Spin-Orbit Interaction and Chaos in Celestial Mechanics

Manuel Maria Murteira Barreira da Cruz

May 2017

Under supervision of Prof. Doutor Rui Manuel A. Dilão
Non-Linear Dynamics Group, University of Lisbon, IST
Department of Physics, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

Abstract

This extended abstract contains a summary of the Thesis it is associated with. We begin by giving a precise definition of the concept of spin-orbit interaction, illustrating along the way with some examples of our Solar System. Afterwards, we briefly review some of the theory developed over the past few decades to explain the mechanism by which two periodic motions become coupled in a certain sense or uncouple, pointing out some of the shortcomings of the models in order to justify our approach to the problem. We consider a convenient toy model for the study of the rotation of a body: a dumbbell-shaped satellite, which has the minimal features of a rigid body, moving in the gravitational field of a point mass, a system that we call the Keplerian Dumbbell system. This model will enable us to analyse the relations between the orbital and the spin motions of a body in a simple environment, with no approximations. Finally, we extend our analysis of the Keplerian Dumbbell system to the framework of the Restricted Three-Body Problem, with the goal of characterising the mechanism responsible for the observed chaotic behaviour of Nix, the small satellite of the Pluto-Charon system that was photographed by the New Horizons spacecraft in 2015, and which tumbles unpredictably in its orbit.

Keywords: Keplerian Dumbbell, Spin-Orbit Interaction, Synchronisation, Resonance, Chaos, Restricted Three-Body Problem

1. The Spin-Orbit Effect in the Solar System

The so-called spin-orbit interaction, or coupling, in Celestial Mechanics refers to the dependence of the rotational period of a satellite on its orbital period around a primary planet or star, and it manifests itself in the form of a synchronisation of two periodic motions, [11, 4]. There are commensurabilities between various types of frequencies or periods; for instance, the orbit-orbit coupling involves the orbital periods of two or more bodies. However, our main focus is on the synchronisation and/or resonance between the rotational and the orbital periods of a single body. The most obvious example of these is the Moon, whose orbital period is exactly equal to its rotational period, which leads to the well-known fact that it keeps the same face towards the Earth at all times: we say that the Moon is in a 1:1 or *synchronous* state. Another example of a synchronous 1:1 rotation of particular interest is that of Charon, the major moon of the dwarf planet Pluto. In this case, not only are the rotational and orbital periods of Charon equal, they are also equal to the period of rotation of Pluto, which results in the satellite being seen always in the same position in the sky from the planet.[1] In fact, most of the major natural satellites in the solar system are rotating in a 1:1 synchronous state, as shown in Table I. The notable exceptions to this spin-orbit state among the satellites are Hyperion and Nereid; the rotation period of the former is chaotic [20, 9], while the rotation period of the latter seems to be near a 750:1 state [7]. Nereid is a mystery, and its rotation appears to be rather irregular [15]. Mercury, on the other hand, was found to be in a 3:2 synchronous spin-orbit state as it revolves around the Sun, thereby completing three rotations on its axis while making two revolutions around the Sun [14].

[1] This is called a state of mutual “tidal-locking" [11].

2. Capture into Synchronisation

The effective mechanism of capture into synchronisation remains relatively unknown. However, it is widely accepted that one has to invoke dissipative torques in order to explain the state of rotation of a satellite or planet [1]. One way of accomplishing this is by assuming that the initial spin period of the body was short and introducing tidal torques, which act to brake this spin. The body will then pass through several spin-orbit states, and may be captured into any one of them if the value of the spin rate is below a critical value when the angle γ

Planet	Satellite	e	T (d)	Rotation State
Mercury		0.206	87.97	3:2
Earth	Moon	0.054900	27.321661	1:1
Mars	Phobos	0.0151	0.318910	1:1
	Deimos	0.00033	1.262441	1:1
Jupiter	Io	0.0041	1.769138	1:1
	Europa	0.0101	3.551810	1:1
	Ganymede	0.0015	7.154553	1:1
	Callisto	0.007	16.689018	1:1
Saturn	Epimetheus	0.009	0.694590	1:1
	Enceladus	0.0045	1.370218	1:1
	Titan	0.0292	15.945421	1:1
	Hyperion	0.1042	21.276609	chaotic
Uranus	Miranda	0.0027	1.413	1:1
	Ariel	0.0034	2.520	1:1
	Umbriel	0.0050	4.144	1:1
	Oberon	0.0008	13.463	1:1
Neptune	Proteus	0.000	1.122315	1:1
	Triton	0.0004	-5.876854	1:1
	Nereid	0.7512	360.13619	$T_{\rm rot} = 0.48$ d†
Pluto	Charon	0.0076	6.387223	1:1

Table 1: Orbital period (T) and synchronisation state of some of the major, natural satellites in the solar system. We also show the eccentricity (e) of the orbit of the satellites. Data taken from [11]. † The value of the rotation period of Nereid $T_{\rm rot}$ has been taken from [7].

in

$$\ddot{\gamma} = -\mathrm{sgn}(H(p,e))\frac{1}{2}\omega_0^2 \sin 2\gamma + \frac{|\langle N_s \rangle|}{C} \quad (1)$$

enters its first libration, in analogy with a pendulum [6]. This is the content of the so-called averaging theory for the spin-orbit coupling. In equation (1), the $H(p, e)$ are power series in the eccentricity e that also depend on the synchronisation state[2] p, which, by hypothesis of the theory, is a rational number. The constant ω_0 is the libration frequency and depends on the shape of the body, $|\langle N_s \rangle|$ is a mean tidal torque, C is the largest principal moment of inertia of the body, and $\mathrm{sgn}(\cdot)$ is the sign function. Equation (1) itself is known as the pendulum equation. In general, though, one cannot know *a priori* the initial conditions of the body and, consequently, if this requirement will be satisfied. So, a suitable probability distribution over the initial angular velocity should be introduced in order to calculate capture probabilities. Goldreich and Peale [5, 6], for instance, define these capture probabilities as the ratio of the range of the energy integral of Eq. (1),

$$E = \frac{1}{2}C\dot{\gamma}^2 - \frac{3}{4}(B-A)\, n^2 H(p,e)\cos 2\gamma, \quad (2)$$

for which capture results, to the whole range of E for which γ becomes an angle of libration, where A, B, C are the principal moments of inertia of the body and n its average orbital angular velocity. However, they also argue that the mean tidal torque, $\langle N_s \rangle$, has to somehow vary with $\dot{\gamma}$ for capture to occur, otherwise the body will simply pass through synchronisation and continue to despin [5, 6].

Estimating capture probabilities is then just a matter of incorporating a model of tidal dissipation that accounts for the variation of $\langle N_s \rangle$ with $\dot{\gamma}$ into the theory. This has been done in [5, 6] for two models in which the tidal potential is expanded in a Fourier time series and each component of the tide given a phase lag. In the first model, which assumes that the phase lags depend on the tidal frequencies, a very low value of about 7% was calculated for the probability of capture of Mercury into the 3:2 state, for $e = 0.206$, and this probability was found to vary with the value of $(B - A)/C$ and the eccentricity. On the other hand, the second model, in which only the signs, but not the magnitudes of the lags, depend on frequency, gave a much higher value of about 70% for the corresponding probability, and this was found to be determined by the synchronisation state p and the eccentricity alone. This model has some shortcomings though, as it leads to unrealistic constant torques and doesn't account for the damping of the librations of the body [11].

An alternative mechanism for Mercury's capture into the 3:2 state has been proposed in [2], where it is argued that for any eccentricity, the spin rate of a body is naturally driven towards an equilibrium value which depends on its current eccentricity. Then, since the eccentricity varies due to the chaotic evolution of Mercury's orbit, the spin rate can be raised and lowered, thus making it possible for a synchronisation to be repeatedly crossed, and increasing Mercury's probability of becoming trapped. Correia and Laskar [2] have computed a probability of capture into the 3:2 state for Mercury of 55.4%.

3. The Keplerian Dumbbell

As we saw, the averaging theory adopts the tidal dissipation hypothesis to justify the capture into synchronisation. However, this hypothesis seems to be weak. Not only the models of tidal dissipation rely on poorly determined parameters which are related to the internal structure of the bodies involved in the spin-orbit interaction, tides on small, solid bodies like some moons or asteroids are much weaker than those that act on larger bodies in hydrostatic equilibrium [4].

We follow a different approach here. Let us consider a system comprised of a primary point mass m_1 and a dumbbell, interacting through their mutual gravitational force. We call this system the Keplerian Dumbbell (KD). The dumbbell is formed by two point masses m_2 and m_3, connected by a rigid massless rod of length ℓ. To describe the motion of the KD system, we consider the inertial reference frame $\mathcal{S} = (Cxyz)$ centred at the centre of mass of

[2] The synchronisation state p of a satellite is defined as the ratio of the satellite's spin rate to its average orbital angular velocity or *mean motion*.

the three masses (Figure 1).

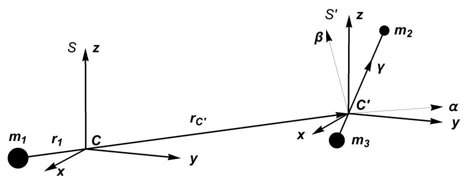


Figure 1: Reference frames of the Keplerian Dumbbell (KD) problem. In the reference frame $\mathcal{S}$ of the centre of mass of the three-body system, the centre of mass C' of the dumbbell has cylindrical coordinates (r, θ, z). In $\mathcal{S}'$, the orientation of the dumbbell is specified by the angles ψ (polar) and ϕ (azimuthal). The orientations of the coordinate axes of $\mathcal{S}$ and $\mathcal{S}'$ are the same.

The dumbbell is allowed to rotate in the three-dimensional ambient space, and the configuration manifold of each of the dumbbell masses is a sphere $\mathbb{S}^2$, centred at the centre of mass C' of the dumbbell. To describe the attitude dynamics of the dumbbell relative to the reference frame $\mathcal{S}' = (C'xyz)$, we consider the azimuthal and the polar spherical angles ϕ and ψ, respectively. The distances of the masses m_2 and m_3 to the centre of mass C' of the dumbbell are $\ell_2 = \ell m_3/(m_2+m_3)$ and $\ell_3 = \ell m_2/(m_2+m_3)$. The unit vectors of the coordinate axes (x, y, z) are $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. We denote by $\boldsymbol{\gamma}$ the unit vector directed along the dumbbell towards mass m_2. In spherical coordinates, $\boldsymbol{\gamma} = \cos\phi\sin\psi\,\boldsymbol{e}_1 + \sin\phi\sin\psi\,\boldsymbol{e}_2 + \cos\psi\,\boldsymbol{e}_3$. The projection of the rod on the (x, y)-horizontal plane of $\mathcal{S}'$ is $\boldsymbol{p}(\phi) = \cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2$, then we define a new unit vector $\boldsymbol{\alpha} = \boldsymbol{p}(\phi+\pi/2) = -\sin\phi\,\boldsymbol{e}_1 + \cos\phi\,\boldsymbol{e}_2$ (Figure 1). As $\boldsymbol{\gamma}\cdot\boldsymbol{\alpha} = 0$, $\boldsymbol{\alpha}$ is perpendicular to $\boldsymbol{\gamma}$. On the other hand, as $\boldsymbol{\alpha}\cdot\boldsymbol{e}_3 = 0$, the two vectors $\{\boldsymbol{\gamma}, \boldsymbol{e}_3\}$ define a plane perpendicular to $\boldsymbol{\alpha}$ and, therefore, the angular velocity of the dumbbell around the instantaneous direction of rotation $\boldsymbol{\alpha}$ is $\dot{\psi}$. Let $\boldsymbol{\beta} = \boldsymbol{\gamma}\wedge\boldsymbol{\alpha} = -\cos\psi\cos\phi\,\boldsymbol{e}_1 - \cos\psi\sin\phi\,\boldsymbol{e}_2 + \sin\psi\,\boldsymbol{e}_3$ be a third unit vector. Then the unit vectors $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ are mutually perpendicular and define the principal axes of inertia of the dumbbell.

In the reference frame $\{C'\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}\}$, the inertia tensor of the dumbbell is a diagonal matrix, whose diagonal elements are

$$I_{\boldsymbol{\alpha}} = I_{\boldsymbol{\beta}} = m_2\ell_2^2 + m_3\ell_3^2 = \frac{m_2 m_3}{m_2+m_3}\ell^2 \text{ and } I_{\boldsymbol{\gamma}} = 0. \tag{3}$$

3.1. Equations of motion

After transforming the variables of the problem into new, dimensionless variables $(u,\ \theta,\ v,\ \phi,\ \psi)$, the Lagrangian of the KD system in the new variables becomes

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(u'^2 + v'^2 + u^2\theta'^2\right)\\ &+\frac{1}{2}\frac{(1-\delta)\delta}{\mu}\varepsilon^2\left(\psi'^2 + \phi'^2\sin^2\psi\right)\\ &+\frac{1-\delta}{\|\boldsymbol{\rho_{12}}\|} + \frac{\delta}{\|\boldsymbol{\rho_{13}}\|},\end{aligned} \tag{4}$$

where $\delta = m_3/(m_2+m_3)$ is a parameter that measures the relative weight of the masses of the dumbbell. For a symmetric dumbbell with $m_2 = m_3$, $\delta = 1/2$. The parameter ε measures the length of the rod of the dumbbell in units of r_0, the approximate radius of the trajectory of the dumbbell. For a dumbbell satellite or asteroid, ε is close to zero. The mass parameter $0 < \mu \leq 1$, given by $\mu = m_1/(m_1+m_2+m_3)$, measures the relation between the masses of the primary and of the dumbbell. For a small dumbbell satellite, $\mu \simeq 1$. The dimensionless distances between the masses, $\|\boldsymbol{\rho_{12}}\|$ and $\|\boldsymbol{\rho_{13}}\|$, are defined by

$$\begin{aligned}\|\boldsymbol{\rho_{12}}\|^2 &= u^2 + v^2 + \delta^2\varepsilon^2 + 2\,\delta\,\varepsilon\,(u\sin\psi\\ &\quad\times\cos(\theta-\phi) + v\cos\psi)\\ \|\boldsymbol{\rho_{13}}\|^2 &= u^2 + v^2 + (1-\delta)^2\,\varepsilon^2 - 2\,(1-\delta)\,\varepsilon\\ &\quad\times\,(u\sin\psi\cos(\theta-\phi) + v\cos\psi)\;.\end{aligned} \tag{5}$$

From (4), the equations of motion of the KD system in the $(u,\ \theta,\ v,\ \phi,\ \psi)$ coordinates are

$$\left\{\begin{aligned}&u'' - u\theta'^2 = -(1-\delta)\frac{u + \delta\,\varepsilon\sin\psi\cos(\theta-\phi)}{\|\boldsymbol{\rho_{12}}\|^3}\\ &+\delta\frac{(1-\delta)\,\varepsilon\sin\psi\cos(\theta-\phi) - u}{\|\boldsymbol{\rho_{13}}\|^3}\\ &u^2\theta'' + 2uu'\theta' = u(1-\delta)\delta\,\varepsilon\sin\psi\sin(\theta-\phi)\\ &\left(\frac{1}{\|\boldsymbol{\rho_{12}}\|^3} - \frac{1}{\|\boldsymbol{\rho_{13}}\|^3}\right)\\ &v'' = -(1-\delta)\frac{v + \delta\,\varepsilon\cos\psi}{\|\boldsymbol{\rho_{12}}\|^3} - \delta\frac{v - (1-\delta)\,\varepsilon\cos\psi}{\|\boldsymbol{\rho_{13}}\|^3}\\ &\varepsilon\left(\phi''\sin^2\psi + 2\,\phi'\,\psi'\sin\psi\cos\psi\right) = -\mu\,u\sin\psi\\ &\sin(\theta-\phi)\left(\frac{1}{\|\boldsymbol{\rho_{12}}\|^3} - \frac{1}{\|\boldsymbol{\rho_{13}}\|^3}\right)\\ &\varepsilon\left(\psi'' - \phi'^2\,\sin\psi\cos\psi\right) = -\mu\,(u\cos\psi\cos(\theta-\phi)\\ &-v\sin\psi)\left(\frac{1}{\|\boldsymbol{\rho_{12}}\|^3} - \frac{1}{\|\boldsymbol{\rho_{13}}\|^3}\right).\end{aligned}\right. \tag{6}$$

These equations depend on the three parameters δ, ε and μ. The left hand side of the second equation in (6) may be written as $\dfrac{\mathrm{d}\left(u^2\theta'\right)}{\mathrm{d}\tau}$, and so $u^2\theta'$ is proportional to the angular momentum of the motion of the centre of mass of the dumbbell with respect to the centre of mass of the three-body system. Therefore, the second equation in (6) can be understood as a condition of conservation of angular momentum.

3.2. Steady states and stability analysis
By construction, the fixed points of the system of equations (6) are the steady states of the KD system. Here, we call steady states to periodic trajectories of the dumbbell. To calculate the coordinates of the fixed points of the system of equations (6), we impose that some of the components of the vector field defined by (6) are zero.

3.2.1 Steady state 1

With the choice $u = 0$ in the first equation in (6), the centre of mass of the KD system coincides with the position of the primary mass m_1. Therefore, to fulfill a fixed point condition we must also have $v = 0$, together with the two speed conditions $u' = 0$ and $v' = 0$. In this case, by (5), $\|\boldsymbol{\rho_{12}}\| = \delta\varepsilon$ and $\|\boldsymbol{\rho_{13}}\| = (1-\delta)\varepsilon$, and the right hand side of the first equation in (6) is zero only if $\delta = 1/2$, implying that $m_2 = m_3$ and $\|\boldsymbol{\rho_{12}}\| = \|\boldsymbol{\rho_{13}}\|$. The second equation in (6) is identically zero, the orbital angular momentum of the dumbbell is also zero, and the dumbbell masses rotate synchronously with arbitrary constant angular speed. Therefore, in the configuration space, the KD system is constrained to move in a sphere of arbitrary radius, with centre at the primary body (Figure 2). If $m_2 \neq m_3$, this steady state does not exist.

Under these conditions, by (6), the attitude of the dumbbell system around the primary is described by the equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\phi' \sin^2\psi\right) = 0 \\ \psi'' - \phi'^2 \sin\psi\cos\psi = 0. \end{cases} \tag{7}$$

The first equation in (7) is a conservation law, implying that $\phi' \sin^2\psi = c$, where c is constant. If $c = 0$, then $\phi(t)$ is constant, for every $t \in \mathbb{R}$, and the second equation in (7), reduces to $\psi'' = 0$. In this case, the dumbbell system rotates around the primary, with fixed azimuthal angle ϕ =constant. In Figure 2a), we depict a solution trajectory of the KD system for $\phi(t) = 0$, for every $t \in \mathbb{R}$.

If $c \neq 0$, the second equation in (7) reduces to

$$\psi'' - c^2 \frac{\cos\psi}{\sin^3\psi} = 0. \tag{8}$$

In the two-dimensional (ψ, ψ') phase space, equation (8), with $\psi \in (0, \pi)$, has a unique fixed point with coordinates $(\psi = \pi/2, \psi' = 0)$. The equation (8) is derivable from the Hamiltonian $\mathcal{H}^* = \psi'^2/2 + c^2/(2\sin^2\psi)$. For the fixed point $(\psi = \pi/2, \psi' = 0)$, $\mathcal{H}^* = c^2$, $\phi' = c$, and the dumbbell has a three-body Eulerian type solution (Figure 2b)), [13].

It is easily shown that the fixed point $(\psi = \pi/2, \psi' = 0)$ is Lyapunov stable of the centre type. For any initial condition away from the fixed point in the (ψ, ψ') phase space, the dumbbell librates around $(\psi = \pi/2, \psi' = 0)$. Due to the conservation law — first equation in (7) — $\phi(t)$ is also periodic, with the same period as $\psi(t)$, and the trajectory in the configuration space is also periodic (Figure 2c)).

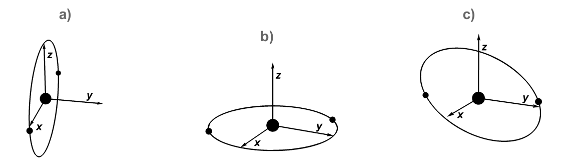


Figure 2: Steady solutions in the three-dimensional configuration space of the dumbbell with equal masses ($m_2 = m_3$). In a), $c = 0$ and, in the configuration space, the KD rotates around the primary body. This trajectory has been calculated for the azimuthal angle $\phi = 0$. In b) and c), $c = 1$. In b), the dumbbell has been obtained with the initial condition $(\psi = \pi/2, \psi' = 0)$ and $\phi' \neq 0$ is a constant. In c), the dumbbell librates around the fixed point $(\psi = \pi/2, \psi' = 0)$ in the (ψ, ψ') phase space, which corresponds to a closed trajectory in the configuration space.

The solutions of the KD system depicted in Figure 2 are also Eulerian type solutions of the general three-body problem, provided $m_2 = m_3$, [13, 12]. They occur for the following conditions:

$$u = 0;\ v = 0;\ u' = 0;\ v' = 0;\ m_2 = m_3, \tag{9}$$

and they are unstable. In fact, the first equation in (6), linearised around this steady state is $u'' = (\omega^2 + 16/\varepsilon^3)u$, showing that the dumbbell Eulerian trajectories are unstable. Moreover, as this fixed point only exists for $m_2 = m_3$, any infinitesimal perturbation on the masses destroys the periodic point. This implies that the dynamical system (6), with $m_2 = m_3$, is structurally unstable, [8].

3.2.2 Steady state 2

For $m_2 = m_3$ i.e. $\delta = 1/2$, the equations (6) have the following equilibrium solution

$$\begin{aligned} u = u_0 > 0;\ v = 0;\ \psi = 0,\ \pi; \\ u' = v' = \psi' = 0;\ m_2 = m_3, \end{aligned} \tag{10}$$

according to which the dumbbell is always oriented vertically. By the second equation in (6), the angular momentum $L_z = u^2\theta'$ is conserved, and the distances $\|\boldsymbol{\rho_{12}}\|$ and $\|\boldsymbol{\rho_{13}}\|$ are equal.

This steady state is unstable. In fact, the linear approximation of the first equation in (6) is $u'' = (u - u_0)96u_0^2/(4u_0^2 + \varepsilon^2)^{5/2}$. Furthermore, as this fixed point only exists for $m_2 = m_3$, any infinitesimal perturbation on the masses destroys the steady state.

In Figure 3a), we show the orbit in configuration space of steady state 2.

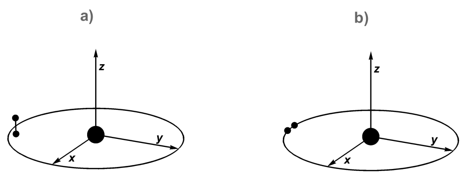


Figure 3: Circular orbits (steady states 2 and 3) in the three-dimensional configuration space of the KD system, with $m_2 = m_3$. The positions of the dumbbell masses m_2 and m_3 at time $\tau = 29$ are shown. In a), we show a circular orbit for the steady state 2, where $\psi(t) = 0$ or $\psi(t) = \pi$, for every $t \geq 0$. In b), we show the stationary circular orbit for the steady state 3, where $\psi(t) = \pi/2$ and $\theta(t) - \phi(t) = \pm\pi/2$, for every $t \geq 0$. The rotational angular velocity of the dumbbell has the same sign as the angular velocity of translation.

3.2.3 Steady state 3

This steady state is similar to the previous one, but occurs for a different orientation of the dumbbell:

$$\begin{aligned} &u = u_0 > 0;\ v = 0;\ \theta - \phi = \pm\pi/2;\ \psi = \pi/2; \\ &u' = v' = \psi' = 0;\ \theta' = \phi';\ m_2 = m_3. \end{aligned} \tag{11}$$

In Figure 3b), we show the orbit in configuration space of this circular steady state. As $m_2 = m_3$, the angular momentum is also conserved. This particular solution of the KD system only exists for $m_2 = m_3$. From the point of view of an observer at the origin of the coordinate system $\mathcal{S}$, m_2 and m_3 maintain the same relative orientation, and the rotation of the dumbbell in the local reference frame $\mathcal{S}'$ is locked with the translation. This corresponds to spin-orbit coupling in a $1 : 1$ synchronisation. However, as $m_2 = m_3$, any infinitesimal perturbation on the masses destroys the periodic point as in the previous steady states. As before, linearising the first equation in (6) around (11), we obtain $u'' = (u - u_0)3u_0^2/(u_0^2 + \varepsilon^2/4)^{5/2}$, and, therefore, this steady state is also unstable.

3.2.4 Steady states 4 to 6

The system of equations (6) has three more steady states:

$$\begin{aligned} &u = u_0 > 0;\ v = 0;\ \theta - \phi = 0, \pi;\ \psi = \pi/2; \\ &u' = v' = \psi' = 0;\ \theta' = \phi', \end{aligned} \tag{12}$$

numbered according to the conditions:

$$\begin{aligned} &\text{steady state 4: } m_2 < m_3, \phi = \theta \\ &\text{steady state 4b: } m_2 > m_3, \phi = \theta + \pi \\ &\text{steady state 5: } m_2 > m_3, \phi = \theta \\ &\text{steady state 5b: } m_2 < m_3, \phi = \theta + \pi \\ &\text{steady state 6: } m_2 = m_3, \phi = \theta \text{ or } \phi = \theta + \pi. \end{aligned} \tag{13}$$

As $\phi(t) = \theta(t)$, the dumbbell rotates in the direction of the translational motion, and the two masses m_2 and m_3 are aligned with the direction defined by the origins of the reference frames $\mathcal{S}$ and $\mathcal{S}'$, Figure 1. This equilibrium solution corresponds to the $1 : 1$ synchronisation of the translational and rotational motions. In Figure 4, we show a sequence of dumbbell positions along a circular orbit in configuration space $\mathcal{S}$.

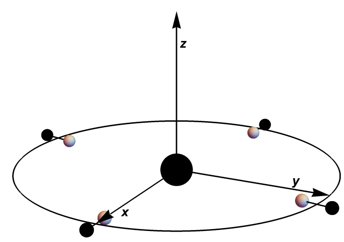


Figure 4: Circular orbit in the three-dimensional configuration space of the KD system, with $m_2 = m_3$, corresponding to steady state 6. The period of rotation of the dumbbell around its centre of mass is the same as the period of translation of the dumbbell, corresponding to a $1 : 1$ synchronisation.

By considering the planar subspace of the phase space and by applying a suitable canonical transformation, one can transform the equations of motion (6) into a reduced Hamiltonian system of equations. Moreover, analysing the eigenvalues of the Hessian matrix of the Hamiltonian function associated to this system, it may be demonstrated that for sufficiently large radius u_0 of the dumbbell circular trajectory, steady states (12)–(13) are Lyapunov unstable. In fact, numerical analysis suggests that this is true for all ranges of the radius u_0. Consequently, all the steady states of the KD system are unstable, and the dynamical system is structurally unstable for equal masses of the dumbbell.

4. Restricted Three-Body Problem with Dumbbell Satellite

Nix is a small natural satellite of the Pluto-Charon system, discovered in 2005 along with Hydra, another moon of the binary system [19]. In recent years, interest has risen in these small moons, not only because the New Horizons spacecraft flew through the double planet system and captured images of their surfaces, but also because it was discovered that they spin and wobble unpredictably [16]. The effect is likely due to the constantly shifting gravitational field produced by the larger bodies, Pluto and Charon, and is only enhanced by the prolate spheroidal shape of the moons, which are the subject of asymmetric gravitational torques. In Chapter 3 of the associated Thesis, the dynamics of the three-body system constituted by Pluto, Charon and Nix is analysed in detail, by using a convenient toy model for Nix — that of a dumbbell satellite. This system may be strictly regarded as a special case of the four-body problem, for which there is a constraint on the distance between two of the bodies. This is the most simple and minimal model of a rigid body that enables us to analyse the mechanism of resonance and the transition to chaos of the small moon. Furthermore, since the mass of Nix is much smaller than that of Pluto or Charon, we treat the case in study in the framework of the Restricted Three-Body Problem (R3BP), [18].

In this section we go over and summarise some of the main ideas and results obtained in Chapter 3 of the associated Thesis.

The dumbbell model discussed in Section 3 of the present paper was incorporated into the framework of the R3BP, with the goal of modelling the dynamics of the satellite Nix under the gravitational influence of Pluto and Charon. This problem may be analysed either in the inertial reference frame $\mathcal{S}$ centred at the barycentre of the massive bodies (the so-called primaries), or in a reference frame $\mathcal{R}$ in corotation with these bodies (see Figure 5), which trace out Keplerian ellipses in the inertial space. The reference frame $\mathcal{R}$ is called *synodic* [3].

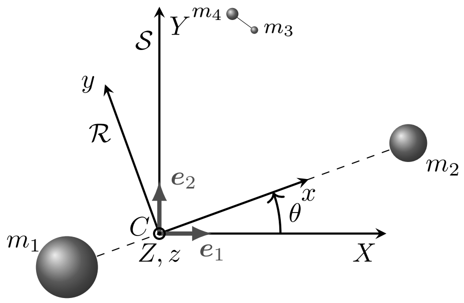


Figure 5: Inertial $\mathcal{S} = (C\,XYZ)$ and synodic $\mathcal{R} = (Cxyz)$ reference frames used in the study of the R3BP with dumbbell satellite. The reference frame $\mathcal{R}$ corotates with the primaries, masses m_1 and m_2, which move along ellipses in inertial space, around their centre of mass, point C. The variable θ is the angular coordinate that sweeps those ellipses in a counterclockwise direction. The dumbbell satellite has masses m_3 and m_4.

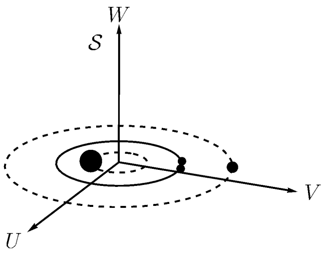
We derived the exact equations of motion of the R3BP *with dumbbell satellite* and focused on two special cases, one in which the dumbbell's centre of mass is constrained to move in the Lagrange plane of the orbits of the primaries — the Planar Circular Restricted Three-Body Problem (PCR3BP) with dumbbell satellite — and the other in which it is constrained to move along an axis orthogonal to this plane and that passes through the barycentre of the primaries — the Sitnikov Problem (SP), [17].

Necessary conditions for these two motions to occur were obtained and, based on that, the steady states of the system were found. Some of these steady states are the direct analogues or extensions of the Lagrangian points in the (conventional) Circular Restricted Three-Body Problem (CR3BP) ([10, 18]). While in the conventional CR3BP there are three equilibrium points over the "x" axis, when the satellite has a dumbbell shape, there may be 1, 2, 3, 4 or 5 equilibrium points over this axis. On the other hand, there are still only two "equilateral", symmetric, equilibrium points in the problem with dumbbell satellite, as in the case of the conventional CR3BP.

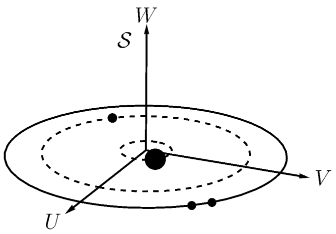
We have determined that these steady states only occur for special configurations of the dumbbell, namely, when it lies on the Lagrange plane, making either a right angle, or being aligned, with the line that connects the primaries, or when it is aligned orthogonally to this plane (Figure 6). Moreover, the new equilateral equilibrium points only exist for configurations in which the dumbbell either lies on the Lagrange plane, aligned with the line that joins the primaries, or is aligned orthogonally to the plane (Figure 7). And they don't occur for different masses of the primaries, unless the dumbbell is aligned vertically to the plane.

Finally, these equilibrium points only exist for equal masses of the dumbbell, therefore we conclude that the R3BP with dumbbell satellite is a structurally unstable problem, much like the KD system.

The special case of the PCR3BP with dumbbell satellite in which the spin axis of the dumbbell is orthogonal to the Lagrange plane was also studied numerically, with values of the parameters appropriate for the Pluto-Charon-Nix system. These parameters are similar to the ones introduced in Section 3, in the context of the KD system. We verified that most, if not all, initial conditions will



(a)

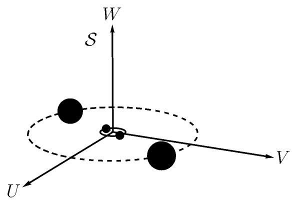


(b)

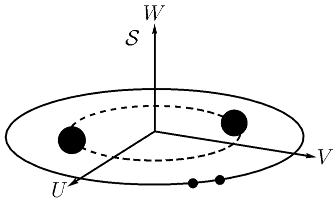
Figure 6: Two steady states of the PCR3BP with dumbbell satellite represented in the inertial $\mathcal{S}$ frame at $\theta = 3\pi/5$ (a) and at $\theta = 5\pi/4$ (b), for which the dumbbell lies in the Lagrange plane and maintains a right angle to the line that joins the primaries. The variables (U, V, W) are dimensionless, cartesian coordinates in $\mathcal{S}$. The dashed lines show the paths of the primaries in inertial space, whereas the solid line illustrates the trajectory of the dumbbell's centre of mass. (a) The centre of mass of the dumbbell describes a circular trajectory in the region between the primaries; (b) the centre of mass of the dumbbell describes a circular trajectory in the region outside the primaries.

lead to either a collision with a primary or ejection of the dumbbell from the system in finite time. This could mean that Nix can be just passing by our solar system right now. For instance, we obtained a solution in which Nix would eventually collide with Charon in a little over 7 years. Some trajectories show definitely signs of chaoticity or, at least, of non periodicity.

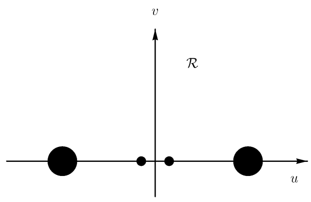
Lastly, we point out that, unlike the conventional CR3BP, the CR3BP with dumbbell satellite doesn't in general possess any invariant. We derived sufficient conditions for the conservation of an effective Hamiltonian in the CR3BP with dumbbell satellite. For instance, whenever the motion of the dumbbell is such that it is either always perpendicularly oriented to the Lagrange plane or confined to move



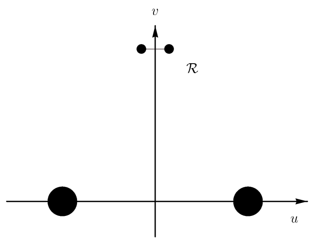
(a)



(b)



(c)



(d)

Figure 7: Steady states of the PCR3BP with dumbbell satellite, in which the dumbbell lies in the Lagrange plane and is aligned along the direction of the segment line that joins the primaries, represented in the inertial $\mathcal{S}$ ((a), (b)) and synodic $\mathcal{R}$ ((c), (d)) reference frames. The variables (u, v) are dimensionless, cartesian coordinates in $\mathcal{R}$. (a), (c): the dumbbell and primaries are depicted at $\theta = 4\pi/3$ (also represented in (a) is the trajectory of the masses of the dumbbell in the inertial space); (b), (d): the dumbbell and primaries are depicted at $\theta = 7\pi/4$. This steady state is an extension of the equilateral Lagrangian points of the CR3BP ([10, 18]) to the case where the satellite has a dumbbell shape.

and rotate in the mediating plane of the line that connects the primaries, there will be conservation of an effective Hamiltonian.

5. Conclusions

In the Thesis we proposed to study the problem of spin-orbit interaction using an approach other than the Averaging Theory, or the Tidal Theory, which rely on poorly determined parameters related to the internal structure of celestial bodies. Taking that into account, we decided to consider the problem where a satellite, modelled as a dumbbell, revolves around a point mass, interacting with it through the gravitational force. This is what we called the Keplerian Dumbbell (KD) system. Despite not being new, an analysis without approximations of the full dynamics of the KD was lacking. We derived the exact equations of motion for this system, and then found and analysed its steady states or stationary orbits. We showed that all the steady states of the KD system are unstable to small variations on the initial conditions and that the KD is a structurally unstable problem for equal masses of the dumbbell. For the case where the two masses of the dumbbell are equal and the motion of the centre of mass of the dumbbell is planar, the steady states are Lyapunov unstable. For the case where the dumbbell is aligned with the direction connecting its centre of mass to the centre of mass of the KD system and the motion of the centre of mass of the dumbbell is planar, we have proved that for a sufficiently large trajectory radius, these steady states are also Lyapunov unstable. Numerical analysis of the eigenvalues of the Hessian matrix of the effective Hamiltonian associated to these steady states suggests that they are always unstable, irrespective of the radius of the trajectory. The effective Hamiltonian is at least a $1-$saddle near the steady states.

As all the steady states are Lyapunov unstable, we expect the KD to exhibit chaos. Only the steady states 4 and 5, in which the dumbbell is aligned with the direction connecting its centre of mass to the centre of mass of the KD, exist for different masses of the dumbbell. Interestingly, some of the unstable steady states are the Eulerian solutions of the General Three-Body Problem. In this way, we provided a link between the KD system and this problem.

In the limit when the length of the dumbbell goes to zero, the Kepler problem is recovered. Nevertheless, the stability of the system changes abruptly in this limit, since the fixed point of the Kepler problem is stable of the centre type.

In the future we plan to extend the analysis of the dumbbell rigid body to a body constituted by multiple dumbbells, each aligned along one of the coordinate axes. We also wish to study the interaction between two dumbbell rigid bodies, in order to investigate other types of synchronisations, and to replace eventually the dumbbell by an axisymmetric body.

References

[1] A. Celletti, G. Della Penna, and C. Froeschlé. Analytical approximation of the solution of the dissipative standard map. *International Journal of Bifurcation and Chaos*, 8:2471–2479, December 1998.

[2] A. C. M. Correia and J. Laskar. Mercury's capture into the 3/2 spin-orbit resonance as a result of its chaotic dynamics. *Nature*, 429:848–850, June 2004.

[3] L. Euler. De moto rectilineo trium corporum se mutuo attrahentium. *Novo Comm. Acad. Sci. Imp. Petrop.*, 11:144–151, 1767.

[4] R. Fitzpatrick. *An Introduction to Celestial Mechanics*. Cambridge University Press, 2012. ISBN: 978-1107023819.

[5] P. Goldreich and S. J. Peale. Spin-orbit coupling in the solar system. *Astronomical Journal*, 71:425–438, 1966.

[6] P. Goldreich and S. J. Peale. Dynamics of planetary rotations. *Annu. Rev. Astron. Astrophys.*, 6:287–320, September 1968.

[7] T. Grav et al. The short rotation period of nereid. *The Astrophysical Journal Letters*, 591:L71, June 2003.

[8] M. W. Hirsch and S. Smale. *Differential equations, Dynamical systems, and Linear Algebra*. Academic Press, San Diego, 1974. ISBN: 978-0123495501.

[9] J. J. Klavetter. Rotation of hyperion. I. observations. *Astronomical Journal*, 97:570–579, February 1989.

[10] L. Lagrange. Essai sur le problème des trois corps. *Œuvres*, 6:229–331, 1772.

[11] C. D. Murray and S. F. Dermott. *Solar System Dynamics*. Cambridge University Press, 1999. ISBN: 978-0521575973.

[12] Z. E. Musielak and B. Quarles. The three-body problem. *Reports on Progress in Physics*, 77:065901 (30pp), June 2014.

[13] L. A. Pars. *A Treatise on Analytical Dynamics*. Heinemann, London, 1965.

[14] G. H. Pettengill and R. B. Dyce. A radar determination of the rotation of the planet mercury. *Nature*, 206:1240, 1965.

[15] B. E. Schaefer et al. Nereid: Light curve for 1999–2006 and a scenario for its variations. *Icarus*, 196:225–240, July 2008.

[16] M. R. Showalter and D. P. Hamilton. Resonant interactions and chaotic rotation of pluto's small moons. *Nature*, 522:45–49, June 2015.

[17] K. Sitnikov. The existence of oscillatory motions in the three-body problem. *Doklady Akademii Nauk SSSR*, 133:303–306, 1960.

[18] V. G. Szebehely. *Theory of Orbits, The Restricted Problem of Three Bodies*. Academic Press, 1967.

[19] H. A. Weaver, S. A. Stern, M. J. Mutchler, A. J. Steffl, M. W. Buie, W. J. Merline, J. R. Spencer, E. F. Young, and L. A. Young. Discovery of two new satellites of pluto. *Nature*, 439:943–945, 2006.

[20] J. Wisdom, S. J. Peale, and F. Mignard. The chaotic rotation of hyperion. *Icarus*, 58:137–152, May 1984.