by Proposition 4.8, it suffices to check $\psi_{l',k'}^*(\mathfrak{v}_{l,k}) = 0$ in two cases:

1. $l' > l$ and

2. $l' \leq l$, $k < k'$.

By the formula in Lemma 4.19,

$$\psi_{l',k'}^*(\mathfrak{v}_{l,k}) = det \begin{pmatrix} \{(wd/v)_{2n-l-i+j}\}_{i,j=1}^{i=2m-l, j=2(m-k)} \\ \\ \{(wvc)_{l-2k-2i+j}\}_{i=1, j=1-2m+l}^{i,j=l-2k} \end{pmatrix} \tag{4.29}$$

where $w_i = 0$ for $i < 0$ or $i > l' - 2k'$, $v_i = 0$ for $i < 0$ or $i > 2m - l'$, $d_i = 0$ for $i < 0$ or $i > n - l' + k'$ and $c_i = 0$ for $i < 0$ or $i > k'$.

1. $l' > l$:

   The class wd/v is the only class $x \in H^*((S_{l',k'})_h)$ that satisfies the equation $vx = wd$. Since the coefficient group for cohomology is $\mathbb{Z}_2$, the i-th component of wd/v is given by

$$(wd/v)_i = (wd)_i + v_1(wd/v)_{i-1} + v_2(wd/v)_{i-2} + ... + v_{2m-l'}(wd/v)_{i-(2m-l')}.$$

   In particular, the elements of the first row of the matrix in (4.29) are given by

$$(wd/v)_{2n-l-1+j} = (wd)_{2n-l-1+j} + (wd/v)_{2n-l-2+j}v_1 + ... + (wd/v)_{2n-l-2m+l'+j}v_{2m-l'}.$$

   Note that $(wd)_i = 0$ for $i > l' - 2k' + 2(n - l' + k') = 2n - l'$. Since $2n - l' < 2n - l$, it follows that $(wd)_{2n-l-1+j} = 0$ for all j. As $2m - l' < 2m - l$, it follows that the first row is a linear combination of the $2m - l'$ rows below. Therefore, the determinant in (4.29) is zero.

2. $l' \leq l, k < k'$:

   Let us call the matrix in (4.29) by M. Firstly, observe that, similarly to (3.19) in the proof of Theorem 3.17, the class c in the lower submatrix of M can be taken out. That is because M is obtained as the product

$$M = \begin{pmatrix} I_{2m-l} & & \mathbf{0} & & \\ & 1 & c_1 & \cdots & c_{l-2k-1} \\ \mathbf{0} & 0 & 1 & \cdots & c_{l-2k-2} \\ & \vdots & \cdots & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \{(wd/v)_{2n-l-i+j}\}_{i,j=1}^{i=2m-l,j=2(m-k)} \\ \\ \{(wv)_{l-2k-2i+j}\}_{i=1,j=1-2m+l}^{i,j=l-2k} \end{pmatrix}. \tag{4.30}$$

   The matrix on the left has determinant equal to 1 so, to compute the determinant in (4.29) we may assume M is the second factor in (4.30). Let us write M in six blocks:

$$M = \begin{pmatrix} A & B \\ E & F \\ G & H \end{pmatrix}$$

   where,

$$A = \{(wd/v)_{2n-l-i+j}\}_{i,j=1}^{i=2m-l,j=2m-l'}, \quad B = \{(wd/v)_{2n-l-i+j}\}_{i=1,j=2m-l'+1}^{i=2m-l,j=2(m-k)}, \quad E = \{(wv)_{l-2k-2i+j}\}_{i=1,j=1-2m+l}^{i=l-l',j=l-l'}$$
$$F = \{(wv)_{l-2k-2i+j}\}_{i=1,j=l-l'+1}^{i=l-l',j=l-2k}, \quad G = \{(wv)_{l-2k-2i+j}\}_{i=l-l'+1,j=1-2m+l'}^{i=l-2k,j=l-l'}, \quad H = \{(wv)_{l-2k-2i+j}\}_{i=l-l'+1,j=l-l'+1}^{i,j=l-2k}$$

   Denote also the i-th column of A by a_i, the i-th column of B by b_i and do the same for the other blocks. Recall that the i-th component of wd/v is

$$(wd/v)_i = (wd)_i + v_1(wd/v)_{i-1} + ... + v_{2m-l'}(wd/v)_{i-(2m-l')}.$$

All elements of B are of the form $(wd/v)_i$ with $i \geq 2n - l - (2m - l) + (2m - l') + 1 = 2n - l' + 1 > 2n - l'$. Since $(wd)_i = 0$ for $i > 2n - l'$, an element $(wd/v)_i$ of B is written as

$$(wd/v)_i = v_1(wd/v)_{i-1} + ... + v_{2m-l'}(wd/v)_{i-(2m-l')}.$$

Since A has $2m - l'$ columns, it follows that the first column of B is a linear combination of the columns of A:

$$b_1 = v_1 a_{2m-l'} + v_2 a_{2m-l'-1} + ... + v_{2m-l'} a_1$$

The other columns of B are also linear combinations of the $2m - l'$ previous columns. For instance,

$$b_2 = v_1 b_1 + v_2 a_{2m-l'} + ... + v_{2m-l'} a_2.$$

Therefore, multiplying M on the right by

$$X = \begin{pmatrix} & v_{2m-l'} & 0 & 0 & \cdots & 0 \\ & v_{2m-l'-1} & v_{2m-l'} & 0 & \cdots & 0 \\ I_{2m-l'} & \vdots & \vdots & \ddots & \cdots & \vdots \\ & v_1 & v_2 & \cdots & \cdots & v_{l'-2k} \\ & 1 & v_1 & \cdots & \cdots & v_{l'-2k-1} \\ & 0 & 1 & v_1 & \cdots & \vdots \\ 0_{l'-2k \times 2m-l'} & 0 & 0 & 1 & \cdots & \vdots \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

results in a matrix $\tilde{M}$ given by

$$\tilde{M} = \begin{pmatrix} A & 0 \\ E & \tilde{F} \\ G & \tilde{H} \end{pmatrix}$$

Note that $det(X) = 1$ so $det(M) = det(\tilde{M})$. If $\tilde{f}_i$ denotes the i-th column of $\tilde{F}$, then observe that

$$\tilde{f}_1 = f_i + v_1 e_{2m-l'} + v_2 e_{2m-l'-1} + ... + v_{2m-l'} e_1$$

The other columns of $\tilde{F}$ are, in the same way, linear combinations of the $2m - l'$ previous columns. For instance,

$$\tilde{f}_2 = f_2 + v_1 f_1 + v_2 e_{2m-l'-1} + ... + v_{2m-l'} v_2.$$

Likewise, if $\tilde{h}_i$ denotes the i-th column of $\tilde{H}$, then

$$\tilde{h}_1 = h_1 + v_1 g_{2m-l'} + v_2 g_{2m-l'-1} + ... + v_{2m-l'} g_1.$$

And the other columns of $\tilde{H}$ are, in the same way, linear combinations of the $2m - l'$ previous columns. By the product formula, one has

$$(wv^2)_i = (wv \cdot v)_i = (wv)_i + v_1 (wv)_{i-1} + ... + v_{2m-l'} (wv)_{i-(2m-l')}.$$

Therefore,

- $\tilde{F} = \{(wv^2)_{l-2k-2i+j}\}_{i=1, j=l-l'+1}^{i=l-l', j=l-2k}$ and
- $\tilde{H} = \{(wv^2)_{l-2k-2i+j}\}_{i=l-l'+1, j=l-l'+1}^{i,j=l-2k}$.

Observe that, since coefficients are in $\mathbb{Z}_2$, one has

$$v^2 = v_1^2 + v_2^2 + ...$$

Hence, $(v^2)_i = 0$ for odd i. Thus, one can write $\tilde{M}$ as a product similar to the one obtained in (4.30) with v^2 playing the role of c:

$$\tilde{M} = \begin{pmatrix} I_{2m-l} & & & \mathbf{0} & \\ & 1 & (v^2)_1 & \cdots & (v^2)_{l-2k-1} \\ \mathbf{0} & 0 & 1 & \cdots & (v^2)_{l-2k-2} \\ & \vdots & \cdots & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ E' & F' \\ G' & H' \end{pmatrix}$$

where now

$$E' = \{w_{l-2k-2i+j}\}_{i=1, j=1-2m+l'}^{i=l-l', j=l-l'}, \quad F' = \{w_{l-2k-2i+j}\}_{i=1, j=l-l'+1}^{i=l-l', j=l-2k}$$
$$G' = \{w_{l-2k-2i+j}\}_{i=l-l'+1, j=1-2m+l'}^{i=l-2k, j=l-l'}, \quad H' = \{w_{l-2k-2i+j}\}_{i=l-l'+1, j=l-l'+1}^{i,j=l-2k}.$$

Any element of F' is of the form w_i with $i \geq l - 2k - 2(l - l') + (l - l') + 1 = l' - 2k + 1 > l' - 2k'$ since $k' > k$. As $w_i = 0$ for $i > l' - 2k'$, $F' = 0$. Moreover, any element of the first column of H' is of the form w_i with $i \geq l - 2k - 2(l - l' + 1) + (l - l' + 1) = l' - 2k - 1 = l' - 2k' + 2(k' - k) - 1 > l' - 2k'$ since $k' - k \geq 1$. Therefore, all elements of the first column of H' are zero, thus $det(H') = 0$. This finishes the proof because

$$det(\tilde{M}) = det \begin{pmatrix} A & 0 \\ E' & 0 \\ G' & H' \end{pmatrix} = det \begin{pmatrix} A \\ E' \end{pmatrix} det(H') = 0.$$

□

We saw that the classifying map of $Hom(TM, i^*TN)$ is $(\tau_M, \tau_N \circ i)$. Therefore, under the identification $H^*(X_h) \cong H^*(BU(n) \times BO(2m))$, one has $f^*\mathfrak{v}_{l,k} = (\tau_M^*, i^*\tau_N^*)\mathfrak{v}_{l,k}$. Hence, condition $f^*\mathfrak{v}_{l,k} = 0$ translates into the following:

Theorem 4.21. *Let $i : M \to N$ be a smooth map between a $2m$-manifold M and a $2n$-manifold N endowed with an almost symplectic form ω. Then, the following equation is a necessary condition for the existence of a section s of $Hom(TM, i^*TN)$ such that $rank(s(x)) > l$ and $rank(s(x)^*\omega) > 2k$ for all $x \in M$.*

$$det \begin{pmatrix} \{(i^*c(N)/w(M))_{2n-l-i+j}\}_{i,j=1}^{i=2m-l, j=2(m-k)} \\ \\ \{w(M)_{l-2k-2i+j}\}_{i=1, j=1-2m+l}^{i,j=l-2k} \end{pmatrix} = 0$$

Remark 4.22. Note that when $l = 2m$, one has

$$(\tau_M, \tau_N \circ i)^*\mathfrak{v}_{2m,k} = \tau_M^*\mathfrak{v}_k$$

where $\mathfrak{v}_k$ is the class in Theorem (3.17). One could also prove this fact using the following results:

- The sets $\{\overline{S_{2m,k}}\}_{k=0,\dots,m}$ form a stratification of $Mono(TM, i*TN)$ - the open sub-bundle of $Hom(TM, i^*TN)$ of injective homomorphisms. Moreover, if $s : M \to Mono(TM, i*TN)$ is a section transversal to the spaces $S_{2m,k}$, then

$$D([s^{-1}(\overline{S_{l,k}})]) = (\tau_M, \tau_N \circ i)^*\mathfrak{v}_{2m,k}.$$

- The following map is a submersion:

$$Mono(TM, i^*TN) \xrightarrow{F} \Lambda^2 T^*M$$
$$\phi \mapsto \phi^*\omega$$

  and $F^{-1}(R_k) = S_{2m,k}$.

It follows that if $s : M \to Mono(TM, i^*TN)$ is a section transversal to the spaces $S_{2m,k}$, then $F \circ s$ is a section transversal to the spaces R_k. Therefore, one has

$$(\tau_M, \tau_N \circ i)^*\mathfrak{v}_{2m,k} = D([(F \circ s)^{-1}(S_{2m,k})]) = D([s^{-1}(R_k)]) = \tau_M^*\mathfrak{v}_k.$$

On the other hand, if $l = 2k$, then

$$(\tau_M, \tau_N \circ i)^*\mathfrak{v}_{l,l/2} = det\left(\{i^*c(N)/w(M)\}_{i,j=1}^{i,j=2m-l}\right)$$

is the Giambelli-Thom-Porteous class of the degeneracy locus of points $x \in M$ where $rank((di)_x) \le l$ (see (1.1)).

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Thom Polynomials for Degeneracy Loci of 2-forms and Maps to an Almost Symplectic Manifold

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