A Nonminimal Coupling Model and its Short-Range Solar System Impact

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(Dated: June 20, 2014)

The objective of this work is to present the effects of a nonminimally coupled model of gravity on a Solar System short range regime as studied in Ref. [1]. The action functional of the model involves two functions $f^1(R)$ and $f^2(R)$ of the Ricci scalar curvature R. Using a Taylor expansion around $R = 0$ for both functions, it was found that the metric around a spherical object is a perturbation of the weak-field Schwarzschild metric. The perturbation of the tt component of the metric, a Newtonian plus Yukawa term, is constrained using the available observational results. Besides, this effect vanishes when the characteristic mass scales of each function are identical, the conclusion is that the Starobinsky model for inflation is not experimentally constrained. Moreover, the geodetic precession effect, obtained also from the radial perturbation of the metric, reveals to be of no relevance for the constraints.

Keywords: $f(R)$ theories; Nonminimal Coupling; Yukawa modifications on inverse-square law; Solar System

I. INTRODUCTION

General Relativity (GR) was developed by Einstein in 1916 and since then it has been able to demonstrate itself as a solid theory of physics, at a theoretical and an observational level [2]. The theory is mathematically represented by the Einstein-Hilbert action,

$$S = \int \left[\kappa \left(R - 2\Lambda\right) + \mathcal{L}_m\right] \sqrt{-g}\ \mathrm{d}^4x, \qquad (1)$$

where R is the Ricci scalar, $\mathcal{L}_m$ is the matter Lagrangian density, Λ stands for the Cosmological Constant, g is the determinant of the metric and $\kappa = c^4/\left(16\pi G\right)$, with G as Newton's constant of gravity and c as the velocity of light in a vacuum. The variation of the action with respect to the metric yields the Einstein Field Equations (EFE),

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}, \qquad (2)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter, clearly showing the relation between matter and geometry.

A. Standard Cosmology and Inflation

Following the Cosmological Principle, the universe can be said to have a Friedmann-Lemaître-Robertson-Walker metric [3]

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right), \qquad (3)$$

where $a(t)$ is the cosmological scale factor, $d\Omega^2$ is the line element for the 2-sphere and K is related to the curvature of the spatial section of spacetime, where the spatial coordinates are comoving coordinates.

At very large scales the universe can be described as being a perfect fluid, endowed with an energy-momentum tensor

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu + p g_{\mu\nu}, \qquad (4)$$

where ρ is the energy density, p is the pressure and u_μ is the 4-velocity vector. Inserting this tensor and the metric (3) into the EFE (2) gives the cosmology structure equations:

$$\dot{a}^2 + Kc^2 = \frac{a^2}{3}\left(8\pi G\rho + \Lambda c^2\right), \qquad (5)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \qquad (6)$$

where $\dot{a} = da/dt$. These equations yield a full description of the structure and development of the universe if an equation of state (EOS) $p = p(\rho)$ is known.

Initially, the standard Big-Bang model had some problems to be solved, so inflation theory [5] was developed. Defining critical density as $\rho_c(t) = 3H^2/8\pi G$, where $H = \dot{a}/a$, and neglecting the cosmological constant term, the Friedmann equation (5) can be written as

$$|\Omega - 1| = \frac{|K|c^2}{a^2 H^2}, \qquad (7)$$

where Ω is the density parameter defined as $\Omega = \rho/\rho_c$. To solve the flatness problem, the $r.h.s.$ should decrease with time, which means that $a^2H^2 = \dot{a}^2$ should increase, so $\ddot{a} > 0$, which is the inflation condition.

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The origin of inflation is explained by means of a scalar field named inflaton, ϕ, with a potential $V(\phi)$. The inflation condition means that,

$$\ddot{a} > 0 \Leftrightarrow p < -\rho c^2/3 \Leftrightarrow \dot{\phi}^2 < V(\phi), \tag{8}$$

This allows to adopt the so-called Slow-Roll Approximation (SRA), $V(\phi) \gg \dot{\phi}^2$. Some very recent experimental observations of inflation are mentioned in Ref. [6].

B. Modern problems

Modern physics uses the concepts of dark matter and dark energy to advance an explanation for the astrophysical problem of missing matter (seen for example in the flattening of galactic rotation curves) and the cosmological problem of the accelerated expansion of the universe, respectively. There are many proposals for dark energy, as the so-called quintessence models [7] and the existence of scalar fields that account for both dark matter and dark energy [8].

More recent approaches start from the idea of the incompleteness of the fundamental laws of GR, involving, for example, corrections to the Einstein-Hilbert action. Such theories involve a nonlinear correction to the geometry part of the action, thus being called $f(R)$ theories. In the last decade, the study of $f(R)$ theories has been very profitable, as thoroughly discussed in Ref. [9]. These can be extended to also include a nonminimum coupling (NMC) between the scalar curvature and the matter Lagrangian density.

II. $f(R)$ THEORIES

This standard modification of GR resorts to the modified action functional

$$S = \int \left[\frac{1}{2} f(R) + \mathcal{L}_m\right] \sqrt{-g}\, \mathrm{d}^4x, \tag{9}$$

where the standard Einstein-Hilbert action is recovered if $f(R) = 2\kappa\,(R - 2\Lambda)$. This action yields the modified field equations,

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + \Box f_R g_{\mu\nu} = T_{\mu\nu}, \tag{10}$$

where $f_R \equiv df/dR$, $R_{\mu\nu}$ is the Ricci tensor and $\Box$ is the D'Alembertian operator.

$f(R)$ theories have an equivalence with scalar-tensor theories, which is suggested by considering the trace of the field equations (10),

$$f_R R - 2f(R) + 3\Box f_R = T, \tag{11}$$

hinting that $f_R \equiv \varphi$ acts as an additional degree of freedom. Considering the following action for a scalar-field

$$S_\chi = \int \left(2\kappa\left[\varphi R - U(\varphi)\right] + \mathcal{L}_m\right) \sqrt{-g}\, \mathrm{d}^4x, \tag{12}$$

where $\varphi = f_\chi(\chi) = df/d\chi$ and the field potential is

$$U(\varphi) = \chi(\varphi)\varphi - f(\chi(\varphi)), \tag{13}$$

one notices that when $\chi = R$, the action S_χ reduces to the standard $f(R)$ action S from (9). This form of the action is important because of its similarity with the Brans-Dicke (BD) theory of gravity [10], with a vanishing BD parameter $\omega_{BD} = 0$.

A. Starobinsky Inflation and Reheating

An early model for inflation was posited by Starobinsky [11], with

$$f(R) = 2\kappa \left(R + \frac{R^2}{6M^2}\right), \tag{14}$$

where $M \sim 10^{13}$ GeV$/c^2$ is a constant with dimensions of mass. Inflation is best described by the Starobinsky version of the cosmological structure equations (5), where one of them is

$$\ddot{R} + 3H\dot{R} + M^2 R = 0. \tag{15}$$

The end of inflation is thus followed by a phase of reheating of the universe. It is in this phase that most of the matter that is observed today in the Universe was formed. This phenomenon is introduced with a scalar field χ with mass m_χ, whose action includes a nonminimal coupling with strength ξ of the field with the scalar curvature. As χ is a quantum field, it can be decomposed into Fourier modes χ_k, whose equation of motion is

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 + \xi R\right)\chi_k = 0, \tag{16}$$

where $k = |\mathbf{k}|$ is a comoving wavenumber. The case where $|\xi| > 1$ allows to perform a series of approximations [12] so that the above equation becomes $\ddot{\chi}_k + \omega_k^2 \chi_k \simeq 0$.

Particle production is thus obtained with the resonance of this oscillator - a phenomenon of parametric resonance which can be more detailed by defining z with $M(t - t_0) = 2z \pm \pi/2$, where the opposite signs are related to the sign of ξ. The new variable leads to what is called the Mathieu equation

$$\frac{d^2\chi_k}{dz^2} + \left[A_k - 2q\cos(2z)\right]\chi_k \simeq 0, \tag{17}$$

with A_k and q determining the strength of parametric resonance and defined by

$$A_k = \frac{4k^2}{a^2M^2} + \frac{4m_\chi^2}{M^2}, \qquad q = \frac{8|\xi|}{M(t-t_0)}. \tag{18}$$

The oscillatory aspect of the field implies that the resonance growth changes with the expansion of the Universe, which also causes the parameters A_k and q to depend on time. This process of particle production due to parametric resonance of the field is called preheating. If $\xi = 0$, which is the standard reheating regime, there still is a parametric oscillation with $q = 0$, due to the expansion of the Universe alone. Nonetheless, the preheating scenario is much more efficient than standard reheating.

B. Solar System Effect

The validity of $f(R)$ gravity at the Solar System scale should be assessed. The long-range impact of $f(R)$ is taken into account by considering the flat FLRW metric with spherically symmetric perturbations

$$\begin{aligned} ds^2 = -[1{+}2\Psi(r)]dt^2 \\ + a(t)^2\left([1+2\Phi(r)]\,dr^2 + r^2 d\Omega^2\right), \end{aligned} \tag{19}$$

where Ψ and Φ are perturbing functions such that $|\Psi(r)| \ll 1$ and $|\Phi(r)| \ll 1$.

The background curvature is nonzero, $R_0(t) \neq 0$, thus it can be decomposed into a sum of two components:

$$R(r,t) \equiv R_0(t) + R_1(r), \tag{20}$$

where $R_1(r)$ is considered as a time-independent perturbation of the background curvature. The validity of the following treatment assumes that $R_1 \ll R_0$ and this analysis follows the work of Ref. [13]. Considering also that all derivatives of $f(R)$ are well defined at $R_0(t_0)$, where t_0 is the present time, a Taylor expansion around R_0 can be used to write the functions $f(R_0 + R_1)$ and $f_R(R_0 + R_1)$. As the framework is a weak-field regime, the expansion terminates by neglecting terms non-linear in the perturbation R_1, provided that the following conditions are taken into account:

$$\begin{aligned} f_0 + f_{R0}R_1 &\gg \frac{1}{n!}f^{(n)}(R_0)R_1^n, \\ f_{R0} + f_{RR0}R_1 &\gg \frac{1}{n!}f^{(n+1)}(R_0)R_1^n, \end{aligned} \tag{21}$$

for all $n > 1$, where

$$f_{RR0} \equiv d^2f/dR^2\big|_{R=R_0}, \tag{22}$$

$$f^{(n)}(R_0) = d^nf/dR^n\big|_{R=R_0}. \tag{23}$$

Notice that $f_R = df/dR$ and $f_0 \equiv f(R_0)$.

Applying these expansions into the trace equation yields

$$f_R R - 2f + 3\Box f_R = T^{\cos} + T^{\rm s}, \tag{24}$$

where $T^{\rm s}$ is the trace of the energy-momentum tensor of a spherically symmetric mass source. Considering that $R_1(r)$ is time-independent, so that the operator $\Box$ becomes $\Box \approx \nabla^2$ and that the matter at the local perturbation is considered with no pressure, such that $T^{\rm s} = -c^2\rho(r)$ (with $c = 1$), then the linearised equation for the spatial dependent curvature can be written as

$$\nabla^2 R_1 - m^2 R_1 = -\frac{\rho}{3f_{RR0}}, \tag{25}$$

with

$$m^2 \equiv \frac{1}{3}\left(\frac{f_{R0}}{f_{RR0}} - R_0 - 3\frac{\Box f_{RR0}}{f_{RR0}}\right). \tag{26}$$

This equation can be solved by resorting to Green functions.

1. *Long Range Regime*

Considering first the long range regime $mr \ll 1$, the Green function can be approximated by $-1/(4\pi r)$. This means that the term linear in R_1 from (25) may disappear and the solution for the curvature perturbation becomes

$$R_1 = \frac{1}{12\pi f_{RR0}}\frac{M_S}{r}, \tag{27}$$

where M_S is the total mass of the source.

Adopting the arguments for linearization from (21), using the curvature equation and neglecting all the terms that are not linear in R_1, Φ and Ψ, the solutions for these perturbations are taken from the time (tt) and radial (rr) components of the field equations. The linearization of the tt component of (10) has the form

$$f_{R0}\nabla^2\Psi + \frac{1}{2}f_{R0}R_1 - f_{RR0}\nabla^2 R_1 = \rho. \tag{28}$$

Inserting equation (25), where the linear term in R_1 was neglected, reduces the above to

$$f_{R0}\nabla^2\Psi = \frac{2}{3}\rho - \frac{1}{2}f_{R0}R_1, \tag{29}$$

which can be decomposed according to $\Psi = \Psi_0 + \Psi_1$. Then, the first equation $f_{R0}\nabla^2\Psi_0 = 2\rho/3$, when integrated by Gauss's Theorem, becomes

$$\Psi_0'(r) = \frac{1}{6\pi f_{R0}} \frac{m(r)}{r^2}, \tag{30}$$

where the prime represents differentiation with respect to r and $m(r)$ is the mass of a sphere with radius r. Assuming that $\lim_{r\to\infty} \Psi_0 = 0$, the solution for Ψ_0 is

$$\Psi_0 = -\frac{1}{6\pi f_{R0}} \frac{M_S}{r}. \tag{31}$$

On the other hand, using the long-range approximation $mr \ll 1$ and a cosmological constraint, it can be shown that $|\Psi_1| \ll |\Psi_0|$, then Ψ_1 may be neglected, thus $\Psi \approx \Psi_0$.

Following the same procedures, the linearization of the rr component of (10) ends up yielding

$$\Phi = \frac{1}{12\pi f_{R0}} \frac{M_S}{r}. \tag{32}$$

Comparing the solution Φ with Ψ, it is easy to conclude that $\Psi = -2\Phi$, thus implying that the PPN parameter $\gamma = -\Phi/\Psi$ takes the value $\gamma = 1/2$, which contradicts the observational results that yield $\gamma \sim 1$ — implying that, if the additional degree of freedom is long-ranged ($mr \ll 1$), $f(R)$ gravity is incompatible with experiment.

2. Short Range Regime

To study the short range regime of $f(R)$ gravity in the Solar System, it is more adequate to consider a perturbation of a Minkowski metric of the form

$$\begin{aligned} ds^2 = &-[1 + 2\Psi(r)]c^2 dt^2 \\ &+ [1 + 2\Phi(r)]\, dr^2 + r^2 d\Omega^2, \end{aligned} \tag{33}$$

where Ψ and Φ are again perturbing functions such that $|\Psi(r)| \ll 1$ and $|\Phi(r)| \ll 1$, since one assumes that the background cosmological dynamics play no short-range role.

In the following, it is assumed that matter behaves as dust, so that its energy-momentum tensor has the same form of (4), but with $p = 0$. The trace of the energy-momentum tensor is $T = -\rho c^2$. The Lagrangian density of matter is considered as $\mathcal{L}_m = -\rho c^2$ (see Ref. [14] for a discussion). The function $\rho = \rho(r)$ is that of a spherically symmetric body with a static radial mass density $\rho = \rho(r)$ and the function $\rho(r)$ and its first derivative are assumed to be continuous across the surface of the body,

$$\rho(R_S) = 0 \quad \text{and} \quad \frac{d\rho}{dr}(R_S) = 0, \tag{34}$$

where R_S denotes the radius of the spherical body. These conditions play a relevant role in the following sections, where integrals that have R_S as an integration limit appear.

This analytical setup allows to define the function $f(R)$ as a Taylor expansion around $R = 0$ [15], similar to the Starobinsky action functional from (14),

$$f(R) = 2\kappa \left(R + \frac{R^2}{6m^2} \right) + \mathcal{O}(R^3). \tag{35}$$

In this case, the linearization is done according to the expansion of R, Ψ and Φ in powers of c^{-1} and all the terms of the order $\mathcal{O}(c^{-3})$ or less are neglected. Inserting (35) into the curvature equation (11) and linearizing it, yields

$$\nabla^2 R - m^2 R = -\frac{8\pi G}{c^2} m^2 \rho, \tag{36}$$

which will end up with a twofold solution for the curvature, whose outer ($r > R_S$) version is

$$R^{\uparrow}(r) = \frac{2GM_S}{rc^2} m^2 A(m, R_S) e^{-mr}, \tag{37}$$

with $A(m, R_S)$ a form factor, discussed in a subsequent section, defined as

$$A(m, R_S) = \frac{4\pi}{mM_S} \int_0^{R_S} \sinh(mr) r\rho(r) dr. \tag{38}$$

Neglecting all terms smaller than $\mathcal{O}(1/c^2)$ and combining it with the same linearisation of the Ricci tensor, the linearisation of the tt and rr components of the field equations (10), yields the solution for Ψ and Φ,

$$\Psi(r) = -\frac{GM_S}{c^2 r} \left[1 + \frac{1}{3} A(m, R_S) e^{-mr} \right], \tag{39}$$

$$\Phi(r) = \frac{GM_S}{c^2 r} \left[1 - \frac{1}{3} A(m, R_S) e^{-mr} (1 + mr) \right]. \tag{40}$$

In the GR limit, $m \to \infty$, the exponential term in both Ψ and Φ vanishes and the weak-field approximation of the Schwarzschild metric is obtained, as expected from a weak spherical perturbation on a Minkowski metric.

III. NONMINIMAL COUPLING

The remarkable range of applications of $f(R)$ gravity, and in particular the Starobinsky inflationary model, prompted physicists to generalize these theories even further. This generalization is made by a NMC between geometry and matter, through the product of the matter Lagrangian density with another function of the curvature, as posited by the action

$$S = \int \left[\frac{1}{2} f^1(R) + \left[1 + f^2(R)\right] \mathcal{L}_m \right] \sqrt{-g} d^4x. \quad (41)$$

The field equations, obtained through the variation of the action with respect to the metric, are

$$\begin{gathered} \left(f_R^1 + 2f_R^2\mathcal{L}_m\right) R_{\mu\nu} - \frac{1}{2} f^1 g_{\mu\nu} = \\ \left(1+f^2\right) T_{\mu\nu} + \left(\Box_{\mu\nu} - g_{\mu\nu}\Box\right) \left(f_R^1 + 2f_R^2\mathcal{L}_m\right), \end{gathered} \quad (42)$$

where $f_R^i \equiv df^i(R)/dR$. If $f^1(R) = 2\kappa(R - 2\Lambda)$ and $f^2(R) = 0$, the action (41) collapses to the Einstein-Hilbert action from (1). In opposition to GR and pure $f(R)$ gravity, these field equations do not respect the covariant conservation of energy, $\nabla^\mu T_{\mu\nu} \neq 0$.

A NMC model is also equivalent to scalar-tensor theories, though with two independent scalar fields, introduced through the action

$$S = \int \left[\psi R - V(\phi,\psi) + \left[1 + f^2(\phi)\right] \mathcal{L}_m\right] \sqrt{-g} d^4x, \quad (43)$$

with the potential

$$V(\phi,\psi) = \phi\psi - \frac{1}{2} f^1(\phi), \quad (44)$$

and where the fields are defined as

$$\phi = R, \qquad \psi = f_R^1(\phi)/2 + f_R^2(\phi)\mathcal{L}_m. \quad (45)$$

A. Preheating and Inflation

In section II A, the process of particle production after inflation was explained by means of a preheating mechanism, whose main characteristic was on the parametric resonance of a quantum field χ. The action of this field included a NMC term between the curvature and the field itself, of the type $\xi R \chi^2$, which amounts to a variable mass for the scalar field. This prompts for the more general assumption of the action (41) with the functions

$$f^1(R) = 2\kappa \left(R + \frac{R^2}{6M^2} \right), \qquad f^2(R) = 2\xi \frac{R}{M^2}. \quad (46)$$

Logically, this specific action has to be able to explain inflation at the standard SRA, which means that the NMC model only works as part of a perturbative regime with $f^2(R) \sim 0$.

The NMC parameter ξ is dimensionless and subject to the range $1 < \xi < 10^4$, where the lower bound comes from the SRA and the upper bound is obtained by resorting to the initial inflationary temperature [12]. As in Eq. (16), the decomposition of the scalar field from the NMC in Fourier modes yields that each mode is governed by a second order differential equation. Indeed, by a couple of transformations, one ends up reaching a Mathieu equation like Eq. (17) from the pure $f(R)$ case, though with parameter $q = 4\xi/z$.

The results are, therefore, the same as the ones mentioned in section II A.

B. Solar System Long Range Regime

To assess the NMC effect on the Solar System, the perturbed FLRW metric from Eq. (19) is used. As the curvature can also be decomposed in the form (20), then the approximations detailed in Eq. (21) are also applied, with the exception that now both $f^1(R)$ and $f^2(R)$ are subject to it. The curvature solution is obtained through the linearisation of the trace of the FE, which yields the equation

$$\begin{aligned} \nabla^2 U - m^2 U = &-\frac{1}{3}\left(1 + f_0^2\right)\rho^{\mathrm{s}} \\ &+ \frac{2}{3} f_{R0}^2 \rho^{\mathrm{s}} R_0 + 2\rho^{\mathrm{s}} \Box f_{R0}^2 + 2 f_{R0}^2 \nabla^2 \rho^{\mathrm{s}}, \end{aligned} \quad (47)$$

with

$$\begin{aligned} m^2 = \frac{1}{3}\Bigg[&\frac{f_{R0}^1 - f_{R0}^2 \mathcal{L}_m}{f_{RR0}^1 + 2f_{RR0}^2\mathcal{L}_m} - R_0 \\ &- \frac{3\Box\left(f_{RR0}^1 - 2f_{RR0}^2\rho^{\cos}\right) - 6\rho^{\mathrm{s}}\Box f_{RR0}^2}{f_{RR0}^1 + 2f_{RR0}^2\mathcal{L}_m}\Bigg], \end{aligned} \quad (48)$$

where the potential $U(r,t)$ is defined as

$$U(r,t) = \left[f_{RR0}^1(t) + 2f_{RR0}^2(t)\mathcal{L}_m(r,t)\right] R_1(r), \quad (49)$$

and $f_{RR0}^1 + 2f_{RR0}^2\mathcal{L}_m \neq 0$ is assumed. Naturally, this expression becomes the one obtained for the pure $f(R)$ theories, Eq. (25), when $f^2(R) = 0$. To work out the solution of the above equation outside the star, it can be shown that it is possible to neglect the linear term in U. Then, that equation (47) becomes

$$\nabla^2 U = \eta(t)\rho^{\mathrm{s}}(r) + 2f_{R0}^2 \nabla^2 \rho^{\mathrm{s}}, \quad (50)$$

$$\eta(t) = -\frac{1}{3}\left(1 + f_0^2\right) + \frac{2}{3} f_{R0}^2 R_0 + 2\Box f_{R0}^2. \quad (51)$$

Integrating and manipulating this equation, the curvature R_1 is given by

$$R_1(r,t) = \frac{\eta(t)}{4\pi\left(2f_{RR0}^2\rho^{\cos} - f_{RR0}^1\right)} \frac{M_{\mathrm{S}}}{r}, \quad (52)$$

which obviously reduces to Eq. (27) when $f^2(R) = 0$. The $R_1(r)$ solution for $r < R_S$ is not relevant for the following analysis, so it is not presented here. Nonetheless, one finds it in Ref. [16], as well as all details of the above computation.

The same linearisation procedure that was used for the curvature may be used for the tt component of the field equations (42). The computation of Ψ once more relies on the decomposition of the type $\Psi = \Psi_0 + \Psi_1$. The final solution for Ψ outside the spherical body is,

$$\Psi(r,t) = -\frac{1 + f_0^2 + f_{R0}^2 R_0 + 3\Box f_{R0}^2}{6\pi \left(f_{R0}^1 - 2 f_{R0}^2 \rho^{\cos}\right)} \frac{M_S}{r}. \tag{53}$$

The Newtonian limit requires that $\Psi(r)$ should be proportional to M_S/r, leading to the following constraint:

$$\left|2 f_{R0}^2\right| \rho^{s}(r) \ll \left|f_{R0}^1 - 2 f_{R0}^2 \rho^{\cos}(t)\right|, \quad r \leq R_S. \tag{54}$$

The linearization of the rr component of the field equations and a subsequent integration gives the solution

$$\Phi(r,t) = \frac{1 + f_0^2 + 4 f_{R0}^2 R_0 + 12\Box f_{R0}^2}{12\pi \left(f_{R0}^1 - 2 f_{R0}^2 \rho^{\cos}\right)} \frac{M_{\mathrm{S}}}{r}. \tag{55}$$

The development of these computations may be found in Ref. [16].

Having the solutions of both potentials Ψ and Φ, the PPN parameter γ is then

$$\gamma = \frac{1}{2}\left[\frac{1 + f_0^2 + 4 f_{R0}^2 R_0 + 12\Box f_{R0}^2}{1 + f_0^2 + f_{R0}^2 R_0 + 3\Box f_{R0}^2}\right], \tag{56}$$

showing that it is completely defined by the background metric R_0, whose value is obtained from the cosmological solution of NMC gravity, developed in Ref. [17]. The case $f^2(R) = 0$ obviously yields the value $\gamma = 1/2$ from the computations of the previous sections.

Concluding, all of what was said above means that for a NMC model in a long range regime to be compatible with Solar System, (i) either the present analysis is not adequate (the conditions $|mr| \ll 1$ and $|R_1| \ll R_0$ are not satisfied) or (ii) the validity condition of the Newtonian limit has to satisfy the Cassini constraint $\gamma \sim 1$ [18].

The next step is to explore condition (ii), computing the parameter of mass m^2 in a specific cosmological solution, *i.e.* in the case where the functions $f^i(R)$ assume power-law forms compatible with the description of the current phase of accelerated expansion of the Universe. Indeed, applying the conditions listed in (i), it becomes clear that the procedure used does not exclude or even constrain the NMC model explored in Ref. [17]. It is a null-result that nevertheless prompts to inspect more closely the inverse situation of a relevant mass parameter and a negligible cosmological background.

IV. EXPERIMENTAL CONSTRAINTS ON MODIFICATIONS OF GR

It is relevant to introduce some ways to constrain possible modifications of GR. The first modification is a Yukawa contribution to the Inverse Square Law (ISL), whose potential is then

$$U(r) = -G\frac{m_i}{r}\left(1 + \alpha e^{-r/\lambda}\right), \tag{57}$$

where α is the Yukawa strength parameter and λ is a characteristic length scale.

These parameters α and λ are constrained by limits imposed from experimental results. Figure 2 shows how large α can be in the different length scales dictated by λ. The overall data in this graphic is a composition from different experiments, ranging from torsion balance experiments [20] to laser ranging tests between the Earth, the Moon (LLR) and the LAGEOS satellites (Keplerian tests) [21]. The values of α above the curve, in the yellow shaded area, are excluded as physically possible values with a 95%-confidence level.

Another constraint that can be made to modifications of GR is related to the post-Newtonian limit of GR. This name is very appropriate because the post-Newtonian limit is simply a way to write a general theory of gravitation in the form of lowest-order deviations from the Newton law of gravitation. Mathematically, this limit can be written as an expansion of the Minkowski metric in terms of small gravitational potentials, that are defined in terms of matter variables like the matter density $\rho(\mathbf{x})$, where $\mathbf{x}$ is the position vector. Throughout the years, this was generalized by Will in the standard form known today as the Parameterized Post-Newtonian (PPN) formalism [22].

Accordingly, the current version of the PPN formalism is written with ten parameters which are the coefficients of the metric potentials. These parameters may take different values according to the theory under study. Each of the parameters measures or indicates general properties of the specific theories of gravity. Interestingly enough, the GR theory only includes the parameters $\gamma = \beta = 1$, all the others are null; therefore, the metric in a resting frame takes the form

$$g_{00} = -1 + 2U - 2\beta U^2, \quad g_{ij} = \left(1 + 2\gamma U\right)\delta_{ij}, \tag{58}$$

where $i, j = \{1, 2, 3\}$ and U is the gravitational potential defined as $U\left(r,t\right) \equiv \int \rho\left(r', t'\right) / \left|r - r'\right| d^3x'$.

The experimental data of the Solar System validates the GR theory, so a different theory needs to have very similar PPN parameters as GR. The best way to start would be to compute the value of γ, determined by the Cassini experiment as $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ [18].

Another way to check if a modified theory of GR is viable is if it also predicts the geodetic precession. This effect appears when computing the conservation of the angular momentum of a gyroscope with an intrinsic angular momentum vector $S^\mu = \left(S^0, \mathbf{S}\right)$ orbiting a spherical body like the Earth. The equation becomes

$$\nabla_\nu S_\mu = 0 \Leftrightarrow \frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau}, \tag{59}$$

where τ is the coordinate of proper time and the angular momentum vector S_μ is defined so as to be orthogonal to the velocity vector. Defining a new spin vector $\boldsymbol{\mathcal{S}}$ as

$$\boldsymbol{\mathcal{S}} = (1+\phi)\,\mathbf{S} - \frac{1}{2}\mathbf{v}\,(\mathbf{v}\cdot\boldsymbol{S})\,, \tag{60}$$

and considering the weak field approximation with the metric according to the PPN formalism[1],

$$g^{00} = -1-2\phi, \quad g^{i0} = \zeta_i, \quad g^{ij} = (1-2\phi)\,\delta_{ij}, \tag{61}$$

yields, after many manipulations,

$$\frac{d\boldsymbol{\mathcal{S}}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{\mathcal{S}}, \qquad \boldsymbol{\Omega} = \frac{1}{2}\nabla\times\boldsymbol{\zeta} - \frac{3}{2}\mathbf{v}\times\nabla\phi, \tag{62}$$

showing that $\boldsymbol{\mathcal{S}}$ precesses with angular velocity $\boldsymbol{\Omega}$. Following Ref. [4], if the central spherical body is the rotating Earth at rest, then

$$\boldsymbol{\Omega} = 3G\frac{\mathbf{r}\,(\mathbf{r}\cdot\mathbf{J}_\oplus)}{r^5} - G\frac{\mathbf{J}_\oplus}{r^3} + 3GM_\oplus\frac{\mathbf{r}\times\mathbf{v}}{2r^3}, \tag{63}$$

where $M_\oplus$ and $\mathbf{J}_\oplus$ are the mass and angular momentum of the Earth. The last term depends only on the mass of the earth and not on its spin and it is called geodetic precession.

The geodetic precession effect was observationally checked by the Gravity Probe B (GPB) experiment. This experiment consisted in a satellite orbiting the Earth with extremely precise gyroscopes. The spin axis of the gyroscopes was set to be aligned with a guide star during a certain period of time. Then, the change in the precession of the spin axis alignment reveals the predicted geodetic effect. The GPB confirmed the geodetic effect with 0.28% of accuracy [23].

V. SHORT RANGE SOLAR SYSTEM IMPACT

The contents of the following chapters include original work and are completely based in Ref. [1].

In this section, the procedure applied in the short range regime of the Solar System impact of pure $f(R)$ theories will be followed, though considering a NMC model. The action functional used is again the one from Eq. (41). It is once more assumed that matter behaves as dust, *ie* Eq. (4) with $p = 0$. A spherically symmetric body with mass density $\rho = \rho(r)$ that respects conditions (34) is also considered.

The metric used is the one from (33). For the purpose of the present work the functions Ψ and Φ will be computed at order $\mathcal{O}(1/c^2)$. The functions $f^i(R)$ are assumed to admit Taylor expansions around $R = 0$,

$$f^1(R) = 2\kappa\left(R + \frac{R^2}{6m^2}\right), \quad f^2(R) = 2\xi\frac{R}{m^2}, \tag{64}$$

It should be noted that the Cosmological Constant is dropped, consistent with the assumption that the metric is asymptotically flat.

A. Solutions for R, Ψ and Φ

As in the previous sections, the trace of the FE yields the curvature equation

$$\nabla^2 R - m^2 R = -\frac{8\pi G}{c^2}m^2\left[\rho - 6\left(\frac{2\xi}{m^2}\right)\nabla^2\rho\right], \tag{65}$$

which by applying a Green function method, has a twofold solution, whose expression outside the star is

$$R^\uparrow(r) = \frac{2GM_S}{c^2 r}m^2\,(1-12\xi)\,A(m,R_S)e^{-mr}, \tag{66}$$

with M_S the mass of the spherical body and $A(m,R_S)$ a form factor defined as

$$A(m,R_S) = \frac{4\pi}{mM_S}\int_0^{R_S}\sinh(mr)r\rho(r)dr. \tag{67}$$

The expression (66) vanishes as $r\to\infty$ and it is considered to be valid only at Solar System scales, since spacetime should assume a De Sitter metric with curvature $R_0 \neq 0$ at cosmological scales. It must also be noted that in the limit $m\to 0$ then $R^\uparrow(r)\to 0$ for any $r > R_S$.

As for the solutions of Ψ and Φ, they are obtained through the linearization of the tt and rr components of the FE (42), neglecting all terms smaller than $\mathcal{O}(1/c^2)$. The computation of Ψ will once more pass through $\Psi_0 + \Psi_1$, and several integrations end up with

$$\Psi(r) = -\frac{GM_S}{c^2 r}\left[1 + \left(\frac{1}{3} - 4\xi\right)A(m,R_S)e^{-mr}\right], \tag{68}$$

$$\Phi(r) = \frac{GM_S}{c^2 r}\left[1 - \left(\frac{1}{3} - 4\xi\right)A(m,R_S)e^{-mr}(1+mr)\right]. \tag{69}$$

In the GR limit, $\xi = 0$ and $m\to\infty$, the exponential term in both Ψ and Φ vanishes and the weak-field approximation of the Schwarzschild metric is recovered, as expected.

[1] In these computations for the geodetic precession, all the computations are made considering $c = 1$.

B. Discussion of Yukawa potential

From the tt component of the metric, it is possible to identify a Newtonian potential plus a Yukawa perturbation:

$$U(r) = -\frac{GM_S}{r}\left(1 + \alpha A(m, R_S) e^{-r/\lambda}\right), \tag{70}$$

defining the characteristic length $\lambda = 1/m$ and the strength of the Yukawa addition

$$\alpha = \frac{1}{3} - 4\xi, \tag{71}$$

so that, if $\xi = 0$, the Yukawa strength for pure $f(R)$ theories is obtained, $\alpha = 1/3$; it should also be noticed that a positive NMC yields $\alpha \leq 1/3$. Strikingly, a NMC with $\xi = 1/12$ cancels the Yukawa contribution.

The dimensionless form factor, defined in (67), was found by integrating the field equations of NMC gravity, but is not specific of it nor of $f(R)$ theories. This form factor can then be evaluated in several ways, according to the function of mass density $\rho(r)$. Taking the limit of a point source, $r \to 0$ allows to expand around $mr \ll 1$, so that $\sinh(mr) \approx mr[1 + (mr)^2/6]$ and then $A(m, R_S) \sim 1$. This can be verified explicitly by making all computations with a test mass density (such as a uniform profile) and, in the end, taking the limit $R_S \to 0$. Indeed, if $\rho_0 = 3M_S/(4\pi R_S^3)$, then $A(m, R_S)$ admits the limiting cases

$$\begin{aligned} A(m, R_S) &\approx 1 + \frac{(mR_S)^2}{10} \sim 1 \quad , \quad mR_S \ll 1, \\ A(m, R_S) &\approx \frac{3}{2}\frac{e^{mR_S}}{(mR_S)^2} \quad , \quad mR_S \gg 1. \end{aligned} \tag{72}$$

If the central body is the Sun (with radius $R_\odot$), a more accurate density NASA polynomial profile [24] can instead be considered, obtaining the limiting cases

$$\begin{aligned} A(m, R_\odot) &\approx 1 + 6 \times 10^{-2}(mR_\odot)^2 \sim 1 \quad , \quad mR_\odot \ll 1, \\ A(m, R_\odot) &\approx 7.3\frac{e^{mR_\odot}}{(mR_\odot)^3} \quad , \quad mR_\odot \gg 1. \end{aligned} \tag{73}$$

Both forms for $A(m, R_\odot)$ are plotted in Fig. 1, showing that it grows with $mR_\odot$.

It should also be noted that the PPN formalism is incompatible with the presence of a Yukawa term in the gravitational potential, since the latter cannot be expanded in powers of $1/r$. Nevertheless, for consistency, what happens if the condition $mr \ll 1$ is valid throughout the Solar System region can be considered: in this case, the metric (33) with the above limits for the form factor of ~ 1 is well approximated by

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2GM_S}{c^2 r}\left(\frac{4}{3} - 4\xi\right)\right] c^2 dt^2 \\ &+ \left[1 + \frac{2GM_S}{c^2 r}\left(\frac{2}{3} + 4\xi\right)\right] dr^2 + r^2 d\Omega^2, \end{aligned} \tag{74}$$

which yields the PPN parameter

$$\gamma = \frac{1}{2}\frac{1 + 6\xi}{1 - 3\xi}. \tag{75}$$

In the absence of a NMC, $\xi = 0$, this yields $\gamma = 1/2$, a strong departure from GR that is disallowed by current experimental bounds, as seen before. Nonetheless, this result appears to show that a NMC allows $f(R)$ theories to remain compatible with observations, as long as $\xi = 1/12$ — which is just a restatement of the previously obtained result. Again, the path towards obtaining the γ parameter depicted above is presented for illustration only.

C. Experimental constraints to NMC gravity parameters

As it was shown before, the Yukawa potential (70) is not new in physics as an alternative way to account for deviations from Newtonian gravity or other forces of nature. Fig. 2 shows the exclusion plot for the phase space (λ, α), which is used to constrain the phase space of the model (41) under scrutiny. It is seen that a NMC with $\xi = 1/12$ cancels this contribution, as shown by the values of $|\alpha| \to 0$ overlaid on the exclusion plot. Also, it should be noticed that large values of ξ lead to a large, negative strength $\alpha \sim -4\xi$.

Further insight is obtained by casting the NMC presented in Eq. (64) as $f^2(R) = R/(6M^2)$, so that it is characterized by a distinct mass scale M, instead of the relative strength parameter ξ: by making the transformation $\xi = (m/M)^2/12$, the suggestive form

$$\alpha = \frac{1}{3}\left[1 - \left(\frac{m}{M}\right)^2\right] \tag{76}$$

is thus obtained, which, inverting, allows to plot the exclusion plot for the phase space (m, M) in Fig. 3.

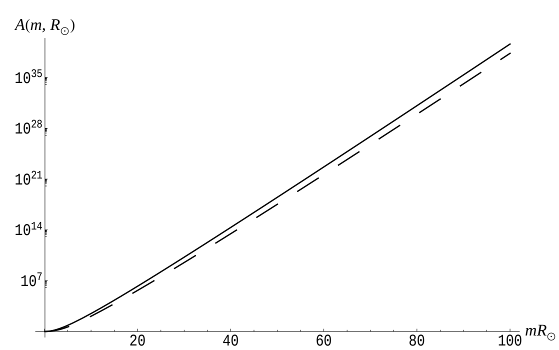

Figure 1. Form factor $A(m, R_\odot)$ for a constant density (full) and fourth-order density profile for the Sun.

These two figures 2 and 3 show that, if m falls within the range 10^{-22} eV $< m <$ 1 meV (corresponding to lengthscales λ ranging from the millimeter to Solar System scales), then the strong constraints available on the

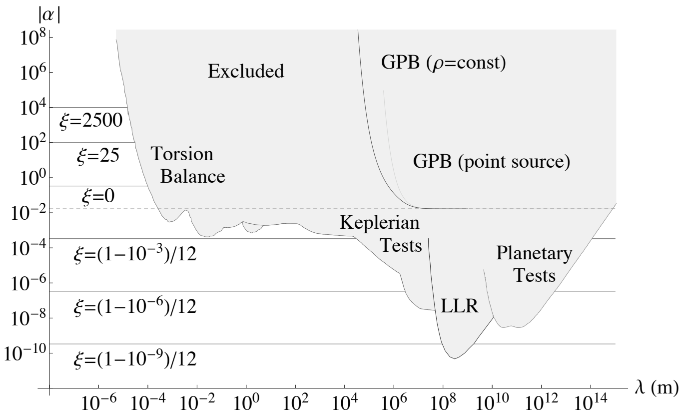

Figure 2. Yukawa exclusion plot for α and λ. Adapted from Refs. [25].

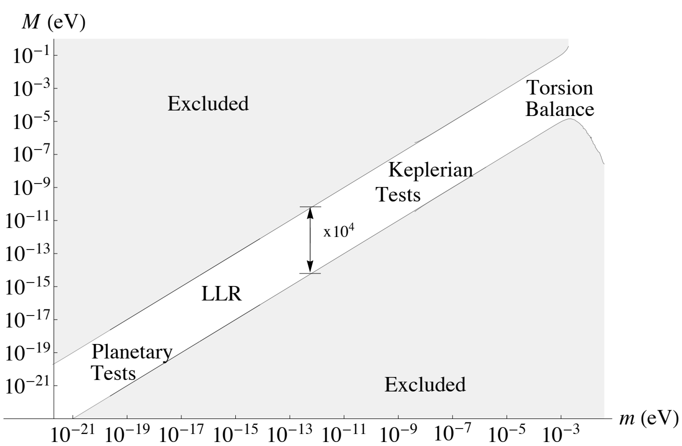

Figure 3. Exclusion plot for the characteristic mass scales M and m.

Yukawa strength, $|\alpha| \ll 1$, require that $\xi \sim 1/12$ — or, equivalently, that both characteristic mass scales are very similar, $m \sim M$.

1. Geodetic precession

In this section it is assumed that the Earth can be approximated as a spherically symmetric body. Following the computations from section IV on geodetic precession and since $(d\mathbf{S}/dt)_{\rm sec} = \mathbf{\Omega}_G \times \mathbf{S}$, the angular velocity vector $\mathbf{\Omega}_G$ of geodetic precession is given by

$$\begin{aligned}\mathbf{\Omega}_G =& \frac{3}{2}\frac{(GM_S)^{3/2}}{c^2 r^{5/2}} \left[1 + \alpha A(m,R_S)(1+mr)e^{-mr}\right]^{1/2} \\ &\times \left[1 - \frac{\alpha A(m,R_S)}{3}(1+mr)e^{-mr}\right]\mathbf{n},\end{aligned} \tag{77}$$

where $\mathbf{n}$ is the unit vector perpendicular to the plane of the orbit. If $\xi = 0$, the above expression reduces to the case for $f(R)$ models, as expected [15].

The final results of the GPB experiment report an accuracy of 0.28% in the measurement of geodetic precession [23], which corresponds to the following constraint on NMC gravity parameters:

$$\left|\frac{\Omega_G - \Omega_G^{\rm GR}}{\Omega_G^{\rm GR}}\right| < 0.0028, \tag{78}$$

where only the modulus of angular velocity is considered, and $\Omega_G^{\rm GR}$ denotes the value of geodetic precession in GR. Substituting the expression of NMC geodetic precession in this constraint and knowing that $x \equiv \alpha A(m,R_S)(1+mr)e^{-mr}$ has to respect $x \ll 1$, then the condition that remains to be analysed is

$$|\alpha| < \frac{0.0168}{1+mr}\frac{e^{mr}}{A(m,R_S)}. \tag{79}$$

From the assumption (34) of continuity of mass density and its derivative across the surface of the Earth, it is possible to consider the form factor $A(m,R_S)$ to converge to the value corresponding to the uniform density model, Eq. (72), hence the inequality (79) reads

$$\begin{aligned}&|\alpha| < 0.0168 \quad , \quad mR_\oplus \ll 1 \quad , \\ &|\alpha| < 0.0112\frac{mR_\oplus^2}{r}e^{m(r-R_\oplus)} \quad , \quad mR_\oplus \gg 1 \quad ,\end{aligned} \tag{80}$$

where $R_\oplus \approx 6371$ km is the radius of the Earth. Knowing that the GPB satellite orbits the Earth at a height of $r \sim 650$ km, this condition can be seen in the (λ,α) exclusion plot, as shown in Fig. 2. It is found that the condition is well-within the already excluded phase space, so that the current bounds on geodetic precession do not add any new constraint on the model parameters.

VI. DISCUSSION AND OUTLOOK

In this work it has been computed the effect of a NMC model, specified by (64), in a perturbed weak-field Schwarzschild metric, as depicted in Eq. (68) and (69). In the weak-field limit, this translates into a Yukawa perturbation to the usual Newtonian potential, with characteristic range and coupling strength

$$\lambda = \frac{1}{m} \quad , \quad \alpha = \left(\frac{1}{3} - 4\xi\right) = \frac{1}{3}\left[1 - \left(\frac{m}{M}\right)^2\right] \quad . \tag{81}$$

This result is quite natural and can be interpreted straightforwardly: a minimally coupled $f(R)$ theory introduces a new massive degree of freedom, leading to a Yukawa contribution with characteristic lengthscale $\lambda = 1/m$ and coupling strength $\alpha = 1/3$.

The introduction of a NMC has no dynamical effect in the vacuum, as there is no matter to couple the scalar curvature to. As a result, it is not expected any modification in the range of this Yukawa addition. Conversely, a NMC has an impact on the description of the interior of the central body leading to a correction to the latter's coupling strength, which has a negative sign since $\mathcal{L}_m = -\rho$.

Using the available experimental constraints, it was found that, for 10^{-22} eV $< m <$ 1 meV, the NMC parameter must be $\xi \sim 1/12$ or, equivalently, both mass scales m and M of the non-trivial functions $f^1(R)$ and $f^2(R)$ must be extremely close.

If this is the case, the latter relation is not interpreted as an undesirable fine-tuning, but instead is suggestive of a common origin for both non-trivial functions $f^1(R)$ and $f^2(R)$.

Conversely, for values of m (or λ) away from the range mentioned above the Yukawa coupling strength, α can be much larger than unity, so that ξ can assume any value and the mass scales m and M can differ considerably. In particular, the Starobinsky inflationary model, which requires the much heavier mass scale $m \approx 3 \times 10^{13}$ GeV $\sim 10^{-6} M_P$, manifests itself at a lengthscale $\lambda \sim 10^{-29}$ m. This implies that the generalized preheating scenario posited previously, which requires $1 < \xi < 10^4$, is thus completely allowed by experiment and unconstrained by this work.

Finally, by computing the perturbation induced on geodetic precession, it was found that no significant new constraint arises, as this is already included in the existing Yukawa exclusion plot.

Higher-order computations (on $1/c^2$) might reveal other effects, mainly when one begins to take into account the competing effect of the cosmological background dynamics (modelled for example by a Cosmological Constant).

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