# Relativistic tidal Love numbers
## Tests of strong-field gravity

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The tidal Love numbers (TLNs) relate the induced multipolar structure of a body with the perturbing tidal environment. Interestingly, in general relativity (GR), TLNs of black holes (BHs) are precisely zero. This intriguing result motivate us to analyze this property under more general scenarios. In this thesis we study the case of wormholes finding that, even in ultracompact configurations, their TLNs are non-zero. We are also interested in some BH solutions in gravity theories beyond vacuum GR. In this context our results show that the "zero-Love" property is extensive to BHs described in Brans-Dicke and Einstein-Maxwell gravity. Still in this perspective, we study the TLNs of non-rotating BHs in Chern-Simons gravity and, interestingly, we find that they are non-zero. This result provides the first evidence that TLNs of BHs can be non-trivial in modified gravity. We argue that this result may be used as a novel method to test strong-field gravity and to identify possible exotic compact objects with future precise gravitational wave measurements. Our methodology starts with the perturbation of our BH solutions using a linear perturbation formalism. We then impose the Regge-Wheeler gauge and solve the perturbed field equations of the theory using appropriate boundary conditions. From the resulting solutions, we identify the induced multipole moments and the tidal fields which allow us to compute the TLNs. Some of these results and further research on TLNs of exotic compact objects and BHs in modified gravity theories can be found in Ref. [1].

## I. INTRODUCTION

### A. Motivation

Tidal interactions play a fundamental role in astrophysics across a broad range of scales, from stellar objects like ordinary stars and neutron stars to large celestial systems such as galaxies. Several astrophysical structures (e.g., binaries and tidal tails [2, 3]) are consequences of tidal interactions. Tidal effects can be particularly strong and important in the regime that characterizes compact objects, giving rise to extreme phenomena such as tidal disruptions.

The deformability of a self-gravitating object immersed in an external tidal field is measured in terms of its tidal Love numbers (TLNs) [4, 5]. These can leave a detectable imprint in the gravitational-wave (GW) signal emitted by a neutron-star binary in the late stages of its orbital evolution [6–8]. So far, a relativistic extension [7, 9, 10] of the Newtonian theory of tidal deformability has been mostly motivated by the prospect of measuring the TLNs of neutron stars through GW detections and, in turn, understanding the behavior of matter at ultranuclear densities [11–17]. The purpose of this thesis is to show that tidal effects can also be used to explore more fundamental questions related to the nature of compact objects and to the behavior of gravity in the strong-field regime.

An intriguing result in classical general relativity (GR) is the fact that the TLNs of a black hole (BH) are precisely zero. This property has been originally demonstrated for small tidal deformations of a Schwarzschild BH [9, 10] and has been recently extended to arbitrarily strong tidal fields [18] and to the spinning case [19–21], at least in the axisymmetric case to quadratic order in the spin [20] and generically to linear order in the spin [21].

The identical vanishing of the TLNs of BHs within Einstein's theory poses a problem of "naturness" in classical GR [22], the resolution of which could indicate the presence of new physics. Since TLN are encoded in the GW signal, this property can be tested with future GW observations.

### B. ECOs

The current state-of-art of stellar structure evolution suggests that matter, even in extreme forms, cannot support the self-gravity present in massive compact objects and, naturally, these ultracompact objects tend to collapse to BH states. However, if we consider other types of objects, composed by other forms of matter, and that rely on different supporting mechanisms, we can construct objects that are almost as compact as BHs and do not possess an event-horizon. We call them exotic compact objects (ECOs) – such as boson stars [24–26], gravastars [27, 28], wormholes [29]. ECOs might be formed from the collapse of exotic fields or by quantum effects at the horizon scale, and represent the prototypical example of exotic GW sources [30–32] which might be searched for with ground- or space-based detectors.

In this thesis we will restrict this analysis to the TLNs of wormholes and their study is of extreme interest to modern astrophysics and gravitation physics. From an observational point of view, it would be interesting to conduct searches for this types of objects in order to understand if they are real astrophysical objects or just hypothetical, exotic solutions of the Einstein's equations.

### C. Modified gravity

Alternatively, GR might not be a good description of the geometry close to horizons. BHs other than Kerr arise in theories beyond GR which are motivated by both theoretical arguments and by alternative solutions to the dark matter and the dark energy problems (for recent reviews on strong-field tests of gravity in the context of GW astronomy, see Refs. [33, 34]). Arguably, the simplest hairy BHs arise in Einstein-Maxwell theory and are described by the Reissner-Nordstrom solution. Although astrophysical BHs are expected to be electrically neutral [35], Reissner-Nordstrom BHs can be studied as a proxy of BHs beyond vacuum GR and could also emerge naturally in models of minicharged dark matter and dark photons [36]. In several scalar-tensor theories, BHs are uniquely described by the Kerr solution, as in GR [37]. However, due to the introduction of a scalar degree of freedom (nonminimally) coupled to gravity, the linear response of BHs to external perturbations is generically different from GR. Finally, in quadratic theories of gravity the Einstein-Hilbert action is considered as the first term of a possibly infinite expansion containing all curvature invariants, as predicted by some scenarios related to string theory and to loop quantum gravity [34]. To leading order in the curvature corrections, stationary BHs in these theories belong to only two families [39, 40], usually named the Einstein-dilaton-Gauss-Bonnet (EdGB) solution [41–43] and the Chern-Simons solution [44, 45].

The research developed in this Master thesis, added with future works and comparison with data obtained from GW observations might lead to new constraints in these theories and new insights on how GR may be modified in the strong field regime.

## II. TIDAL LOVE NUMBERS OF STATIC COMPACT OBJECTS

Let us consider a compact object immersed in a tidal environment [5]. Following Ref. [9], we define the symmetric and trace-free polar and axial[1] tidal multipole moments of order $l$ as $\mathcal{E}_{a_1\ldots a_l} \equiv [(l-2)!]^{-1}\langle C_{0a_10a_2;a_3\ldots a_l}\rangle$ and $\mathcal{B}_{a_1\ldots a_l} \equiv [\frac{2}{3}(l+1)(l-2)!]^{-1}\langle \epsilon_{a_1bc}C^{bc}_{a_20;a_3\ldots a_l}\rangle$, where $C_{abcd}$ is the Weyl tensor, a semicolon denotes a covariant derivative, $\epsilon_{abc}$ is the permutation symbol, the angular brackets denote symmetrization of the indices $a_i$ and all traces are removed. The polar (respectively, axial) moments $\mathcal{E}_{a_1\ldots a_l}$ (respectively, $\mathcal{B}_{a_1\ldots a_l}$) can be decomposed in a basis of even (respectively, odd) parity spherical harmonics. We denote by $\mathcal{E}^{lm}$ and $\mathcal{B}^{lm}$ the amplitudes of the polar and axial components of the external tidal field with harmonic indices $(l, m)$, where $m$ is the azimuthal number ($|m| \leq l$). The structure of the external tidal field is entirely encoded in the coefficients $\mathcal{E}^{lm}$ and $\mathcal{B}^{lm}$ (cf. Ref. [9] for details).

As a result of the external perturbation, the mass and current multipole moments[2] ($M_l$ and $S_l$, respectively) of the compact object will be deformed. In linear perturbation theory, these deformations are proportional to the applied tidal field. In the nonrotating case, mass (current) multipoles have even (odd) parity, and therefore they only depend on polar (axial) components of the tidal field. Hence, we can define the (polar and axial) TLNs as [7, 9]

$$\begin{aligned} k_l^E &\equiv -\frac{1}{2}\frac{l(l-1)}{M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{M_l}{\mathcal{E}_{l0}}, \\ k_l^B &\equiv -\frac{3}{2}\frac{l(l-1)}{(l+1)M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{S_l}{\mathcal{B}_{l0}}, \end{aligned} \tag{1}$$

where $M$ is the mass of the object, whereas $\mathcal{E}_{l0}$ (respectively, $\mathcal{B}_{l0}$) is the amplitude of the axisymmetric component of the polar (respectively, axial) tidal field. The factor $M^{2l+1}$ was introduced to make the above quantities dimensionless. It is customary to normalize the TLNs by powers of the object's radius $R$ rather than by powers of its mass $M$. Here we adopted the latter nonstandard choice to be in coherence with [1]. Thus, our definition is related to those used by Hinderer, Binnington and Poisson (HBP) [7, 9] through

$$k_{l\,\mathrm{ours}}^{E,B} = \left(\frac{R}{M}\right)^{2l+1} k_{l\,\mathrm{HBP}}^{E,B}\,. \tag{2}$$

Modified theories of gravity and ECOs typically require the presence of extra fields which are (non)minimally coupled to the metric tensor. Here we shall consider some representative example of both scalar and vector fields. A full and complete treatment of this problem would require allowance for an extra degree of freedom, the external external scalar and electromagnetic (EM) applied fields.

We decompose an external EM field in its electric and magnetic components ($\mathfrak{E}_{lm}$ and $\mathfrak{B}_{lm}$, respectively), which can induce an electric and magnetic multipole moment ($Q_l$ and $J_l$, respectively) on a charged body. Similarly to the gravitational case, we can define an analogous of the gravitational TLNs as

$$\begin{aligned} \mathcal{K}_l^E &:= -\frac{1}{2}\frac{l(l-1)}{M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{Q_l}{\mathfrak{E}_{l0}}, \\ \mathcal{K}_l^B &:= -\frac{3}{2}\frac{(l-1)l}{(l+1)M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{J_l}{\mathfrak{B}_{l0}}, \end{aligned} \tag{3}$$

[1] It is slightly more common to use the distinction electric/magnetic components rather than polar/axial. Since we shall discuss also electromagnetic fields, we prefer to use the former distinction.

[2] We adopt the Geroch-Hansen definition of multipole moments [46, 47], equivalent [48] to the one by Thorne [49] in asymptotically mass-centered Cartesian coordinates.

where $\mathfrak{E}_l$ and $\mathfrak{B}_l$ are the amplitudes of the azimuthal component of the applied EM field with polar and axial parity, respectively. Similarly, an applied scalar field $\mathcal{E}^S_{lm}$ can induce a scalar multipole moment $\Phi_l$ associated to a scalar TLN,

$$k^S_l := -\frac{1}{2}\frac{l(l-1)}{M^{2l+1}}\sqrt{\frac{4\pi}{2l+1}}\frac{\Phi_l}{\mathcal{E}^S_l}\,, \tag{4}$$

where $\mathcal{E}^S_l$ is the amplitude of the azimuthal component of the external scalar field with harmonic index $l$.

We expand the metric, the scalar field, and the Maxwell field in spherical harmonics as presented in Appendix A. Since the background is spherically symmetric, perturbations with different parity and different harmonic index $l$ decouple. In the following we discuss the polar and axial sector separately; due to the spherical symmetry of the background, the azimuthal number $m$ is irrelevant and we drop it.

## III. LOVE NUMBERS IN GR

In this section we describe some representative models of compact objects described in Einstein's theory of GR and discuss their TLNs. Technical details are given in Appendix.

### A. Uncharged BHs

We consider static BHs which are solution of Einstein-Maxwell theory, although our results are valid for any $U(1)$ field minimally coupled to gravity, as in the case of dark photons or the hidden $U(1)$ dark-matter sector [36]. The Einstein-Maxwell field equations read

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{5}$$
$$\nabla_\mu F^{\mu\nu} = 0, \tag{6}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the field strength and

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(g^{\mu\gamma}F_{\alpha\mu}F_{\beta_\gamma} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right). \tag{7}$$

The solution of field equations (5) and (6) describing this a BH without electric charge is the Schwarzschild metric, whose line element read as Eq. (A2) where

$$e^\Gamma = e^{-\Lambda} = 1 - \frac{2M}{r} \equiv f, \tag{8}$$

where the quantity $M$ can be identified as the BH's mass.

#### 1. Polar TLNs

The polar functions of the metric are decoupled from the EM function $u_1$. We find the following decoupled equations,

$$r^2(r-2M)^2H_0'' + 2r(r-2M)(r-M)H_0' - \left(l(l+1)r^2 - 2l(l+1)Mr + 4M^2\right)H_0 = 0. \tag{9}$$

$$u_1'' - \frac{l(l+1)u_1}{r(r-2M)} = 0\,, \tag{10}$$

This system allows for a closed-form solution. For simplicity, we impose the absence of electric tidal fields, which requires that the function $u_1$ does not contain $r^3$ terms at large distance. In this case, the regular solution at the horizon for $l = 2$ reads

$$\begin{aligned} H_0 &= -r^2\mathcal{E}_2 + 2Mr\mathcal{E}_2 \equiv -\mathcal{E}_2 r^2 f\,, \\ u_1 &= -r^3\mathfrak{E}_2 + 3Mr^2\mathfrak{E}_2 - 2M^2 r\mathfrak{E}_2\,, \end{aligned} \tag{11}$$

The solutions in (11) do not contain asymptotically decaying terms and we can immediately conclude that the presence of the tidal environment does not induce any multipolar response on the body. Thus, by comparing with expansions (A9) and (A11), the body's multipole moments $M_2$, $Q_2$ are zero, and, according to Eqs. (1)–(3), so are their respective TLNs. Since this procedure can be generalized to higher values of $l$ we conclude that,

$$k^E_l = 0\,, \qquad \mathcal{K}^E_l = 0\,. \tag{12}$$

#### 2. Axial TLNs

The calculations for gravitational axial TLNs and magnetic TLNs are simpler and the final system reads

$$\begin{aligned} h_0 &= \frac{r^3\mathcal{B}_2}{3} - \frac{2}{3}Mr^2\mathcal{B}_2 \equiv \frac{1}{3}\mathcal{B}_2 r^3 f\,, \\ u_4 &= -2r^3\mathfrak{B}_2 + 3Mr^2\mathfrak{B}_2\,, \end{aligned} \tag{13}$$

which proves that also the axial TLNs of a uncharged BH are zero, $k^B_2 = 0$. It is straighforward to extend this result to higher multipoles,

$$k^B_l = 0\,, \qquad \mathcal{K}^B_l = 0\,. \tag{14}$$

To conclude, we obtain the result that the "zero-Love" property found BHs in vacuum GR is extensive to uncharged BHs in the Einstein-Maxwell theory.

### B. Charged BHs

The background spacetime metric is the well-known Reissner-Nordström metric, whose line element reads as in Eq. (A2) with

$$e^\Gamma = e^{-\Lambda} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \equiv f, \tag{15}$$

where $M$ and $Q$ denote the mass and the charge of the BH, respectively. The background Maxwell 4-potential reads

$$A^{(0)}_\mu = (-Q/r, 0, 0, 0)\,. \tag{16}$$

Because the background is electrically charged, gravitational and EM perturbations are coupled to each other. To compute the tidal deformations, we expand the metric as in Eqs. (A3)–(A4) and the Maxwell field as in Eqs. (A5) and (A8). As before, we consider the polar and the axial sectors separately.

### 1. Polar TLNs

The polar functions of the metric are coupled to the EM function $u_1$ through the field equations. We find the following coupled equations,

$$\mathcal{D}_1^{(2)} H_0 + \frac{4Q}{r^3 - 2Mr^2 + Q^2 r}\mathcal{D}_1^{(1)} u_1 = 0 \,, \qquad (17)$$

$$\mathcal{D}_2^{(2)} u_1 + \frac{Q}{r}\mathcal{D}_2^{(1)} H_0 = 0 \,, \qquad (18)$$

where we defined the operators,

$$\mathcal{D}_1^{(2)} = \frac{d^2}{dr^2} - \frac{2(M-r)}{r^2 f}\frac{d}{dr} + \\ + \frac{Q^2 r(4M - (\lambda-2)r) - r^2\left(4M^2 - 2\lambda M r + \lambda r^2\right) - 2Q^4}{r^6 f^2} \,,$$

$$\mathcal{D}_1^{(1)} = \frac{d}{dr} + \frac{\left(Q^2 - r^2\right)}{r\left(r(r-2M) + Q^2\right)} \,,$$

$$\mathcal{D}_2^{(2)} = \frac{d^2}{dr^2} + \frac{4Q^2 - \lambda r^2}{r^4 f}\frac{d}{dr} \,,$$

$$\mathcal{D}_2^{(1)} = \frac{d}{dr} + \frac{2\left(Mr - Q^2\right)}{r^3 f} \,.$$

with $\lambda := l(l+1)$. This system allows for a closed-form solution. For simplicity, we impose the absence of electric tidal fields, which requires that the function $u_1$ does not contain $r^3$ terms at large distance. In this case, the regular solution at the horizon for $l = 2$ reads

$$H_0^{l=2} = -\mathcal{E}_2 r^2 f \,, \qquad (19)$$

$$u_1^{l=2} = \frac{\mathcal{E}_2 r^2 Q f}{2} \,. \qquad (20)$$

As expected, due to the gravito-EM coupling an external tidal field induces a Maxwell perturbation which is proportional to the BH charge $Q$. A simple comparison between the above results and the expansions in Eqs. (A9) and (A11) shows that the multipole moments are all vanishing.

Although the full solutions for $l > 2$ are cumbersome, it can be shown that for any $l > 2$ the large-distance expansion of the solutions for $H_0$ and $u_1$ which are regular at the horizon is truncated at the $1/r$ term and it is an exact solution of the coupled system. Therefore, the above result directly extends to any $l$,

$$k_l^E = 0 \,. \qquad (21)$$

In other words, we obtain that the TLNs of a charged BH are zero in the polar sector, as in the Schwarzschild case.

### 2. Axial TLNs

The calculations for gravitational axial TLNs and magnetic TLNs are simpler. The $r\varphi$-component of Einstein's equations leads to $h_1 = 0$ which automatically satisfies the $\theta\varphi$-component. The $r$- and $\theta$-components of Maxwell's equations impose a relation between $u_2$ and $u_3'$ which must be settled with a gauge choice (usually the Lorentz gauge). The final system reads

$$r^2 f(r^2 h_0'' - 4Qu_4') - \left[2Q^2 + r\left(-4M + rl(l+1)\right)\right] h_0 = 0 \,,$$

$$r^2 f u_4'' + Qh_0' + 2Mu_4' - l(l+1)u_4 - \frac{2Q}{r}(Qu_4' - h_0) \,.$$

Also in the axial sector the coupled system admits an analytic, closed-form solutions. In the absence of EM tidal fields, the solutions which are regular at the horizon read

$$h_0^{l=2} = \frac{r^3}{3} f \mathcal{B}_2 \,, \qquad (22)$$

$$u_4^{l=2} = \frac{r^2}{2} Q\mathcal{B}_2(1 - Q^2/r^2) \,. \qquad (23)$$

which proves that also the (gravitational) axial TLNs of a charged BH are zero, $k_2^B = 0$. It is straighforward to extend this result to higher multipoles,

$$k_l^B = 0 \,. \qquad (24)$$

To conclude, we obtain the interesting results that all TLNs or a static BH in Einstein-Maxwell theory are identically zero, as in the uncharged case.

## C. Wormholes

A natural extension of the procedure developed in the previous sections is the calculation of tidal Love numbers of wormholes. The Schwarzschild solution written in the usual coordinates is valid in the range $r \in ]2M, +\infty[$ , however, this solution can be extended using appropriate coordinate transformations to describe the maximal extension of the spacetime. The maximal analytical solution describes the existence of a wormhole spacetime composed by two qualitatively identical universes connected by a bridge [50, 51]. One appropriate and intuitive method to construct wormhole solutions [29] consists in taking two copies of the ordinary Schwarzschild solution and remove from them the four-dimensional regions described by $r_{1,2} \leq r_0$. With this procedure, we obtain two manifolds whose geodesics terminate at the timelike hypersurfaces

$$\partial\Omega_{1,2} \equiv \{r_{1,2} = r_0 \,|\, r_0 > 2M\} \,. \qquad (25)$$

The two copies are now glued together by identifying these two boundaries, $\partial\Omega_1 = \partial\Omega_2$, such that the resulting spacetime is geodesically complete and comprises of two distinct regions connected by a wormhole with a throat at $r = r_0$. Since the wormhole spacetime is composed by two Schwarzschild metrics, the stress-energy tensor vanishes everywhere except on the throat of the wormhole. The patching at the throat requires a thin-shell of matter with surface density and surface pressure

$$\sigma = -\frac{1}{2\pi r_0}\sqrt{1-\frac{2M}{r_0}}\,, \qquad p = \frac{1}{4\pi r_0}\frac{1-M/r_0}{\sqrt{1-2M/r_0}}\,, \tag{26}$$

which imply that the weak and the dominant energy conditions are violated, whereas the null and the strong energy conditions are satisfied when $r_0 < 3M$. To cover the two patches of the spacetime, we use the radial tortoise coordinate $r_*$, which is defined by

$$\frac{dr}{dr_*} = \pm\left(1-\frac{2M}{r}\right)\,, \tag{27}$$

where we distinguish between the two universes using the upper or lower sign. Without loss of generality, we can assume that the tortoise coordinate at the throat is zero $r_*(r_0) = 0$, such that the domain of one universe corresponds to $r_* > 0$ whereas the other domain corresponds to $r_* < 0$.

For boundary conditions we use

$$[[K]] = 0\,, \qquad [[dK/dr_*]] = 0\,, \tag{28}$$

for the polar sector, and

$$[[h_0]] = 0\,, \qquad [[dh_0/dr_*]] = 0\,, \tag{29}$$

for the axial sector. Furthermore, we consider that in the other universe there are no tidal fields.

In Fig. 1, we show the (polar and axial) TLNs with $l = 2, 3$ as functions of $r_0$. Interestingly, in this case the TLNs have the opposite sign to those of a neutron star and a Newtonian fluid star. Furthermore, they vanish in the BH limit, i.e. when $r_0 \to 2M$. After introducing $\xi := r_0/(2M) - 1$, the behavior of the TLNs in the BH limit ($\xi \to 0$) reads

$$k_2^E \sim \frac{4}{5(8+3\log\xi)}\,, \tag{30}$$

$$k_3^E \sim \frac{8}{105(7+2\log\xi)}\,, \tag{31}$$

$$k_2^B \sim \frac{16}{5(31+12\log\xi)}\,, \tag{32}$$

$$k_3^B \sim \frac{16}{7(209+60\log\xi)}\,, \tag{33}$$

where we have omitted subleading terms of $\mathcal{O}\left(\frac{\xi}{(\log\xi)^2}\right)$. The logarithmic dependence of the TLNs is very interesting, because it implies that the deviations from zero (i.e., from the BH case) are relatively large even when the throat is located just a Planck length $\ell_P$ away from the would-be horizon $r_0 - 2M \sim \ell_P \approx 1.6\times10^{-33}$ cm. In this case, the above results yield

$$k_2^E \approx -3\times10^{-3}\,, \qquad k_2^B \approx -6\times10^{-3}\,,$$
$$k_3^E \approx -4\times10^{-4}\,, \qquad k_3^E \approx -9\times10^{-4} \tag{34}$$

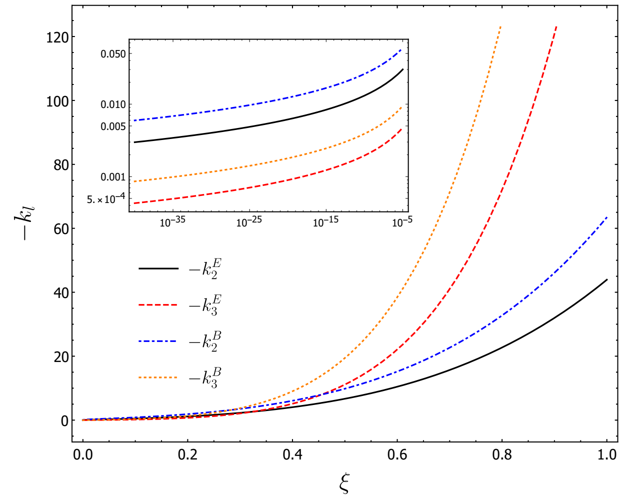


FIG. 1. The $l = 2$ and $l = 3$, axial- and polar-type TLNs for a stiff wormhole constructed by patching two Schwarzschild spacetimes at the throat radius $r = r_0 > 2M$. The TLNs are negative and all vanish in the BH limit, $r_0 \to 2M$.

for a wormhole in the entire mass range $M \in [1, 100]\,M_\odot$.

## IV. TIDAL PERTURBATIONS OF BHS BEYOND GR

In this section we discuss the TLNs of BHs in other theories of gravity. Technical details are given in Appendix A.

### A. Tidal TLNs in scalar-tensor theories

We start with scalar-tensor theories, which generically give rise to stationary BH solutions which are identical to those of GR [34, 52]. Therefore, the background solution which we deal with is still described by the Schwarzschild geometry. In the Jordan frame, neglecting the matter Lagrangian, this theory is described by the action (cf., e.g., Ref. [34])

$$S = \frac{1}{16\pi}\int dx^4\sqrt{-g}\left(\Phi R - \frac{\omega_{\rm BD}}{\Phi}\partial_\mu\Phi\partial^\mu\Phi\right)\,, \tag{35}$$

where $\omega_{\rm BD}$ is a dimensionless coupling constant and $\Phi$ is a scalar field characteristic of the theory. Action (35) yields the equations of motion,

$$G_{\mu\nu} = \frac{\omega_{\rm BD}}{\Phi^2}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\Phi\partial^\lambda\Phi\right) + \frac{1}{\Phi}\nabla_\mu\nabla_\nu\Phi\,, \tag{36}$$

$$\Box\Phi = 0\,. \tag{37}$$

As mentioned above, the background solution is Schwarzschild with a vanishing scalar field.

Following the procedure described in Sec. II, we consider metric perturbations given by Eqs. (A3) and (A4) for the polar and axial sector, respectively, and a scalar field perturbation given by Eqs. (A6) and (A7). Since the scalar perturbations are even-parity, axial gravitational perturbations do not couple to them, implying that this sector is governed by equations identical to those of vacuum GR. Therefore, the axial-type TLNs of a nonrotating BH in Brans-Dicke gravity are zero,

$$k_l^B = 0\,. \tag{38}$$

On the other hand, in the polar sector, scalar perturbations can be obtained from Eq. (37) by using the decomposition in Eqs. (A6) and (A7),

$$\delta\phi'' - \frac{2(r-M)\delta\phi' - l(l+1)\delta\phi}{r(2M-r)} = 0\,, \tag{39}$$

The solution which is regular at the horizon is

$$\delta\phi = C_l P_l\left(\frac{r}{M} - 1\right)\,, \tag{40}$$

where $P_l$ is a Legendre polynomial, $C_l$ an integration constant, and we have expanded the scalar field as in Eq. (A6). By comparing the above expression with the scalar-field expansion in Eq. (A13), we conclude that $C_l \propto \mathcal{E}_l^{\rm S}$ and that the induced scalar multipoles $\Phi_l$ are zero. Therefore, Eq. (40) represents an external scalar tidal field and the scalar TLN defined by Eq. (4) is identically zero.

By substituting Eq. (40) in Eq. (36) we obtain an inhomogeneous differential equation for $H_0$. For $l = 2$, we can identify $C_2 \equiv -2/3M^2\mathcal{E}_2^{\rm S}$ and we get

$$H_0'' + \frac{2(r-M)}{r(r-2M)}H_0' - \frac{2\left(2M^2 - 6Mr + 3r^2\right)}{r^2(r-2M)^2}H_0 - \frac{4M^2\left(2M^2 - 6Mr + 3r^2\right)}{3r^2(r-2M)^2}\mathcal{E}_2^{\rm S} = 0\,. \tag{41}$$

Imposing regularity at the horizon, we find

$$H_0 = -r^2\mathcal{E}_2 + 2Mr\mathcal{E}_2 - \frac{2}{3}M^2\mathcal{E}_2^{\rm S}\,. \tag{42}$$

Therefore, the induced quadrupolar moment is zero, and the polar-type TLN is zero, $k_2^E = 0$, just as in the GR case. This result generalizes to higher multipoles,

$$k_l^E = 0\,. \tag{43}$$

In conclusion, although in Brans-Dicke theory the BH metric perturbations depend on scalar tides, all TLNs of a static BH vanish, as in the case of GR.

### B. BHs in Chern-Simons gravity

In this section we compute the TLNs of a nonrotating BH in Chern-Simons theory [45],

$$S_{\rm CS} = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + \frac{\alpha_{\rm CS}}{4}\Phi\,{}^*RR\right]\,,$$

,

$$\tag{44}$$

where $\alpha_{\rm CS}$ is the coupling constant of the theory and ${}^*RR$ is the Pontryagin scalar,

$${}^*RR = \frac{1}{2}R_{abcd}\epsilon^{baef}R^{cd}_{ef}\,. \tag{45}$$

The field equations arising from action (44) are

$$R_{ab} = \frac{1}{2}\partial_a\Phi\partial_b\Phi - \alpha_{\rm CS}C_{ab}\,, \tag{46}$$

$$\Box\Phi = -\frac{\alpha_{\rm CS}}{4}{}^*RR, \tag{47}$$

where $C_{ab} = \nabla_c\Phi\epsilon^{cde(a}\nabla_e R^{b)}_d + \nabla_d\nabla_c\Phi^*R^{dabc}$. We will focus on spherically symmetric background solutions to Eqs. (46)–(47). In these conditions, the Pontryagin scalar vanishes and the background is described by the Schwarzschild metric with a vanishing scalar field [53].

#### 1. *Polar TLNs*

In Chern-Simons gravity, the polar perturbations of a Schwarzschild BH are equivalent to GR [54]. Therefore, the analysis for the polar TLNs is identical to that discussed in Ref. [9], and one can conclude that the polar TLNs of a nonrotating BH in Chern-Simons gravity are zero for any value of $l$,

$$k_l^E = 0\,. \tag{48}$$

The polar TLNs are the dominant correction to the inspiral waveform [14]. Thus, the simple fact that in Chern-Simons gravity these TLNs are vanishing already suggests that it would be very difficult to constraint this theory with GW measurements of the BH tidal deformability.

#### 2. *Axial TLNs*

On the other hand, the field $\Phi$ transforms as a pseudoscalar and is therefore part of the axial sector. We can thus expect that nontrivial axial-type TLNs in Chern-Simons gravity may exist. In the stationary limit, we find $h_1 = 0$, and the field equations for the axial sector reduce to a system of two coupled second-order differential equations for $h_0$ and $\delta\phi$. This system can be solved numerically for a generic coupling $\alpha_{\rm CS}$ or perturbatively when $\zeta_{\rm CS} := \alpha_{\rm CS}/M^2 \ll 1$. The latter case is consistent with the theory (44) being an effective field theory [45]. We have adopted both procedures, described below.

In the perturbative limit, we expand the metric and scalar perturbations in powers of the coupling $\zeta_{\rm CS} \ll 1$,

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \zeta^2_{\rm CS} h^{(2)}_{\mu\nu} + ... \tag{49}$$

$$\delta\phi = \zeta_{\rm CS}\delta\phi^{(1)} + ... \tag{50}$$

and solve the perturbation equations order by order in $\zeta_{\rm CS}$.

For $l = 2$ and to $\mathcal{O}(\zeta^0_{\rm CS})$, the metric perturbation which is regular at the horizon reads

$$h^{(0)}_0 = \frac{\mathcal{B}_2}{3} r^3 \left(1 - \frac{2M}{r}\right) , \tag{51}$$

as in the GR case. The advantage of the perturbative approach is that the equations decouple from each other. To $\mathcal{O}(\zeta^1_{\rm CS})$, the only correction is in the scalar-field equation, which reads

$$\mathcal{D}^{(2)}_S \delta\phi^{(1)} = \frac{12\mathcal{B}_2 M}{r^2(r-2M)} , \tag{52}$$

where $\mathcal{D}^{(l)}_S := d^2/dr^2 - \frac{1}{r(r-2M)}\left[l(l+1) + 2(M-r)\frac{d}{dr}\right]$. Again for simplicity, we impose the absence of scalar tidal fields, i.e. we require that $\delta\phi$ does not contain a divergent $r^l$ term at large distance [cf. Eq. (A13)]. In this case, the solution which is regular at $r = 2M$ reads

$$\begin{aligned}\delta\phi^{(1)} = -\frac{\mathcal{B}_2 M^2}{2}\Big\{&54 - 36y + \pi^2(2+3(y-2)y) \\ &+3\log\left[\frac{y}{2}\right]\left[12(1-y) + (2+3(y-2)y)\log\left[\frac{y}{2}\right]\right) \\ &+6(2+3(y-2)y)\,\mathrm{polylog}\left[2, 1-\frac{y}{2}\right]\Big\} ,\end{aligned} \tag{53}$$

where we defined $y = r/M$.

Using the previous solution, the $\mathcal{O}(\zeta^2_{\rm CS}, \epsilon^1)$ equation for the axial perturbation is an inhomogeneous differential equation that reads

$$\mathcal{D}^{(2)}_A h^{(2)}_0 = \mathcal{S}^{(2)}_A , \tag{54}$$

with

$$\begin{aligned}\mathcal{S}^{(2)}_A(r) := \frac{3\mathcal{B}_2 M}{(y-2)y^4}\Big\{&(2-y)\left[\pi^2\left(3y^2-2\right) - 18(1+2y)\right] \\ &-6(y-2)\left(3y^2-2\right)\mathrm{polylog}\left[2, 1-\frac{y}{2}\right] + 3\log\left[\frac{y}{2}\right] \\ &\times\left[4(3(y-1)y-4) - (y-2)\left(3y^2-2\right)\log\left[\frac{y}{2}\right]\right]\Big\} ,\end{aligned} \tag{55}$$

and $\mathcal{D}^{(l)}_A := d^2/dr^2 + \frac{4M + l(l+1)r}{r^2(r-2M)}$.

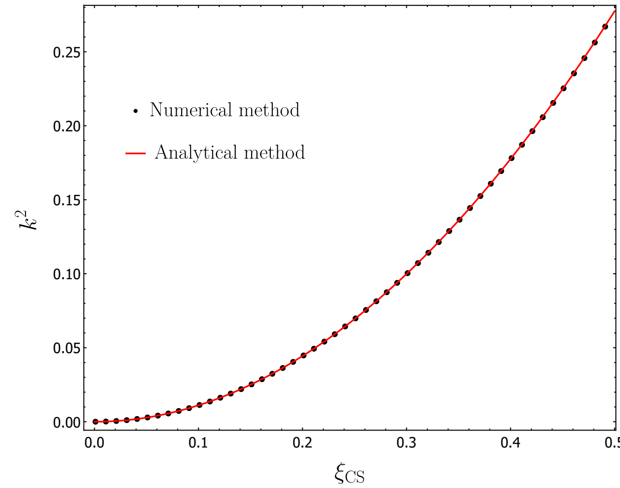


FIG. 2. Axial TLNs of a BH in Chern-Simons gravity for $l = 2$ perturbations calculated for different values of $\alpha_{\rm CS}$. The dots correspond to the values obtained directly from a numerical integration and the line corresponds to analytical result in Eq. (62).

It is convenient to solve the above inhomogeneous equation through Green's function techniques. The solution with the correct boundary conditions is

$$\begin{aligned}h^{(2)}_0(r) = &\frac{\Psi_+(r)}{W}\int_{2M}^r dr' \mathcal{S}^{(2)}_A(r')\Psi_-(r') \\ &+ \frac{\Psi_-(r)}{W}\int_r^\infty dr' \mathcal{S}^{(2)}_A(r')\Psi_+(r') ,\end{aligned} \tag{56}$$

where the two linearly independent solutions of the homogeneous problem read

$$\Psi_-(r) = C_1 r^2(r-2M) , \tag{57}$$

$$\begin{aligned}\Psi_+(r) = \frac{C_2}{24M^5 r}\Big(&2M\left(2M^3 + 2M^2 r + 3Mr^2 - 3r^3\right) \\ &+3r^3(2M-r)\log\left(1 - \frac{2M}{r}\right)\Big) ,\end{aligned} \tag{58}$$

and

$$W \equiv \Psi'_+(r)\Psi_-(r) - \Psi'_-(r)\Psi_+(r) = C_1 C_2 , \tag{59}$$

is the Wronskian. We notice that, in the absence of scalar tidal field, the source term, $\Psi_-$ and $\Psi_+$ behave as

$$\begin{aligned}\mathcal{S}^{(2)}_A &\sim \mathcal{B}_2/r^5 , \\ \Psi_-(r) &\sim C_1 r^3 , \\ \Psi_+(r) &\sim -\frac{C_2}{5r^2}\end{aligned} \tag{60}$$

at large distances, the first integral in Eq. (56) converges, whereas the second integral does not contribute to the current quadrupole $S_2$. Fortunately, it is possible to compute these integrals in closed form and obtain the asymptotic behavior of $h_0^{(2)}(r)$:

$$h_0^{(2)}(r) \to -\frac{9\left(\mathcal{B}_2 M^5(8\zeta(3)-9)\right)}{5r^2} + \mathcal{O}(M^3/r^3)\,, \quad (61)$$

where $\zeta$ is the Riemann's $\zeta$ function. By comparing the above result with Eq. (A10) and using Eq. (1), it is straighforward to obtain

$$k_2^B = \frac{9(8\zeta(3)-9)}{5}\zeta_{\rm CS}^2 \approx 1.11\zeta_{\rm CS}^2\,, \quad (62)$$

Interestingly, we find that the axial TLN is nonzero and proportional to $\zeta_{\rm CS}^2$, as expected. We have also confirmed this result by integrating numerically the field equations for arbitrary values of $\alpha_{\rm CS}$ and by extracting the quadratic correction in the small-coupling limit. Figure 2 shows a comparison between the analytical result (62) and the numerical one.

## V. CONCLUSIONS

The theory of tidal deformability of compact objects has acquired increasingly more attention over the last few years. So far, most of the applications of this theory have been limited to astrophysics and to the possibility of constraining the equation of state of neutron stars with GW observations. In this thesis and related works [1], we argued that tidal effects can also be used to explore more fundamental questions related to the existence of ECOs, and to the behavior of gravity in the strong-field regime.

Our main results can be summarized as follows:

- In the framework of GR, the TLNs of a wormhole are generically nonzero and, in the BH limit, all TLNs vanish but only logarithmically.

- The TLNs of BHs in Einstein-Maxwell theory and of an uncharged static BH in Brans-Dicke theory identically vanish as in GR, providing an extension to the existing "zero-Love" rule. Besides these trivial cases, our results strongly support the idea that the TLNs of a BH are nonzero in extensions of GR. In particular, we have explicitely shown that the axial TLNs of a Schwarzschild BH in Chern-Simons gravity are nonzero. The analysis of the TLNs of BHs in EdGB gravity is left for future work [55].

At the same time, our work is intended to be only a first step in understanding the tidal deformability of ECOs and of BHs beyond GR; as such, it can be extended in several interesting ways:

- We neglected the presence of tidal fields of different nature, for example EM tidal fields in Einstein-Maxwell theory or scalar fields in Brans-Dicke theory or in Chern-Simon gravity. We anticipate that the presence of extra tidal fields will give rise to new families of TLNs, which are related to the mass/current multipole moments induced by a (scalar or vector) extra tidal field [55].

- We focused on nonrotating objects. The spin of the individual components of a neutron-star binary system are typically small, but this might not be the case for ECOs and for BHs. In general, subleading spin effects might be included by applying the formalism developed in Refs. [19–21, 56] to the systems studied in this work.

- Our computation for BHs in extension of GR greatly simplifies if the metric is Schwarzschild, as in Chern-Simons gravity. However, in general BHs in modified gravity might have extra charges. This is the case of static BHs in Einstein-dilaton-Gauss-Bonnet theory or of spinning BHs in Chern-Simons gravity. Although our procedure can be extended to those cases, the presence of extra charges makes the multipolar expansion more involved. Nonetheless, our results strongly support the idea that the TLNs of a BH in extensions of GR are generically nonzero.

### Appendix A: Determination of TLNs

In order to compute the TLNs we need to calculate the expressions for the induced mass and current multipole moments as a function of the external tidal field. We use a linear perturbation theory approach to disturb slightly the spacetime metric,

$$g_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + h_{\mu\nu}, \quad (A1)$$

where $\overset{(0)}{g}_{\mu\nu}$ is the background spacetime metric and $h_{\mu\nu}$ is a small perturbation to the spacetime metric. As we mentioned, we focus on static, spherically symmetric background metrics which are described by,

$$\overset{(0)}{g}_{\mu\nu} = -e^{\Gamma}dt^2 + e^{\Lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \quad (A2)$$

We decompose $h_{\mu\nu}$ in spherical harmonics allowing us to separate the perturbation in even and odd parts, $h_{\mu\nu} = h_{\mu\nu}^{\rm even} + h_{\mu\nu}^{\rm odd}$. Choosing the Regge-Wheeler gauge [57] the most general form for $h_{\mu\nu}$ is,

$$h_{\mu\nu}^{\rm even}=\begin{pmatrix} e^{\Gamma}H_0^{lm}(t,r)Y^{lm} & H_1^{lm}(t,r)Y^{lm} & 0 & 0\\ H_1^{lm}(t,r)Y^{lm} & e^{\Lambda}H_2^{lm}(t,r)Y^{lm} & 0 & 0\\ 0 & 0 & r^2K^{lm}(t,r)Y^{lm} & 0\\ 0 & 0 & 0 & r^2\sin^2\theta K^{lm}(t,r)Y^{lm}\end{pmatrix}, \tag{A3}$$

$$h_{\mu\nu}^{\rm odd}=\begin{pmatrix} 0 & 0 & h_0^{lm}(t,r)S_\theta^{lm} & h_0^{lm}(t,r)S_\varphi^{lm}\\ 0 & 0 & h_1^{lm}(t,r)S_\theta^{lm} & h_1^{lm}(t,r)S_\varphi^{lm}\\ h_0^{lm}(t,r)S_\theta^{lm} & h_1^{lm}(t,r)S_\theta^{lm} & 0 & 0\\ h_0^{lm}(t,r)S_\varphi^{lm} & h_1^{lm}(t,r)S_\varphi^{lm} & 0 & 0\end{pmatrix}, \tag{A4}$$

with $\left(S_\theta^{lm}, S_\varphi^{lm}\right)\equiv\left(-Y_{,\varphi}^{lm}/\sin\theta,\, \sin\theta\, Y_{,\theta}^{lm}\right)$.

In the presence of scalars or vectors, spacetime fluctuations are accompanied by corresponding fluctuations in these fields,

$$A_\mu = A_\mu^{^{(0)}} + \delta A_\mu, \tag{A5}$$

$$\Phi = \Phi^{^{(0)}} + \delta\Phi, \tag{A6}$$

where $A^{^{(0)}}$ and $\Phi^{^{(0)}}$ are background quantities while $\delta A_\mu$ and $\delta\Phi$ are small perturbations. We expand the scalar perturbation $\delta\Phi$ and 4-potential $\delta A_\mu$ as [58, 59],

$$\delta\Phi = \delta\phi_{lm}Y^{lm}, \tag{A7}$$

$$\delta A_\mu = \left(\frac{u_1^{lm}}{r}Y^{lm}, \frac{u_2^{lm}\,e^{-\Gamma}}{r}Y^{lm}, \frac{u_3^{lm}}{l(l+1)}Y_b^{lm} + u_4^{lm}S_b^{lm}\right), \tag{A8}$$

with $Y_b^{lm}\equiv\left(Y_{,\theta}^{lm}, Y_{,\varphi}^{lm}\right)$. Henceforward we shall drop the $(lm)$ superscripts on all the definitions with the exception of multipole moments.

Solving the appropriate field equations for the theory in study will give us expressions for the metric functions in (A3) and (A4), for the electromagnetic functions in (A8) and for the scalar fields. The remaining task consists in extracting the multipole moments and tidal fields from the spacetime metric. Thorne developed a method to define the multipole coefficients of any spacetime metric given in coordinates which are asymptotically Cartesian and mass centered (ACMC) [49]. Another definition of multipole moments was given in a Geroch-Hansen construction [46, 47]. These two distinct definitions of moments were shown to be equivalent [48].

The multipole moments can be extracted from the asymptotic behavior of the spacetime metric and fields:

$$g_{00}=-1+\frac{2M}{r}+\sum_{l\geq 2}\left(\frac{2}{r^{l+1}}\left[\sqrt{\frac{4\pi}{2l+1}}M_lY^{l0}+(l'<l\,\text{pole})\right]-\frac{2}{l(l-1)}r^l\left[\mathcal{E}_lY^{l0}+(l'<l\,\text{pole})\right]\right), \tag{A9}$$

$$g_{0\varphi}=\frac{2J}{r}\sin^2\theta+\sum_{l\geq 2}\left(\frac{2}{r^l}\left[\sqrt{\frac{4\pi}{2l+1}}\frac{S_l}{l}S_\varphi^{l0}+(l'<l\,\text{pole})\right]+\frac{2r^{l+1}}{3l\,(l-1)}\left[\mathcal{B}_lS_\varphi^{l0}+(l'<l\,\text{pole})\right]\right), \tag{A10}$$

$$A_t=-\frac{Q}{r}+\sum_{l\geq 1}\left(\frac{2}{r^{l+1}}\left[\sqrt{\frac{4\pi}{2l+1}}Q_lY^{l0}+(l'<l\,\text{pole})\right]-\frac{2}{l(l-1)}r^l\left[\mathfrak{E}_lY^{l0}+(l'<l\,\text{pole})\right]\right), \tag{A11}$$

$$A_\varphi=\sum_{l\geq 1}\left(\frac{2}{r^l}\left[\sqrt{\frac{4\pi}{2l+1}}\frac{J_l}{l}S_\varphi^{l0}+(l'<l\,\text{pole})\right]+\frac{2r^{l+1}}{3l\,(l-1)}\left[\mathfrak{B}_lS_\varphi^{l0}+(l'<l\,\text{pole})\right]\right), \tag{A12}$$

$$\Phi=\Phi_0+\sum_{l\geq 1}\left(\frac{1}{r^{l+1}}\left[\Phi_lY^{l0}+(l'<l\,\text{pole})\right]-r^l\left[\mathcal{E}_l^{\rm S}+(l'<l\,\text{pole})\right]\right). \tag{A13}$$

The decomposition of the scalar field and the vector potentials were chosen so they could be easily compared with the metric decomposition. An appropriate comparison between the solution of the field equations and the expansions (A9)–(A13) gives us a method to extract the multipole moments and consequently the TLNs.

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