Mechanical and thermodynamical properties of matter in strong gravitational fields

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In this work we study a mechanical property for compact stars and thermodynamical properties of thin matter shells in the context of general relativity. For compact stars made of perfect fluid with a small electrical charge we obtain the interior Schwarzschild limit, i.e., the limiting radius to mass relation. We also obtain the maximum mass that the star can attain. As for thin matter shells we study a rotating shell in (2+1)-dimensional AdS spacetime. We start by using the junction conditions to obtain the dynamical relevant quantities: rest mass, pressure and angular velocity. Then we use the first law of thermodynamics to obtain the shell's entropy differential, in the slowly rotating limit. The entropy is parametrized by a phenomenological function with free parameters. We also take the shells to their gravitacional radius and we obtain the Bekenstein-Hawking entropy of a black hole.

I. INTRODUCTION

In 1915, Einstein presented a geometric theory of gravitation known as general relativity or the general theory of relativity. This theory is the description of gravitation in modern physics and generalizes Newton's gravitation by interpretating gravity as a geometric property of spacetime. It is described by Einstein's field equations, a set of non-linear partial differential equations, which tellsl us that the spacetime is modeled by the distribution of matter. [2]. The field equations, also known as Einstein's field equations, are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where Greek indices are spacetime indices running from 0 to d, with 0 as the time index and d as the number of spatial dimensions and where the velocity of light in vacuum equals one ($c = 1$). The Einstein tensor $G_{\mu\nu}$ is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, and R is the Ricci scalar. On the left side of Eq. (1) we have the Einstein tensor which is a purely geometric quantity, whereas on the right side we have the energy-momentum tensor, $T_{\mu\nu}$, which provides the information about the matter. G is the gravitational constant.

Strong field regime, as the name indicates, refers to a strong gravitational field. Basically, it is a gravitational field that causes large deviations from flat spacetime. Unfortunately we do not know if general relativity is valid in this regime, due to lack of experimental data. The sought alternatives theories must verify general relativity as a special case for sufficiently weak fields. These alternative theories are based on the principle of least action, by choosing the appropriate lagragian density for gravity [4].

Strong gravitational fields can be found in compact objects, which is a term used in astrophysics to refer to the most dense objects in nature. We separate compact objects into two categories: continuous matter, i.e., compact stars, and thin matter shells. In this kind of objects it is imperative to use general relativity as the gravitation theory, since Newtonian theory of gravitation does not provide a realistic description for gravitational fields.

White dwarfs, neutron stars and black holes are examples of compact stars. Also exotic stars, which are a compact star composed of something other than baryons as darkmatter. Compact stars are stellar objects with small volume for their mass, which means very high density. For instance, neutron stars typically have masses of the same order of magnitude as the sun's mass, $M \sim 1.4 M_\odot$, in radii of only $R \sim 10$km. Thus, we can have densitys $\rho \sim 10^{15}$g cm^{-3}, even larger than the nuclear ground state density, $\rho_0 \sim 10^{14}$g cm^{-3} [3].

Now, the first compact star ever displayed was a star made of an incompressible perfect fluid, i.e., $\rho(r) =$ constant, and isotropic pressure (where $\rho(r)$ is the energy density at the radius r) [28][36]. This interior Schwarzschild star solution is spherically symmetric and has a vacuum exterior. An incompressible equation of state is interesting from various aspects, since one can extract clean results and it also provides compactness limits. Furthermore, this incompressible fluid applies to both fermion and boson particles, as long as the fluid is at an incompressible state. As a drawback for such an equation of state, one can mention that the speed of sound through such a medium is infinite, but generically the overall structure is not majorly changed. Schwarzschild found that there was a limit, when the central pressure p_c goes to infinity and that the star's radius to the mass limit is $R/M = 9/4 = 2.25$ ($G = 1$), indeed a very compact star [28]. Volkoff [5] and Misner [6] rederived the Schwarzschild interior limit of $R/M = 9/4$ using the propitious Tolman-Oppenheimer-Volkoff (TOV) equation, a differential equation for the pressure profile as a function of the other quantities [7] [8]. In addition, Misner [6] even found a maximum mass for a given density of the incompressible fluid, the Misner mass.

One can ask if the Schwarzschild limit can be modified, allowing for instance a lower R/M relation. One way is to have some kind of repulsive matter or new field in the star. Another way is by resorting to some alternative theory of gravity. It is also well known that the introduction of new matter fields can be mimicked by modifications of the gravitational field. Rather than introducing an alternative theory of gravitation we here opt to study the case in which we add a matter field to the existent matter. We consider matter with a small electric charge, introducing thus an additional electric field in addition to the usual matter and gravitational fields. This addition of an electric charge and an electric field to the Schwarzschild incompressible matter configurations brings insight to the configurations overall structure in

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more complex situations and its study in stars mimics other fields and possible alterations in the gravitational field. The important quantity in knowing how much electric charge a star can support is the ratio of the mass m to the charge q of the main fundamental constituents of the star [9]. For normal matter the net electric matter in a star is utterly negligible as the ratio of the proton mass m_p to the proton charge e is $m_p/e = 10^{-18}$, giving thus $Q/M \simeq (m_p/e)^2 \simeq 10^{-36}$, where Q is the star's total charge [9], see also [10]. However, stars can contain some dark matter in their interior, and of the several dark matter fluid candidates some could be electrically charged. Indeed, natural candidates to compose the dark matter are supersymmetric particles. The lightest supersymmetric particles that makes the bulk of dark matter should be neutral. One possible candidate is the neutralino [11], however, some of these particles could be electrically charged. The mass m to charge q ratio of these supersymmetric particles are much higher than the baryonic mass to charge ratio, indeed current supergravity theories indicate that some particles can have a ratio of one. For a $m/q \sim 0.1 - 0.3$ one has $Q/M \simeq 0.01 - 0.1$, a small but non-negligible electric charge. Thus, if dark matter populates the interior of stars, and some of it is made of electrically charged particles there is the possibility that stars have a tiny but non-negligible electric charge. In this case the radius-mass relations for the corresponding stars should get modifications and have an influence in the structure of the compact star.

A possible way to simulate compact objects, while keeping the most important physical aspects, is using thin matter shells. A thin matter shell is a hypersurface which separates spacetime into two regions: the interior region and the exterior region. Due to the development of a singularity in spacetime there are some conditions that must be satisfied to ensure that the entire spacetime is a valid solution of Einstein's equations. These conditions are called junctions conditions (see [12]). In this kind of system, the material degrees of freedom of the shell are related to the gravitational degrees of freedom by the Einstein equations, which implies that the thermodynamics of the shell is acutely related to the structure of spacetime.

The usefulness of thin matter shells is also evident from the fact that we can take the black hole limit, i.e., the shells can be taken to their gravitational radius. Once we obtained the entropy from the thermodynamics approach, we can take the black hole limit and obtain the black hole entropy. Thus, the black hole thermodynamic properties can be attained using a much more simplified computation than the usual black hole mechanics. This idea was developed by Martinez [13] (see also [14]).

Although black holes are sometimes considered as compact stars, they are a different and special object. They are formed by the gravitational collapse of a massive star or a cluster of stars. As long as more mass is gathered, the star reaches its breaking point. The pressure is not sufficient to outweigh gravity and the star collapses. Black holes are objects of extreme density with such strong gravitational attraction that even light cannot escape. The boundary of the region from which nothing can escape is denominated event horizon.

Astronomers have discovered two types of astrophysical black holes, i.e., candidates to black holes with strong experimental evidence: the stellar-mass black holes, with masses from 5 to 30 solar masses and supermassive black holes with masses from 10^6 to 10^{10} solar masses. There is also evidence of another class of black holes, the intermediate-mass black hole, with masses between the stelar-mass black holes and supermassive black holes, that range from 10^2 to 10^5 solar masses.

Black holes have a finite temperature, $T_{\rm BH}$, and entropy [15, 16, 18]. They also emit radiation known as Hawking radiation. This radiation, which is also acknowledged as black hole evaporation, reduces the mass and the energy of the black hole and is caused by quantum effects. If black holes did not have entropy, $S_{\rm BH}$, one would violate the second law of thermodynamic by throwing some mass into the black hole. Bekenstein proposed that the entropy was proportional to the area of its event horizon, by comparing the first law of thermodynamics to the energy conservation law of a black hole [15], also known as the first law of black hole dynamics. On the other hand the second law of black hole mechanics states that the black hole area never decreases ($\Delta A \geq 0$), similar to the second law of thermodynamics ($\Delta S \geq 0$). The second law of black hole mechanics is a mathematically rigorous consequence of general relativity, while the second law of thermodynamics is a law that stands for systems with many degrees of freedom. However, this relation between both laws proved to be of a fundamental nature. The black hole entropy, also known as Bekenstein-Hawking entropy is given by

$$S_{\rm BH} = \frac{1}{4}\frac{A_+}{l_{\rm p}^2}\,, \tag{2}$$

in units with $k_B = 1$ (to be used in all this work), where $l_{\rm p} = \sqrt{G\hbar}$ is the Planck length, $\hbar$ is the Planck constant, k_B the Boltzmann constant and A_+ is the area of the event horizon. The black hole temperature is known as the Hawking temperature and it is

$$T_{\rm H} = \frac{\hbar\kappa}{2\pi}\,, \tag{3}$$

where κ is the surface gravity.

There are other ways to obtain the black hole entropy. Hawking also derived the black hole entropy, but from a path integral approach of quantum field theory in curved spacetime [19]. York [20] obtained the black hole entropy using the grand canonical ensemble. The last two approaches are: through quasi-black holes [21–23] and through thin matter shells [13, 14, 31]. In this work this approach of thin matter shells will be used.

Despite the fact that most of engineering problems which conciliate heat with work and energy can be solved with the theory of thermodynamics, one must remark that the full understanding of the macroscopic physics of a given system is not enough on a physical perspective. One needs to perceive the macroscopic behaviour of the system by understanding its microscopic dynamics, as one does on statistical mechanics where the laws of thermodynamics are obtained from first principles through the microscopic analysis of the system, concerning the notion of degree of freedom and its influence on the phase space. The entropy of a macroscopic system contains information about the degrees of freedom, since it is related to the number of ways in which the system can be formed. So if there is no microscopic theory the entropy can be used to find some clues about that theory. A black hole is an example of a system where we do not know the microscopic theory describing the system, despite having an expression for the entropy [15–18]. This is because we need a theory for quantum gravity to describe the microscopics of a black hole. Whereas in statistical mechanics the degrees of freedom are well-known, for a black hole that is not true, since all the information is lost through the event horizon ("no-hair theorem" [2]). Thus, the black hole entropy quantifies the information of the system but does not clarify about the nature of the degrees of freedom, which can be gravitational or material. So we expect that a quantum theory of gravity should not create any type of distinction between material and gravitational degrees of freedom. However this is still a theme for phenomenological studies. Quantum gravity becomes relevant at radii, $R \sim l_{\rm p} \sim 10^{-33}$cm. This kind of investigation may clear up some features of the thermodynamics of the gravitational field which can lead us to some aspects of a quantum theory of gravity. [24].

Sometimes it is interesting to reduce the spatial dimension to 2 and study general relativiy in (2+1)-dimensions. This decrease in dimensionality reduces the degrees of freedom and simplifies the calculations, although maintaining the essential physical features. The interest in (2+1)-dimensional spacetimes suffered an increment after the discovery of a black hole solution in spacetimes asymptotically AdS, the Bañados-Teitelbom-Zanelli (BTZ) black hole [25] [26]. The BTZ black hole is a semi classical system with a Bekenstein-Hawking entropy, Eq. (2). This black hole has a Hawking temperature given by $T_{\rm H} = \dfrac{\hbar}{2\pi l^2}\dfrac{r_+^2 - r_-^2}{r_+}$ and an angular velocity $\omega_{\rm BH} = \dfrac{r_-}{l r_+}$ [27].

This abstract is organized as follows. Sec. II presents an analytical scheme to investigate the limiting radius to mass relation and the maximum mass of relativistic stars made of an incompressible fluid with a small electric charge, thus generalizing the Schwarzschild interior limit [28]. The investigation is carried out by using the hydrostatic equilibrium equation, i.e., the Tolman-Oppenheimer-Volkoff (TOV) equation [7, 8], together with the other equations of structure, with the further hypothesis that the charge distribution is proportional to the energy density and that the energy density is constant. Sec. III is dedicated to the analysis of the dynamics of a rotating thin matter shell in a (2+1)-dimensional asymptotically AdS spacetime, i.e., a ring dividing two vacuum regions, the interior region and the exterior region. The exterior spacetime is BTZ and thus asymptotically AdS and inside the ring the spacetime is flat AdS. Our approach to the dynamics of the shell will be equivalent to the ones of [29], [30] and [12], for a (3+1)-dimensional rotating shell. Futhermore, we study the thermodynamics of that thin matter shell for the slowly rotating limit. The approach for Sec. III is the following. We integrate the first law of thermodynamics to obtain the entropy of the thin shell using its pressure, rest mass and angular velocity forthcoming from the junction conditions. The entropy is obtained up to a function of the gravitational radius of the shell. Finally we push the shell to its gravitational radius, i.e., we take the black hole limit and we arrive to the Bekenstein-Hawking entropy of the BTZ black hole [31].

II. COMPACT STARS WITH A SMALL ELECTRIC CHARGE: THE LIMITING RADIUS TO MASS RELATION AND THE MAXIMUM MASS FOR INCOMPRESSIBLE MATTER

A. General relativistic equations

1. Basic equations

We are interested in analyzing highly compacted charged spheres as described by the Einstein-Maxwell equations with charged matter. In this section we set $G = 1$ and $c = 1$. The field equations are Eq. (1) together with

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu , \tag{4}$$

with the same convention for indexes. The Faraday-Maxwell tensor $F_{\mu\nu}$ is defined in terms of an electromagnetic four-potential A_μ by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and J_μ is the electromagnetic four-current. $T_{\mu\nu}$ is written here as a sum of two terms,

$$T_{\mu\nu} = E_{\mu\nu} + M_{\mu\nu} \, . \tag{5}$$

$E_{\mu\nu}$ is the electromagnetic energy-momentum tensor, which is given in terms of the Faraday-Maxwell tensor $F_{\mu\nu}$ by the relation

$$E_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\gamma F_{\nu\gamma} - \frac{1}{4} g_{\mu\nu} F_{\gamma\beta} F^{\gamma\beta} \right) . \tag{6}$$

$M_{\mu\nu}$ represents the matter energy-momentum tensor and we assume to be the energy-momentum tensor of a perfect fluid, namely,

$$M_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu} , \tag{7}$$

with ρ and p being the energy density and the pressure of the fluid, respectively, and U_μ is the fluid four-velocity. For a charged fluid, this current is given in terms of the electric charge density ρ_e by

$$J^\mu = \rho_e U^\mu \, . \tag{8}$$

The other Maxwell equation $\nabla_{[\alpha} F_{\beta\gamma]} = 0$, where [...] means antisymmetrization, is automatically satisfied.

2. Equations of structure

One finds that one of Einstein equations gives a differential equation for $m(r)$, i.e.,

$$\frac{dm(r)}{dr} = 4\pi\rho(r) r^2 + \frac{q(r)}{r}\frac{dq(r)}{dr} \, . \tag{9}$$

Since $m(r)$ represents the gravitational mass inside the sphere of radial coordinate r, Eq. (9) represents then the energy conservation as measured in the star's frame. The only non-vanishing component of the Maxwell equation (4) is given by

$$\frac{dq(r)}{dr} = 4\pi\rho_e(r)\, r^2 \sqrt{1 - \frac{2m(r)}{r} + \frac{q^2(r)}{r^2}} , \tag{10}$$

Finally, for the pressure it yields

$$\frac{dp}{dr} = -\,(p+\rho)\frac{\left(4\pi p r + m/r^2 - q^2/r^3\right)}{\left(1 - 2m/r + q^2/r^2\right)} + \rho_e \, \frac{q/r^2}{\sqrt{1 - 2m/r + q^2/r^2}} \, . \tag{11}$$

where to simplify the notation we have dropped the functional dependence. Eq. (11) is the TOV equation modified by the inclusion of electric charge [32].

3. Equation of state and the charge density profile

Here we assume an incompressible fluid, i.e.,

$$\rho(r) = \text{constant}\,, \tag{12}$$

so the energy density is constant along the whole star.

Following [33] (see also [34]), we assume a charge density proportional to the energy density,

$$\rho_e = \alpha\rho\,, \tag{13}$$

where, in geometric units, α is a dimensionless constant which we call the charge fraction. The charge density along the whole star is thus constant as well.

4. The exterior vacuum region to the star and the boundary conditions

The conditions at the center of the star are that $m(r=0)=0$ and $q(r=0)=0$, to avoid any type of singularities, and that $p(r=0)=p_c$, $\rho(r=0)=\rho_c$, and $\rho_e(r=0)=\rho_{ec}$, where p_c is the central pressure, ρ_c is the central energy density, and ρ_{ec} is the central charge distribution, the two latter having the same constant values throughout the star (see Eqs. (12-13)).

The interior solution is matched at the surface to the exterior Reissner-Nordström spacetime, with metric given by

$$ds^2 = -F(r)\,dT^2 + \frac{dr^2}{F(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,, \tag{14}$$

where

$$F(r) = 1 - 2M/r + Q^2/r^2\,, \tag{15}$$

with the outer time T being proportional to the inner time t, and M and Q being the total mass and the total charge of the star, respectively.

At the surface of the star one has a vanishing pressure, i.e., $p(r=R)=0$. The boundary conditions at the surface of the star are then $m(R)=M$, $q(R)=Q$, besides $p(R)=0$.

An important quantity for the exterior metric is the gravitational or horizon radius r_+ of the configuration. The Reissner-Nordström metric, given through Eqs. (14)-(15), then has

$$r_+ = M + \sqrt{M^2 - Q^2}\,. \tag{16}$$

as the solution for its own gravitational radius.

B. The interior Schwarzschild limit: The zero charge case

1. Equations

Before we treat the small charge case analytically, we consider the exact Schwarzschild interior solution as given by [5] and displayed later in Misner's lectures [6]. For this we put $q=0$ in Eqs. (9)-(11). Integrating Eq. (9) gives

$$m(r) = \frac{4}{3}\pi\rho\, r^3\,, \quad 0 \le r \le R\,. \tag{17}$$

Defining a characteristic length R_c as

$$R_c^2 = \frac{3}{8\pi\rho}, \tag{18}$$

we can rewrite the mass function, Eq. (17), as

$$m(r) = \frac{1}{2}\frac{r^3}{R_c^2}\,, \quad 0 \le r \le R\,. \tag{19}$$

Defining a new radial coordinate χ by

$$r = R_c \sin\chi\,, \tag{20}$$

and integrating Eq. (11) yields the pressure

$$p = \rho\frac{\cos\chi - \cos\chi_s}{3\cos\chi_s - \cos\chi}. \tag{21}$$

2. The interior Schwarzschild limit: The R and M relation and the minimum radius

The central pressure, p_c is the pressure computed at zero radius $r=0$, i.e., $\chi=0$, so that

$$p_c = \rho\frac{1-\cos\chi_s}{3\cos\chi_s - 1}\,. \tag{22}$$

This blows up,

$$p_c \to \infty \quad \text{when} \quad \cos\chi_s \to 1/3\,. \tag{23}$$

This is equivalent to

$$\frac{R}{M} = \frac{9}{4}, \tag{24}$$

which is the Schwarzschild limit found by [28].

3. Misner mass bound

Following [6] we can also display a mass bound. By eliminating R in Eqs. (24) and (19) at star radius, and noting that $p_c \le \infty$, one gets the mass bound

$$M \le \frac{1}{2}\left(\frac{8}{9}\right)^{3/2} R_c\,. \tag{25}$$

To get a mass we have to have a density and thus an R_c. One can make sense of a constant density if one takes it as the density at which matter is almost incompressible and the pressure throughout the star is very high. If the fluid is an ideal gas this happens when the particles have relativistic velocities of the order 1. For fermions this happens when the Fermi levels are near the rest mass m_n of the fermions, neutrons say, while for bosons this means that the gas temperatures are of the order of the rest mass m_b of the particles. This gives, for both fermions and bosons, a density of one particle per cubic Compton wavelength. I.e., for a particle with mass m and Compton wavelength λ given by $\lambda = \hbar/m$ the density is $\rho \sim m^4/\hbar^3$. In the case of a star composed of neutrons, [6] obtains

$$M \leq 1.5 M_\odot\,, \tag{26}$$

where $M_\odot$ is the Sun's mass. This bound is similar to the Chandrasekhar limit $M_{\rm Chandrasekhar} = 1.44 M_\odot$ [35], or the Oppenheimer-Volkoff mass, $M_{\rm OV} \simeq 1 M_\odot$ [7], both found for equations of state different from the one used here and through totally different means.

C. The electrically charged interior Schwarzschild limit: The small charge case

1. Equations: perturbing with a small electric charge

(i))Expansion in the electric charge parameter α
We are going to solve equations (9), (10), and (11), treating the charge $q(r)$ as a small perturbation, thus assuming α small. To do so, we note that the solutions for the mass and the charge will be of the form

$$q(r) = q_1(r)\,, \tag{27}$$
$$m(r) = m_0(r) + m_1(r)\,, \tag{28}$$

where we are assuming that the non-perturbed charge is zero $q_0(r) = 0$, $m_0(r)$ is the mass of the uncharged star given by Eq. (17), and $q_1(r)$ and $m_1(r)$ are the perturbed small charge and mass functions to be determined. The pressure is also assumed to be given by the expansion

$$p(r) = p_0(r) + p_1(r)\,, \tag{29}$$

where p_0 is the pressure in the uncharged case, given by Eq. (21), and $p_1(r)$ is the perturbation induced in the pressure when a small charge is considered.

At this point, it will prove useful to introduce the dimensionless variable

$$x = \frac{r}{R_c}\,, \tag{30}$$

where R_c is the characteristic length defined in Eq. (18). The expressions for the mass, charge, and pressure in this new variable are generically defined as

$$m(x) = \frac{m(r)}{R_c}\,, \quad q(x) = \frac{q(r)}{R_c}\,, \quad p(x) = \frac{p(r)}{\rho}\,. \tag{31}$$

From Eq. (30) we defined x_s as the x at the surface. Note that, while the boundary condition for the non-charged star was simply $p(x_s) = p_0(x_s) = 0$, the boundary condition for the charged star becomes

$$p_0(x_s) + p_1(x_s) = 0\,. \tag{32}$$

(ii)Calculation of the perturbed charge distribution q_1
Expanding Eq. (10) for small α and solving for $q_1(x)$ results in

$$q_1(x) = \frac{3}{4}\,\alpha\,\left(\arcsin x - x\sqrt{1-x^2}\right)\,. \tag{33}$$

(iii) Calculation of non-perturbed and perturbed masses
The unperturbed mass m_0 can now be expressed simply as

$$m_0(x) = \frac{x^3}{2}\,. \tag{34}$$

One can also find an expression for m_1, namely,

$$m_1(x) = \frac{3}{8}\,\alpha^2\,\left(3x - x^3 - 3\sqrt{1-x^2}\arcsin x\right)\,. \tag{35}$$

(iv) Equations for the pressures, solution for the zeroth order pressure, and calculation of the perturbed pressure at the star's radius
To find the equations for the pressures $p_0(x)$ and $p_1(x)$, we expand Eq. (11) in powers of α and retain the two lowest terms, thus obtaining two differential equations:

$$\frac{dp_0(x)}{dx} = -\frac{(1+p_0(x))\left(3p_0(x)x/2 + m_0(x)/x^2\right)}{1 - 2m_0(x)/x}\,, \tag{36}$$

and

$$\begin{aligned}\frac{dp_1}{dx} =& -\frac{p_1(3p_0x/2 + x/2) + (1+p_0)(3p_1x/2 + f_2)}{(1-2m_0/x)} \\ &- \frac{(1+p_0)(3p_0x/2 + x/2)f_1}{(1-2m_0/x)^2} + \\ &+ \frac{\alpha q_1}{x^2\sqrt{1-2m_0/x}}\,,\end{aligned} \tag{37}$$

where we have defined the auxiliary functions $f_1 = f_1(x)$ and $f_2 = f_2(x)$ by

$$f_1(x) = \frac{2m_1(x)}{x} - \frac{q_1^2(x)}{x^2}\,, \tag{38}$$

and

$$f_2(x) = \frac{m_1(x)}{x^2} - \frac{q_1^2(x)}{x^3}\,. \tag{39}$$

Ultimately, we want to obtain an equation for the radius R for which the central pressure blows up. From Eq. (29), the central pressure is $p(0) = p_0(0) + p_1(0)$. It is possible to show that $p_1(0)$ is always finite (see [36]). So we have to find a solution for the radius R at which $p_0(0)$ blows up.

Then, the expression for p_0, analogous to Eq. (21), subjected to the boundary condition given by Eq. (32) becomes

$$p_0(x) = \rho\,\frac{\sqrt{1-x^2} - \sqrt{1-x_s^2}\,(1 + 2p_1(x_s))}{3\sqrt{1-x_s^2}\,(1+2p_1(x_s)) - \sqrt{1-x^2}}. \tag{40}$$

Since $p_0(x)$ depends on $p_1(x_s)$ in the denominator, we have to calculate the perturbed pressure at the star's radius.

The equation for p_1, Eq. (37), cannot be solved analytically for all x. However, we are only interested in the value of p_1 at the surface of the star. At this particular radius it is possible to obtain the exact value of the perturbed pressure without ever solving Eq. (37). The reason for this is the fact that at the star's radius the pressure $p(x_s) = p_0(x_s) + p_1(x_s)$ is zero. Therefore, we can insert the boundary condition $p(x_s) = 0$ in the exact derivative of the pressure given by Eq. (11), written in terms of x instead of r, and expand the resulting equation for small α. Using the expansion (29) and equating terms of the same order in α one can obtain a compatibility condition which must be physically required. The unique valid solution to this condition must satisfy the boundary condition (32) so it is

$$p_0(x_s) = p_1(x_s) = 0\,. \tag{41}$$

(iv) Equation for the minimum radius
The central pressure $p_0(0)$ given in Eq. (40) is divergent if

$$\frac{x_s}{m(x_s)} = \frac{9}{4} - \left(\frac{9}{8}p_1(x_s) + x_s\,\frac{m_1(x_s)}{m_0^2(x_s)}\right)\,, \qquad (42)$$

valid in first order in α^2.

Now, the minimum star radius R will not be just $\sqrt{8/9}\,R_c$ but will have corrections of order α^2. These corrections will induce changes of the order α^4 in Eq. (42). Thus, we can set $x_s = \sqrt{8/9}$ in Eqs. Eqs. (34), (35), (41) and (42) and obtain

$$\frac{x_s}{m_s} = \frac{9}{4} - 1.529\,\alpha^2\,, \qquad (43)$$

up to order α^2.

In addition, it can also be interesting to express α in terms of the total charge Q and mass M. The following relation valid for small q_1, or small α, can be found $\frac{Q}{M} = \frac{q_1(x_s)}{m_0(x_s)+m_1(x_s)} = \frac{q_1(x_s)}{m_0(x_s)}$ so that

$$1.641\,\alpha = \frac{Q}{M}\,. \qquad (44)$$

2. The electric interior Schwarzschild limit: The R, M and Q relation for small charge

Therefore the desired ratio (43) in terms of the quantities R, M and Q is

$$\frac{R}{M} = \frac{9}{4} - 0.568\,\frac{Q^2}{M^2}\,, \qquad (45)$$

valid up to order Q^2/M^2.

We can also express the limit in terms of the horizon radius, r_+, for the Reissner-Nordström metric. Up to order Q^2/M^2 one has, $r_+ = M + \sqrt{M^2 - Q^2} = 2M\left(1 - \frac{1}{4}\frac{Q^2}{M^2}\right)$. So,

$$\frac{R}{r_+} = \frac{9}{8} - 0.003\frac{Q^2}{M^2}\,, \qquad (46)$$

up to order Q^2/M^2. These two last equations are different forms of the Schwarzschild interior limit for small charge. All of them show that, in comparison with the uncharged case Eq. (24), the star can be more compact. In particular, Eq. (46) shows that in the charged case the radius of the star can be a bit nearer its own horizon.

In [37] these compact stars were studied numerically. An $R/M \times Q/M$ relation was given numerically for $0 \leq Q/M \leq 1$. For small charge, $Q/M \ll 1$, one can extract from the numerical calculations in [37] that $\frac{R}{M} \simeq 2.25 - 0.6\,\frac{Q^2}{M^2}$. This should be compared to our analytical calculation valid in first order of Q/M, given here in Eq. (45), i.e., $\frac{R}{M} = 2.25 - 0.568\,\frac{Q^2}{M^2}$. It shows that the numerical code used in [37] is compatible with the analytical calculation. In that work [37] it was also shown numerically that in the other extreme, namely, $Q/M = 1$, one would obtain a star at its own gravitational radius, $R/M = 1$, i.e., an (extremal) quasiblack hole.

A related theme is the Buchdahl and the Buchdahl-Andréasson bounds. [38] by imposing a simple set of assumptions, namely, the spacetime is spherically symmetric, the star is made of a perfect fluid, and the density is a nonincreasing function of the radius, found that the radius to mass relation is $R/M \geq 9/4$. Thus the Schwarzschild limit [28], i.e., the limiting R/M configuration when the central pressure goes to infinity, is an instance that saturates the Buchdahl bound. Following the line of reason of Buchdahl, [39] obtained a bound for the minimum radius of a star using the following energy condition $p + 2p_T \leq \rho$ where p_T is the tangential pressure, p is the radial pressure and ρ is the energy density. This bound, the Buchdahl-Andréasson bound, is given by $\frac{R}{M} \leq \frac{9}{\left(1+\sqrt{1+3Q^2/R^2}\right)^2}$. Retaining terms in first order in Q^2 one gets $\frac{R}{M} = \frac{9}{4} - 0.667\,\frac{Q^2}{M^2}$. Thus, our configurations of constant density and a charge distribution proportional to the energy density having $\frac{R}{M} = \frac{9}{4} - 0.568\,\frac{Q^2}{M^2}$, see Eq. (45), does not saturate the bound. This raises the question of whether there are other types of charged matter that can saturate the bound. One type is thin shells with an appropriate relation between surface energy density and surface pressure [39]. Are there continuous (non-thin-shell) distribution configurations that saturate the bound? It seems that, as the configurations analyzed here, the configurations studied in [40–42] do not saturate the bound. It remains to be seen if the electrically charged configurations analyzed in [43, 44] saturate the bound. For further study on bounds of electrically charged stars see [45–47].

3. A mass bound

We can adapt the mass bound from section II B 3 to the small charge case yielding

$$M \leq \frac{1}{2}\left(\frac{8}{9}\right)^{3/2} R_c\,\left(1 + 0.679\,\alpha^2\right)\,. \qquad (47)$$

In the case of a compact star composed of neutrons in the incompressible state speckled with some charged particles, we obtain

$$M \leq 1.5\,M_\odot\,\left(1 + 0.679\,\alpha^2\right)\,, \qquad (48)$$

Comparing equation (48) with equation (26) we see that we can attain bigger mass on a charged star. This is expected since the electrostatic repulsion is opposite to the gravitational force, which means that we can put more mass on the star without it collapsing.

III. DYNAMICS AND THERMODYNAMICS OF A ROTATING THIN SHELL IN A (2+1)-DIMENSIONAL ASYMPTOTICALLY ADS SPACETIME AND THE BTZ SLOWLY ROTATING BLACK HOLE LIMIT

A. The thin shell spacetime

In 2+1 dimensions, Einstein's equation with cosmological constant Λ is

$$G_{\mu\nu} = 8\pi G_3 T_{\mu\nu} + \Lambda g_{\mu\nu}\,, \qquad (49)$$

where G_3 is the gravitacional constant in 2+1 dimensions. We will choose units with the velocity of light equal to one, $c=1$, so that G_3 has units of the inverse of mass. The greek indices run from $0,1,2$, with 0 begin the time index. Since we want to work in an AdS spacetime, with negative cosmological constant, we define the AdS length l as $-\Lambda=\frac{1}{l^2}$.

We consider a timelike shell with radius R, i.e., a ring, in a (2+1)-dimensional spacetime. This ring divides spacetime into two regions, an inner region and an outer one.

The exterior metric is given by the BTZ line element [25]

$$\begin{aligned} ds_+^2 = g^+_{\alpha\beta}dx_+^\alpha dx_+^\beta = &-\left(\frac{r^2}{l^2}-8G_3m\right)dt_+^2 \\ &+\frac{dr^2}{\left(\frac{r^2}{l^2}-8G_3m+\frac{16J^2G_3^2}{r^2}\right)} \\ &-8G_3Jdt_+d\phi+r^2d\phi^2, \quad r\leq R, \end{aligned} \tag{50}$$

written in exterior coordinate system $x_+^\alpha=(t_+,r,\phi)$, where J is the spacetime angular momentum and m is the Arnowitt-Deser-Misner (ADM) mass. On the other hand, the interior region is flat AdS spacetime with metric given by

$$ds_-^2 = g^-_{\alpha\beta}dx^\alpha dx^\beta = -\frac{\rho^2}{l^2}\,dt_-^2+\frac{l^2}{\rho^2}d\rho^2+\rho^2d\psi^2\,, \tag{51}$$

written in interior coordinate system $x_-^\alpha=(t_-,\rho,\psi)$.

B. The thin shell gravitational junction conditions: exact solution

Now we analyze exactly the dynamic of this rotating shell. To remove the off-diagonal term in the induced metric, as viewed from the exterior region, obtained by setting $r=R$ in Eq. (50), we go to a co-rotating frame, defining the new polar coordinate $\psi=\phi-\Omega t_+$. This makes the induced metric diagonal if Ω is choosen to be

$$\Omega=\frac{4G_3J}{R^2}\,. \tag{52}$$

Therefore, in coordinates (t,ψ), where we chose $t\equiv t_+$, the induced metric is written as

$$ds_\Sigma^2=-\left(\frac{R^2}{l^2}-8G_3m+\frac{16J^2G_3^2}{r^2}\right)dt^2+R^2d\psi^2\,. \tag{53}$$

On the other hand the induced metric, as viewed from the interior region, is obtained by setting $\rho=R$ in Eq. (51). Applying the first junction condition, which states that the induce metric must be the same on both sides of the shell [12], yields

$$\left(\frac{R^2}{l^2}-8G_3m+\frac{16J^2G_3^2}{R^2}\right)dt^2=\frac{R^2}{l^2}\,dt_-^2\,, \tag{54}$$

so that we can write the interior metric as

$$\begin{aligned} ds_-^2 = &-\frac{\rho^2}{l^2}\frac{\left(R^2/l^2-8G_3m+\frac{16J^2G_3^2}{r^2}\right)}{R^2/l^2}\,dt^2 \\ &+\frac{l^2}{\rho^2}d\rho^2+\rho^2d\psi^2\,. \end{aligned} \tag{55}$$

On the other hand the second junction condition is

$$S^a_{\ b}=-\frac{\epsilon}{8\pi G_d}\left([K^a_{\ b}]+[K]h^a_{\ b}\right)\,, \tag{56}$$

where $K_{ab}=n_{\alpha;\beta}e^\alpha_a e^\beta_b$, with n_α the normal vector to the shell and e^α_a the tangent vectors to the shell, ";" denotes the covariant derivative, $\epsilon=n^\alpha n_\alpha$ and takes values $+1$ or -1 if the hypersurface is timelike or spacelike, respectively. The latin indices are spacetime indices with values 0 or 1, associated with coordinates (t,ψ), respectively [12]. Therefore, the components of the surface stress-energy tensor are given by

$$S^t_{\ t}=\frac{-1}{8\pi G_3 l}\left(1-\frac{l}{R}\sqrt{-8G_3m+\frac{R^2}{l^2}+\frac{16J^2G_3^2}{R^2}}\right)\,, \tag{57}$$

$$S^t_{\ \psi}=\frac{-J\sqrt{-8G_3m+\frac{R^2}{l^2}+\frac{16J^2G_3^2}{R^2}}}{2\pi R^3}\,, \tag{58}$$

$$S^\psi_{\ t}=\frac{J}{2\pi R\sqrt{-8G_3m+\frac{R^2}{l^2}+\frac{16J^2G_3^2}{R^2}}}\,, \tag{59}$$

$$S^\psi_{\ \psi}=\frac{1}{8\pi G_3 l}\left(-1+\frac{R}{l}\frac{1}{\sqrt{-8G_3m+\frac{R^2}{l^2}+\frac{16J^2G_3^2}{R^2}}}\right)\,. \tag{60}$$

We will attempt to write the surface stress-energy tensor in a perfect fluid form $S^a_{\ b}=(\sigma+p)u^a u_b+ph^a_{\ b}$, where h_{ab} is the induced metric from either outer or inner regions, λ is the surface density and p is the superficial pressure. Also the shell must move rigidly in the ψ direction with an uniform angular velocity ω implying that the velocity vector is expressed as

$$u^a=\gamma(t^a+\omega\psi^a)\,. \tag{61}$$

At this point is useful to introduce the gravitacional radius r_+ and the Cauchy radius r_- of the shell, with subsequent explicit expression

$$r_\pm=2l\sqrt{G_3m\pm\sqrt{G_3^2m^2-\frac{J^2G_3^2}{l^2}}}\,. \tag{62}$$

It will be also useful to define the quantity

$$k=\sqrt{\left(1-\frac{r_+^2}{R^2}\right)\left(1-\frac{r_-^2}{R^2}\right)}\,. \tag{63}$$

Therefore, one obtains

$$\begin{aligned} \lambda=\frac{M}{2\pi R}=&\frac{1}{8\pi G_3 l}\left(1-k\right) \\ &+\frac{r_+^2r_-^2}{R^4}\frac{\left(1-R^2/r_+^2\right)}{8\pi G_3 lk}\,, \end{aligned} \tag{64}$$

where Eq. (63) was used and M is the shell's rest mass,

$$\begin{aligned} p=&\frac{1}{8\pi G_3 l}\left[\frac{1}{k}\left(1-\frac{r_+^2r_-^2}{R^4}\right)-1\right] \\ &+\frac{r_-^2}{R^2}\frac{\left(R^2-r_+^2\right)}{8\pi G_3 lk\left(r_+^2-r_-^2\right)}\left(-\frac{2r_-^2r_+^2}{R^4}+\frac{r_+^2+r_-^2}{R^2}\right)\,, \end{aligned} \tag{65}$$

$$\omega = \frac{r_-}{r_+ l} - \frac{r_- r_+}{l R^2}\,. \tag{66}$$

The shell's angular velocity measured in the nonrotating frame is $\Omega_{shell} = \omega + \Omega$, given by

$$\Omega_{shell} = \frac{r_-}{l r_+}\,, \tag{67}$$

which coincides with the BTZ spacetime angular velocity [27].

C. Thermodynamics for the thin shell

At this moment we have to consider that the shell as a temperature T as measured locally, i.e., is a hot shell, and has an entropy S.

In the formalism stated in [48], the entropy S of a system is given in terms of the state independent variables, which we must designate, in order to obtain the complete thermodynamics description of the system. The natural choice of variables is (M, A, J), where $A = 2\pi R$ is the ring perimeter.

With this variables, we can write the first law of thermodynamics as

$$T dS = dM + p\, dA - \omega dJ\,, \tag{68}$$

where T, p and ω are the temperature, the pressure and the angular velocity, respectively.

In order to Eq. (68) for dS to be an exact differential, there are three integrability condition to be satisfied, as follow (see [14, 48])

$$\left(\frac{\partial\beta}{\partial A}\right)_{M,J} = \left(\frac{\partial\beta p}{\partial M}\right)_{A,J}\,. \tag{69}$$

$$\left(\frac{\partial\beta}{\partial J}\right)_{M,A} = -\left(\frac{\partial\beta\omega}{\partial M}\right)_{A,J}\,. \tag{70}$$

$$\left(\frac{\partial\beta p}{\partial J}\right)_{M,A} = -\left(\frac{\partial\beta\omega}{\partial A}\right)_{M,J}\,. \tag{71}$$

D. The three equations of state: equation for the pressure, equation for the temperature and equation for angular velocity

1. *The three independent thermodynamic variables*

The thermodynamics of the shell is only analyzed up to order J^2, i.e., a slowly rotating shell, such that

$$M = \frac{R}{4G_3 l}\,(1-k) \tag{72}$$

2. *The pressure equation of state*

Futhermore expanding Eq. (65) and combining with Eq. (72) we arrive to the seeked equation of state for the pressure

$$p(M,R,J) = p(M,R) = \frac{1}{8\pi G_3 l}\left(\frac{1}{1-\frac{4G_3 M l}{R}} - 1\right)\,, \tag{73}$$

where A has been swapped with R as thermodynamic variable. This equation follows exclusively from the spacetime structure and does not depend on the matter composing the shell. This can be observed since the equation is derived only by using gravitational considerations and the junction conditions on the ring, without ever specifying the matter fields. It is the same as obtained in [31].

3. *The temperature equation of state*

From the the first integrability condition, Eq. (69), by changing from variables (M, A, J) to (R, r_+, r_-), yields

$$\left(\frac{\partial\beta}{\partial R}\right)_{r_+,r_-} = \frac{\beta}{R k^2}\,. \tag{74}$$

The analytic solution of this equation for $\beta(R, r_+, r_-) \simeq \beta(R, r_+)$ is

$$\beta(R, r_+) = \frac{R}{l}\, k(R, r_+)\, b(r_+)\,, \tag{75}$$

up to order J^2, where $b(r_+)$ is an arbitrary function of the gravitational radius r_+.

The function $b(r_+)$ is the Tolman relation for the temperature in a (2+1)-dimension gravitational system [50]. It has units of inverse temperature and is the inverse temperature of the shell if located at radius R such that $R = \sqrt{r_+^2 + l^2}$.

Although b depends only on r_+, it is forced to depend on the state variables (M, R, J) through the functions $r_+(m(M, R, J) \simeq r_+(m(M, R))$. Also the integrability condition does not specify b, which is expected, as discussed in [13] (see also [49]). To yield a precise form for b one must specify the matter fields of the shell.

4. *The angular velocity equation of state*

For the sake of finding the equation of state for the angular velocity, Eqs. (69)-(71) can be combined to obtain

$$\frac{1}{2\pi}\left(\frac{\partial\omega}{\partial R}\right)_{r_+,r_-} + \omega\left(\frac{\partial p}{\partial M}\right)_{R,J} + \left(\frac{\partial p}{\partial J}\right)_{M,R} = 0\,, \tag{76}$$

where we mixed terms in variables (R, M, J) and (R, r_+, r_-) for computational simplicity.

The solution of this differential equation is given by

$$\omega(R, r_+, r_-) = \frac{\omega_0(r_+, r_-) l}{R k} - \frac{r_+ r_-}{R^3 k}\,, \tag{77}$$

where $\omega_0(r_+, r_-)$ is an arbitrary function. Recalling the dynamics we obtained the angular velocity of Eq. (66), which is precisely the angular velocity from Eq. (77) blueshifted, i.e., divided by $R/l\,k$ and with $\omega_0(r_+, r_-)$ fixed by

$$\omega_0(r_+, r_-) = \frac{r_-}{l r_+}\,. \tag{78}$$

This is analog to the Tolman relation for the temperature [50]. In this case we pull the angular velocity from infinite, which is the result from dynamics, Eq. (66), to the shell's radius, by blueshift.

E. Entropy of the thin shell

Now we have perceived all we need to find the entropy S. Eq. (68) gives the differential of the entropy

$$dS = \frac{b(r_+) r_+}{4Gl^2} dr_+ \,. \tag{79}$$

Integration of Eq. (79) yields the entropy, $S(r_+)$ and introduces an integration constant, S_0. The entropy is a function of r_+ alone, so it is a special function of (M, R) given by $S = S(r_+(M, R, J)) \simeq S(r_+(M, R))$. To get a specific expression for the entropy we need to prescribe the function $b(r_+)$. Therefore the entropy of the shell is

$$S = \int \frac{b(r_+) r_+}{4Gl^2} dr_+ + S_0 \,, \tag{80}$$

where S_0 is an integration constant. It seems physically reasonable to impose that the entropy must be zero when the shell vanishes ($M = 0$ and $J = 0$ so that $r_+ = 0$), which implies that $S_0 = 0$.

F. The black hole limit

The differential of the entropy, Eq. (79), is valid for $R \geq r_+$ so we can take the black hole limit $R \to \infty$. In this limit we wave to consider quantum and their back-reaction will diverge unless one take the temperature of the shell to be the Hawking temperature [13, 14, 31]

$$T_{\rm H} = \frac{\hbar}{2\pi l^2} \frac{r_+^2 - r_-^2}{r_+} \simeq \frac{\hbar}{2\pi l^2} r_+ \,. \tag{81}$$

This fixes the function $b = 1/T_{\rm H}$ and therefore the entropy of the shell at its own gravitational radius is

$$S(r_+) = \frac{\pi r_+}{2 l_{\rm p}^2} = \frac{A_+}{4 l_p^2} \,, \tag{82}$$

where $l_{\rm p}^2 = 1/(G\hbar)$ is the Planck length. One can notice that Eq. (82) is precisely the Bekenstein-Hawking entropy for a black hole, forasmuch as $A_+ = A_{\rm h} = 2\pi r_+$ is the black hole horizon perimeter.

IV. CONCLUSIONS

We have tracked down some mechanical properties of compact stars and thin matter shells, as well as some thermodynamic properties of thin matter shells which led us to the black hole entropy. On the subject of compact stars, we have obtained the interior Schwarzschild limit of spherically symmetric star configurations composed by a fluid with constant energy density and with a small electrical charge distribution proportional to its energy density. This analysis is opportune, since stars containg some type of dark matter may possess a small but non-negligble electric charge. We have found that the limiting star configuration can have more mass on a smaller radius due to eletric charge distribution, which is an expected result from its repulsive effect. The numerical code used in [37] is compatible with our analytical calculation. Futhermore our configurations of constant density and a charge distribution proportional to the energy density do not saturate the Buchdahl-Andréasson bound [39]. This raises the question of whether or not there are other types of charged matter that can saturate the bound. Thin shells with an appropriate relation between surface energy density and surface pressure do saturate the bound [39]. Are there continuous matter (non-thin-shell) distributions that saturate the bound? This question is an open problem to solve.

As for thin matter shells, we analyzed a (2+1)-dimensional shell in an asymptotically AdS rotating spacetime obtaining the shell's rest mass, the pressure and the angular velocity by using the junction conditions (see Sec. III). Note that due to rotation, the shell's surface energy-stress tensor was not in the form of a perfect fluid. We overcame this difficulty by attempting to write the tensor in a perfect fluid form such as [12, 29, 30]. Futhermore, we have generalized the Lemos and Quinta [31] work on the thermodynamics of a self-gravitating (2+1)-dimensional thin matter shell by considering rotation, yet slowly rotation. We arrived to a differential for the entropy, by using the first law of thermodynamics. Since the entropy must be an exact differential we obtained integrability equations which gave us the shell's inverse temperature and a equation for the thermodynamic's angular velocity. This angular velocity is precisely the angular velocity obtained by dynamics blueshifted, since this last one is defined at infinity and we need to pull it to the shell. Finally we took the shell to its own gravitacional radius, which by quantum arguments fixed the inverse temperature equal to the inverse of the Hawking temperature and we found that the entropy was equal to the Bekenstein-Hawking entropy of a BTZ black hole. However, we were only able to solve the thermodynamic problem for the slowly rotating limit. The general case is an open problem to solve. The difficulty relies on interpretating the pressure terms that appear in the first law of thermodynamics.

In the thesis it is also analyzed the problem of the dynamics and thermodynamics of a thin matter shell in a (3+1)-dimensional asymptotically AdS spacetime, similar to what was done in Sec. III. The thermodynamics of a self-gravitating (d+1)-dimensional thin matter shell in a asymptotically AdS spacetime for $d > 3$ is also an open problem. In addition, we also study the thermodynamical stability of those shells, following [48].

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