Strategic decisions in offshore petroleum production

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Along this work, we will focus our attention in an investment problem considering two factors of uncertainty, that will be treated as an optimal stopping problem. These two factors will be more precisely, two possible feedstocks used in a production line. We will consider a switching model, assuming two possible dynamics for the prices behaviour: geometric Brownian motion and an arithmetic Brownian motion.

For these two dynamics considered, a switching model where there is only one switch available is presented. The general switching model using these two dynamics presents a classic indeterminacy and thus it cannot be solved analytically. Considering a single opportunity switch allows both an achievement of an analytical solution and a proximity to a more realistic scenario. One particular scenario considered is the position of a company responsible for oil and natural gas extraction, although only when we consider the arithmetic Brownian motion dynamic.

I. INTRODUCTION

This article has its main focus on the thematic of investment under uncertainty. We consider that due to the increasing level of uncertainty that companies have to deal with in the economical environment as well as financial markets, the study of this sort of investments couldn't be more relevant. The main goal of this analysis is, of course, to be able to help firms make their investment decisions in the best possible way, once such decisions are, in most cases, irreversible. Additionally, the ability to treat flexibility in the used techniques, is really relevant to the study of this problem. Real option theory compared to the traditional capital budgeting techniques allows to account for the value of flexibility.

The problem of optimal investment time will be modelled as an optimal stopping problem, whose formulation can be consulted in [6]. The main goal will be to determine threshold values for the dynamics process, at which a certain course of action should be performed, following the same methodology used in [3], [4] and [1]. Additionally, and once more as in [3], [4] and [1], we will also be interested in obtaining the firm's value associated with the referred investment. The type of investments we will dedicate our study to are those which consider two factors of uncertainty. We shall present a methodology that, considering two different feedstock sources in a production line, aims to determine the time at which is more profitable to switch from the feedstock in use to a substitute feedstock. This procedure using switching options is presented in [4] for the geometric Brownian motion dynamic.

An important consideration that is made in what we mentioned above, and that wasn't yet referred, is the irreversibility of the considered investment. As it is mentioned in [3] and [1], once an investment is made, there is no turning back, and the capital that was paid can no longer be reimbursed. [9] and [10] state that irreversibility, together with the uncertainty feature and the flexibility, that are all described above and identify completely the type of investments we are analysing, define a real option problem.

Inside the range of real options, [4] considers a quite specific type, the switching options. It also derives a particular restricted model using switching options, that later on we will be quite interested in studying. Such a model is the single opportunity switch model, as it is referred to in [4]. [9] states that these type of options are quite used to describe investments involving energy resources, such as fuel oil, gas, electricity, among others. The reason presented for this to happen is the flexibility required by these sort of markets, to change between the alternative energy resources, that is provided by the switching options.

This explains the reason for our choice of applying this models to a scenario involving energy resources extraction, that is presented in [7]. Adopting the perspective taken in [7], we decided to assume the position of a company responsible for the extraction of both oil and natural gas, who would be implementing a switching model similar to the one presented in [4] and including the respective restriction already mentioned above.

II. ENERGY-SWITCHING OPTIONS

As we referred above, we will be studying investment under uncertainty, although now from a different perspective from [3]. Similarly to what is presented in [4] we will start by considering the production of some saleable output. Later on we will look at a specific application considering energy production. The uncertainty will then be related to the feedstock source that one should choose. The optimality of the feedstock's source in use might change with several factors. It can, for example, vary with cost, season, etc. In this sense, our main goal is to decide which feedstock is optimal to use in each moment, and make a switch, if necessary, according to an optimal criteria to be defined later, following the same procedure used in [4].

The first approach assumes the behaviour of the feedstock prices to follow a geometric Brownian motion dynamic, as it is done in [4], and following this reference in a lot of other aspects as well.

Posteriorly, in a second approach, we make a few changes to our initial assumptions. First of all, a change of the dynamic for the feedstock prices is considered. Instead of the geometric Brownian motion, we assume that the prices behave according to an arithmetic Brownian motion. The choice for the new dynamic is inspired in [7], where such behaviour is also considered for the long-term feedstock prices.

Also based on [7], we will adopt the position of a company responsible for the extraction of both oil and natural gas, which will now become our two possible feedstocks, and the development of this next model shall be based on their perspective. The main idea of this new approach is to consider a concrete scenario, based on the one presented in [7] and adapt the first established model to it, that has been developed following the reasoning presented in [4], considering a different price dynamic.

Additionally, and independently of the dynamic considered, we will consider a simplification of the general switching model, where the restriction of one single opportunity to switch feedstocks is implemented. As we will see later on, this simplified model will prove itself quite useful in terms of both adaptability to the assumed scenario and simplicity to reach a solution.

A. Geometric Brownian Motion Dynamic

We shall then consider the production line of a certain item, and the existence of two possible sources for the feedstock in use, as described in [4]. We are going to denote the feedstock coming from the first source as feedstock 1 and the one coming from the second source as feedstock 2, and we will refer to the one in use as the incumbent and to the the one that is not being used at the moment as the substitute. We will also fix switching costs, for whenever we consider the switch between the incumbent feedstock and the substitute.

Let X_1 denote the price of feedstock 1 and X_2 the price of feedstock 2. In this first approach we will assume both variables to follow a geometric Brownian motion as we can see below in (1) and once again as it is done in [4]. Thus the dynamic for the price of feedstock I, for $I \in \{1, 2\}$ is given by,

$$dX_I = \alpha_I X_I dt + \sigma_I X_I dZ_I \tag{1}$$

where α_I is the risk-adjusted drift rate for feedstock I, σ_I the volatility rate and dZ_I the increment of a standard Wiener process. Also, we have that $Cov[dX_1, dX_2] = \rho\sigma_1\sigma_2 dt$, where $|\rho| \leq 1$.

We must now define F_I, as the firm's value when feedstock I is the incumbent. The firm's value is composed by one term representing the option value, more precisely switching option value, and one term denoting the incumbent asset value. We must also define D_0 as the output or selling price of our final product. In this sense, and according to [4], the net cash flow for this case scenario is going to be given by $D_0 - X_I$, meaning that we have the price at which the final product is being sold minus the price paid for the feedstock used. So, resulting from the application of Ito's lemma together with the dynamic programming principle to the discounted value of F_I, given by $e^{-rt}F_I(X_1, X_2)$, we obtain the following dynamic programming equation,

$$\frac{1}{2}\sigma_1^2 X_1^2 \frac{\partial^2 F_I}{\partial X_1^2} + \frac{1}{2}\sigma_2^2 X_2^2 \frac{\partial^2 F_I}{\partial X_2^2} + \rho\sigma_1\sigma_2 X_1 X_2 \frac{\partial^2 F_I}{\partial X_1 \partial X_2} + \alpha_1 X_1 \frac{\partial F_I}{\partial X_1} + \alpha_2 X_2 \frac{\partial F_I}{\partial X_2} - rF_I + (D_0 - X_I) = 0 \tag{2}$$

where r denotes the risk-free interest rate, and F_I is always a short term for $F_I(X_1, X_2)$.

1. *Characteristic Root Equation*

Following the same course of action used in [4], next we propose a solution to equation (2), compute its derivatives and replace them in (2).

According to [4], the proposed solution, that we will denote by G_I once we can only consider it to be a true solution of (2) after we prove it verifies all the conditions we will impose, is going to be given by,

$$G_I(X_1, X_2) = A_I X_1^{\beta_I} X_2^{\eta_I} + \frac{D_0}{r} - \frac{X_I}{r - \alpha_I} \tag{3}$$

for $I \in \{1, 2\}$, and where A_I, β_I and η_I are parameters still to be determined. It is worth mentioning that the term $A_I X_1^{\beta_I} X_2^{\eta_I}$ in (3) corresponds to the switching option value and is also the solution for the homogeneous part of (2), whereas the term $\frac{D_0}{r} - \frac{X_I}{r-\alpha_I}$ corresponds to the incumbent value in the absence of a switching option.

Computing all the derivatives of (3) and replacing them in (2) will originate an ellipse equation. Such an equation is denoted in [4] as the *characteristic root equation*, and we can defined a function Q_I, for each $I \in \{1, 2\}$, of β_I and η_I, such that,

$$\begin{aligned} Q_I(\beta_I, \eta_I) &= \frac{1}{2}\sigma_1^2 \beta_I(\beta_I - 1) + \frac{1}{2}\sigma_2^2 \eta_I(\eta_I - 1) \\ &\quad + \rho\sigma_1\sigma_2\beta_I\eta_I + \alpha_1\beta_I + \alpha_2\eta_I - r \\ &= 0 \end{aligned} \tag{4}$$

The plot of $Q_I(\beta_I, \eta_I) = 0$ originates an ellipse. We present the graphic in figure 1, where the values for the fixed constants are presented in table I, and are the same ones used in [4].

The homogeneity-degree-1 condition, given by $\beta_I + \eta_I = 1$ and mentioned in [4], is represented in figure 1, even though it won't be used for now. However for some

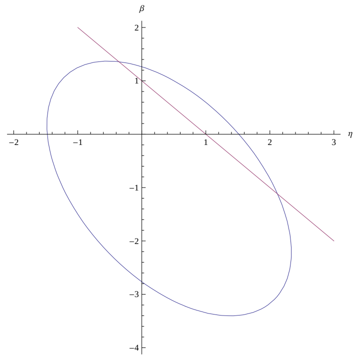

Figura 1: Graphic of the ellipse defined by $Q_I(\beta_I, \eta_I) = 0$ and the line $\beta_I + \eta_I = 1$

Parameters	Symbol	Value
Feedstock 1 volatility	σ_1	0.2
Feedstock 2 volatility	σ_2	0.25
Feedstock 1 risk-adjusted drift rate	α_1	0.05
Feedstock 2 risk-adjusted drift rate	α_2	0.03
Risk-free interest rate	r	0.07
Correlation between X_1 and X_2	ρ	0.5

Tabela I: Parameter Values - Ellipse equation - Energy switch model

simplifications presented in [4] such an assumption is, in fact, made.

The procedure used from now on is going to be to consider the four usual quadrants for the values of β and η, divide the term of G_I corresponding to the option value into those four quadrants and afterwards identify which quadrant will be relevant to use in the future, relying on economical arguments to do so.

Let us start by supposing that the incumbent feedstock is feedstock 1, and thus we shall treat G_1. So, we know that whenever the price of feedstock 1 tends to zero, the corresponding option value presented in the firm's value, let's denote it as G_{H1}, should do so as well. This can be translated into the following $\lim_{X_1 \to 0} G_{H1} = 0 \Leftrightarrow \lim_{X_1 \to 0} A_1 X_1^{\beta_1} X_2^{\eta_1} = 0 \Rightarrow \beta_1 > 0$. Similarly, whenever the price for feedstock 2 is infinitely large, the corresponding option value in G_{H1}, will tend to zero. This means that $\lim_{X_2 \to \infty} G_{H1} = 0 \Leftrightarrow \lim_{X_2 \to \infty} A_1 X_1^{\beta_1} X_2^{\eta_1} = 0 \Rightarrow \eta_1 \leq 0$.

This implies that whenever feedstock 1 is the incumbent we should choose the second quadrant for G_1. In this sense, and adding in index 2 to the parameters in question to emphasize that the considered quadrant is the second, we will have,

$$G_1(X_1, X_2) = A_{12} X_1^{\beta_{12}} X_2^{\eta_{12}} + \frac{D_0}{r} - \frac{X_1}{r - \alpha_1} \tag{5}$$

Proceeding the same way in the case where feedstock 2 is the incumbent, obviously adapting the arguments for this case, leads us the choice of the fourth quadrant for this second scenario. Thus, and adding in index 4 to the parameters in question to emphasize that the considered quadrant is the fourth, we have that,

$$G_2(X_1, X_2) = A_{24} X_1^{\beta_{24}} X_2^{\eta_{24}} + \frac{D_0}{r} - \frac{X_2}{r - \alpha_2} \tag{6}$$

Before we move any further we must note three aspects that the reader should take into account. Firstly, the notation G_{HI}, $I \in \{1, 2\}$ for the option value term figuring in the firm's value expression whenever feedstock I is the incumbent, was adopted because the option value term in G_I also corresponds to the solution of the homogeneous part of (2), as we mentioned above. Secondly, it must be taken into account that all these arguments, and the consequent choice of the quadrants for each G_I, follow the perspective of someone interested in acquiring the final product. Clearly, if we were considering the perspective of the company responsible for the production the whole reasoning would be opposite. We choose to take this point of view to follow the reasoning presented in [4]. Later on, this perspective will be reversed, and we will analyse the company's side as well. At last, we must observe that the quadrant choice that was made above will influence directly the definition of the *characteristic root equation*, presented in (4). If we are going to consider from now on, that when the incumbent feedstock is 1, we'll have the values of β_1 and η_1 belonging to the second quadrant, and when the incumbent feedstock is 2, we'll have those values belonging to the fourth quadrant, then for $I = 1$ we shall consider $Q_1(\beta_{12}, \eta_{12}) = 0$, and when $I = 2$ we will have $Q_2(\beta_{24}, \eta_{24}) = 0$.

2. *Value-Matching Relationship and Smooth Pasting Conditions*

Once more following the same course of action as in [4], we shall now establish the value-matching relationships and smooth pasting conditions for this case. However, we should first define $\hat{X}_{IJ}$ and K_{IJ}.

$\hat{X}_{IJ}$, when $I \neq J$, denotes the threshold price for feedstock I, whenever I is the incumbent and J is the substitute. However, when $I = J$, we get the case where there is only an incumbent feedstock and no substitute. This means that when we refer to a certain price as a threshold price, we are evaluating it in the period of time immediately before a switch is considered. In this sense, for $I, J \in \{1, 2\}$ we have the following,

- $\hat{X}_{12}$ is the threshold price for feedstock 1, being it the incumbent, when 2 is the substitute;
- $\hat{X}_{21}$ is precisely the opposite of the previous one, thus the threshold price for feedstock 2 when it is the incumbent and 1 is the substitute;

- $\hat{X}_{11}$ brings up the case where there is no other substitute feedstock and thus, in this case, it is the threshold price for feedstock 1 whenever the substitute feedstock is also 1;
- $\hat{X}_{22}$ similarly, is the threshold price for feedstock 2 whenever the substitute feedstock is also 2.

K_{IJ} is more simple to define, as it denotes the cost of switching from feedstock I to feedstock J, whenever I is the incumbent and J is the substitute. K_{IJ} can only be defined when we have $I \neq J$.

We are now in conditions of presenting the value-matching relationships and posteriorly the smooth pasting conditions.

As the function G_I originates two different functions, one for each value of $I \in \{1,2\}$, this will produce two value-matching relationships. According to [4], they state that,

1. for the case where feedstock 1 is the incumbent and feedstock 2 is the substitute, the difference between the value of the substitute $G_2(\hat{X}_{12}, \hat{X}_{22})$ and the value of the incumbent $G_1(\hat{X}_{12}, \hat{X}_{22})$, immediately before a switch, is equal to the switching cost K_{12}, meaning that $G_1(\hat{X}_{12}, \hat{X}_{22}) = G_2(\hat{X}_{12}, \hat{X}_{22}) - K_{12}$
2. for the case where feedstock 2 is the incumbent and feedstock 1 is the substitute, the difference between the value of the substitute $G_1(\hat{X}_{11}, \hat{X}_{21})$ and the value of the incumbent $G_2(\hat{X}_{11}, \hat{X}_{21})$, immediately before a switch, equals the switching cost K_{21}, which means that $G_2(\hat{X}_{11}, \hat{X}_{21}) = G_1(\hat{X}_{11}, \hat{X}_{21}) - K_{21}$

If we take now the expressions for G_1 and G_2 from (5) and (6) and substitute them in the two indicated equations above, we obtain the following,

$$A_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}} + \frac{D_0}{r} - \frac{\hat{X}_{12}}{r-\alpha_1} = A_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}} + \frac{D_0}{r} - \frac{\hat{X}_{22}}{r-\alpha_2} - K_{12} \tag{7}$$

$$A_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}} + \frac{D_0}{r} - \frac{\hat{X}_{21}}{r-\alpha_2} = A_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}} + \frac{D_0}{r} - \frac{\hat{X}_{11}}{r-\alpha_1} - K_{21} \tag{8}$$

Once the value-matching relationships were already presented, we should now proceed in order to obtain the smooth pasting conditions associated with them. As we have two variables considered in each value-matching relationship, they will originate two smooth pasting conditions each, one concerning each variable, which makes a total of four smooth pasting conditions. These conditions essentially guarantee the smoothness of G_1 and G_2 in the threshold points. They also make the problem in question a free boundary problem. They are given by,

$$\left.\frac{\partial G_1}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} \tag{9}$$

$$\left.\frac{\partial G_1}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} \tag{10}$$

$$\left.\frac{\partial G_2}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{11}\\ X_2=\hat{X}_{21}}} = \left.\frac{\partial G_1}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{11}\\ X_2=\hat{X}_{21}}} \tag{11}$$

$$\left.\frac{\partial G_2}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{11}\\ X_2=\hat{X}_{21}}} = \left.\frac{\partial G_1}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{11}\\ X_2=\hat{X}_{21}}} \tag{12}$$

At the light of what is done in [4], working with the smooth pasting conditions in sets of two equations (the first two conditions and the last two), we are able to isolate the terms $A_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}}$, $A_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}}$, $A_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}}$ and $A_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}}$, that appear in the smooth pasting conditions, as it is presented in (13), (14), (15) and (16).

$$A_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}} = \frac{1}{\Delta}\left[\frac{\eta_{24}\hat{X}_{12}}{r-\alpha_1} + \frac{\beta_{24}\hat{X}_{22}}{r-\alpha_2}\right] \tag{13}$$

$$A_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}} = \frac{1}{\Delta}\left[\frac{\eta_{12}\hat{X}_{12}}{r-\alpha_1} + \frac{\beta_{12}\hat{X}_{22}}{r-\alpha_2}\right] \tag{14}$$

$$A_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}} = \frac{1}{\Delta}\left[\frac{\eta_{24}\hat{X}_{11}}{r-\alpha_1} + \frac{\beta_{24}\hat{X}_{21}}{r-\alpha_2}\right] \tag{15}$$

$$A_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}} = \frac{1}{\Delta}\left[\frac{\eta_{12}\hat{X}_{11}}{r-\alpha_1} + \frac{\beta_{12}\hat{X}_{21}}{r-\alpha_2}\right] \tag{16}$$

where $\Delta = \beta_{14}\eta_{22} - \beta_{22}\eta_{14}$.

If we substitute now the last four results in the original value-matching relationships, given by (7) and (8), we obtain the following equations, as in [4].

$$\frac{\hat{X}_{12}}{r-\alpha_1}\left[1 - \frac{\eta_{24}-\eta_{12}}{\Delta}\right] - \frac{\hat{X}_{22}}{r-\alpha_2}\left[1 - \frac{\beta_{12}-\beta_{24}}{\Delta}\right] = K_{12} \tag{17}$$

$$\frac{\hat{X}_{21}}{r-\alpha_2}\left[1 - \frac{\beta_{12}-\beta_{24}}{\Delta}\right] - \frac{\hat{X}_{11}}{r-\alpha_1}\left[1 - \frac{\eta_{24}-\eta_{12}}{\Delta}\right] = K_{21} \tag{18}$$

It is worth mentioning that these two last equations are really useful simplifications of the value-matching relationships, once we have eliminated two unknown parameters A_{12} and A_{24}. Also, as G_I verified all boundary conditions we imposed, we can now assume that it truly represents the firm's value given by F_I.

The model constituted by the two *characteristic root equations* $Q_1(\beta_{12}, \eta_{12}) = 0$ and $Q_2(\beta_{24}, \eta_{24}) = 0$, the two value-matching relationships (7) and (8) and the four smooth pasting conditions (9) - (12), constitute our general model for investments concerning switching options. One can observe that this model contains eight equations, four unknown variables $\hat{X}_{12}$, $\hat{X}_{22}$, $\hat{X}_{21}$ and $\hat{X}_{11}$, and six unknown parameters A_{12}, A_{24}, β_{12}, β_{24}, η_{12} and η_{24}. Clearly, a solution to this model cannot be derived analytically. One could always try to reach a solution numerically, however we adopted another approach. We will consider a simplification to this general model, also mentioned in [4], that is going to be presented next, and where an analytical solution will be easier to reach.

3. *Single Opportunity Switch Model*

According to [4], this simplified model considers exclusively scenarios where one has only one opportunity to switch from feedstock 1 to feedstock 2, or from feedstock 2 to feedstock 1. Naturally, this new feature will induce some changes in both the value-matching relationships and smooth pasting conditions. These changes will be quite profitable to us, once the complexity of the model will decrease tremendously, allowing us to solve it using a method we are already familiar with. Similarly to [4], we will use the index s in all the computations from now on, to emphasize the fact that we are dealing with a single opportunity switch model.

Let us start by evaluating the effects on the first value-matching relationship, (7). We should recall that we have feedstock 1 as the incumbent and feedstock 2 as the substitute. What will happen now is that once we switch from feedstock 1 to 2, there will no longer be the possibility of switching back to feedstock 1. In this sense, we need to select the term in the first value-matching relationship that represents this possibility of change and eliminate it from the equation. The term in (7) that corresponds to the possible switch back from feedstock 2 to feedstock 1 is $A_{24s}\hat{X}_{12s}^{\beta_{24s}}\hat{X}_{22s}^{\eta_{24s}}$, as it is described in [4]. This term represents precisely the switching option value element in the composition of the firm's value when the incumbent feedstock is 2, G_2, in the points $\hat{X}_{12}$ and $\hat{X}_{22}$. Eliminating this term we get the "new" value-matching relationship for this specific model:

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} - \frac{\hat{X}_{12s}}{r-\alpha_1} = -\frac{\hat{X}_{22s}}{r-\alpha_2} - K_{12} \qquad (19)$$

Consequently, we can now obtain two new associated smooth pasting conditions. They are the following,

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} = \frac{\hat{X}_{12s}}{\beta_{12s}(r-\alpha_1)} \qquad (20)$$

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} \qquad (21)$$

This implies that,

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} = \frac{\hat{X}_{12s}}{\beta_{12s}(r-\alpha_1)} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} \qquad (22)$$

If we analyse the model presented in [3], we can observe that equation (22) as precisely the same form as the second equation figuring in that model. In this sense, we can follow the same reasoning here and derive a function H, as it is done in [3], consistent with our present case scenario. So, considering that from (22) we have $A_{12s} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}}}$ and $\hat{X}_{12s} = -\frac{\hat{X}_{22s}\beta_{12s}(r-\alpha_1)}{\eta_{12s}(r-\alpha_2)}$, we can take (19), and using these results, it is possible to derive the following equation

$$\frac{\hat{X}_{22s}}{(r-\alpha_2)}\left[\frac{\eta_{12s}+\beta_{12s}-1}{\eta_{12s}}\right] + K_{12} = 0 \qquad (23)$$

Thus, if we fix a value for $\hat{X}_{22s}$ we can define a function of β_{12s} and η_{12s} that together with the *characteristic root equation* will allow us to determine values for these two parameters, following the same methodology used in [3]. Such a function. H, is defined in (24):

$$H(\beta_{12s}, \eta_{12s}|\hat{X}_{22s}) = \frac{\hat{X}_{22s}}{(r-\alpha_2)}\left[\frac{\eta_{12s}+\beta_{12s}-1}{\eta_{12s}}\right] + K_{12} = 0 \qquad (24)$$

where we use the notation $H(.|x)$ to denote that x is fixed.

Now, we proceeding exactly the same way for the second value-matching relationship, we rewrite it as

$$A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} - \frac{\hat{X}_{21s}}{r-\alpha_2} = -\frac{\hat{X}_{11s}}{r-\alpha_1} - K_{21} \qquad (25)$$

From the associated smooth pasting conditions, we can obtain the following result,

$$A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} = -\frac{\hat{X}_{11s}}{\beta_{24s}(r-\alpha_1)} = \frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} \qquad (26)$$

Once again from (26), we can outdraw that $A_{24s} = \frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}}}$ and $\hat{X}_{11s} = -\frac{\hat{X}_{21s}\beta_{24s}(r-\alpha_1)}{\eta_{24s}(r-\alpha_2)}$. Using these results in (25) leads us to a function H, specified in (27), that for a fixed value of $\hat{X}_{21s}$, gives us an expression of β_{24s} and η_{24s}. This expression together with the corresponding *characteristic root equation* will allow us to determine values for β_{24s} and η_{24s}.

$$H(\beta_{24s}, \eta_{24s}|\hat{X}_{21s}) = \frac{\hat{X}_{21s}}{(r-\alpha_2)}\left[\frac{1-\eta_{24s}-\beta_{24s}}{\eta_{24s}}\right] + K_{21} = 0 \qquad (27)$$

Summarizing, the single opportunity switch model is constituted by the following elements: two *characteristic root equations*, that were also present in the more general model and that did not suffer any alteration when we consider a single opportunity to switch, besides the presence of an index s, whose purpose was already referred above. They are $Q_1(\beta_{12s}, \eta_{12s}) = 0$ and $Q_2(\beta_{24s}, \eta_{24s}) = 0$; two relationships given by (22) and (26), outdrawn from the corresponding smooth pasting conditions; two H functions, one given by (24), that we will denote by H_{22}, and the other one given by (27), that we denote by H_{21}.

This is exactly the same composition of the model presented in [3]. The unknowns we wish to determine with this model are β_{12s}, η_{12s}, β_{24s}, η_{24s}, $\hat{X}_{12s}$ and $\hat{X}_{11s}$, once $\hat{X}_{21s}$ and $\hat{X}_{22s}$ are being considered as fixed.

As in [4], we can also observe that from the moment we eliminated the terms in the value-matching relationships, corresponding to either the switch back from feedstock 2 to 1 or from 1 to 2, and as soon as the values for $\hat{X}_{21s}$ and $\hat{X}_{22s}$ were fixed, the problem concerning the determination of $\beta_{12s}, \eta_{12s}, \hat{X}_{12s}$, and the problem of determining β_{24s}, η_{24s}, $\hat{X}_{11s}$, became completely separated. In this sense, and with all the tools derived so far, we can now present an algorithm that one can use to determine each set of (three) values, separately. Such an algorithm is going to be given by,

1. Start by fixing $\hat{X}_{22s}$ ($\hat{X}_{21s}$).

2. Solve the system

$$\begin{cases} Q_1(\beta_{12s}, \eta_{12s}) = 0 \quad (Q_2(\beta_{24s}, \eta_{24s}) = 0) \\ \\ H(\beta_{12s}, \eta_{12s}|\hat{X}_{22s}) = 0 \quad (H(\beta_{24s}, \eta_{24s}|\hat{X}_{21s}) = 0) \end{cases}$$

   in order to find β_{12s}, η_{12s} (β_{24s}, η_{24s}).

3. Solve $\frac{\hat{X}_{12s}}{\beta_{12s}(r-\alpha_1)} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)}$ $\left(-\frac{\hat{X}_{11s}}{\beta_{24s}(r-\alpha_1)} = \frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} \right)$ in order to find $\hat{X}_{12s}$ ($\hat{X}_{11s}$).

The algorithm described above finishes the presentation of this specification of the general energy-switching model. The assumption of a single switch opportunity was made because such a scenario is observed quite often, as it is stated in [4]. Also, later on when we consider different dynamics for the feedstock prices, and apply this model to a more concrete scenario, this single switch assumption will be required, and so all this reasoning will be quite useful then.

B. Arithmetic Brownian Motion Dynamic

In the approach that follows next, we will adopt a few changes in our initial assumptions. First of all, the dynamic used to describe the selling prices behaviour will be the arithmetic Brownian motion, instead of the geometric Brownian motion considered previously. Also, we will assume a particular scenario, where the single switch model developed before, and presented in [4] will be applied. This new scenario is the production of oil and natural gas. Based on the problem described in [7], we will adopt the position of a company responsible for the extraction of both oil and gas, and we shall adapt the next approach to this perspective.

Before we move on to the description of the model itself, we must familiarize ourselves with the process behind the extraction of these two feedstocks. For obvious reasons, we do not provide a full description of such production process, and we just explain briefly and in a quite informal way how it is done. According to [7], in order to extract oil, one should first inject natural gas, and once all the oil has been extracted we then proceed to the natural gas extraction. In this sense, there is only one switch available for us to make, and so the single opportunity switch model described in [4] will be the one that suits our interest best. Nevertheless, the general model should always be derived first, before developing the specific cases, which we present next.

At last, we must also evaluate the impact of these new assumptions in the saleable output considered. In these scenario, considering this kind of energy markets, we will no longer have two feedstocks for one outcoming product. We will now have two feedstocks that are being extracted, and that will be sold as two distinct energy resources. So, instead of only one output price given by D_0, we shall now define two extraction prices, E_{oil} and E_{gas}, denoting the extraction price of oil and gas, respectively.

We are now in conditions to start dealing with the model itself. So, let X_I, $I \in \{1, 2\}$, denote the selling price for feedstock I, where now we actually know what will each value of I refer to. We shall consider oil to be feedstock 1 and natural gas to be feedstock 2. Assuming that both of these prices follow an arithmetic Brownian motion, for $I \in \{1, 2\}$, we have the following,

$$dX_I = \alpha_I dt + \sigma_I dZ_I \tag{28}$$

Naturally, α_I is the risk-adjusted drift rate, σ_I is the the volatility rate, and dZ_I is the increment of a standard Wiener process. Additionally, we can describe the dependence between the two variables by its covariance term $\rho\sigma_1\sigma_2$ where $Cov[dX_1, dX_2] = \rho\sigma_1\sigma_2 dt$, with $|\rho| \leq 1$.

Once again, let $F_I = F_I(X_1, X_2)$ be the function that represents the firm's value when feedstock I is the incumbent feedstock. The next step, similarly to what has been done so far, is to take the discounted value of F_I, $e^{-rt}F_I(X_1, X_2)$, and apply the dynamic programming principle together with Ito's lemma and obtain the corresponding dynamic programming equation. However we need to define first the net cash flow for this case.

As we are extracting two different feedstocks, oil and gas, that will originate two different saleable outputs instead of just one, and so the net cash flow, when feedstock

I is the incumbent is going to be given by $X_I - E_I$, instead of $D_0 - X_I$. We are then in conditions to obtain the partial differential equation considered in the dynamic programming equation, that shall be given by,

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 F_I}{\partial X_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 F_I}{\partial X_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 F_I}{\partial X_1 \partial X_2} + \alpha_1\frac{\partial F_I}{\partial X_1} + \alpha_2\frac{\partial F_I}{\partial X_2} - rF_I + (X_I - E_I) = 0 \quad (29)$$

where r denotes the risk-free interest rate, as usual.

1. *Characteristic Root Equation*

Now, we shall propose a solution to this equation, meaning that a concrete function denoted by $G_I(X_1, X_2)$ that satisfies (29) will be suggested. Then, this function is going to be placed in the equation, and the required computations will be made, in order to obtain a condition that will later integrate our final model.

The function G_I that we propose is the following,

$$G_I(X_1, X_2) = A_I e^{X_1\beta_I + X_2\eta_I} + \frac{X_I}{r} + \frac{\alpha_I}{r^2} - \frac{E_I}{r} \quad (30)$$

We should once more notice that in this function the term $A_I e^{X_1\beta_I + X_2\eta_I}$ corresponds to the solution of the homogeneous part of (29) and it represents the switching option value. The term $\frac{X_I}{r} + \frac{\alpha_I}{r^2} - \frac{E_I}{r}$ denotes the incumbent asset value associated with the investment.

Computing its derivatives and replacing them in (29) we obtain an ellipse equation. Similarly to what we did before, we can define a function Q of β_I and η_I, defined in (31), and which is also presented in [3] and [4], where it is called *characteristic root equation*:

$$\begin{aligned} Q_I(\beta_I, \eta_I) &= \frac{1}{2}\sigma_1^2\beta_I^2 + \frac{1}{2}\sigma_2^2\eta_I^2 + \rho\sigma_1\sigma_2\beta_I\eta_I \\ &\quad + \alpha_1\beta_I + \alpha_2\eta_I - r \\ &= 0 \end{aligned} \quad (31)$$

The plot of $Q_I(\beta_I, \eta_I) = 0$ originates an ellipse that can be seen in figure 2, where the values for the fixed constants are the same ones considered in the first section, and thus in [4].

In this case the homogeneity-degree-1 condition, given by $\beta_I + \eta_I = 1$, does not appear in figure 2. However, the condition $\beta_I + \eta_I = 0$ does, as it will appear later on when we consider the single opportunity switch simplification.

The next step will be to take the term in G_I, that corresponds to the solution of the homogeneous part of (29), and consider the four usual quadrants to the parameters β_I and η_I figuring there. The second step will be to use economical arguments in order to choose which quadrant is relevant to integrate the structure of each G_I, for $I \in \{1, 2\}$.

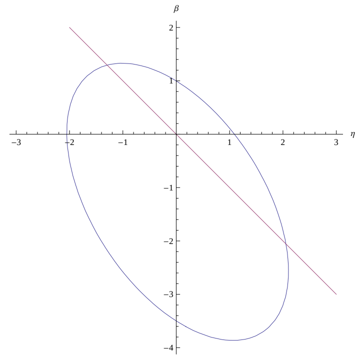

Figura 2: Graphic of the ellipse defined by $Q_I(\beta_I, \eta_I) = 0$ and the line $\beta_I + \eta_I = 0$

The economical arguments that will lead us to the choice of the right quadrant in each case are presented below. We should take into account that now our perspective has changed. We recall that in the previous section we assumed the perspective from the final consumer of the product, as happens in [4], which we no longer assume in the present problem. Once we are now assuming the position of a company responsible for the extraction of these two feedstocks, we are no longer thinking as the buyer of the final product, which in this case would be an energy resource.

a. Let us start with the scenario where the incumbent feedstock is oil, thus feedstock 1. Whenever the oil price tends to zero, we are interested in switching from 1 to 2, as it will no longer be profitable for us to keep extracting, and posteriorly selling oil. So, the price of the segment in the firm's value that corresponds to the option value, denoted by G_{H1}, will increase infinitely, meaning that $\lim_{X_1 \to 0} G_{H1} = \infty$. This doesn't actually impose any restrction regarding the value of β_1, but once we consider the condition presented in figure 2, we can observe that $\beta_1 \leq 0$. On the other hand, whenever we observe that the price at which we sell the natural gas is very high, we also want to make the switch from 1 to 2, once we will have more profit if we start extracting gas. So, G_{H1} will assume extremely high values. This means that $\lim_{X_2 \to \infty} G_{H1} = \infty$, which implies that η_1 has to be non-negative and thus, $\eta_1 \geq 0$. Then, we can conclude that the right quadrant to choose when feedstock 1 is the incumbent is the fourth. Thus, G_1 can be written as follows,

$$G_1(X_1, X_2) = A_{14} e^{X_1\beta_{14} + X_2\eta_{14}} + \frac{X_1}{r} + \frac{\alpha_1}{r^2} - \frac{E_1}{r} \quad (32)$$

Below this, the corresponding *characteristic root equation* for $I = 1$ will be given by $Q_1(\beta_{14}, \eta_{14}) = 0$.

b. Now we should consider the case where the incumbent feedstock is gas, meaning feedstock 2. The reasoning here is very similar to what was used in the previous case. So, whenever the gas price tends to 0, the option value must tend to infinity, $\lim_{X_2 \to 0} G_{H2} = \infty$.

This happens because as we are selling the feedstock at a really low price, we will not be making a lot of profit from its extraction and so the best decision for us is to switch. Again this doesn't necessarily imply any restrictions regarding the values that η_2 can take, however when the restriction $\beta_I + \eta_I = 0$ is considered, it follows that $\eta_2 \leq 0$. Similarly, whenever the oil price increases infinitely, we will be really interested in switching from 2 to 1 as it will be more profitable for us to start extracting oil. In this sense, the option value must tend to infinity and thus, $\lim_{X_1 \to \infty} G_{H2} = \infty$. This implies that $\beta_2 \geq 0$. Therefore the right quadrant is going to be the second one,

$$G_2(X_1, X_2) = A_{22} e^{X_1 \beta_{22} + X_2 \eta_{22}} + \frac{X_2}{r} + \frac{\alpha_2}{r^2} - \frac{E_2}{r} \quad (33)$$

With this result in mind, we can now obtain the *characteristic root equation* when $I = 2$, $Q_2(\beta_{22}, \eta_{22}) = 0$. Next we present the corresponding value-matching and smooth pasting conditions.

2. *Value-Matching Relationships and Smooth Pasting Conditions*

Before we move on and present the value-matching relationships and the smooth pasting conditions, there is the need to introduce some new variables and constants to the problem - $\hat{X}_{IJ}$, where I denotes the incumbent feedstock and J the substitute, and K_{IJ}, for $I \neq J$. As a similar introduction was already made for the geometric Brownian motion case, we will refer the reader to that section to familiarize himself with these new variables.

Let us now present the value-matching relationships, as it is stated in [4]:

1. for the case where feedstock 1 is the incumbent and feedstock 2 is the substitute, the difference between the value of the substitute $G_2(\hat{X}_{12}, \hat{X}_{22})$ and the value of the incumbent $G_1(\hat{X}_{12}, \hat{X}_{22})$, immediately before a switch, is equal to the switching cost K_{12}, which means that $G_1(\hat{X}_{12}, \hat{X}_{22}) = G_2(\hat{X}_{12}, \hat{X}_{22}) - K_{12}$

2. for the case where feedstock 2 is the incumbent and feedstock 1 is the substitute, the difference between the value of the substitute $G_1(\hat{X}_{11}, \hat{X}_{21})$ and the value of the incumbent $G_2(\hat{X}_{11}, \hat{X}_{21})$, immediately before a switch, equals the switching cost K_{21}, meaning that $G_2(\hat{X}_{11}, \hat{X}_{21}) = G_1(\hat{X}_{11}, \hat{X}_{21}) - K_{21}$

Once we are now considering a scenario where, according to [7], one has to start with oil extraction and only afterwards we can proceed to the gas extraction. In this sense, we don't think it is necessary to consider the case where we start with gas as the incumbent feedstock. Thus, from now on, we will only consider the first value-matching relationship presented.

Substituting then (32) in the first value-matching relationship we obtain the following,

$$A_{14} e^{\hat{X}_{12}\beta_{14} + \hat{X}_{22}\eta_{14}} + \frac{\hat{X}_{12}}{r} + \frac{\alpha_1}{r^2} - \frac{E_1}{r} = A_{22} e^{\hat{X}_{12}\beta_{22} + \hat{X}_{22}\eta_{22}} + \frac{\hat{X}_{22}}{r} + \frac{\alpha_2}{r^2} - \frac{E_2}{r} - K_{12} \quad (34)$$

Also, it should be noticed that once we consider this restriction, and consequently we ignore the second value-matching relationship, which eliminates the variables $\hat{X}_{11}$ and $\hat{X}_{21}$ from the problem.

As before, the smooth-pasting conditions are given by:

$$\left.\frac{\partial G_1}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12} \\ X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12} \\ X_2=\hat{X}_{22}}} \quad (35)$$

$$\left.\frac{\partial G_1}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12} \\ X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12} \\ X_2=\hat{X}_{22}}} \quad (36)$$

Isolating the terms $A_{14} e^{\hat{X}_{12}\beta_{14} + \hat{X}_{22}\eta_{14}}$ and $A_{22} e^{\hat{X}_{12}\beta_{22} + \hat{X}_{22}\eta_{22}}$ that will appear in (35) and (36) we obtain expressions (37) and (38).

$$A_{14} e^{\hat{X}_{12}\beta_{14} + \hat{X}_{22}\eta_{14}} = \frac{1}{\Delta}\left(-\frac{\eta_{22} + \beta_{22}}{r}\right) \quad (37)$$

$$A_{22} e^{\hat{X}_{12}\beta_{22} + \hat{X}_{22}\eta_{22}} = \frac{1}{\Delta}\left(-\frac{\eta_{14} + \beta_{14}}{r}\right) \quad (38)$$

where $\Delta = \beta_{14}\eta_{22} - \beta_{22}\eta_{14}$.

Taking these expressions and using them in the value-matching relationship, in order to simplify it, will lead us to the following result:

$$\hat{X}_{12} - \hat{X}_{22} + \frac{\beta_{14} + \eta_{14} - \beta_{22} - \eta_{22}}{\Delta} = E_1 - E_2 - \frac{1}{r}(\alpha_1 - \alpha_2) - rK_{12} \quad (39)$$

This is, in fact, a simplified version of the value-matching relationship, as the dependence on A_{14} and A_{22} was eliminated. Such a result will be quite valuable in the next section. Also, and similarly to what happened in the previous section, we can observe that G_I verified the boundary conditions we imposed and thus it can be considered from now on as the true firm's value, initially represented by F_I.

Following the same reasoning used in the previous chapter we can conclude that the final model will be constituted by five equations: two *characteristic root equations*: $Q_1(\beta_{14}, \eta_{14}) = 0$ and $Q_2(\beta_{22}, \eta_{22}) = 0$, one value-matching relationship given by (34), and two smooth-pasting conditions given by (35) and (36); where six unknown parameters shall be determined: $A_{14}, A_{22}, \beta_{14}, \eta_{14}, \beta_{22}, \eta_{22}$, and the value of the two remaining unknown variables $\hat{X}_{12}$ and $\hat{X}_{22}$ should be found.

Once again, we have more unknown variables and parameters than equations, which constitutes an indetermination. Also, the model we have developed so far doesn't consider the restriction of a single switch opportunity, and that is a very important assumption to make when dealing with oil and gas extraction. With this in mind, we will consider now the simplification of this general model, where the single opportunity switch is considered.

3. Single Switch Opportunity Model

Let us now proceed to the implementation of the single opportunity switch model for the arithmetic Brownian motion. As we mentioned before, according to [7], when we are extracting oil and natural gas, we start with oil extraction and then we switch to natural gas. In this process, there is only one possible change and it has to be from oil to gas. In this sense, we only have one opportunity to change, and the decision to switch is irreversible. So, we must realize that this scenario will induce quite some changes in all the components of our model, from the *characteristic root equations* to the value-matching and smooth pasting conditions.

We shall then take the first value-matching relationship, given by (34) and start implementing this single opportunity feature in it, similarly to what we did in the last section. Again we will use an s in the index to point out that we are now developing the single opportunity switch model, as it's done in [4]. Thus, taking (34), we need to select the term that is allowing to switch back from gas to oil, once the first switch from oil to gas is made. This term is $A_{22s}e^{\hat{X}_{12s}\beta_{22s}+\hat{X}_{22s}\eta_{22s}}$, and as we identified it, we must now eliminate it. We will obtain the following "rewritten" value-matching relationship,

$$A_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}} + \frac{\hat{X}_{12s}}{r} + \frac{\alpha_1}{r^2} - \frac{E_1}{r} = \frac{\hat{X}_{22s}}{r} + \frac{\alpha_2}{r^2} - \frac{E_2}{r} - K_{12} \qquad (40)$$

Once we removed the term $A_{22s}e^{\hat{X}_{12s}\beta_{22s}+\hat{X}_{22s}\eta_{22s}}$ from the first value-matching relationship we also eliminated from the problem the following unknown parameters: A_{22}, β_{22} and η_{22}. In this sense, it would no longer be reasonable to consider the second *characteristic root equation* $Q_2(\beta_{22s}, \eta_{22s}) = 0$, and so this equation shall be eliminated as well.

Taking now (40) and deriving the smooth pasting conditions associated with it, we will obtain

$$A_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}} = -\frac{1}{\beta_{14s}r} \qquad (41)$$

$$A_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}} = \frac{1}{\eta_{14s}r} \qquad (42)$$

This implies that $-\frac{1}{\beta_{14s}r} = \frac{1}{\eta_{14s}r} \Leftrightarrow -\beta_{14s} = \eta_{14s} \Leftrightarrow \beta_{14s} + \eta_{14s} = 0$.

Remark that this is precisely the condition appearing in figure 2. If we observe now this plot once again, taking into account that β_{14s} and η_{14s} are in the fourth quadrant and that $\beta_{14s} + \eta_{14s} = 0$, we can easily determine the values for this two parameters. Naturally, we will replace $\beta_{14s} = -\eta_{14s}$ in the *characteristic root equation*, and we will obtain the following solution to β_{14s}.

$$\beta_{14s} = \frac{-(\alpha_1-\alpha_2)-\sqrt{(\alpha_1-\alpha_2)^2+(2r\sigma_1^2+2r\sigma_2^2-4r\rho\sigma_1\sigma_2)}}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} \qquad (43)$$

Now, the next step would be to proceed to the value-matching relationship (40) and determine the unknown variables. However, and similarly to what happened before, we now have two unknown variables and only one value-matching relationship to use. Once more, we will fix a value for one of the variables, which is going to be again $\hat{X}_{22s}$.

As (40) also depends on A_{14s} and we cannot determine its value yet with the information we've got so far. We have to betake equation (39) as it doesn't depend on this parameters. As $\beta_{14s} + \eta_{14s} = 0$ then we set that (39) becomes

$$\hat{X}_{12s} - \hat{X}_{22s} = E_1 - E_2 - \frac{1}{r}(\alpha_1 - \alpha_2) - rK_{12} \qquad (44)$$

So, fixing the value for $\hat{X}_{22s}$, we may derive $\hat{X}_{12s}$ using (44).

Summarizing the whole procedure we can now establish an algorithm to solve the single opportunity switch model. Start by obtaining the value for β_{14s} using (43), then obtain the value for η_{14s} using the equality $\beta_{14s} = -\eta_{14s}$, after this fix a value for $\hat{X}_{22s}$, and finally obtain the value for $\hat{X}_{12s}$ using (44).

Additionally, we could obtain the value for A_{14s} using one of the smooth pasting conditions, (41) or (42), and after obtain the firm's value using (32).

III. CONCLUSIONS

In this section we would like to make a few final considerations to what have been stated before. We shall present the main conclusions one can outdraw from the work presented above, as well as a selection of the aspects that should be improved in a future approach.

Concerning the switching model presented, we observed that, independently of the dynamic considered, treating the switching model without any extra considerations leads us to a classic case of indeterminacy. However, again independently of the dynamic assumed, when we consider the restriction of a single opportunity to perform the switch, originally developed in [4] for the geometric Brownian motion dynamic and referred to as the single opportunity switch model, the problem could then be solved, in both cases (meaning for the two dynamics considered). Furthermore, in the case where the dynamic considered is the arithmetic Brownian motion, and in addition to that, a position of a company responsible for the of oil and natural gas is adopted, such considerations will simplify even more the problem. So, in this second case, the single switch opportunity model is even more simple to solve, compared to the first case.

Additionally, we consider that a practical application of this models can be very interesting to observe. Thus, in this sense, we hope to apply actual real data to this problem and analyse the behaviour of each of the presented models.

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[1] Dixit A. K. and Pindyck R. S., "Investment Under Uncertainty," Princeton University Press (1994).

[2] Peskir G. and Shiryaev A., "Optimal Stopping and Free-Boundary Problems," Springer (2006).

[3] Adkins R. and Paxson D., "Renewing Assets with Uncertain Revenues and Operating Costs," Journal of Financial and Quantitative Analysis (2011) **46(3)** 785-813.

[4] Adkins R. and Paxson D., "Reciprocal energy-switching options," The Journal of Energy Markets (2011) **4(1)** 91-120.

[5] McDonald R. L. and Siegel D. R., "The Value of Waiting to Invest," Quarterly Journal of Economics (1986) **101** 707-728.

[6] Ross K., "Stochastic Control in Continuous Time," (2008) *http://www.swarthmore.edu/NatSci/kross1/Stat220notes.pdf.*

[7] Hem Ø. D. and Svendsen A., "Real Option Analysis of Strategic Decisions in Offshore Petroleum Production," *Master thesis* Norwegian University of Science and Technology (2010).

[8] Björk T., "Arbitrage Theory in Continuous Time," Oxford University Press (2004).

[9] Trigeorgis L., "Real Options: Managerial Flexibility and Strategy in Resource Allocation," The MIT Press (1998).

[10] Couto G. and Pimentel P., "Opções Reais: Decisão de Investimento sob Incerteza," Coingra, Lta (2009).