Fractionalization in low-dimensional systems

João Duarte Pereira Machado[1]

[1] *Instituto Superior Técnico, Lisboa, Portugal*
duarte.machado@ist.utl.pt

Resorting to a Fermi-liquid-like approach, the energy bands, the phase shifts and relation between the rapidities and the pseudomomenta for the Bethe ansatz pseudoparticles of the unidimensional Heisenberg and Kondo models are obtained. For the unidimensional particle-hole Hubbard model, using the Östlund-Granath transformation, the electronic degrees of freedom can be described by the quasicharge and quasispin pseudoparticles. Through the Bethe-Salpeter equation, the binding between these pseudoparticles is analyzed. It was found that a quasispin and a quasicharge form a boundstate under certain conditions, the quasicharges do not bind among them in the particle-hole Hubbard model and that the quasispins form inumerous boundstates between them. The transition to the twodimensional case was also analyzed with the introduction of a transverse hopping.
Keywords: unidimensional systems, Bethe ansatz, quasicharge, quasispin, Bethe-Salpeter equation

I. INTRODUCTION

Unidimensional systems provide a new world of unique physical phenomena (such as spin-charge separation) and a source for paradigms of integrable systems (such as Bethe ansatz), legitimizing the undertaken study of the excitations and interactions between them in these particular systems. These unidimensional systems are generally composed of weakly interacting chains whose transverse properties have values several orders of magnitude below the values of the corresponding properties along the chain. In these systems a variety of new phenomena appear due to the restricted phase space. One of these phenomena is spin-charge separation and it was already verified experimentally resorting to angle resolved photoemission spectroscopy [1]. This separation can also be seen in the Bethe ansatz solution of some models, where pseudoparticles related with these degrees of freedom decouple [3]. In the first part, the pseudoparticles of the unidimensional Heisenberg model are derived through the use of Bethe ansatz, along with its energy band in terms of the pseudomomenta, the phase shift between them and the relation between pseudomomentum and rapidities. An analogous derivation is made for the Kondo model as well. This derivation is possible by a Fermi-liquid-like approach already used for the Hubbard model [7],[8],[9].

In a second part, it is introduced the Östlund-Granath transformation for the unidimensional Hubbard model, thus separating the electronic operators in operators acting on empty sites and sites with a spin ↑ or sites with a spin ↓ and doubly occupied sites (Quasicharge operators) and in operators acting on empty and doubly occupied sites or occupied by an electron (Quasispin operators). It has been proposed that the holons and spinons are related to the Östlund-Granath operators but for the the rotated electrons [6]. Resorting to techniques of many-particle systems and the Bethe-Salpeter equation, the production of boundstates between the various combinations of excitations was investigated.

Finally, it is considered how the transition from unidimensional to twodimensional systems changes the behaviour of the boundstates found. This transition is controlled through a tunable parameter responsible for the coupling between different chains.

II. BETHE ANSATZ EXCITATIONS

The Heisenberg model was the first N-particle model for which an exact solution was found. It describes a chain of spins with first neighbour interactions and its Hamiltonian is given by

$$H = J\sum_{r}^{N}\vec{S}_r\vec{S}_{r+\delta} \tag{1}$$

The exact solution was possible due to the Bethe ansatz method. This method consists in obtaining any excited state of the system by the creation of excitations in the form

$$|\Omega_M\rangle = \sum_{\{n_j\}}\Big(\psi_{\{n_j\}}\prod_{j=1}^{M}S^-_{n_j}\Big)|GS\rangle \tag{2}$$

where the coefficients $\psi_{\{n_j\}}$ are assumed to be

$$\psi(\{n_j\}) = \sum_{P} e^{i\sum_j^M k_j n_j + i\sum_{i>j}^M sgn(n_1-n_j)\phi(k_i,k_j)} \tag{3}$$

Imposing periodic conditions, the wavefunction of the system must be the same apart from a phase factor of e^{ikL}, which leads to

$$\left(\frac{\lambda_j+i}{\lambda_j-i}\right)^N = \prod_{l\neq j}^{M}\frac{\lambda_j-\lambda_l+2i}{\lambda_j-\lambda_l-2i} \tag{4}$$

using the auxiliary variable $\lambda = \cot\frac{k}{2}$. The logarithmic form of this equations exhibits some quantum numbers. As in standard quantum theory, the pseudomomenta $\{q\}$

associated with these or the rapidities $\{\lambda\}$ can be used to describe the system. In the macroscopic limit, the continuum Bethe-ansatz equation is

$$q = 2\arctan\big(\lambda(q)\big) - \frac{1}{\pi}\int \mathrm{d}q' N(q')\arctan\left(\frac{\lambda(q)-\lambda(q')}{2}\right) \tag{5}$$

and in this limit, the energy is

$$E = -\frac{JN}{\pi}\int \mathrm{d}q\, N(q)\frac{1}{1+4\lambda^2(q)} \tag{6}$$

where the function $N(q) : [-\pi;\pi[\to \mathbb{R}$ is an extension to the interval $[-\pi;\pi]$ of the function counting the existence (=1) or inexistence (=0) of a given pseudomomentum on that interval. Eq. 5 is enough to fully describe the macroscopic behaviour of the model, but its hermetic structure hinders the extraction of direct information about its properties.

Nevertheless, the low-lying excitations can be retrieved by expanding the energy near its groundstate value, as well as the rapidities and the pseudomomentum distribution ($N(q)$). Hence, $E = E^0 + E^1 + E^2 + ...$, $\lambda(q) = \lambda_0(q)+\lambda_1(q)+\lambda_2(q)+...$ and $N(q) = N^0(q)+\delta(q)$.

$$E^1 = \frac{N}{2\pi}\int \mathrm{d}q\epsilon(q)\delta(q) \tag{7}$$

$$E^2 = \frac{N}{(2\pi)^2}\iint \mathrm{d}q\mathrm{d}q'\frac{1}{2}f(q,q')\delta(q)\delta(q') \tag{8}$$

As in Fermi liquid theory, E^1 is related to the energy band $\epsilon(q)$ and E^2 to the pseudoparticle interaction $f(q,q')$. Writing λ_1 as

$$\lambda_1(q) = \frac{\mathrm{d}\lambda_0(q)}{\mathrm{d}q}\int \mathrm{d}q'\,\delta(q')\phi(q,q') \tag{9}$$

one is able to retrieve an integral equation for the phase shifts ϕ by grouping the terms with the same order and susbtituting λ_1 in the equations by the expression above.

$$\begin{aligned}\phi(\lambda,\lambda') =& \frac{1}{\pi}\arctan\left(\lambda-\lambda'\right) \\ &- \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{d}\lambda''\,\frac{1}{1+(\lambda-\lambda'')^2}\phi(\lambda'',\lambda')\end{aligned} \tag{10}$$

With the use of an auxiliary function $f(\lambda) = \partial_\lambda\phi(\lambda-\lambda')$, the integral equation for the phase shifts becomes an equation for f, and performing a Fourier transform, its solution is found

$$f(\omega) = \frac{1}{1+e^{|\omega|}} \tag{11}$$

Recalling the definition of f, the phase shift is immediately obtained

$$\phi(\lambda) = \frac{i}{2\pi}\log\left(\frac{\Gamma\left[1-\frac{i\lambda}{2}\right]\Gamma\left[\frac{1}{2}+\frac{i\lambda}{2}\right]}{\Gamma\left[1+\frac{i\lambda}{2}\right]\Gamma\left[\frac{1}{2}-\frac{i\lambda}{2}\right]}\right) \tag{12}$$

The relation between pseudomomenta and rapidities can be found in the same manner by defining

$$\sigma(\lambda) = \frac{\mathrm{d}q}{\mathrm{d}\lambda_0} \tag{13}$$

and observing that it also obeys the integral equation

$$\sigma(\lambda) = \frac{4}{1+4\lambda^2} - \frac{1}{\pi}\int \mathrm{d}\lambda'\,\frac{1}{1+(\lambda-\lambda')^2}\sigma(\lambda') \tag{14}$$

and to maintain consistency, σ obeys the relation

$$\int_0^{+\infty} \mathrm{d}\lambda\sigma(\lambda) = k_{F,\downarrow} \tag{15}$$

By the same steps as before, the solution of the equation is

$$\sigma(\lambda) = \pi\frac{1}{\cosh(\pi\lambda)} \tag{16}$$

and integrating σ, the relation between the pseudomomenta and the rapidities is found to be

$$\tan\left(\frac{q}{2}\right) = \tanh\left(\frac{\pi}{2}\lambda_0(q)\right) \tag{17}$$

For the energy band, it is defined the variable

$$\eta(\lambda) = \frac{\mathrm{d}\epsilon(q)}{\mathrm{d}\lambda_0(q)} \tag{18}$$

which also obeys an integral equation given by

$$\eta(\lambda) = -|J|\frac{\lambda}{(\frac{1}{4}+\lambda^2)^2} + \frac{1}{\pi}\int \mathrm{d}\lambda'\,\frac{1}{1+(\lambda-\lambda')^2}\eta(\lambda') \tag{19}$$

By the exact same path, the expression for the pseudoparticle band is

$$\epsilon(q) = -\frac{|J|\pi}{2}\frac{1}{\cosh(\pi\lambda_0(q))} = -\frac{|J|\pi}{2}\cos(q) \tag{20}$$

The pseudoparticle energy is lower for pseudomomenta around zero so that in the groundstate all pseudoparticles will form a Fermi sea centred around zero. The quantum numbers of the system will then pack closely to each other around that value at the groundstate and any excitations can be described by placing holes in the groundstate distribution or adding numbers outside the set.

With the obtention of the energy bands and phase shifts of the model, dynamical properties can be evaluated. They are usually difficult to obtain because they demand an evaluation of matrix elements and the knowledge of the wavefunctions of the model. Bethe ansatz provides the exact N-particle wavefunction, but not only the expressions obtained this way are very complicated, the relation between the Bethe ansatz quantum numbers and the particle operators of the system is in general unknown.

To obtain the dynamical structure factor for the Heisenberg model, one can use the results known for the Hubbard model, and note that in the limit $U \to \infty$, and at half-filling and disregarding the contributions of the charge degrees of freedom to the phase shifts, the Hubbard-model reduces to the Heisenberg model. The spin dynamical structure factor is given by the groundstate correlation functions

$$S_{singlet}(k,\omega) = \frac{1}{N}\sum_{j,l} e^{ikl}\int_{-\infty}^{+\infty} dt e^{i\omega t}\langle GS|S_j^z S_{j+l}^z|GS\rangle \tag{21}$$

or

$$S_{triplet}(k,\omega) = \frac{1}{N}\sum_{j,l} e^{ikl}\int_{-\infty}^{+\infty} dt e^{i\omega t}\langle GS|S_j^+ S_{j+l}^-|GS\rangle \tag{22}$$

involving spin singlet or spin triplet excitations. At zero magnetic field these excitations are degenerate. The key step to evaluate this correlation function is to observe which allowed processes couple the groundstate with the excited states and assess their contribution. The simplest case involves the creation of two spinons for the spin triplet excitation case, and it turns out to be the most important. The remaining allowed processes, involving excitations of a larger number of spinons or strings with complex rapidities (the case of the singlet), have very small contributions to the spectral weight so it is enough to consider the dominant term of two spinons.

The two spinons are holes of the energy band of the spin pseudoparticles both in the Hubbard model and in the Heisenberg model. Labeling the holes' pseudomomenta as q and q', the energy of this excited state is given by

$$\omega(k) = -\epsilon_s(q) - \epsilon_s(q') \tag{23}$$

where k is the momentum of the excitation given by $k = \pi - q - q'$. The momentum π arises because the excitation induces an overall π-shift due to the change of the quantum numbers from half-integers to integers.

Due to the biparametric excitation (since there is a contribution of two spinons), the excitation spectrum is a continuum. The lowest energy line corresponds to keeping one hole at a Fermi point while the other hole sweeps the whole energy band. Below this line there is no spectral weight and above it, the spectral weight decreases algebraically with an exponent that has been determined before for the Hubbard model. This dependence can be written as

$$S_s(k,\omega) \sim (\omega - \omega_s(q))^{-1+\xi_0} \tag{24}$$

By analogy with what has been done before for the Hubbard model, the exponent of this lower line is given by

$$\xi_0 = 2\Delta_s^1 + 2\Delta_s^{-1} \tag{25}$$

where

$$2\Delta_s^1 = \left[\frac{1}{2}\xi^1 + \Phi(q_{FS}^0, q)\right]^2$$
$$2\Delta_s^{-1} = \left[\frac{1}{2}\xi^1 + \Phi(-q_{FS}^0, q)\right]^2 \tag{26}$$

and where Φ is the phase shift, q is the spinon's pseudo-momentum,

$$\xi^1 = 1 + \bar{\Phi}(\infty,\infty) - \bar{\Phi}(\infty,-\infty) \tag{27}$$

and $q_{FS}^0 = k_{F,\downarrow} = k_{F,\uparrow}$. The energy line where the power law is calculated is given by

$$\omega(k) = -\epsilon_s(\pi - k) \tag{28}$$

where $\epsilon_s(q)$ is spinon energy band calculated above. Using Eqs. (62,66,67,68) and Eq. (B1) of [9] and comparing with Eq. (5.3) of [10], the phase shift values at $\lambda = \pm\infty$ are

$$\bar{\Phi}(\infty,-\infty) = \frac{\sqrt{2}}{4} \quad \text{and} \quad \bar{\Phi}(\infty,\infty) = \frac{3\sqrt{2}}{4} - 1 \tag{29}$$

Therefore

$$\xi^1 = \frac{1}{\sqrt{2}} \tag{30}$$

ξ_0 can also be calculatedand its value is found to be $\xi_= \frac{1}{2}$. Therefore the exponent is given by

$$-1 + \xi_0 = -\frac{1}{2} \tag{31}$$

This method can also be used for obtaining the corresponding results for the Kondo model. This model describes an electronic gas interacting with a magnetic impurity. Its Hamiltonian is given by

$$H = \sum_{k,\sigma} \epsilon_k c_{k,\sigma}^\dagger c_{k,\sigma} + J c_{r=0,a}^\dagger \,\vec{\sigma}_{a,b}\, c_{r=0,b} \cdot \vec{\sigma}_0 \tag{32}$$

By an analogous construction, the Bethe-ansatz equations for the Kondo model with a linearized band can be found to be [3]

$$e^{ik_j L} = \prod_{\gamma=1}^{M} \frac{\lambda_\gamma - 1 + \frac{ic}{2}}{\lambda_\gamma - 1 - \frac{ic}{2}} \tag{33}$$

$$-\prod_{\delta\neq\gamma} \frac{\lambda_\delta - \lambda_\gamma + ic}{\lambda_\delta - \lambda_\gamma - ic} = \left(\frac{\lambda_\gamma - 1 - \frac{ic}{2}}{\lambda_\gamma - 1 + \frac{ic}{2}}\right)^{N_e} \left(\frac{\lambda_\gamma - \frac{ic}{2}}{\lambda_\gamma + \frac{ic}{2}}\right) \tag{34}$$

where

$$c = \frac{8J}{4 - 3J^2} \tag{35}$$

The logarithmic form of these equations lead to the logarithmic continuum Bethe ansatz equation for the spin part (the charge part decouples completely and is a trivial non-interacting gas)

$$q = 2D\arctan\left(\frac{2\lambda(q)-2}{c}\right) + \frac{2}{L}\arctan\left(\frac{2\lambda(q)}{c}\right) + \frac{1}{\pi}\int \mathrm{d}q'\, N(q')\arctan\left(\frac{\lambda(q)-\lambda(q')}{c}\right) \quad (36)$$

and the energy

$$E = \frac{N_e}{2\pi}\int \mathrm{d}q\,\Big(-2\arctan\Big(\frac{2\lambda(q)-2}{c}\Big)-\pi\Big) \quad (37)$$

The same steps lead to the integral equation for the phase shifts

$$\phi(\lambda,\lambda') = \frac{1}{\pi}\arctan\left(\frac{\lambda-\lambda'}{c}\right) - \frac{1}{\pi c}\int \mathrm{d}\lambda'' \frac{1}{1+\left(\frac{\lambda-\lambda''}{c}\right)^2}\phi(\lambda'',\lambda') \quad (38)$$

whose solution is

$$\phi(\lambda) = \frac{i}{2\pi}\log\left(\frac{\Gamma\big[1-\frac{i\lambda}{2c}\big]\Gamma\big[1+\frac{i\lambda}{2c}\big]}{\Gamma\big[\frac{1}{2}+\frac{i\lambda}{2c}\big]\Gamma\big[\frac{1}{2}-\frac{i\lambda}{2c}\big]}\right) \quad (39)$$

With the same definition of eq. 13, the corresponding integral equation is

$$\sigma(\lambda) = \frac{4D}{c}\frac{1}{1+(\frac{2}{c}(\lambda-1))^2} + \frac{4}{Lc}\frac{1}{1+(\frac{2}{c}\lambda)^2} - \frac{1}{\pi c}\int \mathrm{d}\lambda'\,\frac{1}{1+(\frac{1}{c}(\lambda-\lambda'))^2}\sigma(\lambda') \quad (40)$$

and the solution is given by

$$\sigma(\lambda) = \frac{\pi}{Lc}\frac{1}{\cosh\frac{\pi\lambda}{c}} + \frac{\pi D}{c}\frac{1}{\cosh\frac{\pi}{c}(\lambda-1)} \quad (41)$$

Through the definition of σ, the relation between the pseudomomentum and the rapidity is

$$q = \frac{2}{L}\arctan\tanh\left(\frac{\pi}{2c}\lambda_0\right) + 2D\arctan\tanh\left(\frac{\pi}{2c}\right) + 2D\arctan\tanh\left(\frac{\pi}{2c}(\lambda_0-1)\right) \quad (42)$$

It should be noted that the expression for the phase shifts is similar to the one for the Heisenberg apart from a c factor. The relation between pseudomomenta and rapidities is also similar apart from an additional impurity term and a shift in the rapidities. This similarity is due to the physics of both systems. The spinon energy band can also be obtained by

$$\eta(\lambda) = -\frac{4D}{c}\frac{1}{1+\big(\frac{2}{c}(\lambda-1)\big)^2} - \frac{1}{\pi c}\int \mathrm{d}\lambda'\,\frac{1}{1+\big(\frac{1}{c}(\lambda-\lambda')\big)^2}\eta(\lambda') \quad (43)$$

using the same definition of eq. 18. The solution is

$$\eta(\lambda) = -\frac{\pi D}{c}\frac{1}{\cosh\left(\pi\frac{\lambda-1}{c}\right)} \quad (44)$$

and can be used to obtain the pseudoparticle band

$$\epsilon(q) = -2D\arctan e^{\frac{\pi}{c}(\lambda_0(q)-1)} \quad (45)$$

III. THE HUBBARD MODEL

The particle-hole symmetric Hubbard model can be written as

$$H = -\sum_{r,\sigma}\left(c^\dagger_{r,\sigma}c_{r+\delta,\sigma}+c^\dagger_{r+\delta,\sigma}c_{r,\sigma}\right)+\frac{u}{2}\sum_r(n_r-1)^2 \quad (46)$$

The Östlund-Granath transformation [2] introduces new operators (the quasicharge c and the quasispins q)

$$\begin{aligned} c_r &= c^\dagger_{\uparrow,r}(1-n_{\downarrow,r}) + (-1)^r c_{\uparrow,r}n_{\downarrow,r} \\ q^+_r &= \big(c^\dagger_{\uparrow,r}-(-1)^r c_{\uparrow,r}\big)c_{\downarrow,r} \\ q^-_r &= \big(q^+_r\big)^\dagger \\ q^z_r &= \tfrac{1}{2}-n_{\downarrow,r} \end{aligned} \quad (47)$$

whose relation to the original electronic operators is given by

$$\begin{aligned} c_{r,\uparrow} &= c^\dagger_r(\tfrac{1}{2}+q^z_r)+(-1)^r c_r(\tfrac{1}{2}-q^z_r) \\ c_{r,\downarrow} &= (c^\dagger_r-(-1)^r c_r)q^+_r \end{aligned} \quad (48)$$

The quasicharges represent spinless fermions since they obey the algebra $\{c_r, c^\dagger_{r'}\} = \delta_{r,r'}$ and $\{c_r, c_{r'}\} = 0$ while the quasispin represent spin $\frac{1}{2}$ due to $[q^i_r, q^j_{r'}] = i\delta_{r,r'}\epsilon_{ijk}q^k_r$. As it is a fraccionalization transformation, they also disengage from each other since the operators commute. The relations between operators can be obtained by seeing their action on the states. The action of this operators is represented on Figures 1 and 2.

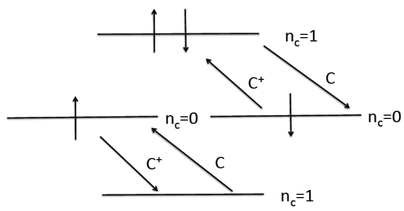

FIG. 1: Quasicharge operator action.

The quasipins can have a bosonic representation with the use of the Holstein-Primakoff transformation,

$$\begin{aligned} q^+_r &= \sqrt{2S}\sqrt{1-\frac{a^\dagger_r a_r}{2S}}\;a_r \approx \sqrt{2S}\,a_r \\ q^-_r &= a^\dagger_r\,\sqrt{2S}\sqrt{1-\frac{a^\dagger_r a_r}{2S}} \approx \sqrt{2S}\,a^\dagger_r \\ q^z_r &= S-a^\dagger_r a_r \end{aligned} \quad (49)$$

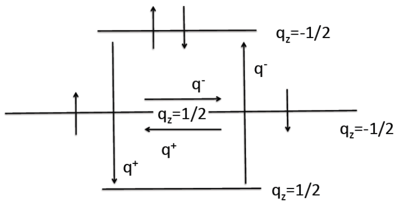

FIG. 2: Quasispin operator action.

Additional terms will be added to the Hamiltonian as an external magnetic field term H_h (aligned in the z direction without any loss of generality), a chemical potential H_μ (recovering the shift made by considering the particle-hole version of the Hubbard model) and a 1^{st} neighbour interaction term H_V.

$$H_h = -h\sum_r s_r^z = -h\sum_r (1-n_r^c)q_r^z \tag{50}$$

$$H_\mu = \left(\frac{u}{2}-\mu\right)\sum_r n_r = \left(\frac{u}{2}-\mu\right)\sum_r (1-2n_r^c q_r^z) \tag{51}$$

$$\begin{aligned} H_V = V\sum_r n_r n_{r+\delta} = VL + 4V\sum_r n_r^c n_{r+\delta}^c q_r^z q_{r+\delta}^z \\ - 4V\sum_r n_r^c q_r^z \end{aligned} \tag{52}$$

Writing the Hamiltonian with the new operators in momentum space,

$$\begin{aligned} H = &\sum_k \epsilon(k)\, c_k^\dagger c_k + \sum_{q,k_1,k_2} U_0(q,k_1,k_2)\, a_{q+k_1-k_2}^\dagger c_{k_2}^\dagger c_{k_1} a_q \\ &+ \sum_{q,k_1,k_2} V_+(q,k_1,k_2)\, a_{q-k_1-k_2+\pi}^\dagger c_{k_1}^\dagger c_{k_2}^\dagger a_q + \sum_q h\, a_q^\dagger a_q \\ &+ \sum_{q,k_1,k_2} V_-(q,k_1,k_2)\, a_{q+k_1+k_2+\pi}^\dagger c_{k_1} c_{k_2} a_q \\ &+ \sum_{k_1,k_2,k_3} U_x(k_1,k_2,k_3)\, c_{k_1+k_2-k_3}^\dagger c_{k_3}^\dagger c_{k_2} c_{k_1} \end{aligned} \tag{53}$$

the fermionic quasicharge band is obtained ($\epsilon(k) = 2\cos k + \frac{u}{2} + B$) along with the interactions

$$U_0(q,k_1,k_2) = 2\big[\cos(q+k_1) + \cos(q-k_2) - B\big] \tag{54}$$

$$\begin{aligned} V_+(q,k_1,k_2) = &2(\cos k_2 - \cos k_1 \\ &+ \cos(k_1-q) - \cos(k_2-q)) \end{aligned} \tag{55}$$

$$\begin{aligned} V_-(q,k_1,k_2) = &2(\cos k_1 - \cos k_2 \\ &+ \cos(k_2+q) - \cos(k_1+q)) \end{aligned} \tag{56}$$

$$U_x(k_1,k_2,k_3) = V\, cos(k_3-k_2) \tag{57}$$

where $B = \frac{h}{2} + \mu - \frac{u}{2} - 2V$ is a useful parameter that encompasses all the physical parameters of the system and spans all $\mathbb{R}$. When $B = 0$ it is as if all those additional terms are not inserted in the Hamiltonian and one reencounters the physics of the particle-hole Hubbard model (with no magnetic field nor 1^{st} neighbour interactions); $B \neq 0$ may be used to probe regimes of intense electronic repulsion or coupling with the magnetic field.

From the quasicharge's and quasispin's energy bands, one can write directly the propagators associated with each excitation as

$$\begin{aligned} \text{Quasicharge } G^0(k,\omega) = &\frac{\aleph(|k|-p_F^c)}{\omega - (2\cos k + \frac{u}{2} + B) + i\eta} \\ &+ \frac{\aleph(p_F^c - |k|)}{\omega - (2\cos k + \frac{u}{2} + B) - i\eta} \end{aligned} \tag{58}$$

$$\text{Quasispin } D^0(k,\omega) = \frac{1}{\omega - h + i\eta} \tag{59}$$

and represented in Figure 3.

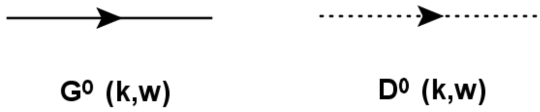

FIG. 3: Diagrammatic representation of the fermionic (G^0) and bosonic (D^0) propagators.

Due to the band's dependence with k, the functions $\aleph(|q|-p_F^c) = \Theta(p_F^c - |q|)$ and $\aleph(p_F^c - |q|) = 1 - \Theta(p_F^c - |q|)$ were used. The associated vertices are directly obtained from the 4 quartic terms and whose respective intensity corresponds to eqs. 54 to 57 and are represented in Figure 4.

IV. BOUNDSTATES OF QUASICHARGES AND QUASISPINS

In analogy with the quantum theory, an indicator of the occurrence of boundstates is the presence of poles in the scattering matrix. These poles do not appear in any finite sum of diagrams so a whole class must be summed. If the coupling constants are small and if the scattering occurs at low energies, the ladder diagrams prevail over the others and their selection can perfectly describe the boundstates' nature [4]. The sum of the whole class of ladder diagrams is

$$\begin{aligned} &\mathcal{M}(p, P-p, p', P-p') = \mathcal{M}_1(p,P,p') \\ &+ \int \mathrm{d}^{d+1}\mathbf{q}\, G^0(p)G^0(P-p)U(p,P,k)\mathcal{M}_1(k,P,p') \\ &+ \int \mathrm{d}^{d+1}\mathbf{k} G^0(p)G^0(P-p)U(p,P,k)\cdot \\ &\cdot \int \mathrm{d}^{d+1}\mathbf{q}\, G^0(k)G^0(P-k)U(k,P,q)\mathcal{M}_1(q,P,p') + \ldots \end{aligned} \tag{60}$$

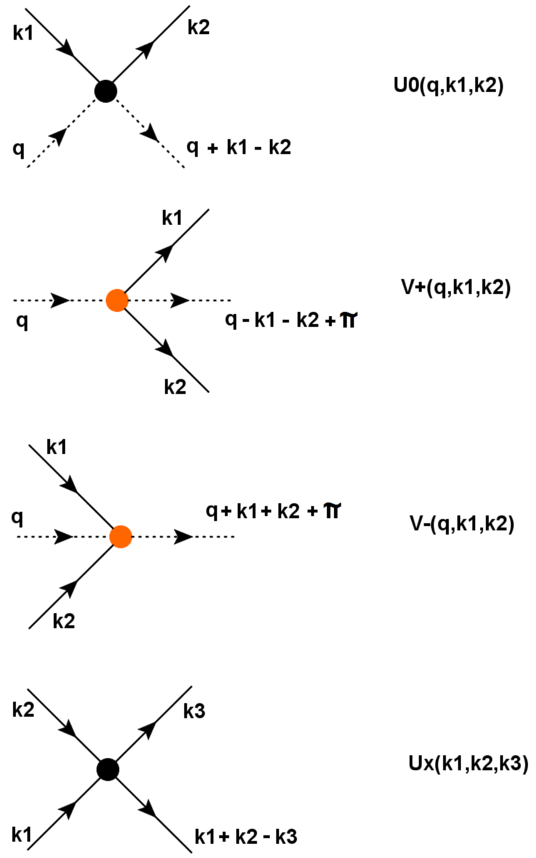

FIG. 4: Vertices.

The infinite sum can be recognized to be the Born series of

$$\mathcal{M}(p, P-p, p', P-p') = \mathcal{M}_1(p, P, p') + \int \mathrm{d}^{d+1}\mathbf{k}\, G^0(p) G^0(P-p) U(p, P, k) \mathcal{M}(k, P-k, p', P-p') \tag{61}$$

where d stands for the spatial dimensions and represented in Figure 5. Defining an effective interaction function Γ

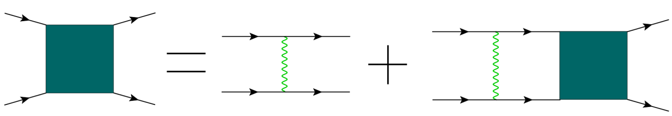

FIG. 5: Integral equation for the scattering matrix.

(represented by the cerulean box), the pole condition is found to be

$$\Gamma(k, P-k, k', P-k') = \int \mathrm{d}^{d+1}\mathbf{q}\, U(k, P, q) G(q, \omega) G(P-q, \Omega-\omega) \Gamma(q, P-q, k', P-k') \tag{62}$$

by noting that the infinite sum is nothing more than the geometrical series. Multiplying eq. 62 by the two-particle propagator G, the entity $G\Gamma$ is proportional to the two-particle wavefunction ψ and integrating in the frequencies, one obtains the Bethe-Salpeter equation for the wavefunction

$$\psi(k) = \frac{1}{\Omega - \varepsilon_1(k) - \varepsilon_2(P-k)} \int \frac{\mathrm{d}\mathbf{q}}{2\pi} V_{eff}(k, q) \psi(q) \tag{63}$$

Writing $\Omega = \Delta + \min\{\varepsilon_1\} + \min\{\varepsilon_2\}$, if $\Delta < 0$ then the system tends to form the boundstate naturally because the remaining energy of the pair is smaller than the free particles' band. If $\Delta > 0$ the binding is still possible but at an energy cost. This leads to the interpretation of Δ as a binding energy for the pair [5]. By algebraic manipulations, the Bethe-Salpeter equation can be reduced to a linear system in the form

$$\Delta\psi_k = \frac{1}{N} \sum_q \Big(N\big(\delta\varepsilon_1(q) - \delta\varepsilon_2(q)\big)\delta_{q,k} + V_{k,q} \Big) \psi_q = \Upsilon_{k,q}\psi_q \tag{64}$$

Eq. 64 provides a method to obtain both Δ and ψ as the eigenvalues and eigenvectors of Υ (and where N is Υ's size). Diagonalizing the Υ matrices will provide a set of eigenvalues and those below the energy bands will be analyzed along with the corresponding eigenvectors.

The simplest Bethe-Salpeter equation for the pseudoparticles considered is for the interaction between a quasispin and a quasicharge (CS case). The diagrammatic representation of the pole condition is found in Figure 6.

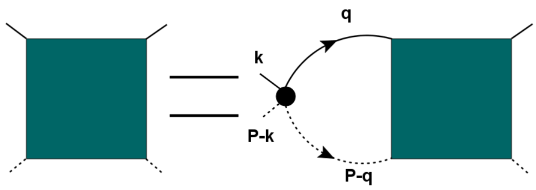

FIG. 6: Diagram associated with the Bethe-Salpeter equation for the 1^{st} CS case

and calculating the associated Bethe-Salpeter equation for the wavefunction leads to

$$\psi(k, P-k) = \frac{1}{\Delta - 2(\cos k + 1)} \int \frac{\mathrm{d}q}{2\pi} V(q, k, P) \psi(q, P-q) \tag{65}$$

where for this 1^{st} case the potential is given by:

$$V(q, k, P) = 2\aleph(|q| - p_F^c)[\cos P + \cos(P-k-q) - B] \tag{66}$$

By the analysis discussed above, a single boundstate is always found except for small p_F^c and very negative B as seen in Figures 7 and 8. The corresponding wavefunction for B=0 and $P = p_F^c = \pi$ is a peak-like structure confirming the boundstate nature. Increasing B will lead to sharper peaks with a more negative binding energy. Using the π-momentum vertices, a ladder sum can also be implemented for the CS scattering and has the diagrammatic representation present in Figure 10.

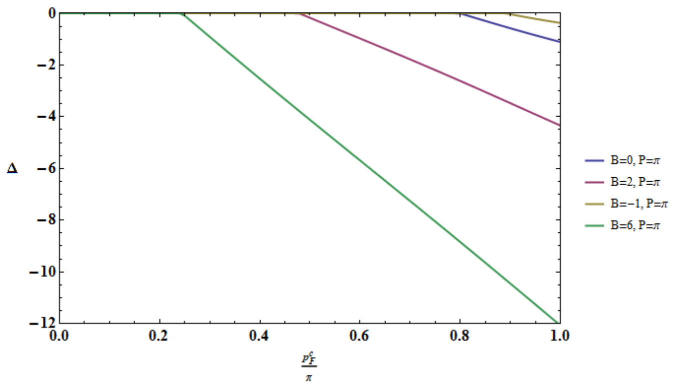

FIG. 7: Δ's variation with p_F^c for various B shows a quasicharge filling threshold after which negative Δ start to emerge. When no negative values exist, it is considered the smallest eigenvalue.

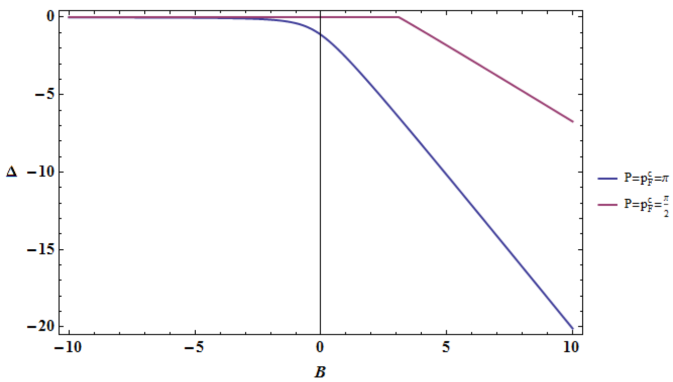

FIG. 8: Δ's variation with B. After B passes a momentum dependent treshold, there is always one $\Delta < 0$.

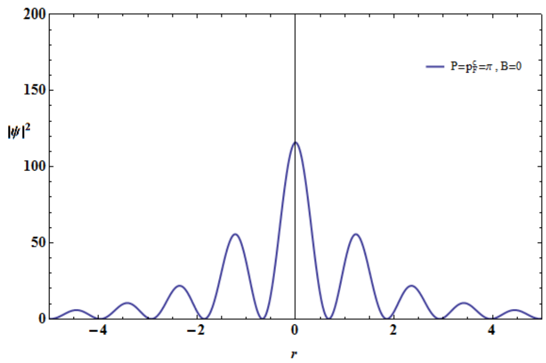

FIG. 9: Square modulus of the pair's wavefunction for the particle-hole Hubbard model.

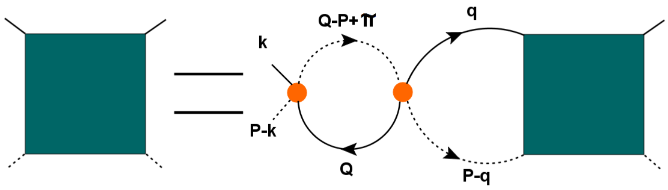

FIG. 10: Diagram associated with the Bethe-Salpeter equation for the 2^{nd} CS case

Through the same algebraic processes, one obtains the same equation for the wavefunction but with a potential for this 2^{nd} case in the form:

$$V(q,k,P) = -8\aleph(|q| - p_F^c) \int \frac{\mathrm{d}Q}{2\pi} \frac{\aleph(|Q| - p_F^c)}{\Delta - 2(\cos Q + 1)} \cdot \big(\cos k - \cos Q - \cos P + \cos(Q + P - k)\big) \cdot \cdot \big(\cos q - \cos(Q) - \cos P + \cos(Q + P - q)\big) \tag{67}$$

As this potential depends on Δ, the roots of the function defined as $f(\Delta) = det(\Xi(\Delta) - \mathbb{I})$, where

$$\Xi_{k,q}(\Delta) = \frac{1}{\Delta - 2(\cos k + 1)} \frac{V_{k,q}(\Delta)}{N} \tag{68}$$

will correspond to the pole condition and mark the possible boundstates. It can be seen in Figure 11 that no boundstates occur with this type of interaction.

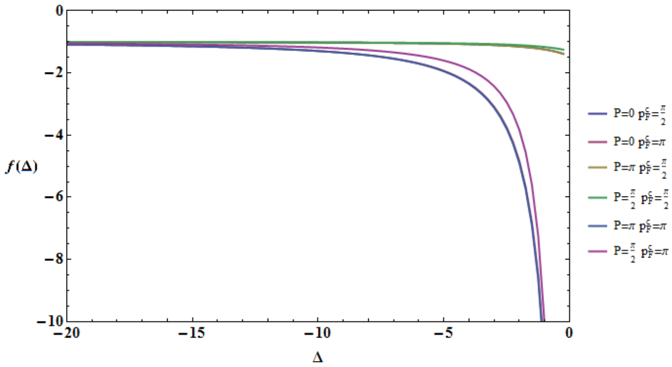

FIG. 11: f's dependence with Δ. The P=0 and the P=π situations overlap with each other.

For the case of the scattering between the quasicharges (CC case), the diagrams associated with the pole condition are represented in Figure 12, 13, 14.

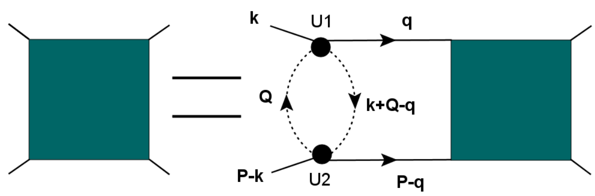

FIG. 12: Diagram associated with the Bethe-Salpeter equation for the 1^{st} CC case

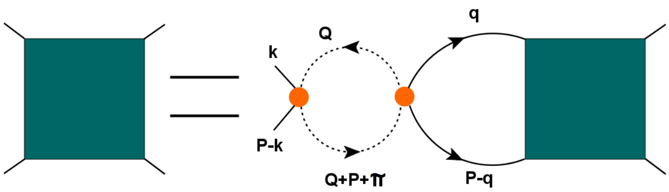

FIG. 13: Diagram associated with the Bethe-Salpeter equation for the 2^{nd} CC case

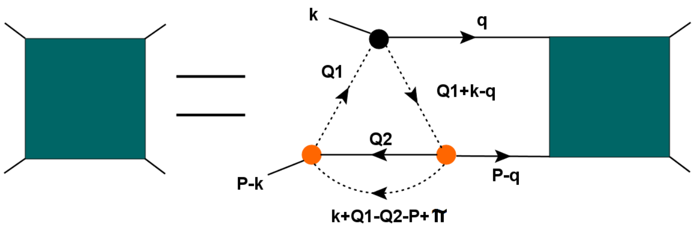

FIG. 14: Diagram associated with the Bethe-Salpeter equation for the 3^{rd} CC case

The calculation of the associated Bethe-Salpeter equation revealed a null interaction between them. Hence, they cannot form boundstates. For the quasispin-quasispin binding (SS case), the pole condition is the one represented in Figure 15

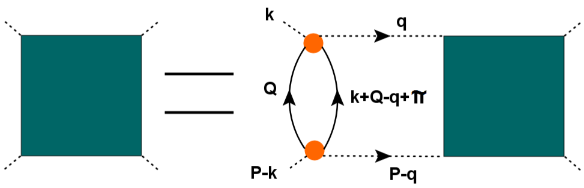

FIG. 15: Diagram associated with the Bethe-Salpeter equation for the 1^{st} SS case

and has the associated Bethe-Salpeter equation

$$\psi(k,P-k)=\frac{1}{\Delta}\int\frac{\mathrm{d}q}{2\pi}V(q,k,P)\psi(q,P-q) \qquad (69)$$

where the potential for this 1^{st} case is

$$\begin{aligned}
V(q,k,P)=8\int\frac{\mathrm{d}Q}{2\pi}\\
\Bigg[\frac{\aleph(|Q|-p_F^c)\aleph(|q-k-Q+\pi|-p_F^c)}{\Delta+2(\cos(q-k-Q)-\cos Q)-\omega_1-2\mu+4V}\\
-\frac{\aleph(p_F^c-|Q|)\aleph(p_F^c-|q-k-Q+\pi|)}{\Delta+2(\cos(q-k-Q)-\cos Q)-\omega_1-2\mu+4V}\Bigg]\\
\cdot\big(\cos(k+Q)+\cos(Q-q)-\cos Q-\cos(q-k-Q)\big)\cdot\\
\cdot\big(\cos Q+\cos(q-k-Q)-\\
\cos(P-q+Q)-\cos(P-k-Q)\big) \qquad (70)
\end{aligned}$$

With a f function analogous to the CS case, it was found that there is always a region where inumerous boundstates occur (as can be seen in Figure .16). A further analysis of this case is too complicated so the attention will fall to another case.

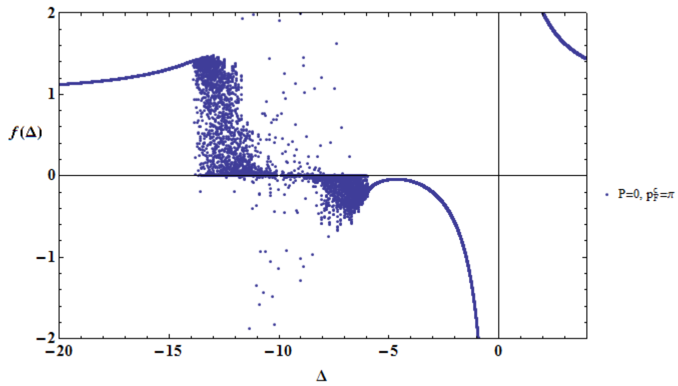

FIG. 16: Evolution of f(Δ). There is an unstable region where boundstates may occur but is not clear the exact corresponding values .

Another possible pole condition is represented in Figure 17

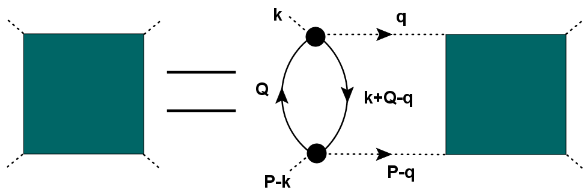

FIG. 17: Diagram associated with the Bethe-Salpeter equation for the 2^{nd} SS case

and has the same Bethe-Salpeter equation but with a potential given by

$$\begin{aligned}
V(q,k,P)=4\int\frac{\mathrm{d}Q}{2\pi}\Bigg[\frac{\aleph(|Q|-p_F^c)\aleph(p_F^c-|k+Q-q|)}{\omega_1-h+2(\cos Q-\cos(Q+k-q))}\\
-\frac{\aleph(p_F^c-|Q|)\aleph(|k+Q-q|-p_F^c)}{\omega_1-h+2(\cos Q-\cos(Q+k-q))}\Bigg]\\
\cdot\big(\cos(k+Q)+\cos(Q-q)-B\big)\\
\cdot\big(\cos(P-q+Q)+\cos(P-k-Q)-B\big) \qquad (71)
\end{aligned}$$

By the eigenvalue analysis method, this interaction also reveals inumerous solutions as seen in Figure 18.

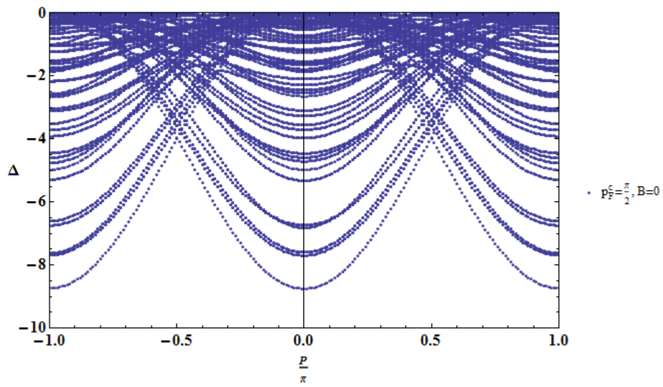

FIG. 18: Diagram associated with the Bethe-Salpeter equation for the 2^{nd} SS case.

A more thorough analysis of the binding requires the corrections of the interactions to the propagators. Through the Dyson's equation, the correction to the bosonic propagator will be

$$D(k,\omega) = \frac{1}{\omega - \varpi(k) - \Sigma^{\star}} \tag{72}$$

and with the approximation for the proper self-energy $\Sigma^{\star}$ given by

$$\Sigma^{\star} \approx \Sigma^{(1)}(k) = 2(1-B)\Big(\frac{p_F^c}{\pi} - 1\Big) - \frac{2}{\pi}\cos k \sin p_F^c \tag{73}$$

the Bethe-Salpeter equation for the CS case becomes

$$\psi_k = \frac{1}{\Delta - 2(\cos k - \frac{1}{\pi}\sin(p_F^c)\cos(P-k) + 1 + \frac{1}{\pi}\sin(p_F^c))} \cdot \int \frac{dq}{2\pi} V_{kq}\psi_q \tag{74}$$

In this degree of approximation, the correction to the fermionic propagator is null. The Bethe-Salpeter equation for the SS case is also immediately obtained.

$$\psi_k = \frac{1}{\Delta + \frac{2}{\pi}\sin(p_F^c)(\cos k + \cos(P-k) - 2)} \int \frac{dq}{2\pi} V_{kq}\psi_q \tag{75}$$

However these corrections result only in changing the threshold value for which the boundstate occurs (as seen in Figure 19).

V. DIMENSIONAL CROSSOVER

The transition to two dimensions can also be analyzed by enlarging the dimension and including a hopping between the chains. The additional terms to the previous Hamiltonian are

$$H_{extra} = -\alpha \sum_{r,\sigma} c^{\dagger}_{x,y,\sigma} c_{x,y+\delta,\sigma} + c^{\dagger}_{x,y+\delta,\sigma} c_{x,y,\sigma} \tag{76}$$

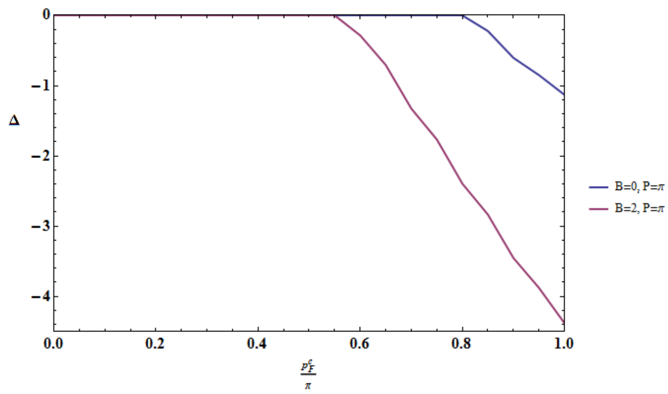

FIG. 19: Δ's quasicharge Fermi momentum depence for the dressed propagator.

and the α parameter can be used to tune the transition from uncoupled chains to a square lattice. For contention reasons only the quasispin-quasicharge case will be analyzed. The correction to the fermionic propagator is

$$G^0(\vec{k},\omega) = \frac{\Theta(\epsilon(k) - E_F)}{\omega - 2(\cos k_x + \alpha\cos k_y + \frac{u}{4}) + i\eta} + \frac{\Theta(E_F - \epsilon(k))}{\omega - 2(\cos k + \alpha\cos k_y + \frac{u}{4}) - i\eta} \tag{77}$$

and the vertex correction

$$U_0 : \quad 2\big[\cos(k'' + k_1) + \cos(k'' - k_2) + \alpha\big(\cos(k'' + k_1) + \cos(k'' - k_2)\big) - B\big] \tag{78}$$

This leads to the Bethe-Salpeter equation

$$\psi_{\vec{k}} = \frac{1}{\Delta - 2(\cos k_x + \alpha\cos k_y + 1 + \alpha)} \int \frac{d\vec{q}}{(2\pi)^2} V_{\vec{q},\vec{k}}\psi_{\vec{q}} \tag{79}$$

with the potential given by

$$V_{\vec{q},\vec{k}} = 2\Theta(\epsilon(k) - E_F)\big[\cos P_x + \alpha\cos P_y + \cos(P_x - k_x - q_x) + \alpha\cos(P_y - k_y - q_y) - B\big] \tag{80}$$

The results did not differ much from the unidimensional case and no new physics emerged. The evolution of the binding energy with the coupling was not significative nor show new boundstates (see Figure 20).

VI. CONCLUSION

In the work developed, it was derived the pseudoparticle energy band, the phase shifts and the relation between pseudomomenta and rapidities for the Heisenberg model and for the Kondo model. These results are useful for obtaining exponents that can be compared with experiments such as the dynamical spin structure factor.

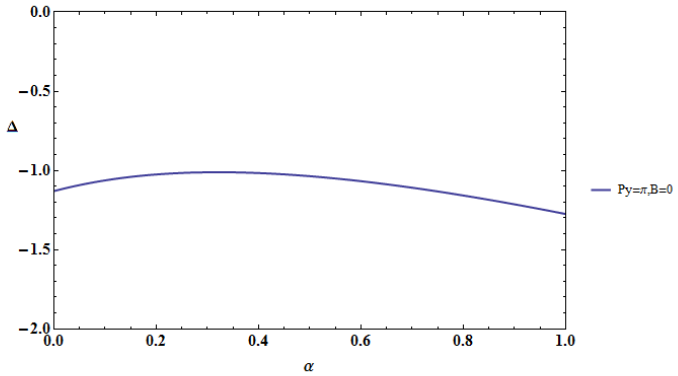

FIG. 20: Δ variation with the interchain hopping parameter α for $P_y = \pi$ and B=0. It is observed that the qualitative behavior is not altered in the transition to the twodimensionality.

This lays the ground for a future calculation of the dynamical structure factor for the Kondo model that can be obtained in a manner analogous to the Heisenberg model. Knowing the physics involved, they must be similar apart from an impurity term. Another possible correlation function to be calculated is the electronic spectral function. Since for the free electron gas it is simply a δ-function spectral function, it would be interesting to see what is the impurity's role. This question has yet not been resolved.

In the study of the scattering between the quasicharges and quasispins was found that the quasicharges and quasispins form boundstates in a vast region of the parameter space. This binding was unexpected since in one dimension there are several examples of fractionalization and the operators representing the pseudoparticles decouple. On the other hand, quasicharges do not bind with each other in the particle-hole Hubbard model but they start to do so when a first neighbour interaction is considered. This absence of binding is a good outcome since the transformation proposed by Östlund and Granath was made with the intent of spliting the quasicharge sector, but this lack of binding is a feature provenient from the Holstein-Primakoff transformation of the quasicharges and it is independent of the dimension. The quasispins always bind with each other and possess inumerous boundstates. These boundstates found may be originated by the use of the ferromagnetic representation. The role of propagator corrections was also analysed and it was not found any significant change in the binding.

Finally, the transition to the bidimensionality did not show any new phenomena, being a continuation of the unidimensional case. As the quasicharges and quasispins already bind in one dimension, the appearance of a boundstate in two dimensions is not surprising. Although the analysis was done for the simplest quasispin-quasicharge interaction, it is not expected that the analyzis of other cases reveal any differences. An undermining problem was the flat quasispin band. The approach taken with the linearization of the Holstein-Primakoff transformation lead to an effective Hamiltonian with the assumption that the quasispin density was negligible. However this is a somewhat drastic approach and there may be other ways to overcome the problem. As the Jordan-Wigner transformation is much more reliable for the quasispin interaction with the quasicharges, the dispersionless quasispin band problem may be bypassed by the simple addition of an electronic spin interaction term $J\vec{S}_r \cdot \vec{S}_{r+\delta}$ instead of the other terms added to the Hubbard Hamiltonian. This interaction might provide a momentum dependent band that will outline a well-defined fermionic propagator for the quasispins. This will also provide a non-trivial interaction between quasicharges.

[1] Kim et al., Distinct spinon and holon dispersions in photoemission spectral functions from one-dimension $SrCuO_2$, Nature Physics, 2

[2] S. Östlund, M. Granath,Exact Transformation for Spin-Charge Separation of Spin-1/2 Fermions without Constraints,PHYSICAL REVIEW LETTERS,96,066404,(2006)

[3] Natan Andrei,Integrable models in condensed matter systems,ICTP Summer course.

[4] Franz Gross,Relativistic quantum mechanics and field theory,WILEY-VCH Verlag GmbH and Co. KGaA, Weinhem, (2004)

[5] J. Smakov, A. Chernyshev, S. White,Binding of Holons and Spinons in the One-Dimensional Anisotropic t-J Model,PRL,98, 266401,(2007)

[6] J.M.P. Carmelo,Nuclear Physics B, 824, 452, (2010)

[7] J.M.P. Carmelo, P.D. Sacramento,Finite-energy Landau liquid theory for the one-dimensional Hubbard model: Pseudoparticle energy bands and degree of localization/delocalization,PHYSICAL REVIEW B,68, (2003)

[8] J. Carmelo, P. Horsch, P.A. Bares, A.A. Ovchinnikov,Renormalized pseudoparticle description of the one-dimensional Hubbard model thermodynamics,PHYSICAL REVIEW B, 44, 18, (1991)

[9] J.M.P. Carmelo, P. Horsch, A.A. Ovchinnikov,Static properties of one-dimensional generalized Landau liquids,PHYSICAL REVIEW B, 45, 14, (1992)

[10] Holger Frahm and V. E. Korepin, Critical exponents for the one-dimensional Hubbard model, Physical Review B 42,10553 (1990)