AN ANALYTICAL APPROACH TO DISPERSION-INDUCED LIMITATIONS ON THE BIT RATE OF FIBER-OPTIC COMMUNICATION SYSTEMS

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ABSTRACT

The main goal of this dissertation is to optimize the bit rate of single mode optical fibers. Closed-form analytical expressions are derived both for the maximum bit rate and the product of the square of the bit rate with the length, to get a figure of merit for the performance of a single-channel fiber-optic communication system (considering only dispersion limitations).

Both unchirped hyperbolic secant pulses and chirped Gaussian (and super-Gaussian) pulses were considered throughout. Apart from these types of pulses, closed-form analytical expressions for pulse width evolution along the fiber involve rather cumbersome calculations. Losses, on the other hand, will be disregarded: they can be easily tackled (e.g., using fiber amplifiers), whence its effect will not be addressed herein. So, dispersion is the only problem under analysis as it imposes severe limitations. Of course, other limitations can affect this figure of merit; however, the focus will be the effects of both group-velocity dispersion and higher-order dispersion.

The main original contribution of this dissertation is to find the optimum values for the input and output pulse widths for any given value of the chirp parameter. Namely, it is shown that there exists a critical value of the chirp parameter for which the input is equal to the output so that, for any given point along the optical fiber, the pulse width is always smaller than the input and output pulse widths.

Keywords: fiber-optic communication systems, single mode fibers, chirped gaussian pulses, group-velocity dispersion, higher-order dispersion, bit-rate, analytical approach.

1. INTRODUCTION

1.1. Motivation

A large demand for the development of fiber-optic communications systems started in the 1970s, since this solution had the potential to allow the transmission of information at high bit-rates and with a potentially low cost of implementation because the raw material that is used for producing optical fibers is silica, which can be easily found and extracted. Some other vantages in using optical fibers that were found during several researches along the years were that in the spectral region between 1,3 μm and 1,55 μm the attenuation was very low (between 0.5 dB/km and 0.2 dB/km), large frequency bandwidth, small size and weight, immunity to electromagnetic interferences, electrical isolation, reliability and ease of maintenance.

Nowadays, the bit-rate in an optical fiber can be up to 100 Gb/s depending on the transmission distance. However, the demand for even higher bit rates never stops and to take a step forward in the speed with which digital information is sent along an optical fiber, the problem of dispersion-induced limitations must be overcomed if it is intended to progress technologically.

It will be crucial to tackle this issue, because every year more users are connected to the internet which requires more effort from the network to give an acceptable quality of experience to everyone connected since more information is requested by the users.

1.2. Objectives

The purpose of this dissertation is to understand how dispersion affects the bit rate on fiber-optic communication systems. It is known that exists a large demand for higher bit-rates, because in quantitative terms, data traffic is increasing every year. This means that it is more important than ever to be aware of dispersion limitations and to know how to deal with them. When travelling a fiber, pulses suffer dispersion and attenuation. Nevertheless, it will be assumed that there is no attenuation meaning that all the energy of the signals on the input of the fibers will be the same as in the output. However, the signals will broaden due to dispersion phenomena. This is a huge problem, since pulses will interfere with their neighbors causing a decrease in the quality of the information that is transmitted along the fibers. The occurrence of dispersion depends on many factors, being the most important ones the following: length of the fiber (L), Group Velocity Dispersion - GVD (β_2), higher-order dispersion (β_3), input pulse width (T_0). The band of wavelengths to be considered in this study is the C band, which reference wavelength is: $\lambda_0 = 1550\,nm$. To have a better understanding on how to tackle this issue, closed-form analytical expressions will be presented to have a qualitative perspective that will allow to analyze every situation that needs to be addressed. Numerical solutions will also be shown to verify the accuracy of the analytical approach. However, a numerical solution for itself only allows to see what happens in a specific situation being that the reason for developing closed-form analytical expressions that are going to give us the tools to address every case in hand.

2. THEORETICAL BACKGROUND

2.1. Pulse Propagation

The guided propagation of electromagnetic waves through an optical fiber occurs due to the physical mechanism called total internal reflection (happening in the interface between the core and the cladding) [1]. For single-mode fibers, any component $\Psi(r,\phi,z,t)$ of the optical fiber electromagnetic field as the form:

$$\Psi(r,\phi,z,t) = F(r,\phi)U(z,t)\,, \tag{1}$$

where $U(z,t)$ will be:

$$U(z,t) = A(z,t)\exp[i(\beta_0 z - \omega_0)t]\,. \tag{2}$$

Since in the course of the study that is going to be made, we are only interested to look at how the information along the optical fiber is transmitted, $U(z,t)$ will be the term of special relevance here, meaning we will not going to analyze the modal function $F(r,\phi)$ further. At the input of the optical fiber, $z = 0$, we get:

$$U(0,t) = g(t)\exp(-i\omega_0 t)\,. \tag{3}$$

In this case, $g(t)$ is the modulating signal (which in fact, contains all the information to being carried) and $\omega_0 = 2\pi f$ is the (angular) frequency of the carrier. So, how may we know $U(z,t)$ knowing $U(0,t)$? To answer this question, we need to apply the Fourier transform. So, after applying the Fourier transform, we get:

$$U(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega-\omega_0)\exp\{i[\beta(\omega)z - \omega t]\}\,d\omega\,. \tag{4}$$

To solve the integral from Equation (4), a variable change needs to take place. For that matter, it is necessary to consider $\Omega = \omega - \omega_0$. This way, it comes that:

$$\beta(\omega) = \beta(\omega_0 + \Omega)\,. \tag{5}$$

To solve Equation (5), we consider a Taylor expansion (around ω_0):

$$\beta(\omega) = \beta(\omega_0+\Omega) = \sum_{m=0}^{\infty}\frac{\beta_m}{m!}\Omega^m\,. \tag{6}$$

Luckily, when the shift in frequency Ω is too small or when $\beta_m \approx 0$ for $m \geq 4$, it is reasonable to consider just $0 \leq m \leq 3$ in the expansion [1]. The outcome is simply:

$$\beta(\omega) = \beta(\omega_0+\Omega) \approx \beta_0 + \beta_1\Omega + \frac{1}{2}\beta_2\Omega^2 + \frac{1}{6}\beta_3\Omega^3. \tag{7}$$

Before saying anything else, let us clarify that β_0 is the longitudinal propagation constant corresponding to the carrier frequency itself and β_1 is the term related with the group-delay, which is something that will be defined further ahead. However, neither β_0 nor β_1 introduce changes in the pulse shape, meaning in the chapters ahead the two terms of greater relevance will be β_2 denominated as the Group-Velocity Dispersion (GVD) and β_3 being the higher-order dispersion term, because both β_2 and β_3 will have a direct influence on the bit-rate of the fiber-optic communication systems. Considering the result seen in Equation (7), we may write:

$$\tilde{A}(z,\Omega) = \tilde{A}(0,\Omega)\exp\left(i\frac{1}{2}\beta_2\Omega^2 z\right)\exp\left(i\frac{1}{6}\beta_3\Omega^3 z\right), \tag{8}$$

$$A(z,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{A}(z,\Omega)\exp(-i\Omega T)\,d\Omega\,. \tag{9}$$

The group-delay is defined as:

$$\tau_g = \beta_1 L\,. \tag{10}$$

The group-velocity is defined as:

$$v_g(\omega) = \frac{1}{\beta'(\omega)} = \frac{1}{\frac{d\beta}{d\omega}}\,. \tag{11}$$

Therefore, the group-velocity dispersion is defined as:

$$\beta_2(\omega) = \frac{d}{d\omega}\left[\frac{1}{v_g(\omega)}\right] = -\frac{1}{v_g^2(\omega)}\frac{dv_g}{d\omega}\,. \tag{12}$$

The relation between the dispersion coefficient D and β_2 is:

$$D = -\frac{2\pi c}{\lambda^2}\beta_2\,. \tag{13}$$

Then, the dispersion slope S is written as:

$$S = \frac{dD}{d\lambda} = \frac{4\pi c}{\lambda^3}\beta_2 + \left(\frac{2\pi c}{\lambda^2}\right)^2\beta_3\,. \tag{14}$$

2.2. Chirped Gaussian Pulses

A pulse is said to be Gaussian if at the input of the fiber, $z = 0$, the pulse has the form [2]:

$$A(0,t) = A_0\exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^2\right], \tag{15}$$

where C is the dimensionless chirp parameter. This signal can also be written in the following form:

$$A(0,t) = A_0\exp\left(-\frac{t^2}{2T_0^2}\right)\exp[-i\phi(t)]\,, \tag{16}$$

in which:

$$\phi(t) = C\,\frac{t^2}{2T_0^2}\,. \tag{17}$$

So, in a pulse where we have a quadratic phase ϕ in time, C is the parameter that influences the slope of the dynamic shifting of frequency $\Omega(t)$ in relation to the carrier wave and it is linear in time. In fact:

$$\Omega(t) = \frac{d\phi}{dt} = C\frac{t}{T_0^2}. \tag{18}$$

To further continue our analysis on the propagation of pulses (in this particular case of chirped Gaussian pulses), it is now relevant to define the dispersion length, L_D, and the normalized distance, ξ. So, both can be defined as [3]:

$$L_D = \frac{T_0^2}{|\beta_2|} \tag{19}$$

$$\xi = \frac{z}{L_D} \tag{20}$$

It is possible to find closed-form analytical expressions for a chirped Gaussian pulse propagating along a fiber under the effect of GVD. However, when the effect of higher-order dispersion must be considered (through β_3) the pulse no longer remains Gaussian. In this case, a proper way to measure the pulse width is to calculate its root mean square (RMS) width $\sigma = \sigma(z)$, such that:

$$\sigma = \sqrt{\langle t^2\rangle - \langle t\rangle^2}\,, \tag{21}$$

where the angle brackets denote averaging with respect to the intensity profile [2]:

$$\langle t^m\rangle = \frac{\int_{-\infty}^{\infty} t^m |A(z,t)|^2 dt}{\int_{-\infty}^{\infty} |A(z,t)|^2 dt}. \tag{22}$$

However, in the lossless case energy (E) does not depend on the point z. This way, it comes that [3]:

$$\langle t^m\rangle = \frac{1}{E}\int_{-\infty}^{\infty} t^m |A(z,t)|^2 dt\,. \tag{23}$$

The RMS width of a chirped Gaussian pulse can be simply be calculated as:

$$\sigma_0 = \frac{T_0}{\sqrt{2}}. \tag{24}$$

As proven in [2], the broadening factor is given by:

$$\left(\frac{\sigma}{\sigma_0}\right)^2 = \left[1 + C\left(\frac{\beta_2 L}{2\sigma_0^2}\right)\right]^2 + \left(\frac{\beta_2 L}{2\sigma_0^2}\right)^2 + (1 + C^2)^2\left(\frac{\beta_3 L}{4\sqrt{2}\sigma_0^3}\right)^2. \tag{25}$$

However, the last expression does not account for the source linewidth and hence is only applicable to optical sources with small spectral width (such as single-mode semiconductor lasers). Therefore, in the next chapters it will always be assumed we are working with optical sources with small spectral width.

2.3. Combined Effects of GVD and Chirp Parameter

Both GVD and chirp parameter will cause the different frequency components of an optical pulse to travel at different velocities. This fact alone may not mean much, but what we want to see is if there is a way to take advantage of such effects when it comes to optimize the bit-rate in a fiber-optic communication system. To begin, let us first look at how GVD alters the velocity of the different frequency components of an optical pulse, by having in mind Equation (12). To make things clear, if $dv_g/d\omega < 0$ then $\beta_2(\omega) > 0$ and one can say we are working in the normal-dispersion regime. In this regime, high-frequency (blue-shifted) components of an optical pulse travel slower than low-frequency (red-shifted) components of the same pulse, i.e., it occurs a shift to the red at the front of the pulse and a shift to the blue at its tail. On the other hand, if $dv_g/d\omega > 0$ then $\beta_2(\omega) < 0$ and one may say we are working in the anomalous-dispersion regime. In this regime, high-frequency (blue-shifted) components of an optical pulse travel faster than low-frequency (red-shifted) components of the same pulse, i.e., it occurs a shift to the blue at the front of the pulse and a shift to the red at its tail, which is the opposite of what happens in the normal-dispersion regime. Now, since we want to know how both GVD and chirp parameter will work together it is necessary to clarify how the chirp parameter will influence the group-velocity of the different frequency components of an optical pulse. To begin with, we need to have in mind Equation (18). From Equation (18), if $C > 0$ the frequency-shift will have a positive dependence with time. Then, for $C > 0$, high-frequency (blue-shifted) components of an optical pulse travel slower than low-frequency (red-shifted) components of the same pulse, i.e., it occurs a shift to the red at the front of the pulse and a shift to the blue at its tail. If $C < 0$ the frequency-shift will have a negative dependence with time. Then, if we consider $C < 0$, high-frequency (blue-shifted) components of an optical pulse travel faster than low-frequency (red-shifted) components of the same pulse, i.e., it occurs a shift to the blue at the front of the pulse and a shift to the red at its tail. From the analysis of (12) and (18), some observations must be made. The first thing that gets our attention is that the effects on the velocity of the different frequency components produced by GVD are the same as the ones of the chirp parameter if their signals are equal, i.e., if they are both positive or both negative. However, if their signals are not equal, the effects produced by both are the opposite. Then, in a fiber-optic communication system where $\beta_2 C > 0$, it is expected to experience pulse broadening since both GVD and chirp parameter are both increasing the difference in the velocity of the different frequency components, which is not an ideal scenario when it comes to optimization of bit-rate in a fiber-optic communication system. However, in a fiber-optic communication system where $\beta_2 C < 0$, we may experience pulse compression since now the combined effects of GVD and chirp parameter are now counteracting each other. Then, this may contribute to reduce the difference between the velocity of the higher-frequency components and the lower-frequency components. This way, it will be ideal that $\beta_2 C < 0$ to allow for the possibility of an effective bit-rate optimization in a fiber-optic communication system. In the next chapters, we will see how having $\beta_2 C > 0$ instead of

$\beta_2 C < 0$ will affect the size of the optical pulses and therefore the maximum bit-rate possible in a fiber-optic communication system.

3. HYPERBOLIC SECANTS PULSES

A Hyperbolic Secant pulse, at $z = 0$, assumes the form [4]:

$$A(0,t) = A_0 \operatorname{sech}\left(\frac{t}{T_0}\right) \exp\left[-i\frac{C}{2}\left(\frac{t}{T_0}\right)^2\right]. \qquad (26)$$

However, in this study, we will always consider Hyperbolic Secant pulses without chirp, meaning the chirp parameter will be: $C = 0$. If that is the case, the previous expression may be rewritten as:

$$A(0,t) = A_0 \operatorname{sech}\left(\frac{t}{T_0}\right). \qquad (27)$$

This approach is taken, because there is no expression in the literature for the broadening factor of Hyperbolic Secant pulses with chirp. Fortunately, there is indeed an expression for the broadening factor of unchirped Hyperbolic Secant pulses which is the following [4]:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \left(\frac{\pi \beta_2 z}{6\sigma_0^2}\right)^2, \qquad (28)$$

being the RMS width of the pulse [4]:

$$\sigma_0 = \frac{\pi}{\sqrt{12}} T_0 \,. \qquad (29)$$

To facilitate our analysis of the broadening factor, the same will be written using normalized variables. But first, let us see some auxiliary steps before we do that:

$$\sigma^2 = \sigma_0^2 + \left(\frac{\pi \beta_2 z}{6\sigma_0^2}\right)^2, \qquad (30)$$

$$\frac{\sigma^2}{|\beta_2| L} = \frac{\sigma_0^2}{|\beta_2| L} + \left(\frac{\pi}{6}\right)^2 \frac{|\beta_2| L}{\sigma_0^2} \left(\frac{z}{L}\right)^2. \qquad (31)$$

Now, knowing that the normalized pulse width is [3]:

$$\chi = \frac{\sigma^2}{|\beta_2| L}, \qquad (32)$$

then:

$$\begin{cases} z = 0 \rightarrow \sigma = \sigma_0 \rightarrow \chi_0 = \dfrac{\sigma_0^2}{|\beta_2| L} \\ z = L \rightarrow \sigma = \sigma_1 \rightarrow \chi_1 = \dfrac{\sigma_1^2}{|\beta_2| L} \end{cases}. \qquad (33)$$

The normalized distance, as seen before, is written as:

$$0 \leq \xi = \frac{z}{L} \leq 1. \qquad (34)$$

From this point, the broadening factor may be rewritten as:

$$\chi = \chi_0 + \left(\frac{\pi}{6}\right)^2 \frac{\xi^2}{\chi_0}. \qquad (35)$$

As such, the pulse width behavior will be defined as:

$$\mu = \mu(\chi_0, \xi) = \frac{\chi}{\chi_0} = 1 + \left(\frac{\pi}{6}\right)^2 \left(\frac{\xi}{\chi_0}\right)^2. \qquad (36)$$

Now, to define what χ_1 and μ_1 will look like, we first state the following:

$$\begin{cases} \xi = 0 \rightarrow \chi = \chi_0 \rightarrow \mu = \mu_0 = 1 \\ \xi = 1 \rightarrow \chi = \chi_1 \rightarrow \mu = \mu_1 \end{cases}. \qquad (37)$$

Having the previous considerations in mind, χ_1 and μ_1 will be rewritten as follows:

$$\chi_1 = \chi_0 + \left(\frac{\pi}{6}\right)^2 \frac{1}{\chi_0}, \qquad (38)$$

$$\mu_1 = 1 + \left(\frac{\pi}{6}\right)^2 \left(\frac{1}{\chi_0}\right)^2. \qquad (39)$$

Now, it is important for us to know what the optimum values are for χ_0 and χ_1. To get there, we must first do the following calculation:

$$\frac{d\chi_1}{d\chi_0} = 1 - \left(\frac{\pi}{6}\right)^2 \frac{1}{\chi_0^2} = 0 \rightarrow \chi_0^{opt} = \frac{\pi}{6}. \qquad (40)$$

Then, the optimum value for χ_1 will be simply:

$$\chi_1^{opt} = \frac{\pi}{3}. \qquad (41)$$

From here, we may finally get the optimum broadening factor and the optimum pulse width behavior, being both written as follows:

$$\chi^{opt} = \frac{\pi}{6}(1 + \xi^2), \qquad (42)$$

$$\mu^{opt} = 1 + \xi^2. \qquad (43)$$

Therefore:

$$\mu_1^{opt} = 2. \qquad (44)$$

To obtain the maximum value for the bit-rate, first we need to have in mind the given practical rule of thumb [1]:

$$\sigma_{max} \leq \frac{T_b}{4} = \frac{1}{4B} \rightarrow B \leq B_0 = \frac{1}{4\sigma_{max}}. \qquad (45)$$

Since the point in the optical fiber where the pulse width is the largest is at its output, $\sigma_{max} = \sigma_1$. With that information, it is possible to get an expression that will give us the optimized bit-rate value:

$$B_0 = \frac{1}{4\sigma_{max}} = \frac{1}{4}\sqrt{\frac{3}{\pi |\beta_2| L}} \approx \frac{0.2443}{\sqrt{|\beta_2| L}}. \qquad (46)$$

But even more important than knowing the B_0 value is to know the product $B_0^2 L$ that will allow us to obtain a figure of merit, which will permit to better evaluate the performance of the fiber-optic communication system. Having a high bit-rate only for short distances will not be good enough for a transoceanic network and that is why this parameter is so important. The bit-rate squared product with the length of the optical fiber is given by:

$$B_0^2 L = \frac{F}{|\beta_2|} \text{ , } with\ F = \frac{3}{16\pi} \approx 0.0597 \,. \quad (47)$$

4. CHIRPED GAUSSIAN PULSES

A chirped Gaussian pulse, at $z = 0$, assumes the form [2]:

$$A(0,t) = A_0 \exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^2\right]. \quad (48)$$

Then, the corresponding broadening factor expression for this kind of pulse (if $\beta_3 = 0$) is given by [2]:

$$\frac{\sigma^2}{\sigma_0^2} = \left(1 + C\frac{\beta_2 z}{2\sigma_0^2}\right)^2 + \left(\frac{\beta_2 z}{2\sigma_0^2}\right)^2 , \quad (49)$$

To facilitate our analysis of the broadening factor, the same will be written using normalized variables. But first, let us see some auxiliary steps before we do that:

$$\sigma^2 = \left(\sigma_0 + C\frac{\beta_2 z}{2\sigma_0}\right)^2 + \left(\frac{\beta_2 z}{2\sigma_0}\right)^2 , \quad (50)$$

$$\sigma^2 = \sigma_0^2 + C\beta_2 z + (1 + C^2)\left(\frac{\beta_2 z}{2\sigma_0}\right)^2 , \quad (51)$$

$$\frac{\sigma^2}{|\beta_2|L} = \frac{\sigma_0^2}{|\beta_2|L} + sgn(\beta_2)C\left(\frac{z}{L}\right) + \frac{1}{4}(1 + C^2)\frac{|\beta_2|L}{\sigma_0^2}\left(\frac{z}{L}\right)^2 . \quad (52)$$

The parameter p is introduced as follows:

$$p = \frac{1}{4}(1 + C^2) \geq \frac{1}{4} . \quad (53)$$

From this point, having in mind Equations (32), (33) and (34), the broadening factor may be rewritten as:

$$\chi = \chi_0 + sgn(\beta_2)C\xi + \frac{p}{\chi_0}\xi^2 . \quad (54)$$

As such, the pulse width behavior will be defined as:

$$\mu = \frac{\chi}{\chi_0} = 1 + \left[sgn(\beta_2)C + p\left(\frac{\xi}{\chi_0}\right)\right]\left(\frac{\xi}{\chi_0}\right). \quad (55)$$

Having in mind Equation (37), χ_1 and μ_1 will be rewritten as follows:

$$\chi_1 = \chi_0 + sgn(\beta_2)C + \frac{p}{\chi_0} , \quad (56)$$

$$\mu_1 = 1 + \left[sgn(\beta_2)C + \frac{p}{\chi_0}\right]\left(\frac{1}{\chi_0}\right). \quad (57)$$

Now, it is important for us to know what the optimum values are for χ_0 and χ_1. To get there, we must first do the following calculation:

$$\frac{d\chi_1}{d\chi_0} = 1 - \frac{p}{\chi_0^2} = 0 \rightarrow \chi_0^{opt} = \sqrt{p} = \frac{1}{2}\sqrt{1 + C^2} . \quad (58)$$

Then, the optimum value for χ_1 will be simply:

$$\chi_1^{opt} = sgn(\beta_2)C + \sqrt{1 + C^2} . \quad (59)$$

From here, we may finally get the optimum broadening factor and the optimum pulse width behavior, being both written as follows:

$$\chi^{opt} = sgn(\beta_2)C\xi + \frac{1}{2}(1 + \xi^2)\sqrt{1 + C^2} , \quad (60)$$

$$\mu^{opt} = 1 + 2\frac{sgn(\beta_2)C}{\sqrt{1 + C^2}}\xi + \xi^2 . \quad (61)$$

To optimize the bit-rate in a fiber-optic communication system that works with chirped Gaussian pulses, we need to find a chirp parameter that will make the output pulse width to be equal to the input pulse width. Such value will be called as the critical value of the chirp parameter. Since $\mu_0 = 1$, we need to find the solution of the following equation:

$$\mu_1^{opt}(C = C_{cr}) = 1 . \quad (62)$$

After solving Equation (62), we obtain for C_{cr} the given equation:

$$C_{cr} = -\frac{1}{\sqrt{3}}sgn(\beta_2) . \quad (63)$$

From here, we may now know what values we will get for χ_0^{opt} and χ_1^{opt} if $C = C_{cr}$. So, by making $C = C_{cr}$ in Equations (58) and (59), we find that regardless of the signal of β_2, the outcome is the following:

$$\chi_0^{opt} = \chi_1^{opt} = \frac{1}{\sqrt{3}} . \quad (64)$$

The result in Equation (64) was expected to happen since in Equation (62) was defined a condition that forced the output width to be equal to the input width. Going back to Equation (32) and knowing from Equation (62) that $\sigma_{max} = \sigma_0 = \sigma_1$, using the result obtained in Equation (62) we can obtain an expression that will define σ_{max}:

$$\sigma_{max} = 3^{-1/4}\sqrt{|\beta_2|L} . \quad (65)$$

This last result will be important to define the bit-rate value of chirped Gaussian pulses.

4.1. Bit-Rate for Chirped Gaussian Pulses

After a clear understanding on how to optimize chirped Gaussian pulses, regardless of the dispersion regime, we saw from Equation (64) that $\chi_0^{opt} = \chi_1^{opt} = 1/\sqrt{3}$ which means that $\sigma_{max} = \sigma_0 = \sigma_1$. Furthermore, being aware of this fact it was also possible to reach an expression for σ_{max} by doing the following on Equation (32):

$$\frac{1}{\sqrt{3}} = \frac{\sigma_{max}^2}{|\beta_2|L} , \quad (66)$$

since the optimized χ value is $1/\sqrt{3}$ as already seen. From here, we get Equation (65). Now, we may define B_0 as:

$$B_0 = \frac{1}{4\sigma_{max}} = \frac{\sqrt[4]{3}}{4}\frac{1}{\sqrt{|\beta_2|L}} \approx \frac{0.3290}{\sqrt{|\beta_2|L}} . \quad (67)$$

At this point, you may be asking yourselves why there is no dependence on the chirp parameter from the expression above. Well, it is logic that was supposed to happen since we found a chirp value which was $C = C_{cr}$ that allowed to optimize σ_{max}. Then, Equation (67) is already the result of such optimization, meaning it is right the way it is. So far, an optimized chirped Gaussian pulse is the one that will give us a higher bit-rate in a single channel. Once again, even more important is to know the product $B_0^2 L$ to get a figure of merit which will allow to better analyze the performance of a fiber-optic communication system giving us a more accurate perception about the performance of such system for different distances. Without further delay, the product $B_0^2 L$ is as follows:

$$B_0^2 L = \frac{F}{|\beta_2|} \text{ , } with\ F = \frac{\sqrt{3}}{16} \approx 0.1083\,. \tag{68}$$

5. CHIRPED GAUSSIAN PULSES: EFFECT OF HIGHER-ORDER DISPERSION

In this chapter, the effect of the higher-order dispersion (β_3) will now be considered in the analysis of the pulse width behavior of chirped Gaussian pulses and corresponding influence on the bit-rate value of a fiber-optic communication system. As a reminder, a chirped Gaussian pulse, at $z = 0$, assumes the form [2]:

$$A(0,t) = A_0 \exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^2\right]. \tag{69}$$

The RMS width of a chirped Gaussian pulse is [2]:

$$\sigma_0 = \frac{T_0}{\sqrt{2}}. \tag{70}$$

Then, the corresponding broadening factor expression for this kind of pulse is given by [2]:

$$\frac{\sigma^2}{\sigma_0^2} = \left(1 + C\frac{\beta_2 z}{2\sigma_0^2}\right)^2 + \left(\frac{\beta_2 z}{2\sigma_0^2}\right)^2 + (1+C^2)^2\left(\frac{\beta_3 z}{4\sqrt{2}\sigma_0^3}\right)^2. \tag{71}$$

However, before going into further analysis it will be useful to rewrite the last expression using normalized variables. As such, some steps must be made first which are the ones below:

$$\begin{aligned}\frac{\sigma^2}{|\beta_2|L} = \frac{\sigma_0^2}{|\beta_2|L} + sgn(\beta_2)C\left(\frac{z}{L}\right) + \frac{1}{4}(1+C^2)\frac{|\beta_2|L}{\sigma_0^2}\left(\frac{z}{L}\right)^2 \\ + \frac{1}{32}(1+C^2)^2\frac{\beta_3^2}{|\beta_2|^3 L}\left(\frac{|\beta_2|L}{\sigma_0^2}\right)^2\left(\frac{z}{L}\right)^2.\end{aligned} \tag{72}$$

Introducing a new adimensional coefficient called a, which is [3]:

$$a^2 = \frac{\beta_3^2}{|\beta_2|^3 L} \neq 0 \ where\ \beta_2 \neq 0. \tag{73}$$

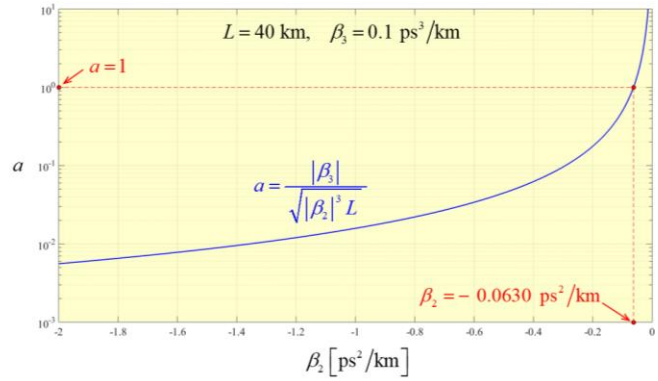

Figure 1: Variation of the adimensional coefficient a with the group-velocity dispersion parameter β_2 for an optical fiber with $L = 40\,km$ and a higher-order dispersion parameter $\beta_3 = 0.1\,ps^3/km$.

In Figure 1, it is presented a concrete example with a fixed value of L and β_3 where we can observe how the value of the adimensional coefficient changes according to β_2. As we can see, with the decreasing of β_2 in module, we see a faster increase of a as β_2 approaches 0. If β_2 is very small (tending to 0), this means that the influence of β_3 in the width of the signal will dominate over the influence of β_2. Having this said, this means that a high value of the adimensional coefficient ($a > 1$) tells the influence of β_3 is greater than the one coming from β_2 in the pulse width of the signal. As such, it is now clear why it is going to be important to discuss the effects of β_3 in chirped Gaussian pulses shape and resulting bit-rate. Now, having in mind Equations (32), (33), (34) and (53), the broadening factor may now be rewritten as:

$$\chi = \chi_0 + sgn(\beta_2)C\xi + \frac{p}{\chi_0}\left(1 + a^2\frac{p}{2\chi_0}\right)\xi^2. \tag{74}$$

Then, the pulse width behavior will be defined as:

$$\mu = \frac{\chi}{\chi_0} = 1 + sgn(\beta_2)C\left(\frac{\xi}{\chi_0}\right) + p\left(1 + a^2\frac{p}{2\chi_0}\right)\left(\frac{\xi}{\chi_0}\right)^2. \tag{75}$$

Now, recalling how χ_1 and μ_1 were defined in Equation (37), we may now write them as:

$$\chi_1 = \chi_0 + sgn(\beta_2)C + \frac{p}{\chi_0}\left(1 + a^2\frac{p}{2\chi_0}\right), \tag{76}$$

$$\mu_1 = 1 + \left[sgn(\beta_2)C + \frac{p}{\chi_0}\left(1 + a^2\frac{p}{2\chi_0}\right)\right]\left(\frac{1}{\chi_0}\right). \tag{77}$$

Now, it is important for us to know what the optimum values for χ_0 and χ_1. To get there, we must first do the following calculation:

$$\frac{d\chi_1}{d\chi_0} = 1 - \frac{p}{\chi_0^2} - a^2\frac{p^2}{\chi_0^3} = 0\,. \tag{78}$$

However, the equation that comes out from the previous expression is the following cubic equation:

$$\chi_0^3 - p\chi_0 - a^2 p^2 = 0\,. \tag{79}$$

The best way to solve the cubic equation that was previously deducted is graphically. As such, we must first rewrite it in a way we can get two curves that will intersect at some point to find a meaningful solution:

$$\chi_0^3 = p(\chi_0 + a^2 p)\,, \tag{80}$$

$$\frac{\chi_0^4}{p} = \chi_0(\chi_0 + a^2 p)\,. \tag{81}$$

If we define $x = \chi_0$, the left side of Equation (81) will look like this:

$$y = \frac{x^2}{\sqrt{p}}. \tag{82}$$

Then, the whole Equation (81) may be written as:

$$y^2 = x(x + a^2 p)\,. \tag{83}$$

At this point, the expressions defining the two curves we need to plot to find a solution for χ_0^{opt} are the ones defined in Equations (82) and (83). So, all we have to do is to find a solution for the given equation:

$$\frac{x^2}{\sqrt{p}} = \sqrt{x(x + a^2 p)}\,, \tag{84}$$

from which we will have a solution which is a point $P(x_0, y_0)$, whose x_0 value will be equal to χ_0^{opt}.

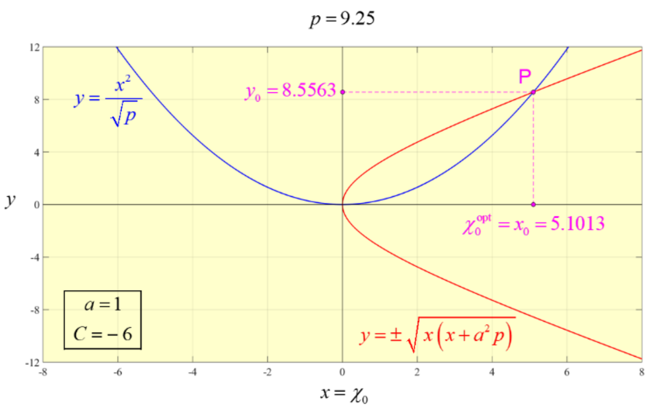

Figure 2: Graphical solution of the cubic equation presented in Equation (79) for $C = -6$ and $a = 1$.

Analyzing Figure 2, two solutions for the cubic equation shown in Equation (79) arise. However, the intersection at the origin if taken as a valid solution would mean it was possible to have pulses whose width was zero, or another to say it is nonexistent. Well, that is not possible physically speaking since it would mean the possibility of having infinite bit-rate values and as we already know, such thing is impossible due to the dispersion effects we are discussing in the course of this study.

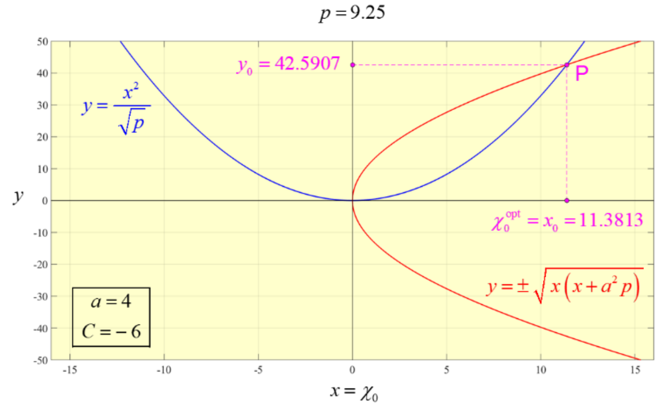

Figure 3: Graphical solution of the cubic equation presented in Equation (79) for $C = -6$ and $a = 4$.

Analyzing Figure 3, we may notice that χ_0^{opt} is more than double than the one obtained in Figure 2. This is no surprise since in Figure 3 we are assuming that the influence of β_3 in the pulse width behavior is much greater than the on assumed in Figure 2. So, the examples shown in those figures confirm why we must not disregard the effects of the higher-order dispersion parameter since it may influence the width of the signal severely, making it much wider than if there was no β_3. Now that we understood how to graphically solve the cubic equation presented in Equation (79), it is now time to show the analytical approach. The first thing we need to know is to have in mind the transition coefficient, which is this one [3]:

$$a_{tr} = \frac{2\sqrt{3}}{3}\left[3(1 + C^2)\right]^{\frac{1}{4}}. \tag{85}$$

Then, after calculating a_{tr} depending on the value of the chirp parameter of the signal we may need to consider, χ_0^{opt} will have one of the two following forms:

$$\begin{cases} a \le a_{tr} \rightarrow \chi_0^{opt} = 2\sqrt{\frac{p}{3}}\cos\left[\frac{1}{3}cos^{-1}\left(\frac{9a^2}{2}\sqrt{\frac{p}{3}}\right)\right] \\ a \ge a_{tr} \rightarrow \chi_0^{opt} = 2\sqrt{\frac{p}{3}}\cosh\left[\frac{1}{3}cosh^{-1}\left(\frac{9a^2}{2}\sqrt{\frac{p}{3}}\right)\right] \end{cases}. \tag{86}$$

To ensure things are clear, for every given C value it will correspond a value for a_{tr}. Then, depending on how great the influence of β_3 is, the value of the adimensional parameter will vary accordingly, i.e, in the same way. So, a being lower or higher than a_{tr} will decide which of the two expressions in Equation (86) will define χ_0^{opt}.

5.1. Critical Value of the Chirp Parameter

In this section, it will be shown how to get the critical value of the chirp parameter, C_{cr}. Firstly, if we recall the process explicated in Equation (62) we know that we need to have in mind the expression that defines μ_1^{opt}. Since we know already μ_1 from Equation (77), we may write μ_1^{opt} as:

$$\mu_1^{opt} = 1 + \left[sgn(\beta_2)C + \frac{p}{\chi_0^{opt}}\left(1 + a^2\frac{p}{2\chi_0^{opt}}\right)\right]\left(\frac{1}{\chi_0^{opt}}\right), \quad (87)$$

being χ_0^{opt} defined according to the expressions shown in Equation (86). Furthermore, since p has a dependence in C there is going to be a critical p value, p_{cr} which will be:

$$p_{cr} = \frac{1}{4}(1 + C_{cr}^2). \quad (88)$$

So, as done in Equation (62), we need to do the same process but now for Equation (87). This means we will need to solve the following equation:

$$sgn(\beta_2)C_{cr} + \frac{p_{cr}}{\chi_0^{opt}}\left(1 + a^2\frac{p_{cr}}{2\chi_0^{opt}}\right) = 0. \quad (89)$$

Solving Equation (89) will give us the value of C_{cr}. However, in that equation, the optimum value χ_0^{opt} is itself a function of C_{cr}. Being aware of such fact, there is only one way to make it easier to get the value of C_{cr} which is by doing a numerical simulation. Trying to solve the last equation analytically would be a tremendous effort.

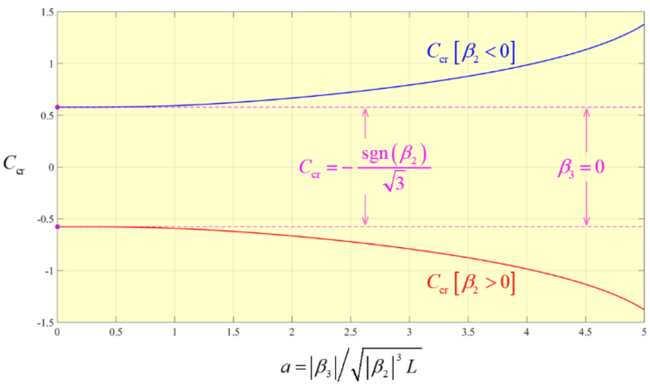

Figure 4: Variation of the chirp critical value, C_{cr} with the adimensional coefficient, a.

Analyzing Figure 4, we see that the chirp critical value will have to be higher if the effects of the third-order dispersion parameter, β_3 are greater than in cases which is not (a equal or close to 0 for example). This means that the presence of chirp will need to higher in order to optimize the bit-rate on the fiber-optic communication systems. Another thing worth mentioning is the symmetry of the chirp critical values curve for both dispersion regimes. The symmetry tells us that the rate of growth in the value of the chirp critical values is the same, being the only difference in the values of C_{cr} their signal which is the opposite between the two dispersion regimes.

5.2. Maximum Bit-Rate Value

Now that we know how the effect of the higher-order dispersion parameter β_3 influences the pulse width of a signal travelling inside the optical-fiber, it is time to show what consequences it will have on the maximum bit-rate of an optical-fiber. From Equation (32), σ_{max} may be written as follows:

$$\sigma_{max} = \sqrt{|\beta_2|L}\sqrt{\chi_{max}}. \quad (90)$$

Defining τ_0 as:

$$\tau_0 = \sqrt{|\beta_2|L}, \quad (91)$$

σ_{max} will be simply:

$$\sigma_{max} = \tau_0\sqrt{\chi_{max}}. \quad (92)$$

χ_{max} is the maximum width for any given value for the adimensional parameter a but considering $C = C_{cr}$. Having in mind the rule of thumb presented in Equation (45), the maximum bit-rate B_0 is given by:

$$B_0 = \frac{1}{4\tau_0\sqrt{\chi_{max}}}. \quad (93)$$

The figure of merit is then:

$$F = \frac{1}{16\chi_{max}}. \quad (94)$$

The bit-rate squared product with the length of the optical fiber will be simply:

$$B_0^2 L = \frac{F}{|\beta_2|}. \quad (95)$$

To get χ_{max}, we need to remind Equation (85). Then, the transition coefficient for $C = C_{cr}$ will be:

$$a_c = \frac{2\sqrt{3}}{3}[3(1 + C_{cr}^2)]^{\frac{1}{4}}. \quad (96)$$

Now, considering the results from Equation (86), χ_{max} will be obtained by the following expressions:

$$\begin{cases} a \le a_c \to \chi_{max} = 2\sqrt{\frac{p_{cr}}{3}}\cos\left[\frac{1}{3}cos^{-1}\left(\frac{9a^2}{2}\sqrt{\frac{p_{cr}}{3}}\right)\right] \\ a \ge a_c \to \chi_{max} = 2\sqrt{\frac{p_{cr}}{3}}\cosh\left[\frac{1}{3}cosh^{-1}\left(\frac{9a^2}{2}\sqrt{\frac{p_{cr}}{3}}\right)\right] \end{cases}. \quad (97)$$

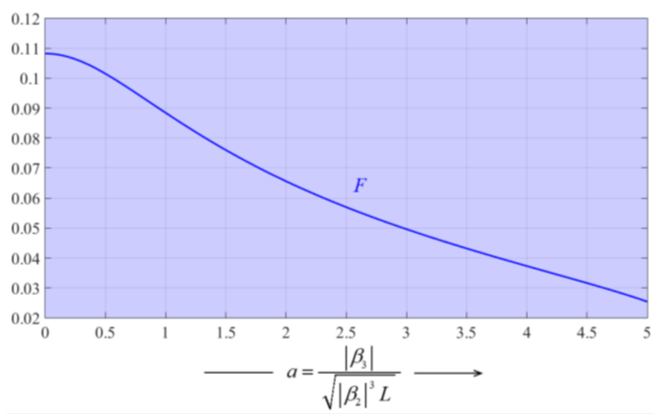

Figure 5: Variation of the figure of merit F with the adimensional coefficient a.

Analyzing Figure 5 and being F a way to measure the efficiency of a fiber-optic communication system, we see that as the influence of β_3 becomes greater in the channel, the efficiency of the mentioned system decreases consequently. This means that if we compare the bit-rate for a given distance where $\beta_3 = 0$ with another one on which $\beta_3 \neq 0$, from Figure 5 we can say that the bit-rate will be lower for the case whose $\beta_3 \neq 0$ in comparison with the one whose $\beta_3 = 0$. Therefore, we conclude once more that β_3 is always not desirable in a fiber-optic communication system. Since β_3 is always undesirable when considering fiber-optic communication systems, it will be nice to have a way to measure the error the effect of higher-order dispersion β_3 introduces in the channel. We have, already noticed that having $\beta_3 = 0$ is the best-case scenario, being the cases when $\beta_3 \neq 0$ the non-desirable ones. That way, we need the error committed in the cases where $\beta_3 \neq 0$. As such, the error is measured as follows:

$$\varepsilon = \frac{B_0(\beta_3 = 0) - B_0(\beta_3 \neq 0)}{B_0(\beta_3 = 0)}, \qquad (98)$$

being $B_0(\beta_3 = 0)$ equal to Equation (67) and $B_0(\beta_3 \neq 0)$ equal to Equation (93). Therefore, the last equation may be rewritten as:

$$\varepsilon = 1 - \frac{1}{\sqrt[4]{3}\sqrt{\chi_{max}}}. \qquad (99)$$

6. SUPER-GAUSSIAN PULSES

A pulse is said to be super-Gaussian if at the input of the fiber, $z = 0$, we have a pulse of the form [4]:

$$A(0,t) = A_0 \exp\left[-\frac{1+iC}{2}\left(\frac{t}{T_0}\right)^{2m}\right], \qquad (100)$$

where m is the parameter that controls the degree of edge sharpness and C is the dimensionless chirp parameter. If $m = 1$, we fall into the category of chirped gaussian pulses. As m increases, the sharpness of the pulse increases accordingly. The RMS width of a super-Gaussian pulse [5] assumes the following form:

$$\sigma_0^2 = \frac{\Gamma(3/2m)}{\Gamma(1/2m)} T_0^2\,. \qquad (101)$$

Then, for any given point z and considering Equations (32), (33) and (34), it is possible to evaluate the broadening factor analytically for super-Gaussian pulses (if $\beta_3 = 0$, since there is no expression in the literature regarding the influence of β_3 in the pulse width) as follows:

$$\chi(\xi) = \chi_0 + sgn(\beta_2)C\,\xi + m^2 f_m\left(\frac{1+C^2}{\chi_0}\right)\xi^2. \qquad (102)$$

where f_m is:

$$f_m = \frac{\Gamma(2-1/2m)\Gamma(3/2m)}{\Gamma^2(1/2m)}. \qquad (103)$$

At this point, it is desirable to know how to obtain the optimal broadening values at each point of the optical fiber, being the input and the output of the optical of special importance. To start, expressions for the optimum value at the input and at the output of the optical fiber are going to be shown:

$$\chi_0^{opt} = m\sqrt{f_m(1+C^2)}\,, \qquad (104)$$

$$\chi_1^{opt} = sgn(\beta_2)C + 2m\sqrt{f_m(1+C^2)}\,. \qquad (105)$$

Then, for every point ξ in the optical fiber, the optimum broadening value may be obtained through the following expression:

$$\chi^{opt}(\xi) = sgn(\beta_2)C\,\xi + m\sqrt{f_m\,(1+C^2)}\,(1+\xi^2)\,. \qquad (106)$$

To compare the width of the pulse at any given point ξ with the width at $\xi = 0$, the upcoming result arises:

$$\mu^{opt}(\xi) = 1 + \frac{sgn(\beta_2)C}{m\sqrt{f_m\,(1+C^2)}}\xi + \xi^2\,. \qquad (107)$$

By applying Equation (62), we may obtain $C = C_{cr}$ which is:

$$C_{cr} = -\frac{\sqrt{m^2 f_m}}{\sqrt{1 - m^2 f_m}} sgn(\beta_2)\,. \qquad (108)$$

However, it will only be possible to find C_{cr} for a certain range of m values. Since $1 - m^2 f_m \geq 0$ is only verified for $1 \leq m \leq 6$, it means for $m \geq 7$ there is no critical value of the chirp parameter. If for $m \geq 7$ we cannot obtain C_{cr}, it also means it will not be possible to achieve $\chi_1^{opt} = \chi_0^{opt}$, therefore we must find another way to achieve optimization. Furthermore, for $m \geq 4$, C_{cr} is no longer the one responsible for determining χ_0^{opt} and χ_1^{opt}. Such role belongs now to C_0 which is the point where the derivative of χ_1^{opt} in relation to the chirp parameter C equals to 0. To calculate C_0, we must do the following:

$$\frac{d\chi_1^{opt}}{dC} = 0\,, \qquad (109)$$

$$C = C_0 = -\frac{sgn(\beta_2)}{\sqrt{4m^2 f_m - 1}}\,, \qquad (110)$$

$$\chi_1^{opt}(C_0) = \sqrt{4m^2 f_m - 1}\,. \qquad (111)$$

6.1. Maximum Bit-Rate Value

Now that we know how the parameter m of a super-Gaussian pulse influences the pulse width of a signal travelling inside the optical-fiber, it is time to show what consequences it will have on the maximum bit-rate of an optical-fiber. From Equation (90), we now we need to define χ_{max} to get σ_{max}. So, χ_{max} is the maximum width for any given value for the adimensional parameter m. However, to get χ_{max} we will have to consider either $C = C_{cr}$ or $C = C_0$ according to the following conditions:

$$\chi_{max} = \begin{cases} \chi_1^{opt}(C = C_{cr}), & 1 \leq m \leq 3 \\ \chi_1^{opt}(C = C_0), & m \geq 4 \end{cases} \qquad (112)$$

Having in mind Equations (93) and (94), we may define the figure of merit for $1 \leq m \leq 3$ and $m \geq 4$. Therefore, the figure of merit for $1 \leq m \leq 3$ will be:

$$\Phi_m = \sqrt{\frac{1 - m^2 f_m}{m^2 f_m}} = \frac{1}{\chi_{max}}, \tag{113}$$

$$F = \frac{\Phi_m}{16}. \tag{114}$$

Moreover, the figure of merit for $m \geq 4$ will be:

$$\Psi_m = \frac{1}{\sqrt{4m^2 f_m - 1}} = \frac{1}{\chi_{max}}, \tag{115}$$

$$F = \frac{\Psi_m}{16}. \tag{116}$$

The bit-rate squared product with the length of the optical fiber will be simply:

$$B_0^2 L = \frac{F}{|\beta_2|}. \tag{117}$$

7. CONCLUSIONS AND FUTURE WORK

The main goal of this work was to find an optimum chirp parameter value for a given pulse, in order to achieve the optimum input and output pulse widths which would allow fiber-optic communication systems to have the best performance possible. In other words, to optimize those systems to be able to work at the maximum bit-rate value according to the optimization performed on the pulses conveying information through the network. With all we have learned from this work, we may now conclude with confidence that the best pulse to achieve optimization in a fiber-optic communication system is a chirped Gaussian pulse, whose parameters are: $m = 1$ and $C_{cr} = -1/\sqrt{3}$ (if $\beta_2 > 0$) or $C_{cr} = 1/\sqrt{3}$ (if $\beta_2 < 0$). From all the pulses analyzed throughout this work, those are the ones that ensure we have the maximum bit-rate achievable by optimization. However, some final notes must be made. Finally, it is now undeniable the presence of chirp is not always a bad thing. It is now proved chirp, possessing the opposite signal of β_2, counteracts the GVD effects and therefore allows the possibility of having fiber-optic communication systems with higher bit-rates, rather than the ones with no chirp at all, as it was proved in the course of this work. From this point, future work perspectives may now be pointed. Deducting a broadening factor expression for hyperbolic secant pulses which includes the chirp parameter (or even more demanding, one that includes the HOD effects), in order to quantify the performance impact such pulses would have on fiber-optic communication systems. Regarding the super-Gaussian pulses, getting an expression for the broadening factor which includes the HOD effects. Although it is obvious the performance is always worst having β_3 in play, being able to analyze such case by means of analytical approach is always better than having to run different simulations for every single case we might find relevant. One final suggestion, being this the most challenging one, would be to include the non-linear effects on pulse behavior.

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