Mixed Models in Time Domain Integration

José Maria Giraldes Tavares da Cunha Rebelo
Departamento de Engenharia Civil, Arquitectura e Georrecursos
Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal

Abstract

This work presents and tests the behaviour of different approximation functions used as the required approximation basis for the time integration algorithm proposed by Freitas [3]. In this model, independent approximations for the displacement, velocity and acceleration fields are defined in the time domain. It is the scope of this dissertation to evaluate the numerical behaviour of a group of orthonormal functions: the Legendre polynomials, the polynomial wavelet system and the Daubechies wavelet systems defined in the interval are tested. With a set of test it will be possible to characterize the comparative behaviour of the different approximation basis. Procedures of *p*-refinement (number of functions used in the approximation basis) and *h*- refinement (number of time steps increment) are tested and characterized.

Keywords: Dynamic Analysis; Time Integration; Finite Elements; wavelet Systems; Legendre polynomials

1. Introduction

The main goal of this work is to evaluate different time integration schemes. In this work two alternative types of formulation are used, the classic formulation and a non-conventional formulation. The first corresponds to the implicit time integration schemes, like Newmark [5] and Wilson-θ [7] methods, where the acceleration is approximated by linear or constant functions and the displacement and velocity are obtained through that acceleration. This type of formulation in mainly used as it presents the advantages of being highly intuitive and also due to the simplicity in the problem formulation. The second type of formulation corresponds to the mixed time integration scheme. This integration is considered mixed due to the fact that the displacement, velocity and acceleration are approximated independently in each time increment. The fundamental conditions of the dynamic problem are not imposed locally, being instead enforced in a weak form. In this formulation is possible to choose a high variety of functions to be used to define the approximation bases for the dynamic quantities. The possibility to choose a basis with very refined approximation functions, allows the consideration of superior time steps, compared to what is possible when classic schemes are used. The two refinement procedures, *p*- and *h*- are possible in this integration scheme, and their efficiency will be tested in this work.

2. Mixed Time Integration Formulation

The first step for the construction of the Mixed Time Integration method (MTI) is the approximations definition. For each time step, independent approximations are defined for the displacement, velocity and acceleration fields.

$$u(t) = T(t)u_n \tag{2.1}$$

$$v(t) = T(t)v_n \tag{2.2}$$

$$a(t) = T(t)a_n \tag{2.3}$$

where $T(t)$, is a row vector with the dimension *N*, that contains the approximation functions. The vectors u_n, v_n e a_n define the correspondent weights.

The row vector, $T(t)$, is defined by:

$$T(t) = \varphi(\tau)Z_T^T \tag{2.4}$$

where $\varphi(\tau)$ is a row vector that contains the approximation functions defined in the local reference system and Z_T is a matrix that defines the linear combination that ensures the uncoupling of the system.

Two operators are constructed based on the approximation functions:

$$h = \int_0^1 \hat{\varphi}^T(\tau)\,\varphi(\tau)dz \tag{2.5}$$

$$g = \hat{\varphi}^T(1)\varphi(1) - \int_0^1 \hat{\varphi}^T(\tau)\varphi(\tau)dz \tag{2.6}$$

The matrix *g* eigenvalue problem is given by:

$$gE_* = E_*\Omega \tag{2.7}$$

where diagonal matrix Ω collects the eigenvalues and matrix E_* gathers the eigenvectors.

When an orthonormal basis is selected, the equation (2.8) can be used.

$$Z_T = {E_*}^T \tag{2.8}$$

The construction of this operator for non-orthonormal functions is presented in [3].

The relations between displacement, velocity and acceleration are not imposed locally. They are enforced in the following weighted residual form:

$$\int_0^{\Delta t} T^*(v - \dot{u})dt = 0 \tag{2.9}$$

$$\int_0^{\Delta t} T^*(a - \dot{v})dt = 0 \tag{2.10}$$

where $T^* = \hat{T}^t$, is the conjugate transpose of row-matrix *T*.

The dynamic equation is also enforced in the following weak form:

$$\int_0^{\Delta t} T^*(Ma + Cv + Ku - F)dt = 0 \tag{2.11}$$

The dynamic equation can be rewritten using the definitions (2.1) to (2.3):

$$\int_0^{\Delta t} T^*(MTa_n + CTv_n + KTu_n - F)dt = 0 \tag{2.12}$$

With the definition of the following operators:

$$H = \frac{1}{\Delta t}\int_0^{\Delta t} T^*\,T dt \tag{2.13}$$

$$G = T^*(\Delta t)T(\Delta t) - \int_0^{\Delta t} \dot{T}^* T dt \tag{2.14}$$

$$\Omega \quad = GH^{-1} \tag{2.15}$$

$$\omega = H^{-1}T^*(0) \tag{2.16}$$

it is possible to express the velocity and the acceleration weights, defined in (2.1) to (2.3), in terms of the displacement weight, u_n, the initial conditions, u_0 and v_0 and the operators defined in (2.15) and (2.16). Is possible to verify that:

$$v_n = \frac{1}{\Delta t}\Omega_n u_n - \frac{1}{\Delta t}\omega u_0 \tag{2.17}$$

$$a_n = \frac{1}{\Delta t^2}{\Omega_n}^2 u_n - \frac{1}{\Delta t^2}\Omega_n \omega u_0 - \frac{1}{\Delta t}\omega v_0 \tag{2.18}$$

Finally, combining equations (2.12), (2.17) and (2.18), it is possible to obtain the equation in its final form:

$$\left(\frac{M}{\Delta t^2}{\Omega_n}^2 + \frac{C}{\Delta t}\Omega_n + K\right)u_n = F_n^0 + F_n \tag{2.19}$$

$$F_n^0 = \frac{M}{\Delta t^2}\Omega_n\omega_n u_0 + \frac{C}{\Delta t}\omega_n u_0 + \frac{M}{\Delta t}\omega_n v_0 \quad ; \quad F_n = \sum_{m=1}^{N} \frac{{H_{nm}}^{-1}}{\Delta t}\int_0^{\Delta t} \hat{T}_m F dt$$

3. Approximation Functions

Several types of orthonormal functions have been tested as approximation basis, such as Legendre polynomials [6], Polynomial wavelets [4] and Daubechies wavelet systems defined on the interval [1].

3.1. Legendre Polynomials

The first functions to be considered are the Legendre polynomials. Using Bonnet recursive formula, the orthonormal Legendre polynomial basis can be defined as follows:

$$\frac{(n+1)}{\lambda_{n+1}}P_{n+1}(x) = \left(\frac{2n+1}{\lambda_n}\right)xP_n(x) - \frac{n}{\lambda_{n-1}}P_{n-1}(x) \tag{3.1}$$

where $P_0(\mathrm{x}) = \lambda_0$ and $P_1(\mathrm{x}) = \lambda_1 x$, and with $\lambda_n = \sqrt{\frac{2n+1}{2}}$.

3.2. Polynomial wavelets

The algorithms required for the definition of a wavelet basis on a closed interval may be found in [9]. Polynomial wavelet systems are based on the definition of linear combinations involving orthogonal Legendre polynomials. The coefficients of that linear combination are obtained using Chebyshev polynomials of the second kind.

The scaling functions are defined as:

$$\phi_{ji}(x) = C_{ji}^{\phi} \sum_{k=0}^{2^j} U_k(y_i^{(2^j+1)}) \sqrt{k+\frac{1}{2}}\, P_k(x) \tag{3.2}$$

with j=0,1... and i=0,1...2^j.

On the other hand, wavelets can be defined as follows:

$$\Psi_{ji}(x) = C_{ji}^{\Psi} \sum_{k=2^j+1}^{2^{j+1}} U_k(y_i^{(2^j)}) \sqrt{k+\frac{1}{2}} P_k(x) \tag{3.3}$$

with j=0,1... and i=0,1...2^j-1.

The Legendre polynomials P_k can be defined as in (3.1). The Chebishev polynomials of the second kind are defined by:

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x) \tag{3.4}$$

with $U_0(x)=1$ and $U_1(x)=2x$.

3.3. Daubechies Wavelets

To generate a complete Daubechies wavelet system defined on the interval, the definition of both scaling functions and wavelets is required:

$$\phi(x) = \sum_{k=0}^{2N-1} a_k \emptyset(2x-k) \tag{3.5}$$

$$\psi(x) = \sum_{k=0}^{2N-1} a_{2N-1-k} \emptyset(2x-k) \tag{3.6}$$

The complete function system is built by considering translation and dilatation operations, either on the scaling function or on the primary wavelet, as defined by:

$$\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k) \tag{3.7}$$

$$\Psi_{j,k}(x) = 2^{\frac{j}{2}} \Psi(2^j x - k) \tag{3.8}$$

4. Convergence Analysis and Validation Tests

It is possible to write the dynamic equilibrium equation in the canonical form:

$$\Delta t^2 a + 2\xi\Omega\Delta t v + \Omega^2 u = \frac{\Omega^2}{k} f \tag{4.1}$$

where $\Omega = \omega\Delta t$ corresponds to the normalized angular frequency.

The solution of the dynamic problem at a time point can be obtained by the expression:

$$x_n = A x_{n-1} + L \tag{4.2}$$

where ${x_n}^T = \{u_n \ \Delta t v_n \ \Delta t^2 a_n\}$ is a row vector with the scaled solution, A is the amplification matrix and L the forcing load vector.

The spectral radius of matrix A, is given by the norm of the highest eigenvalue of the matrix.

$$\rho(A) = \sqrt{\alpha^2 + \beta^2} \tag{4.3}$$

where α and β, are respectively the real and imaginary parts of the highest eigenvalue.

The spectral radius for the theoretical solution is written only in function of the normalized angular frequency and the damping ratio.

$$\rho(A) = e^{-\xi\Omega} \tag{4.4}$$

For a generic approximated solution, the spectral radius is obtained through the computation of the numerical amplification matrix $\bar{A}$ eigenvalues.

$$\bar{\rho}(A) = \sqrt{\bar{\alpha}^2 + \bar{\beta}^2} \tag{4.5}$$

Unconditional stability can be ensured by the following condition:

$$\bar{\rho}(A) \leq 1 \tag{4.6}$$

The values obtained for the spectral radius for the different approximation functions are shown in figures 4.1 to 4.3. for different damping ratios.

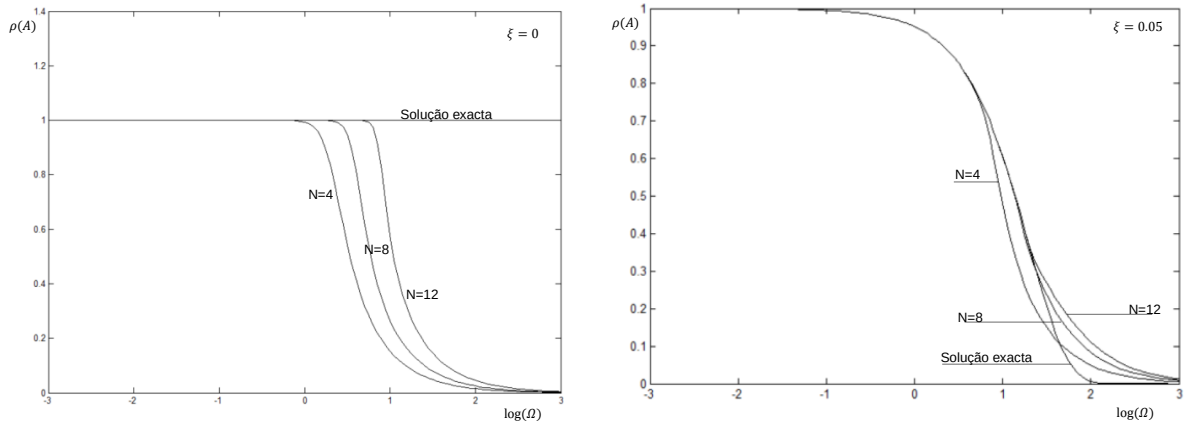

Figure 4.1 – Spectral Radius for Legendre Polynomials .

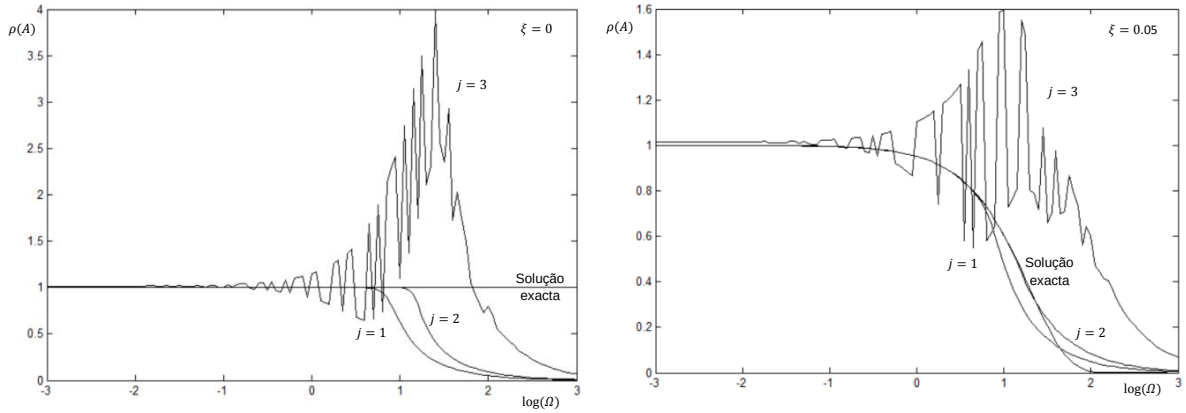

Figure 4.2 – Spectral Radius for polynomial wavelets

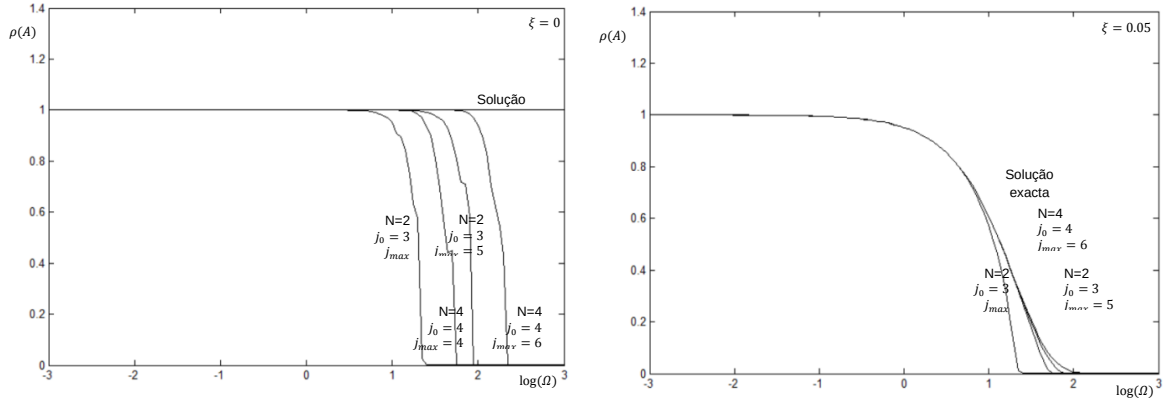

Figure 4.3 – Spectral Radius for Daubechies Wavelets

The results show the unconditional stability for all the functions when lower degree basis are used. The *p*- refinement shows the approximation to the theoretical solution. For Legendre polynomials, stability is ensured for $N < 15$. For polynomial wavelets, stability is ensured for $j < 3$. For Daubechies wavelets, stability is ensured for $j_{max} < 7$.

A single degree of freedom problem is used to test each of the approximation functions and the efficiency of the refinement procedures. The data is shown in table 4.1.

Table 4.1 – Single degree of freedom problem data.

$k = 100;$	$F = 100.sin(100t);$
$c = 1;$	$u_0 = 0;$
$m = 1;$	$v_0 = 0;$
$T = 5\ s$	

The average error is measured.by the expression (4.7)

$$\varepsilon = \frac{\sum_{i=1}^{n}\left(y_i - f(x_i)\right)^2}{n \times \max(y)^2} \tag{4.7}$$

where y_i corresponds to the theoretical solution, $f(x_i)$ corresponds to the approximated solution and n is the number of points where the solution is computed.

The average error variation with the considered time step is shown in figure 4.4 for the Newmark method with $\alpha = \frac{1}{2}$ and $= \frac{1}{6}$.

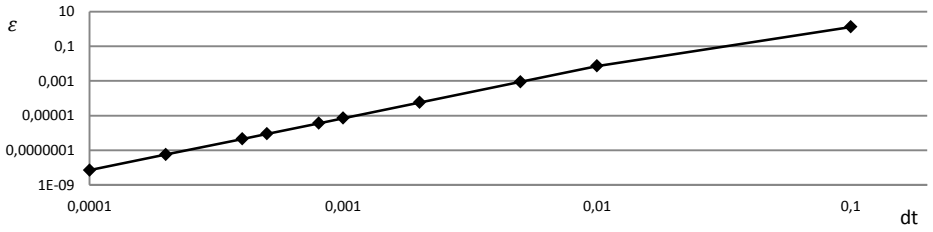

Figure 4.4 – Average error – Newmark method

The same average error is computed for the mixed time integration scheme, using the different types of functions. The error variation with time step considered is shown for Legendre polynomials in figure 4.5, for polynomial wavelets in figure 4.6 and for Daubechies wavelets in figure 4.7.

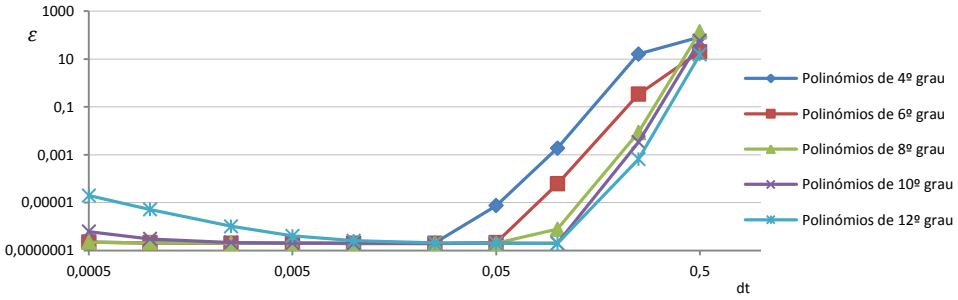

Figure 4.5 – Average error for Legendre polynomials

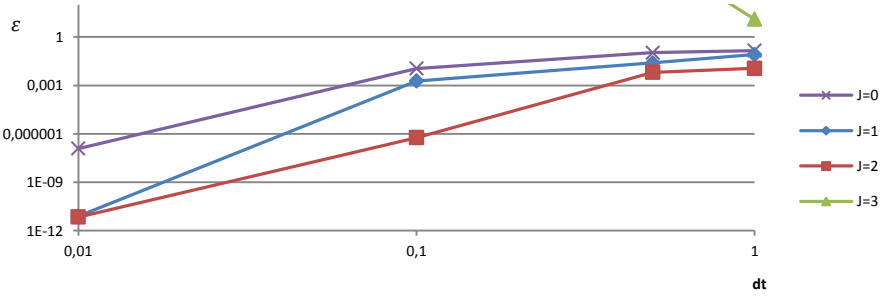

Figure 4.6 – Average error for polynomial wavelets

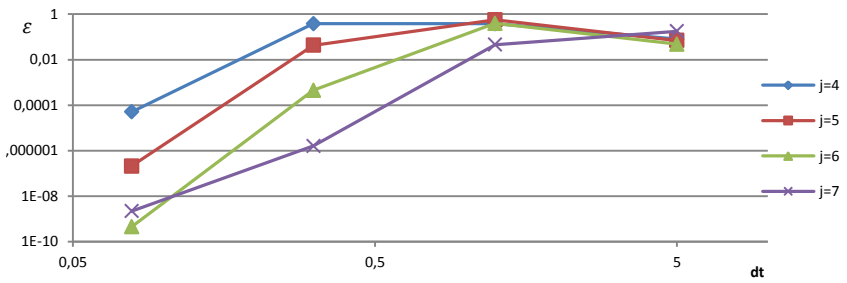

Figure 4.7 – Average error for Daubechies wavelets

The classical Newmark integration scheme presents a simple relation between the time step being considered and the solution accuracy. Interesting solutions are obtained only for small time steps.

Both p- and h- refinement procedures are efficient, although for small time steps the p- refinement induces a decrease in the solution quality, as shown in figure 4.5 for 12th grade Legendre polynomials and in figure 4.7 for refinement degree *j*=7.

For time step $\Delta t = 0.1\,s$, the Newmark method presents an error of $\varepsilon = 1{,}3$ while the Legendre polynomials present ε =t 6,5E-04, the polynomial wavelets present ε =5,7E-07 and Daubechies wavelets present ε =4,4 E-10.

5. Dynamic Analysis

Consider the tapered tower presented in the figure 5.1 (example adapted from [2]) and loaded with a pressure induced by blast.

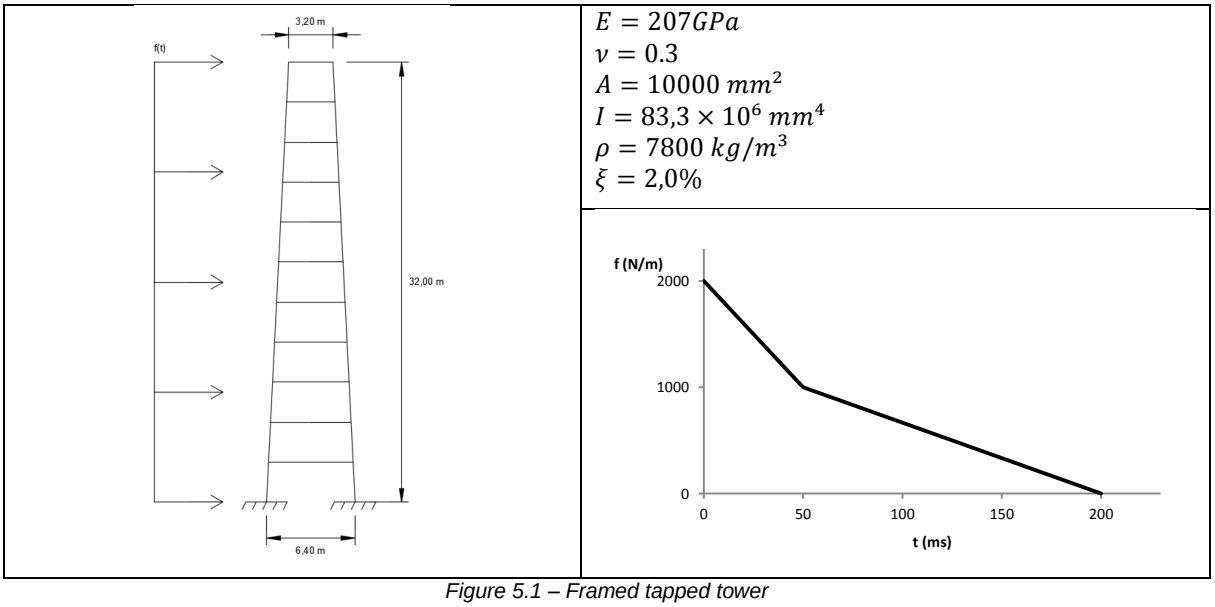

Figure 5.1 – Framed tapped tower

A mesh with 33 finite elements was considered for the spatial discretization, yielding a total of 66 degrees of freedom. Daubechies wavelet systems are used, with family number $N = 4$ and with both base and maximum refinement degree, $j_0 = j_{max} = 5$. A time step, $\Delta t = 0.25s$, was considered.

The figure 5.2 shows the deformed shape of the structure for time instants $t = 0{,}2\ s$ and $t = 0{,}4\ s$. The displacements showed are amplified in order to enhance visibility.

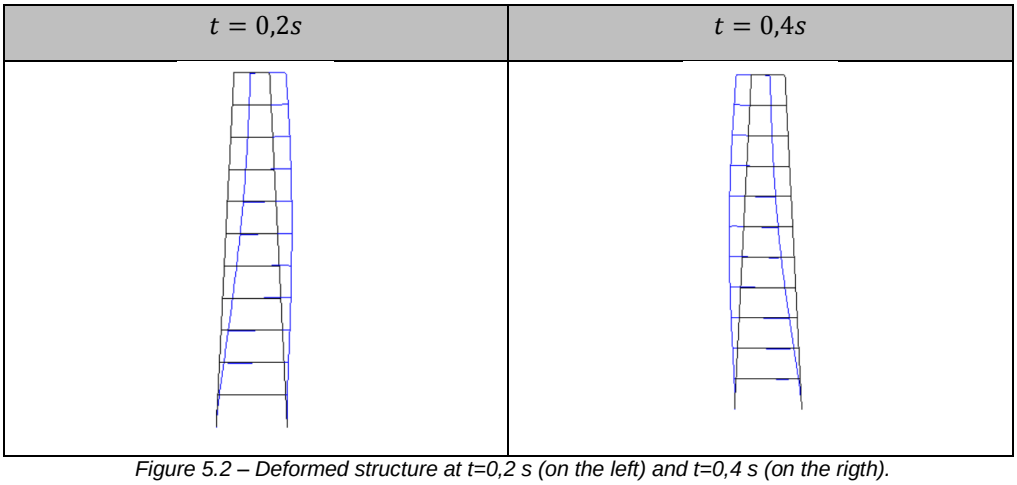

Figure 5.2 – Deformed structure at t=0,2 s (on the left) and t=0,4 s (on the rigth).

The horizontal displacement over time at the top of the tower is plotted in figure 5.3.

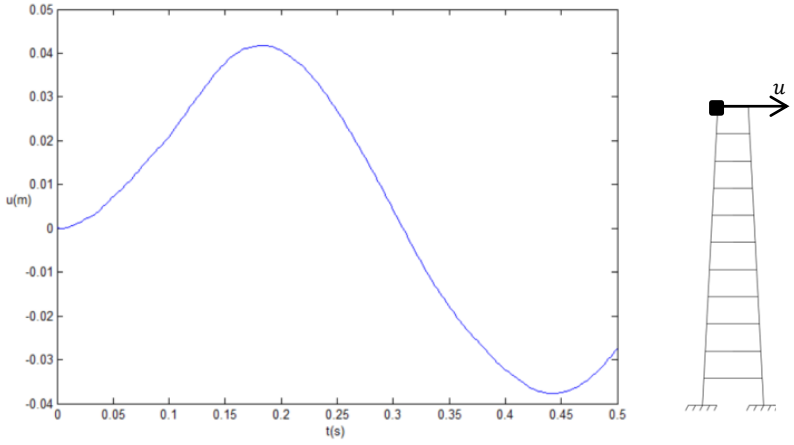

Figure 5.3 – Horizontal displacement at the top.

It is shown in the figure 5.4, the evolution in time of the bending moment at the base of the structure.

0

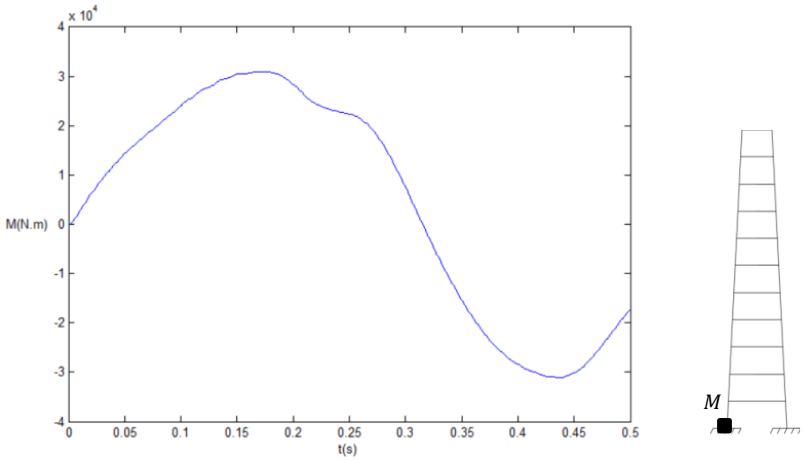

Figure 5.4 – Bending moment at the base.

6. Conclusions

- The mixed time integration method allows the adoption of efficient procedures for both *p*- and *h*- refinements.
- The adoption of high dimension basis allows the use of large time steps.
- For equal value for time increments the Daubechies wavelet systems present the best results.
- When the dimension of the basis is higher than certain value, the stability can not be ensured.

References

[1] Barbosa A.R., *''Wavelets no intervalo em elementos finitos''*, Dissertação para obtenção do Grau de Mestre em Engenharia de Estruturas, Instituto Superior Técnico, Universidade Técnica de Lisboa, 2001.

[2] Bathe K.J., *"Nonlinear Dynamic Response Course",*

[3] Freitas J.A.T., ''Mixed finite element solution of time-dependent problems'', Elsevier B.V., 2008.

[4] Fröhlich J. e Uhlmann M., ''Orthonormal Polynomial Wavelets on the Interval and Applications to the Analysis of Turbulent Flow Fields'', SIAM J Appl. Anal.,1984.

[5] Newmark N.M.,*''A method of computation for structural dynamics''*, ACSE 85 EM85, 1959.

[6] Spiegel M. R. e Abellanas L., *''Fórmulas e Tabelas de Matemática Aplicada''*, Schaum, McGrawHill, (1990).

[7] Wilson E. L., Farhoomand, I., e Bathe, K. J. *"Nonlinear Dynamic Analysis of Complex Structures"* International Journal of Earthquake Engineering and Structural Dynamics, Vol. 1, pp. 241-252, 1973.