Investment Under Two Sources Of Uncertainty
- Strategic Decisions in Offshore Petroleum Production -

Inês Bento Gonçalves Pratas Leitão

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Orientador: Prof. Doutora Cláudia Rita Ribeiro Coelho Nunes Philippart

Júri

Presidente: Prof. Doutor António Manuel Pacheco Pires
Co-orientador: Prof. Doutora Verena Hagspiel
Vogais: Prof. Doutor Stein-Erik Fleten

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Resumo

Ao longo deste trabalho, vamos focar a nossa atenção num problema de investimento considerando duas fontes de incerteza, que será tratado como um problema de paragem óptima. Primeiro, é considerada uma redução para um modelo com um factor. Numa segunda abordagem, será possível lidar com duas fontes de incerteza simultaneamente, usando uma abordagem apelidada de *quase-analítica*.

Ainda tendo dois factores de incerteza, iremos considerar um modelo de troca, assumindo três possíveis dinâmicas para o comportamento dos preços: movimento geométrico *browniano*, movimento aritmético *browniano*, e um processo de reversão à média.

Relativamente às duas primeiras dinâmicas consideradas, é apresentado adicionalmente um modelo de troca onde apenas é permitida uma troca. O modelo de troca geral, usando estas duas dinâmicas apresenta uma indeterminação e por isso não pode ser resolvido analiticamente. Considerando uma única oportunidade de troca permite-nos tanto obter uma solução analítica como a aproximação de um cenário mais realista. Um cenário considerado em particular é a posição de uma companhia responsável pela extracção de petróleo e gás natural, embora apenas quando consideramos a dinâmica do movimento aritmético *browniano*.

Quando consideramos a terceira dinâmica, e usamos o mesmo exemplo relativo à extracção de petróleo e gás natural, apenas conseguimos encontrar parte da solução. Consideramos que a restante parte da solução apenas pode ser encontrada numericamente, e assim sendo esse tratamento é ainda uma opção em aberto para investigação futura.

Palavras Chave: problema de paragem óptima, incerteza, dois factores, abordagem *quase analítica*, opções de troca, oportunidade única.

Abstract

Along this work, we will focus our attention in an investment problem considering two factors of uncertainty, that will be treated as an optimal stopping problem. First, a reduction to a one factor model is considered. In a second approach, it will be possible to deal with the two sources of uncertainty simultaneously, using a so called *quasi analytical* approach.

Still having two factors of uncertainty, we will consider a switching model, assuming three possible dynamics for the prices behaviour: geometric Brownian motion, arithmetic Brownian motion and a mean reverting process.

For the two first dynamics considered, an additional switching model where there is only one switch available is presented. The general switching model using the these two dynamics presents a classic indeterminacy and thus it cannot be solved analytically. Considering a single opportunity switch allows both an achievement of an analytical solution and a proximity to a more realistic scenario. One particular scenario considered is the position of a company responsible for oil and natural gas extraction, although only when we consider the arithmetic Brownian motion dynamic.

When we consider the third dynamic, and we use the same example of oil and gas extraction, we can only find part of the solution. We believe the remaining part of the solution can only be reached numerically, and thus such a treatment is still an open option for future research.

Key words: optimal stopping problems, uncertainty, two factors, *quasi analytical* approach, switching options, single opportunity.

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Chapter 1

Introduction

This thesis has its main focus on the thematic of investment under uncertainty. We consider that due to the increasing level of uncertainty that companies have to deal with in the economical environment as well as financial markets, the study of this sort of investments couldn't be more relevant. The main goal of this analysis is, of course, to be able to help firms make their investment decisions in the best possible way, once such decisions are, in most cases, irreversible. Additionally, the ability to treat flexibility in the used techniques, is really relevant to the study of this problem. Real option theory compared to the traditional capital budgeting techniques allows to account for the value of flexibility.

The problem of optimal investment time will be modelled as an optimal stopping problem, whose formulation can be consulted in Ross (2008). The main goal will be to determine threshold values for the dynamics process, at which a certain course of action should be performed, following the same methodology used in Adkins and Paxson (2011b), Adkins and Paxson (2011a) and Dixit and Pindyck (1994). Additionally, and once more as in Adkins and Paxson (2011b), Adkins and Paxson (2011a) and Dixit and Pindyck (1994), we will also be interested in obtaining the firm's value associated with the referred investment. The type of investments we will dedicate our study to are those which consider two factors of uncertainty. At an initial state, this problematic will actually be solved by reducing the two factor uncertainty to one factor only, following a similar procedure to the one presented in Dixit and Pindyck (1994), which as one can imagine, reduces significantly the complexity of the problem. After this initial treatment, and because the main goal of dealing with two sources of uncertainty is, in fact, to be able to treat them both simultaneously, we shall present a methodology that allows us to achieve our main objectives also for the case where one cannot reduce the number of factors. This procedure is also presented in Adkins and Paxson (2011b), and in Adkins and Paxson (2011a), although in this last one using an approach oriented to the particular case of switching options.

An important consideration that is made in all the approaches mentioned above, and that wasn't yet referred, is the irreversibility of the considered investment. As it is mentioned in Adkins and Paxson (2011b) and Dixit and Pindyck (1994), once an investment is made, there is no turning back, and the capital that was paid can no longer be reimbursed. Trigeorgis (1998) and Couto and Pimentel (2009) state that irreversibility, together with the uncertainty feature and the flexibility, that are all described above and identify completely the type of investments we are analysing, define a real options problem. In this sense, along our study, the type of options we will be dealing with are the real options.

According to Trigeorgis (1998), real options appeared to fix the discrepancy established between the traditional financial theory, where the net present cash flows are predicted, and corporate reality, where exists the need to evaluate the option value with quite some flexibility, due to the markets unpredictability. In this sense, a real option will include both of this characteristics, solving that discrepancy problem. This feature is present in Adkins and Paxson (2011b) and Adkins and Paxson (2011a), and we will make a lot of references to it, once we are defining the firm's value, precisely as an aggregation of the option value and the incumbent asset value.

Inside the range of real options, Adkins and Paxson (2011a) considers a quite specific type, the switching options. It also derives a particular restricted model using switching options, that later on we will be interested in studying. Such a model is the single opportunity switch model, as it is referred to in Adkins and Paxson (2011a). Trigeorgis (1998) states that these type of options are used to describe investments involving energy resources, such as fuel oil, gas, electricity, among others. The reason presented for this to happen is the flexibility required by these sort of markets, to change between the alternative energy resources, that is provided by the switching options.

This explains the reason for our choice of applying this models to a scenario involving energy resources extraction, that is presented in Hem and Svendsen (2010). Adopting the perspective taken in Hem and Svendsen (2010), we decided to assume the position of a company responsible for the extraction of both oil and natural gas, who would be implementing a switching model similar to the one presented in Adkins and Paxson (2011a) and including the respective restriction already mentioned above. Additionally, Hem and Svendsen (2010) approaches the prices dynamics considering short and long term evaluations. For the long term prices, it uses the arithmetic Brownian motion and for the short term prices, the Ornstein Uhlenbeck process. In this sense (as also in in Adkins and Paxson (2011b) and Adkins and Paxson (2011a),) the only dynamic used for the prices is the geometric Brownian motion. Our attempt in this work is to take two new dynamics and try to derive the switching model considered in Adkins and Paxson (2011a), as well as its restricted version, now assuming these new different dynamics for the prices. However, we must make clear that in our approach we did not consider short term or long term prices, we just adopted these two dynamics for a description of the general behaviour of the prices. The arithmetic Brownian motion dynamic was used just like it is presented in Hem and Svendsen (2010), whereas the Ornstein Uhlenbeck process was adapted to a more "geometric" version, that is presented in Dixit and Pindyck (1994).

To conclude, we would like to present the general structure of this document. Initially, it is presented an abstract of the entire work, that intends to introduce the problem to be treated. In the abstract we will refer to the mathematical theory and technical tools that will be used along the document, present the main models and approaches, and state the most relevant conclusions. Then, in chapter 1 we present the problem and mention the main thematics to be explored along the document. In chapter 2, it is presented what we called the *introductory theoretical notions*. This chapter intends to approach more deeply all the mathematical background that will be used in the last two chapters. With this chapter we attempt to give an introduction to the most important concepts and techniques that will be used throughout the rest of the work. Chapter 3 is probably the most relevant one, as it presents the fundamental models developed in this thesis. It starts with a plain approach to the switching model, and its restricted single opportunity switch model, using the geometric Brownian motion as its chosen dynamic, and following the reasoning used in Adkins and Paxson (2011a). The second part of this chapter is an adaptation of the switching model and its restriction, inspired in the scenario considered in Hem and Svendsen (2010), and also using one of its chosen dynamics for the prices, the arithmetic Brownian motion. In chapter 4, we repeat what was done in the second section of chapter 3, but now considering the mean reverting process used in Dixit and Pindyck (1994), to describe the prices dynamics. The considered scenario is still the one presented in Hem and Svendsen (2010). This approach is not complete in the sense of presenting an analytical solution to the full problem but constitutes a very interesting first approach to solve the problem in question. Finally, in chapter 5, we present the main conclusions outdrawn from the whole thesis, and also all the aspects we identify as points for improvement in the future.

Chapter 2

Introductory Theoretical Notions

This chapter will be dedicated to the presentation of several introductory sections that are considered to be important background to the following chapters.

We will start with a section on stochastic control and optimal stopping problems, following the approach presented in Ross (2008) and using some results of Björk (2004) and Peskir and Shiryaev (2006). We begin by presenting the formulation of these two types of problems, and then proceeding with the deduction of the dynamic programming principle and equation for both cases. A verification technique is also presented for the stochastic control problems exclusively, even though the procedure would be similar in the optimal stopping case, and it can be found in detail in Ross (2008).

The second section will concern investment under uncertainty. First, we will present a model describing investment under only one source of uncertainty, using the same reasoning as Dixit and Pindyck (1994). We will refer to that as the one-factor model. Then we will proceed to a type of investment under two sources uncertain sources, which we will refer to as the two-factor model. Our approach will be based on Dixit and Pindyck (1994) and Adkins and Paxson (2011b). In this two-factor model, we will present two different approaches, one where we make a reduction from two factors to one, as in Dixit and Pindyck (1994), and another where we consider a *quasi analytical* approach, and we deal with the two factors simultaneously, as in Adkins and Paxson (2011b).

In both sections we do not intend to make a full and comprehensive presentation, but point some key references, where definitions, main results and proofs can be found.

2.1 Stochastic Control and Optimal Stopping Problems

The main purpose of this first part of the chapter is to introduce the dynamic programming principle and, consequently, the dynamic programming equation associated with it, and apply them to two types of problems, similarly to what is done in Ross (2008) and Björk (2004). The first of them to be presented will be the stochastic control problem, for which we present the complete methodology that includes the formulation of the problem, the dynamic programming principle and equation, and a verification technique.

The second type of problem is the optimal stopping problem, where we make a presentation quite similar to the first one, with the exception of the verification technique, that is not mentioned in this case, as it is similar to the control problem case. As we will see, an optimal stopping problem can, in fact, be seen as a particular case of a stochastic control problem, as it is described in Ross (2008), Björk (2004) or Peskir and Shiryaev (2006).

2.1.1 Stochastic Control Problems

In this section, following a similar course of action to Ross (2008), we will start by presenting the general definition of a stochastic control problem, along with the most important components that figure in its formulation. After that we will deduce the dynamic programming principle and present the correspondent dynamic programming equation that is derived from it. At last, we will present a verification technique, used to assure that a given function, can be in fact considered a viable solution to the problem.

General Formulation of a Stochastic Control Problem

Let us start with the generic formulation of a stochastic control problem. Generally, a stochastic control problem is constituted by several key elements, as it is described in Ross (2008). They are the following:

1. **Time horizon -** Hereby denoted by T, where T can be finite (fixed or random; in the last case it will be assumed to be a stopping time: we will explain this case later on, in this chapter) or infinite.

2. **Controlled state process -** $\{X_t,\, t \leq T\}$, with $X_t \in \mathbb{S}$
It is the process that describes the scenario of interest, and it normally is a solution of the following SDE

$$dX_t = b(t, X_t, U_t)dt + \sigma(t, X_t, U_t)dZ_t, \quad X(0) = x \tag{2.1}$$

where $\{X_t,\, t \leq T\}$ is the controlled process taking values in the state space $\mathbb{S}$, that is typically a subset of $\mathbb{R}^n$, dZ_t is the increment of a standard Brownian motion, x denotes the initial state and $\{U_t,\, t \leq T\}$ is the control process that will be mentioned next.

3. **Control process -** $\{U_t,\, t \leq T\}$, with $U_t \in \mathbb{U}$
It is the process chosen by the investor to influence the behaviour of the controlled process, and can take values in the control space $\mathbb{U}$, normally a subset of $\mathbb{R}^p$ [1].

4. **Admissible controls -** This element intends to restrict the processes that can be defined as controlled or control processes to those who verify certain conditions of admissibility. These conditions can be among others, technical or physical, and their implementation is generally checked in the verification procedure, that shall be mentioned later on.

5. **Cost/Reward function -** $J(x, U)$
This function denotes the cost or reward (depending on the perspective) associated with the investment. It expresses the result loss or gain from the investment, and can depend on the initial state and on the control process. Thus, $J(x, U)$ will denote the cost of taking control U when the initial state of X is x.

[1] Having typically $\mathbb{S} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^p$, we must note that n and p don't necessarily need to have any sort of relation between them, in the sense that we can have either $n = p$ or $n \neq p$ or $n < p$ or $n > p$, etc.

6. **Value function** - $V(x)$
   The value function is the optimized cost/reward function. If we are dealing with a cost function, a minimization is done, if we are dealing with a reward function, a maximization will be performed instead. We emphasize the dependency of the value function on only the the initial state x, as future states, rewards and controls will be random.

The two main goals associated with this problem are to find an optimal control process that optimizes the cost or reward function and determine its correspondent value function. In fact, as we will see later on, in order to define $V(.)$ we will optimize an expected value of such future states, rewards and controls.

Dynamic Programming Principle

The dynamic programming principle is a tool that will help us solve a stochastic control problem. We will now present the problem and the proposed methodology to solve it, similarly to Ross (2008) and Björk (2004). For a matter of simplicity, the dimensions considered from now on, for both the state space and control space, are going to be unidimensional.

Let us have the following probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with underlying filtration $\{\mathcal{F}_t\}$ and $\{Z_t\}$ a d-dimensional $\{F_t\}$ - Wiener process. According to the assumptions made in Ross (2008), for the next part of this section we will consider an infinite discount time horizon and the controlled state process is going to be the following,

$$dX_t = b(X_t, U_t)dt + \sigma(X_t, U_t)dZ_t, \quad X_0 = x \in \mathbb{R} \tag{2.2}$$

with state space defined as $\mathbb{S} = \mathbb{R}$ and including all processes that verify (2.2), where b and σ are fixed given functions such that $b : \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}^d$, and dZ_t will denotes the increment of a Wiener process.

Next, we present the definition of a Markov control process, that can also be found in Ross (2008), which will be the class of controls considered in this chapter.

Definition 2.1 (Markov control process) A Markov control process is a stochastic control process $\{U_t, \, t \leq T\}$ that verifies $U_t = u(X_t)$ for all $t > 0$, where u is denoted a Markov control function that satisfies $u : \mathbb{R} \to \mathbb{U}$

Thus, a Markov control process is a control process that can be seen as a function of the state process. Considering only a Markov control simplifies greatly the mathematical approach, as the Markov property ensures that we just need the actual state of the process (and do not need past information).

In order to emphasize the dependency of the state process on the control function u and on the initial state x, we will denote such state process by X_x^u, solution of (2.2). Furthermore, for any $f \in C^2(\mathbb{S})$ (from now on, in order to ease this explanation, we assume that $\mathbb{S} = \mathbb{R}$) we defne the infinitesimal generator of the process X_x^u by $A \equiv A^u$, such that:

$$Af(x) = b(x, u(x))f'(x) + \frac{1}{2}\sigma^2(x, u(x))f''(x), \quad f \in C^2(\mathbb{R}) \tag{2.3}$$

According to Ross (2008), its law is uniquely determined by the initial state x and the control function u, always assuming b and σ fixed. We remark that in general we need some measurability and adaptability conditions on b and σ, but here we do not mention them. Anyway, in the cases that we will address later on, these conditions are trivially verified.

Next we present the cost function J and the corresponding value function V. We present only the infinite time horizon case, but all the other cases can be derived in a similar way, with the obvious changes. So let J be the cost function, defined as follows:

$$J(x,u) = \mathbb{E}\Big[\int_0^{\infty} e^{-\gamma s} g(X_s, u(X_s))ds\Big] \tag{2.4}$$

where $g(.)$ is a given function (usually called a "utility function", that may depend on the state and the control), and γ is a discount factor. Therefore, $\int_0^{\infty} e^{-\gamma s} g(X_s, u(X_s))ds$ is the overall utility (in the infinite horizon), discounted back to the actual time 0. As future states and controls are unknown, this integral is in fact a random variable and thus it will be nearly impossible to minimize. So we compute instead its expected value, as in Ross (2008), obtaining then the final expression for J. One should notice that this final form of $J(x, U)$ allows us to, at any time, fix x and treat the cost function as a stochastic process U.

Now by definition, the value function of our stochastic control problem is going to be

$$V(x) = \inf_u J(x,u) \tag{2.5}$$

where the infimum is computed over all Markov control functions.

As we mentioned previously, for a question of simplicity, we will ignore all admissibility conditions, and we shall consider that there exists always a function u^* such that $V(x) = J(x, u^*)$, being X^* the state process associated with the optimal control function u^*. Thus, for all $x \in \mathbb{R}$ and $t \in (0, \infty)$ we will have,

$$\begin{aligned} V(x) = J(x,u^*) &= \mathbb{E}\Big[\int_0^{\infty} e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\Big] \\ &= \mathbb{E}\Big[\int_0^{t} e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\Big] + \mathbb{E}\Big[\int_t^{\infty} e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\Big] \end{aligned}$$

Working now only on the term $\mathbb{E}\Big[\int_t^{\infty} e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\Big]$ it follows that,

$$\begin{aligned} \mathbb{E}\Big[\int_t^{\infty} e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\Big] &= e^{-\gamma t}\mathbb{E}\Big[\int_t^{\infty} e^{-\gamma(s-t)} g(X_s^*, u^*(X_s^*))ds\Big] \\ &=_{v=s-t} e^{-\gamma t}\mathbb{E}\Big[\int_0^{\infty} e^{-\gamma v} g(X_{t+v}^*, u^*(X_{t+v}^*))dv\Big] \\ &=_{X_{t+v}^*=\tilde{X}_v^*} e^{-\gamma t}\mathbb{E}\Big[\int_0^{\infty} e^{-\gamma v} g(\tilde{X}_v^*, u^*(\tilde{X}_v^*))dv\Big] \end{aligned}$$

We must note the change of variable $v = s - t$ that was made, as well as the introduction of the stochastic process $\{\tilde{X}_v^*, \quad v > 0\}$. The process $\tilde{X}_v^*$ is going to be a time-homogeneous Markov process, due to the fact that X^* is one too, and so their law is going to be uniquely determined by b, σ (which are fixed), the control function u^*, and the initial state at time 0. Note that $\tilde{X}_v^* \equiv \tilde{X}_t^*$, for t defined previously. Considering $\tilde{X}_0^* = x$ we shall have,

$$\mathbb{E}\left[\int_0^\infty e^{-\gamma v} g(\tilde{X}_v^*, u^*(\tilde{X}_v^*))dv \middle| \tilde{X}_0^* = x\right] = \mathbb{E}\left[\int_0^\infty e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\right] = J(x, u^*) = V(x)$$

As $\tilde{X}_0^* = X_t^*$ it follows that,

$$\begin{aligned}
\mathbb{E}\left[\int_t^\infty e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\right] &= e^{-\gamma t}\mathbb{E}\left[\int_0^\infty e^{-\gamma v} g(\tilde{X}_v^*, u^*(\tilde{X}_v^*))dv\right] \\
&= e^{-\gamma t}\mathbb{E}\left[\mathbb{E}\left[\int_0^\infty e^{-\gamma v} g(\tilde{X}_v^*, u^*(\tilde{X}_v^*))dv \middle| \tilde{X}_0^*\right]\right] \\
&= e^{-\gamma t}\mathbb{E}\left[J(\tilde{X}_0^*, u^*)\right] \\
&= e^{-\gamma t}\mathbb{E}\left[J(X_t^*, u^*)\right] \\
&= e^{-\gamma t}\mathbb{E}\left[V(X_t^*)\right]
\end{aligned}$$

which implies that,

$$V(x) = \mathbb{E}\left[\int_0^t e^{-\gamma s} g(X_s^*, u^*(X_s^*))ds\right] + e^{-\gamma t}\mathbb{E}\left[V(X_t^*)\right] \tag{2.6}$$

This definition of $V(x)$ describes the case where one considers an optimal strategy from the beginning of our time horizon. Below, we will consider a slightly different scenario.

As in Ross (2008) and Björk (2004), let us now consider a new Markov control function $\hat{u}$ that verifies,

$$\hat{u} = \begin{cases} u & \text{for time } (0, t] \\ u^* & \text{for time } (t, \infty) \end{cases}$$

where u is an arbitrary Markov control function with associated state process X. Combining the reasoning used before with the definition of $V(x)$, we will have the following,

$$\begin{aligned}
V(x) \leq J(x, \hat{u}) &= \mathbb{E}\left[\int_0^t e^{-\gamma s} g(X_s, u(X_s))ds\right] + \mathbb{E}\left[\int_t^\infty e^{-\gamma s} g(X_s, u^*(X_s))ds\right] \\
&= \mathbb{E}\left[\int_0^t e^{-\gamma s} g(X_s, u(X_s))ds\right] + e^{-\gamma t}\mathbb{E}\left[V(X_t^*)\right] \qquad (2.7)
\end{aligned}$$

We remark that X_t^* denotes the value of the state process associated with the optimal Markov control u^*.

This definition of the value function denotes the scenario where we consider an arbitrary strategy until time t, and an optimal one afterwards. Now, what the dynamic programming principle states is that the result is exactly the same if we consider an optimal strategy only after time t, or from the very beginning of our time horizon. It appears combining (2.6) and (2.7) and the result presented below,

$$V(x) = \inf_u \mathbb{E}\left[\int_0^t e^{-\gamma s} g(X_s^u, u(X_s^u))ds + e^{-\gamma t} V(X_t^u)\right] \tag{2.8}$$

where the infimum is over all Markov control functions u, and X_s^u denotes the value of the state process associated with an arbitrary control process u.

Then, equation (2.8) represents the dynamic programming principle, stating that the optimal strategy determined for the whole interval $(0, \infty)$ is going to be the same as the optimal strategy calculated just for the interval (t, ∞), with an arbitrary procedure in $(0, t]$.

Dynamic Programming Equation

Now, one can ask how does the dynamic programming principle helps the resolution of our stochastic control problem. This question will be answered with the deduction of the dynamic programming equation, that is done below.

The main idea, also adopted in Ross (2008), is going to be taking the dynamic programming principle (2.8), apply Ito's lemma, presented below as in Björk (2004), integrate the result from 0 to t, consider the expected value of what we obtain and make $t \to 0$.

Lemma 2.1 (Ito's lemma) Consider a diffusion process as described in equation (2.2), and a function $f : \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}$ such that $f(.,.) \in C^{1,2}$. Then, the differential of f is given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + Af\right)dt + \sigma \frac{\partial f}{\partial x} dZ_t \tag{2.9}$$

where in (2.9) we used a shortened notation for the derivatives. For example, $\frac{\partial f}{\partial x}$ is, in fact, $\left.\frac{\partial f(t,x)}{\partial x}\right|_{x=X_t}$.

Following the trends of several books and references, we will use the notation $\left.\frac{\partial f(t,x)}{\partial x}\right|_{x=X_t}$ and $\frac{\partial f}{\partial X_t}$ to denote the same thing.

Before applying Ito's lemma to our problem let us also introduce the definition of martingale, as it is presented in Björk (2004).

Definition 2.2 (Martingale) A stochastic process X is called a martingale if, being it adapted to the respective filtration $\{\mathcal{F}_t\}_{t\geq 0}$, the following conditions hold,

- For all t,

$$\mathbb{E}[|X_t|] < \infty$$

- For all s and t, where $s \leq t$,

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s$$

Now if we have $f(t,x) = e^{-\gamma t}V(x)$, applying Ito's lemma and integrating from 0 to t we obtain what is presented below. We must remark that in order to apply Ito's lemma, we should guarantee that the function in question is $C^{1,2}$. This procedure will be done later on in the next chapters, when we establish the value-matching and smooth pasting conditions, which will guarantee the continuity and smoothness, respectively, of the function in question.

$$\begin{aligned} e^{-\gamma t}V(X_t) - V(x) &= \int_0^t \Big(\frac{\partial}{\partial s}e^{-\gamma s}V(X_s) + e^{-\gamma s}AV(X_s)\Big)ds + \sigma\int_0^t \frac{\partial}{\partial x}\big(e^{-\gamma s}V(X_s)\big)dZ_s \\ &= \int_0^t e^{-\gamma s}[-\gamma V(X_s) + AV(X_s)]ds + \sigma\int_0^t \frac{\partial}{\partial x}\big(e^{-\gamma s}V(X_s)\big)dZ_s \end{aligned}$$

considering naturally that $\int_0^t d(e^{-\gamma s}V(X_s))ds = e^{-\gamma t}V(X_t) - \underbrace{e^{-\gamma \times 0}}_{=1}V(\underbrace{X_0}_{=x})$.

Taking expected values on both sides it follows that,

$$\mathbb{E}[e^{-\gamma t}V(X_t)] = V(x) + \mathbb{E}\Bigg[\int_0^t e^{-\gamma s}[-\gamma V(X_s) + AV(X_s)]ds\Bigg] \tag{2.10}$$

where A is defined, previously, in equation (2.3) and the stochastic term disappears because it is a martingale (in fact, it is an Ito integral , which is known to be a martingale, with expected value equal to zero).

From (2.7) we have that $\mathbb{E}[e^{-\gamma t}V(X_t)] \geq V(x) - \mathbb{E}\Bigg[\int_0^t e^{-\gamma s}g(X_s,u(X_s))ds\Bigg]$ and thus we will obtain,

$$\begin{aligned} &V(x) - \mathbb{E}\Bigg[\int_0^t e^{-\gamma s}g(X_s,u(X_s))ds\Bigg] \leq V(x) + \mathbb{E}\Bigg[\int_0^t e^{-\gamma s}[-\gamma V(X_s) + AV(X_s)]ds\Bigg] \Leftrightarrow \\ &0 \leq \mathbb{E}\Bigg[\int_0^t e^{-\gamma s}[-\gamma V(X_s) + AV(X_s) + g(X_s,u(X_s))]ds\Bigg] \end{aligned}$$

which for an optimal Markov control u^*, and correspondent infinitesimal generator of the process A^*, under the assumption that such a control always exists, leads us to,

$$0 = \mathbb{E}\Bigg[\int_0^t e^{-\gamma s}[-\gamma V(X_s^*) + A^*V(X_s^*) + g(X_s^*, u^*(X_s^*))]ds\Bigg]$$

The next step, will be to divide both sides by t and take $t \to 0$. This implies that $X_t \to x$, $u(X_t) \to u(x)$, and thus,

$$0 \leq [-\gamma V(x) + AV(x) + g(x,u(x))]$$

which for an optimal Markov control u^* is,

$$0 = [-\gamma V(x) + A^* V(x) + g(x, u^*(x))]$$

These two last results are equivalent to the following,

$$0 = \inf_{u \in \mathbb{U}} [-\gamma V(x) + A^u V(x) + g(x, u)] \tag{2.11}$$

Result (2.11) denotes the dynamic programming equation, as stated in Ross (2008), which essentially establishes a mapping between $x \in \mathbb{S}$ and $u \in \mathbb{U}$. Using (2.11), we can find a solution to our stochastic control problem. The procedure is to, for each x, find the control function u that minimizes $-\gamma V(x) + A^u V(x) + g(x, u)$, and then solve the resultant equation in order to find $V(x)$.

Verification

In order to make the resolution of this stochastic control problem simpler, we present now a procedure that goes a bit against the general deduction technique presented while we were deducing both the dynamic programming principle and the dynamic programming equation, usually denoted in the literature as the verification theorem, and that can be found with more detail in Ross (2008).

The procedure consists on taking a function ϕ as the solution, deduce the correspondent control function, verify the admissibility conditions and show that ϕ coincides with the value function of the optimal control problem.

The great advantage of this method is that we can find a solution, and just apply the verification technique to it, avoiding the most difficult part of the problem which would be to actually solve the dynamic programming equation in a deterministic way, in order to find the solution analytically. This way we can use our intuition or past experience to come up with a "lucky guess" that solves our problem.

Let us present once more the stochastic control problem, but now with a different notation, just to separate this approach to the previous deductions. So, one needs to solve the following problem,

$$0 = \inf_{\alpha \in \mathbb{U}} [-\gamma \phi(x) + A^\alpha \phi(x) + g(x, \alpha)], \quad x \in \mathbb{R} \tag{2.12}$$

Now, supposing we found a solution ϕ to this equation, it will be required that we identify the correspondent optimal control function. Such a function can be obtained using the following equation, valid for each x and taken in the control space $\mathbb{U}$.

$$\alpha^*_x = \arg\ \min_{\alpha \in \mathbb{U}} [-\gamma \phi(x) + A^\alpha \phi(x) + g(x, \alpha)] \tag{2.13}$$

So, what we have is that for each $x \in \mathbb{R}$ there exists a unique α_x^* that satisfies (2.13), establishing a kind of mapping relationship between $x \in \mathbb{R}$ and $\alpha_x^* \in \mathbb{U}$. This map is going to be given by $u^* : \mathbb{R} \to \mathbb{U}$ and later will define an optimal Markov control function. However, as we mentioned initially, we can only admit control functions that verify the admissibility conditions. In this sense, we need to establish the definition of an admissible control process, as it is presented in Ross (2008).

Definition 2.3 (Admissible control process) A stochastic process $\{U_t,\, t \leq T\}$ is an admissible control process if,

1. U is $\{\mathcal{F}\}$-adapted;
2. $U_t \in \mathbb{U}$ for all $t \geq 0$;
3. (2.2) has a unique solution;
4. $\int_0^t e^{-\gamma s}\phi'(X_s)\sigma(X_s, U_s)dZ_s$ is a martingale;
5. $e^{-\gamma t}\mathbb{E}[V(X_t)] \to 0$ as $t \to \infty$

Finally, we can state that the set of all admissible controls is denoted by $\mathcal{A}$. It should be noticed that for now the admissible controls are not necessarily Markov controls. Thus, the cost and value functions will be defined as,

$$J(x, U) = \mathbb{E}\bigg[\int_0^\infty e^{-\gamma t} g(X_t, U_t)dt\bigg]$$

$$V(x) = \inf_{U \in \mathcal{A}} J(x, U)$$

The verification procedure follows using the same reasoning used in the deduction of the dynamic programming equation. The first step is to apply Ito's lemma to $e^{-\gamma t}\phi(X_t^*)$ and integrate it from 0 to t,

$$e^{-\gamma t}\phi(X_t^*) = \phi(x) + \int_0^t e^{-\gamma s}[-\gamma\phi(X_s^*) + A^*\phi(X_s^*)]ds + \int_0^t e^{-\gamma s}\phi'(X_s^*)\sigma(X_s^*, U_s^*)dZ_s$$

Then, we add $\int_0^t e^{-\gamma s} g(X_s^*, U_s^*)ds$ to both sides of the equation, and we also take expected values on both sides too. The result is presented next.

$$\mathbb{E}\bigg[\int_0^t e^{-\gamma s} g(X_s^*, U_s^*)ds\bigg] + e^{-\gamma t}\mathbb{E}[\phi(X_t^*)] = \phi(x) + \mathbb{E}\bigg[\int_0^t e^{-\gamma s}[-\gamma\phi(X_s^*) + A^*\phi(X_s^*) + g(X_s^*, U_s^*)]ds\bigg]$$

We can observe that the term in dZ_s disappeared because it is an Ito's integral, which is a martingale, and once we apply an expected value to it, its value becomes zero. One should also notice that if we are assuming that ϕ is solution of (2.13), then the term $-\gamma\phi(X_s^*) + A^*\phi(X_s^*) + g(X_s^*, U_s^*)$ is zero. Finally, we now want to make $t \to \infty$, which will imply that the term $e^{-\gamma t}\mathbb{E}[\phi(X_t^*)]$ will disappear and, in the view of condition 5 of Definition 2.3, the result will be the following,

$$\phi(x) = \mathbb{E}\left[\int_0^\infty e^{-\gamma s} g(X_s^*, U_s^*) ds\right] = J(x, U^*)$$

On the other hand, if we consider now an arbitrary $U \in \mathcal{A}$ we will have $-\gamma\phi(X_s^*) + A^*\phi(X_s^*) + g(X_s^*, U_s^*) \geq 0$ instead of being equal to zero. All the other operations will remain with the same result. In this sense, we shall have that,

$$\phi(x) \leq \mathbb{E}\left[\int_0^\infty e^{-\gamma s} g(X_s, U_s) ds\right] = J(x, U)$$

Since $U \in \mathcal{A}$ is arbitrary, this implies that,

$$\phi(x) \leq \inf_{U \in \mathcal{A}} J(x, U) = V(x)$$

As we already know that $\phi(x) = J(x, U^*)$, we can finally claim that $V(x) = J(x, U^*)$ and thus conclude that U^* is in fact an optimal control, following the same course of action adopted in Ross (2008). This step concludes the verification procedure.

2.1.2 Optimal Stopping Problems in Continuous Time

So far, we have been discussing stochastic control problems, where the main goal was to determine an optimal control function that optimized our cost or reward function. In what follows, similarly to Ross (2008), we assume that we just want to choose a time to take a particular action, in order to minimize an expected cost. For example, when is the best time to make a certain investment? These are the kind of strategies we are going to be worried about in this next section.

General Formulation of an Optimal Stopping Problem

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space with underlying filtration $\{\mathcal{F}_t\}$ and $\{Z_t,\, t \geq 0\}$ is a d-dimensional Wiener process, where Z_t is $\mathcal{F}_t$ - adapted.

As in this case we do not mention explicitly controls, we assume that the dynamics of the $\{X_t,\, t \geq 0\}$ process is simply given by

$$dX_t = b(X_t)dt + \sigma(X_t)dZ_t, \quad X_0 = x \in \mathbb{R} \tag{2.14}$$

which is similar to (2.2), but now b and σ do not depend on the control process U.

Next, we define a stopping time, as in Björk (2004), hereby denoted by τ:

Definition 2.4 ($\{\mathcal{F}_t\}$ - stopping time) A random variable τ taking values in $[0, \infty) \cup \{\infty\}$ is an $\{\mathcal{F}_t\}$ - stopping time if $\{\tau < t\} \in \{\mathcal{F}_t\}$ for all $t \geq 0$. We also denote by $\mathcal{S}$ the set of all $\{\mathcal{F}_t\}$ - stopping times.

Additionally, we must assume that from now on all the considered stopping times will be finite.

In this situation one wants to optimize J choosing not a sequence of controls but only a stopping time, such that

$$J(x,\tau) = \mathbb{E}\bigg[\int_0^\tau e^{-\gamma s} g(X_s)ds + e^{-\gamma\tau} h(X_\tau)\mathbb{1}_{\{\tau<\infty\}}\bigg] \tag{2.15}$$

In (2.15) h is a function, usually called a terminal cost function, and $\tau \in \mathcal{S}$ is a stopping time. Similarly to the general case, x is the initial state, g is the utility function and γ is the discount factor.

Finally, the value function $V : \mathbb{R} \to \mathbb{R}$ associated with this optimal stopping problem is going to be given by,

$$V(x) = \inf_{\tau\in\mathcal{S}} J(x,\tau) \tag{2.16}$$

As we mentioned in the beginning of this section, our main goal is to find an optimal stopping time $\tau^* \in \mathcal{S}$ such that $J(x,\tau^*) = V(x)$. This process can be rather difficult to implement without any sort of guideline, meaning without something that plays the same role as the Markov control did in the previous section. In this sense, what is going to be done now is, for each x, the state space is split in two complementary regions: a continuation and a stopping region, as presented in Ross (2008). So, one should stop as soon as the process exits the continuation region, hereafter denoted by D. Thus, the optimal stopping time for each x will have the form $\tau_x^* = \inf\{t \geq 0 : X_x(t) \notin D\}$, where $X(.)$ denotes, as previously, the stochastic process X with initial state x. The continuation region D will play the same role as the Markov control did before, in the sense that provides us with a strategy to follow. Such a region will be defined as

$$D = \{x \in \mathbb{R} : \quad V(x) < h(x)\} \tag{2.17}$$

So, the optimal strategy presented in (2.17) is to stop as soon as the process exits D, which means that one should stop once the value function equals or exceeds the terminal cost.

Dynamic Programming Principle and Equation for Optimal Stopping

Similarly to what was done before, we will establish the dynamic programming principle but now for this new problem.

Following the same reasoning used in Ross (2008), we should start by considering the continuation region D a subset of $\mathbb{R}$, $D \subset \mathbb{R}$, such that the optimal stopping time can be defined as $\tau_x^* = \inf\{t \geq 0; X_x(t) \notin D\}$, for all $x \in \mathbb{R}$, with $V(x) = J(x,\tau_x^*)$. Fixing $x \in \mathbb{R}$ and letting $\tau \in \mathcal{S}$ be an arbitrary stopping time, let us define $\tilde{\tau} = \inf\{t \geq \tau : X \notin D\}$. It must be noticed that the stopping time $\tilde{\tau}$ only adopts an optimal strategy after τ, before this point the course of action is completely arbitrary. Also we assume that for the stopping time $\tilde{\tau}$ we have $V(x) \leq J(x,\tilde{\tau})$. In this sense,

$$
\begin{aligned}
V(x) \le J(x,\tilde{\tau}) &= \mathbb{E}\Big[\int_0^{\tilde{\tau}} e^{-\gamma s} g(X(s))ds + e^{-\gamma\tilde{\tau}} h(X(\tilde{\tau}))\mathbb{1}_{\{\tilde{\tau}<\infty\}}\Big] \\
&= \mathbb{E}\Big[\underbrace{\int_0^{\tau} e^{-\gamma s} g(X(s))ds}_{\text{arbitrary strategy}} + \underbrace{e^{-\gamma\tau}(\int_\tau^{\tilde{\tau}} e^{-\gamma(s-\tau)} g(X(s))ds + e^{-\gamma(\tilde{\tau}-\tau)} h(X(\tilde{\tau}))\mathbb{1}_{\{\tilde{\tau}<\infty\}})}_{\text{optimal strategy}}\Big] \\
&= \mathbb{E}\Big[\int_0^{\tau} e^{-\gamma s} g(X(s))ds + e^{-\gamma\tau} J(X(\tau),\tau^*_{X(\tau)})\Big] \\
&= \mathbb{E}\Big[\int_0^{\tau} e^{-\gamma s} g(X(s))ds + e^{-\gamma\tau} V(X(\tau))\Big] \qquad (2.18)
\end{aligned}
$$

Clearly, if the arbitrary stopping time τ is such that $\tau < \tau^*_x$, Ross (2008) states that it then follows that $\tau^*_x = \tilde{\tau}$, meaning that $\tilde{\tau}$ is an optimal stopping time which implies that

$$
V(x) = J(x,\tau^*_x) = J(x,\tilde{\tau}) \qquad (2.19)
$$

Naturally, and proceeding the same way we did in the previous section, combining (2.18) and (2.19) we obtain the dynamic programming principle for optimal stopping problems, presented in (2.20), as in Ross (2008).

$$
V(x) = \inf_{\tau\in\mathcal{S}} \mathbb{E}\Big[\int_0^{\tau} e^{-\gamma s} g(X(s))ds + e^{-\gamma\tau} V(X(\tau))\Big] \qquad (2.20)
$$

Next, we would like to derive the dynamic programming equation. Considering a fixed $x \in \mathbb{R}$ and $\tau \in \mathcal{S}$, let us apply Ito's lemma to $e^{-\gamma\tau}V(X(\tau))$, integrate the result from 0 to τ and take expected values on both sides, like it is done in Ross (2008). Then we end up with

$$
\mathbb{E}[e^{-\gamma\tau}V(X(\tau))] = V(x) + \mathbb{E}\Big[\int_0^{\tau} e^{-\gamma t}[AV(X(t)) - \gamma V(X(t))]dt\Big]
$$

where A is the infinitesimal generator defined in (2.3). Then, adding $\mathbb{E}\Big[\int_0^\tau e^{-\gamma t} g(X(t))dt\Big]$ to both members of the equation, we obtain,

$$
\mathbb{E}\Big[e^{-\gamma\tau}V(X(\tau)) + \int_0^{\tau} e^{-\gamma t} g(X(t))dt\Big] = V(x) + \mathbb{E}\Big[\int_0^{\tau} e^{-\gamma t}[AV(X(t)) - \gamma V(X(t))]dt + \int_0^{\tau} e^{-\gamma t} g(X(t))dt\Big] \qquad (2.21)
$$

Now, one should note that if $\tau \le \tau^*_x$, by optimality of τ^*_x, $V(x) = J(x,\tau^*_x) = \mathbb{E}\Big[\int_0^\tau e^{-\gamma s} g(X(s))ds + e^{-\gamma\tau}V(X(\tau))\Big]$ and thus the previous equation is reduced to,

$$
0 = \mathbb{E}\Big[\int_0^{\tau} e^{-\gamma t}[-\gamma V(X(t)) + AV(X(t)) + g(X(t))]dt\Big] \qquad (2.22)
$$

For $x \in D$, as D is an open set and $\tau_x^* > 0$, if we divide both members of the equation by τ and then make $\tau \to 0$, the last equation becomes the following,

$$0 = -\gamma V(X(t)) + AV(X(t)) + g(X(t)), \quad x \in D \tag{2.23}$$

On the other hand, if $x \notin D$, then (2.22) has absolutely no meaning at all. Nevertheless, as τ_x^* is an optimal stopping time we have that $V(x) = J(x, \tau_x^*) = h(x)$, by definition of J, which implies that

$$h(x) - V(x) = 0, x \notin D \tag{2.24}$$

Considering now $\tau \in \mathcal{S}$ an arbitrary stopping time, as in (2.21) $V(x)$ would be such that $V(x) \leq \mathbb{E}\left[\int_0^\tau e^{-\gamma s} g(X(s))ds + e^{-\gamma\tau} V(X(\tau))\right]$, we would have instead of (2.23),

$$0 \leq -\gamma V(X(t)) + AV(X(t)) + g(X(t)), \quad x \in \mathbb{R} \tag{2.25}$$

Similarly, if we consider that we can stop at time zero and pay the terminal cost $V(x) \leq J(x, 0) = h(x)$ and (2.24) becomes,

$$h(x) - V(x) \geq 0, \quad x \in \mathbb{R} \tag{2.26}$$

Combining all the four last results, (2.23), (2.24), (2.25) and (2.26), we obtain the dynamic programming equation for optimal stopping problems presented below, similarly to what figures in Ross (2008).

$$\min\{-\gamma V(X(t)) + AV(X(t)) + g(X(t)), h(x) - V(x)\} = 0, x \in \mathbb{R} \tag{2.27}$$

Finally, we can observe that the region D does not appear explicitly defined in the past derivations. The decision between continuing or stopping, here is represented by the two terms in the minimum operator. Ross (2008) states that the continuation state corresponds to the term $-\gamma V(X(t)) + AV(X(t)) + g(X(t))$ and as soon as we have $\min\{-\gamma V(X(t)) + AV(X(t)) + g(X(t)), h(x) - V(x)\} = h(x) - V(x) = 0$ we decide to stop. This reasoning means that region D will be defined as we initially guessed.

$$D = \{x \in \mathbb{R}^n : \quad V(x) < h(x)\} \tag{2.28}$$

In this section, it is not included the verification technique for optimal stopping problems because the procedure would be quite similar to what was done in the stochastic control problem, and it can be found in detail in Ross (2008).

2.2 Investment Under Uncertainty

In this section we will present a few models that can describe decisions regarding some types of investment under uncertainty. We will start with a one-factor model, following the same procedure used in Dixit and Pindyck (1994), which, as the name points out, intends to describe investments where we have only one source of uncertainty. Then, we move on to investments under two sources of uncertainty. Here, we will present two distinct models, that use different approaches to arrive to the respective solutions. The first one attempts to reduce the number of uncertain factors from two to one, as it is presented in Dixit and Pindyck (1994). In cases where it is possible to reduce the two factor model to a one factor model, we can use the first model we present. The second approach deals with the two factors simultaneously using what is denoted in Adkins and Paxson (2011b) as a *quasi analytical* approach.

2.2.1 One-factor Model

This model intends to describe renewal investments where we have a fixed and known reinvestment or sunk cost K, but the revenue value P is uncertain. Similarly to what is done in Dixit and Pindyck (1994), the main goal of this model is to determine for which value of P, say $\hat{P}$, it is optimal to pay the reinvestment cost K, and thus exercise our option to renew the investment. One must note that this kind of investments is irreversible, and so once you pay the sunk cost K, there is absolutely no turning back. Furthermore, we assume that there are no other kind of costs involved such as operating or transaction costs.

We will also want to determine what is usually called the firm's value, F, as it is defined in Adkins and Paxson (2011b), for that threshold value of P, $\hat{P}$, which will be denoted by $F(\hat{P})$. At this point, we must note that we use F in order to denote the value of the firm. Using the notation of section 2.1, this function F is, in fact, V. We use this notation in the present section because it is the the usual notation for real options literature, whereas V is the usual notation for stochastic control problems. Even though the term firm's value is being used in this context, it doesn't really mean that we are evaluating the value of an actual firm. What happens is that the function F will be constituted by two main terms, the actual option value, which denotes the price of the option, and the incumbent asset value, which refers to the valuation of the remaining assets involved in the investment process. So, in order to avoid misunderstandings and ambiguous terminology, we decided to adopt the expression firm's value to designate the sum of this two terms, even though its meaning doesn't have to be literal in this case.

Here, following the assumptions adopted in Dixit and Pindyck (1994) and Adkins and Paxson (2011b), we assume that the behaviour of P is well described by a geometric Brownian motion, and thus P is the solution of the following stochastic differential equation:

$$dP = \alpha P dt + \sigma P dZ \tag{2.29}$$

where α is the risk-adjusted drift rate, σ is the volatility rate and dZ is the standard increment of a Wiener process.

As in this scenario we are ignoring any kind of operating costs, we can state that the net cash flow is going to be given by P exclusively. Thus the firm's value for this investment is given by $F(P)$. Assuming the necessary conditions for F, we apply Ito's lemma to the discounted value of F, given by $e^{-rt}F(P)$, followed by the dynamic programming principle, and we derive that in the continuation region, F must be such that:

$$\frac{1}{2}\sigma^2 P^2 F''(P) + \alpha P F'(P) - rF + P = 0 \tag{2.30}$$

where r denotes the risk-free interest rate. Once again, we use in this section a different notation for the interest rate (r instead of γ), for similar reasons as the ones for using F instead of V.

We will now present a possible solution to (2.30), that we denote by G, compute its derivatives and replace them in the equation above. After this, we will analyse the result and add some boundary conditions to the problem, that will be mentioned in detail later on.

The proposed solution will be the following,

$$G(P) = AP^{\beta} + \frac{P}{r-\alpha} \tag{2.31}$$

One must note that the term AP^{β} in equation (2.31), solves the homogeneous part of (2.30) and denotes the option value itself, whereas the term $\frac{P}{r-\alpha}$ represents the incumbent asset value for the investor and can be seen as the solution of the non-homogeneous part of (2.30). Computing now the correspondent derivatives one will have the following,

$$G'(P) = A\beta P^{\beta-1} + \frac{1}{r-\alpha} \tag{2.32}$$

$$G''(P) = A\beta(\beta-1)P^{\beta-2} \tag{2.33}$$

Replacing the results in (2.30) will lead us to,

$$\frac{1}{2}\sigma^2 P^2 A\beta(\beta-1)P^{\beta-2} + \alpha P(A\beta P^{\beta-1} + \frac{1}{r-\alpha}) - r(AP^{\beta} + \frac{P}{r-\alpha}) + P = 0 \Leftrightarrow$$

$$AP^{\beta}\left[\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r\right] + \frac{\alpha P}{r-\alpha} - \frac{rP}{r-\alpha} + P = 0 \Leftrightarrow$$

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0 \tag{2.34}$$

In this sense, we will define the resulting expression as a function of β, as in (2.35). This function defines an upward-pointing parabola which is illustrated in figure (2.1). The values for the fixed constants used to obtain this illustrative graphic are presented in table 2.1.

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r \tag{2.35}$$

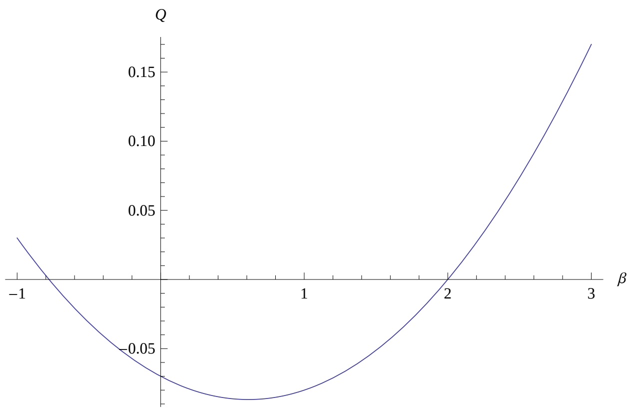

Figure 2.1: Graphic of the parabola defined by $Q(\beta) = 0$

Parameters	Symbol	Value
Revenue volatility	σ	0.3
Revenue risk-adjusted drift rate	α	-0.01
Risk-free interest rate	r	0.07

Table 2.1: Parameter Values - One-factor model

So, according to Kohler and Johnson (2006), if we try to solve the equation $Q(\beta) = 0$, we can fall into three distinct cases: we can have two roots, we can have one double root, or we can have two complex roots. We will ignore the case where there are complex roots, as it makes no sense to assume such a thing in the scope of our problem. When we have one double root, let's call it β', G can be rewritten as:

$$G(P) = A_1 P^{\beta'} ln(P) + A_2 P^{\beta'} + \frac{P}{r - \alpha} \tag{2.36}$$

At last, when we have two distinct roots, let's denote them by β_1 and β_2, the function G can be rewritten as follows:

$$G(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2} + \frac{P}{r - \alpha} \tag{2.37}$$

As we can observe in figure 2.1, that uses the same parameter values as Adkins and Paxson (2011b), the function $Q(.)$ is zero for two possible values of β, one positive, β_1, and one negative, β_2, falling into this last case we mention. In this sense, we will assume from now on, that when we have this type of problem, we would only be interested in considering the case where we have two distinct roots, such that G can be rewritten as (2.37).

These two solutions can be obtained following the same methodology used in Dixit and Pindyck (1994) and they are given by,

$$\begin{cases} \beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \\ \beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \end{cases} \tag{2.38}$$

We must now identify which of these two solution is valid for us to consider from now on. We shall now establish the boundary conditions mentioned before and also presented in Dixit and Pindyck (1994). As we pointed out previously, $\hat{P}$ denotes the threshold or limiting value for the revenue P. Also, we should define K as the sunk cost, which is the monetary value one must pay in order to renew the investment.

The first boundary condition is $F(0) = 0$. This is quite straightforward to observe. It essentially says that if P turns zero then the correspondent firm's value will do so as well. The second boundary condition is $F(\hat{P}) = \hat{P} - K$ and it represents the value-matching relationship, that can be explained as follows: by renewing the investment for the limiting value of P, $\hat{P}$, the firm will receive the correspondent revenue value $(\hat{P})$ minus the sunk cost (K). As we stated in the first section of this chapter this relationship also guarantees the continuity at this threshold point. The last condition is given by $F'(\hat{P}) = 1$ and it is the correspondent smooth pasting condition. As the name states, it is used to guarantee the smoothness of the function at the boundary, that in this case is represented by $\hat{P}$. Due to the fact that $\hat{P}$ is not known in advance, the problem to be solved is a free-boundary problem.

Let us now analyse the boundary conditions we have just presented. The first condition will imply that $G(0) = 0$, meaning that in general β has to be non negative and thus, we should consider β_1 to be the admissible solution of $Q(\beta) = 0$ for us to take from now on. So, G can be rewritten as,

$$G(P) = A_1 P^{\beta_1} + \frac{P}{r - \alpha} \tag{2.39}$$

The value-matching condition implies that,

$$G(\hat{P}) = \hat{P} - K \Leftrightarrow A_1 \hat{P}^{\beta_1} + \frac{\hat{P}}{r - \alpha} = \hat{P} - K \tag{2.40}$$

From the smooth pasting condition it follows that,

$$G'(\hat{P}) = 1 \Leftrightarrow A_1 \beta_1 \hat{P}^{\beta_1 - 1} + \frac{1}{r - \alpha} = 1 \Leftrightarrow A_1 \hat{P}^{\beta_1} = \frac{\hat{P}}{\beta_1}\left(1 - \frac{1}{r - \alpha}\right) \tag{2.41}$$

Substituting this last result in the value-matching condition we have,

$$\frac{\hat{P}}{\beta_1}\left(1 - \frac{1}{r - \alpha}\right) + \frac{\hat{P}}{r - \alpha} = \hat{P} - K \Leftrightarrow \hat{P}\left(\frac{1}{\beta_1} - \frac{1}{\beta_1(r - \alpha)} + \frac{1}{r - \alpha} - 1\right) = -K \Leftrightarrow$$

$$\hat{P} = K\left(\frac{\beta_1(\alpha - r)}{r - \alpha - 1 + \beta_1 - \beta_1(r - \alpha)}\right) \tag{2.42}$$

Finally, we could obtain the value for A_1 by replacing (2.42) again in the value-matching relationship and solving it in order to derive A_1.

Once G verifies all the boundary conditions imposed, it can be considered a true solution of (2.30), denoted by F.

In conclusion, we have tools to completely determine all the unknowns in our original problem. We know that the threshold value for the revenue $\hat{P}$ is given by (2.42), where we can obtain β_1 using the formula presented above. The function F is given by (2.39), where A_1 is derived the way we've just stated.

2.2.2 Two-factor Model: Reduction to One-factor Model

In this section, we will consider a model quite similar to the previous one, but now with two uncertainty variables. As it is done in Adkins and Paxson (2011b) and Dixit and Pindyck (1994), we assume that the uncertainty is due not only to revenues but also to operating costs, hereby denoted by C. Furthermore, they will both follow a geometric Brownian motion as follows:

$$dP = \alpha_P P dt + \sigma_P P dZ_P \tag{2.43}$$

$$dC = \alpha_C C dt + \sigma_C C dZ_C \tag{2.44}$$

Again, α_P and α_C denote the respective risk-adjusted drift rates, σ_P and σ_C denote the volatility rates, and dZ_P and dZ_C denotes the correspondent increments of the standard Wiener processes. We remark that $\{Z_P(t), t \geq 0\}$ and $\{Z_C(t), t \geq 0\}$ may be correlated, and we denote its correlation (stationary) by ρ.

We will consider once more F to be the function denoting the firm's value associated with the investment. In this sense, the dynamic programming equation, that results from the application of Ito's lemma together with the dynamic programming principle to the discounted value of F, given by $e^{-rt}F(P,C)$, will become a bit more complex, due to the presence of two variables. Just like stated in Adkins and Paxson (2011b), it is given by,

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F}{\partial P^2} + \frac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F}{\partial C^2} + \rho\sigma_P\sigma_C PC \frac{\partial^2 F}{\partial P \partial C} + \alpha_P P \frac{\partial F}{\partial P} + \alpha_C C \frac{\partial F}{\partial C} - rF + (P - C) = 0 \tag{2.45}$$

The main goal here is to reduce this two-factor model to a one-factor model, resembling what is done in Dixit and Pindyck (1994), so that we are able to apply the same techniques we used in the previous section. In order to do this, according to Dixit and Pindyck (1994), we assume the following simplifying assumption for the possible solution,

$$G(P,C) = Cf(p), \quad \text{with } p = \frac{P}{C} \tag{2.46}$$

where f is a $C^2(\mathbb{R})$ function still to be determined. Assuming that our function G has this form, let us compute its derivatives and substitute them in (2.45).

$$\begin{aligned} \frac{\partial G}{\partial P} &= f'(p) & \qquad \frac{\partial^2 G}{\partial P^2} &= \frac{f''(p)}{C} \\ \frac{\partial G}{\partial C} &= f(p) - pf'(p) & \qquad \frac{\partial^2 G}{\partial C^2} &= \frac{p^2}{C} f''(p) \\ & & \qquad \frac{\partial^2 G}{\partial P \partial C} &= -\frac{p}{C} f''(p) \end{aligned}$$

Thus, (2.45) becomes

$$\frac{1}{2}\sigma_P^2 P^2 \frac{f''(p)}{C} + \frac{1}{2}\sigma_C^2 C^2 \frac{p^2}{C} f''(p) - \rho\sigma_P\sigma_C PC\frac{p}{C}f''(p) + \alpha_P Pf'(p) + \alpha_C C(f(p) - pf'(p)) - rCf(p)$$
$$+(P-C) = 0 \Leftrightarrow$$
$$\frac{1}{2}\sigma_P^2 pf''(p) + \frac{1}{2}\sigma_C^2 pf''(p) - \rho\sigma_P\sigma_C pf''(p) + \alpha_P f'(p) - \alpha_C f'(p) - \frac{r-\alpha_C}{p}f(p) + (1-\frac{1}{p}) = 0 \Leftrightarrow$$
$$p^2 f''(p)\left[\frac{1}{2}\sigma_P^2 + \frac{1}{2}\sigma_C^2 - \rho\sigma_P\sigma_C\right] + pf'(p)[\alpha_P - \alpha_C] - (r-\alpha_C)f(p) + (p-1) = 0 \Leftrightarrow$$
$$\mathbf{\frac{1}{2}f''(p)p^2}[\sigma_P^2 + \sigma_C^2 - 2\rho\sigma_P\sigma_C] + \mathbf{f'(p)p}(\alpha_P - \alpha_C) - (r-\alpha_C)\mathbf{f(p)} + (p-1) = 0 \tag{2.47}$$

We can observe that the equation above has the same form as (2.30), with the exception of the term -1 in the end. This term should not concern us for the moment, as the solution we found for (2.30), without this term, can easily be adapted to the present case. The part we have detached corresponds to the homogeneous part and, besides the multiplying coefficients, it presents the same structure as before. The non homogeneous part is composed by the main variable minus one. Therefore, and following similar arguments to the ones presented to obtain (2.39), the function f, i.e. solution of (2.47), is given by:

$$f(p) = A_3 p^{\beta_3} + \frac{p}{r-\alpha_P} - \frac{1}{r-\alpha_C} \tag{2.48}$$

where β_3 is equal to β_1, A_3 to A_1, $\sigma^2 = \sigma_P^2 + \sigma_C^2 - 2\rho\sigma_P\sigma_C$, $\alpha = \alpha_P - \alpha_C$ and $r = r - \alpha_C$. We must also note that the term $-\frac{1}{r-\alpha_C}$ corresponds to the adaptation of the solution of (2.30), to the new equation (2.47), where an extra -1 in the end appears.

Moreover, using in a similar way the value-matching and smooth-pasting conditions corresponding to this situation, we end up with the following threshold value for p, $\hat{p}$:

$$\hat{p} = \left(\frac{K(r-\alpha_C)-1}{r-\alpha_C}\right)\left(\frac{\beta_3(r-\alpha_P)}{(r-\alpha_P)-1+\beta_3-\beta_3(r-\alpha_P)}\right) \tag{2.49}$$

The value for A_3 can be found following a similar procedure to what was indicated before.

Our conclusions regarding this model, will concern exclusively the quotient $\frac{P}{C}$, as well as its threshold value $\widehat{(\frac{P}{C})}$, and a function of such quotient $f(p) = f(\frac{P}{C})$. In fact, nothing can really be said regarding either P or C individually, its threshold values $\hat{P}$ or $\hat{C}$, or the function $F(P,C)$, where the two variables figure simultaneously. This is a direct consequence of the reduction that was made in order to solve this problem, if applicable it can simplify the problem quite a lot. Therefore, this analysis is clearly insufficient for the purpose of our study, and we shall move on to an approach that allows us to deal with the two factors simultaneously, as well as to make conclusions about the function $F(P,C)$.

2.2.3 Two-factor Model: Quasi-Analytical Approach

We will now proceed to present the two-factor model, using a new approach, developed in Adkins and Paxson (2011b), that will allow us to deal with the two uncertainty factors at the same time. We will just note that we are still considering an irreversible type of investment, and that in this section an additional deterministic approach will be presented in the end, similarly to what is done in Adkins and Paxson (2011b).

Similarly to the previous case, we assume geometric Brownian motion for the dynamics of the processes. Moreover, following the same course of action as Adkins and Paxson (2011b), P_I and C_I denote the initial values for P and C, respectively. In this sense, assuming that $\alpha_P < 0$, $\alpha_C > 0$, $r - \alpha_P > 0$ and $r - \alpha_C > 0$ as in Adkins and Paxson (2011b), will imply $P < P_I$ and $C > C_I$.

In the same way considered as in Adkins and Paxson (2011b), we will denote $\hat{P}$ and $\hat{C}$ as the optimal threshold levels for P and C. This means that $\hat{P}$ and $\hat{C}$ refer to the values of P and C, immediately after a renewal occurs and the reinvestment cost K is paid. They arise naturally once the boundary conditions are imposed later, in the form of the value-matching and smooth pasting conditions, similarly to what was done before.

Using the same notation we already defined in the last section, and proceeding the exact same way, we will reach once again equation (2.45).

The procedure will be to propose a new solution to equation (2.45), now including both of the considered variables, compute its derivatives and substitute them in the equation, which will lead us to an ellipse equation that will be part of our final model. After that, the value-matching and correspondent smooth-pasting conditions will be introduced and applied to this situation, and with this procedure we should obtain the remaining equations that will integrate our final model.

Elliptical Characteristic Root Equation

Let us start with the proposed solution G to equation (2.45):

$$G(P,C) = AP^{\beta}C^{\eta} + \frac{P}{r-\alpha_P} - \frac{C}{r-\alpha_C} \tag{2.50}$$

where A, β and η are parameters still to be determined. We shall also note that in this solution, Adkins and Paxson (2011b) states that the term $AP^{\beta}C^{\eta}$ represents the renewal option value, which also denotes the solution of the homogeneous part of (2.45), whereas the term $\frac{P}{r-\alpha_P} - \frac{C}{r-\alpha_C}$ denotes the incumbent asset value. Computing the derivatives of the proposed function and replacing them in (2.45) leads us to,

$$\frac{\partial G}{\partial P} = A\beta P^{\beta-1}C^{\eta} + \frac{1}{r-\alpha_P} \qquad \frac{\partial^2 G}{\partial P^2} = A\beta(\beta-1)P^{\beta-2}C^{\eta}$$

$$\frac{\partial G}{\partial C} = A\eta P^{\beta}C^{\eta-1} - \frac{1}{r-\alpha_C} \qquad \frac{\partial^2 G}{\partial C^2} = A\eta(\eta-1)P^{\beta}C^{\eta-2}$$

$$\frac{\partial^2 G}{\partial P \partial C} = A\beta\eta P^{\beta-1}C^{\eta-1}$$

Now, replacing them in (2.45) we get,

$$\frac{1}{2}\sigma_P^2 P^2 A\beta(\beta-1)P^{\beta-2}C^\eta + \frac{1}{2}\sigma_C^2 C^2 A\eta(\eta-1)P^\beta C^{\eta-2} + \rho\sigma_P\sigma_C PCA\beta\eta P^{\beta-1}C^{\eta-1} +$$
$$\alpha_P PA\beta P^{\beta-1}C^\eta + \alpha_P P\frac{1}{r-\alpha_P} + \alpha_C CA\eta P^\beta C^{\eta-1} - \alpha_C C\frac{1}{r-\alpha_C}$$
$$-r\left(AP^\beta C^\eta + \frac{P}{r-\alpha_P} - \frac{C}{r-\alpha_C}\right) + (P-C) = 0 \Leftrightarrow$$

$$AP^\beta C^\eta\left[\frac{1}{2}\sigma_P^2\beta(\beta-1) + \frac{1}{2}\sigma_C^2\eta(\eta-1) + \rho\sigma_P\sigma_C\beta\eta + \alpha_P\beta + \alpha_C\eta - r\right] = 0$$

which, as the $AP^\beta C^\eta$ cannot be zero, we will define $Q(\beta,\eta)$ in (2.51).

$$Q(\beta,\eta) = \frac{1}{2}\sigma_P^2\beta(\beta-1) + \frac{1}{2}\sigma_C^2\eta(\eta-1) + \rho\sigma_P\sigma_C\beta\eta + \alpha_P\beta + \alpha_C\eta - r \tag{2.51}$$

that, consequently, has to be such that $Q(\beta,\eta) = 0$.

The equation we just reached is also denoted in Adkins and Paxson (2011b) by *characteristic root equation*. It defines an ellipse that is represented in figure 2.2. The values for σ_P, σ_C, α_P, α_C, r and ρ, used to illustrate are specified in table 2.2.

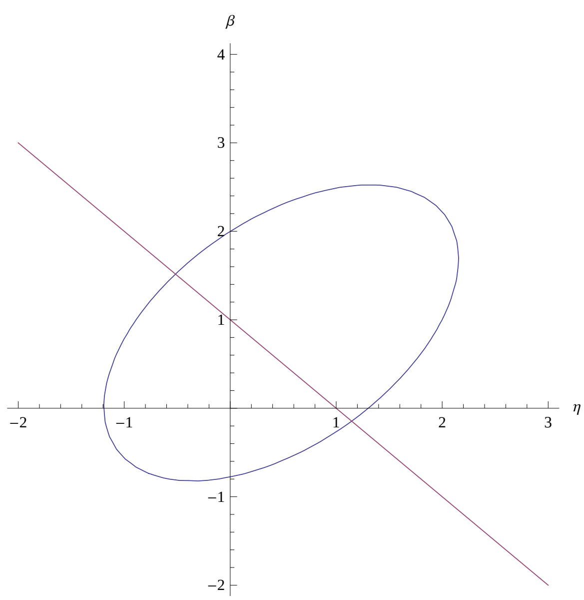

Figure 2.2: Graphic of the ellipse defined by $Q(\beta,\eta) = 0$ and the line $\beta + \eta = 1$

Parameters	Symbol	Value
Revenue volatility	σ_P	0.3
Operating costs volatility	σ_C	0.3
Revenue risk-adjusted drift rate	α_P	-0.01
Operating costs risk-adjusted drift rate	α_C	0.04
Risk-free interest rate	r	0.07
Correlation between P and C	ρ	-0.5

Table 2.2: Parameter Values - Ellipse equation - Two-factor model

The equation $\beta + \eta = 1$ appears in this figure because it represents an assumption made by some authors referred in Adkins and Paxson (2011b), such as McDonald and Siegel (1986). However, we will attempt to solve the problem without using this restriction, even though a similar one will appear naturally later on.

Just by analysing figure 2.2 we can state the following:

1. When $\beta + \eta = 1$, we have two solutions: the two points where the line $\beta + \eta = 1$ intersects the ellipse.
2. When $\beta + \eta \neq 1$, we need further information to determine a solution. In order to do that, we shall now explore the limiting behaviour of P and C.

For that we identify four quadrants where the solution might be and then try to select which one or ones of them is relevant, precisely by analysing the limiting behaviour of the two variables P and C.

$$\begin{aligned}
I: &\quad \{\beta,\eta\} \quad \beta \geq 0; \eta \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_1,\eta_1\} \\
II: &\quad \{\beta,\eta\} \quad \beta \geq 0; \eta \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_2,\eta_2\} \\
III: &\quad \{\beta,\eta\} \quad \beta \leq 0; \eta \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_3,\eta_3\} \\
IV: &\quad \{\beta,\eta\} \quad \beta \leq 0; \eta \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_4,\eta_4\}
\end{aligned}$$

In this sense, (2.50) can be rewritten as follows,

$$G(P,C) = A_1 P^{\beta_1} C^{\eta_1} + A_2 P^{\beta_2} C^{\eta_2} + A_3 P^{\beta_3} C^{\eta_3} + A_4 P^{\beta_4} C^{\eta_4} + \frac{P}{r-\alpha_P} - \frac{C}{r-\alpha_C} \tag{2.52}$$

We will denote by $G_H(P,C)$, the terms in G that correspond to the solution of the homogeneous part of (2.45). Now the analysis of the referred limiting behaviour will concern $G_H(P,C)$ and consists on using the following arguments:

1. $\lim_{P\to\infty} G_H(P,C) = 0$
   If the revenues approach ∞ there will be no reason to renew the asset, and the renewal option value tends to 0.
2. $\lim_{P\to 0} G_H(P,C) = \infty$
   If the revenues are approaching 0 then the renewal is inevitable and so the renewal option value will tend to ∞.
3. $\lim_{C\to\infty} G_H(P,C) = \infty$
   Whenever C becomes infinitely large, the asset should be renewed and so the renewal option value will become exceedingly large.
4. $\lim_{C\to 0} G_H(P,C) = 0$
   When the costs approach 0 there is no reason to renew the asset and so the renewal option value will tend to 0.

Therefore it follows from these equations that we must have $\beta \leq 0$ and $\eta \geq 0$, meaning that the only relevant region is the fourth quadrant, and thus the function G is such that:

$$G(P, C) = A_4 P^{\beta_4} C^{\eta_4} + \frac{P}{r - \alpha_P} - \frac{C}{r - \alpha_C} \tag{2.53}$$

Next we use the value matching and smooth pasting conditions, as in Adkins and Paxson (2011b), to impose other requirements in order to find, if possible, the constants involved in (2.53).

Value Matching Relationship and Smooth Pasting Conditions

Let us consider the renewal event, with P and C both at their threshold levels, $\hat{P}$ and $\hat{C}$. Also, let K represent the renewal reinvestment cost. According to Adkins and Paxson (2011b), the value-matching relationship states is that the difference between the initial value of G and its threshold value will be equal to K, and so

$$G(\hat{P}, \hat{C}) = G(P_I, C_I) - K$$

$$\Leftrightarrow$$

$$A_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4} + \frac{\hat{P}}{r - \alpha_P} - \frac{\hat{C}}{r - \alpha_C} = A_4 {P_I}^{\beta_4} {C_I}^{\eta_4} + \frac{P_I}{r - \alpha_P} - \frac{C_I}{r - \alpha_C} - K \tag{2.54}$$

Also from the smooth-pasting conditions, we set the following relations:

$$\begin{cases} \frac{\partial G}{\partial P}\Big|_{P=\hat{P};\ C=\hat{C}} = 0 \\ \\ \frac{\partial G}{\partial C}\Big|_{P=\hat{P};\ C=\hat{C}} = 0 \end{cases} \tag{2.55}$$

Once more, it must be referred that these two conditions guarantee the continuity and smoothness of G, at the threshold values for P and C.

Computing the correspondent derivatives the previous system of equations leads to the following:

$$\begin{cases} A_4 \beta_4 \hat{P}^{\beta_4 - 1} \hat{C}^{\eta_4} + \frac{1}{r - \alpha_P} = 0 \\ \\ A_4 \eta_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4 - 1} - \frac{1}{r - \alpha_C} = 0 \end{cases} \Leftrightarrow \begin{cases} A_4 = \frac{\hat{P}}{-(r-\alpha_P)\beta_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4}} \geq 0 \\ \\ A_4 = \frac{\hat{C}}{(r-\alpha_C)\eta_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4}} \geq 0 \end{cases} \Leftrightarrow \begin{cases} \hat{P} = -\frac{\hat{C}(r-\alpha_P)\beta_4}{(r-\alpha_C)\eta_4} \\ \\ \hat{C} = -\frac{\hat{P}(r-\alpha_C)\eta_4}{(r-\alpha_P)\beta_4} \end{cases}$$

Therefore:

$$\frac{\hat{P}}{-(r - \alpha_P)\beta_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4}} = \frac{\hat{C}}{(r - \alpha_C)\eta_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4}} \tag{2.56}$$

Nevertheless, the other results presented above will prove themselves quite useful too. Starting by replacing the two derived formulas for A_4, as well as the new expressions for either $\hat{P}$ and $\hat{C}$, in $G(\hat{P},\hat{C})$, we obtain the following,

$$\begin{aligned}
G(\hat{P},\hat{C}) &= \frac{\hat{P}}{-(r-\alpha_P)\beta_4\hat{P}^{\beta_4}\hat{C}^{\eta_4}}\hat{P}^{\beta_4}\hat{C}^{\eta_4} + \frac{\hat{P}}{r-\alpha_P} - \frac{-\frac{\hat{P}(r-\alpha_C)\eta_4}{(r-\alpha_P)\beta_4}}{r-\alpha_C} \\
&= \frac{\hat{P}}{(r-\alpha_P)\beta_4}[-1+\beta_4+\eta_4] > 0 \\
&= \frac{\hat{C}}{(r-\alpha_C)\eta_4\hat{P}^{\beta_4}\hat{C}^{\eta_4}}\hat{P}^{\beta_4}\hat{C}^{\eta_4} + \frac{-\frac{\hat{C}(r-\alpha_P)\beta_4}{(r-\alpha_C)\eta_4}}{r-\alpha_P} - \frac{\hat{C}}{r-\alpha_C} \\
&= \frac{\hat{C}}{(r-\alpha_C)\eta_4}[1-\beta_4-\eta_4] > 0
\end{aligned}$$

From the fact that $\frac{\hat{C}}{(r-\alpha_C)\eta_4} \geq 0$ we can conclude that we need to have $[1-\beta_4-\eta_4] > 0$, which implies $\beta_4+\eta_4 < 1$. Similarly, as we know that $\frac{\hat{P}}{(r-\alpha_P)\beta_4} \leq 0$, we will need to have $[-1+\beta_4+\eta_4] < 0$ which will lead to the same result. It is worth to notice that even though we have not used the homogeneity-degree-1 condition, given by $\beta+\eta=1$ and used by some authors, we had previously stated that a similar condition would arise naturally. That condition is precisely the one we just reached, i.e. $\beta_4+\eta_4 < 1$, also presented in Adkins and Paxson (2011b).

So, we have now a set of restrictions that will help us get closer to the solution we seek. We already know that,

1. $\beta_4 \leq 0$
2. $\eta_4 \geq 0$
3. $\beta_4+\eta_4 < 1$
4. $Q(\beta_4,\eta_4) = 0$

This reduces the range of possible solutions to the points on the dashed line in figure (2.3), presented below.

As we imposed that G verifies all boundary conditions, and once we already stated that it solves (2.45), we can consider it to be truly a solution of (2.45), initially denoted by F.

At this point we have a model containing five unknowns: A_4, β_4, η_4, $\hat{P}$, $\hat{C}$. If we consider equation (2.56) to integrate our model, we can ignore A_4, at least initially, and determine it once we have the values for the remaining parameters. Regarding β_4 and η_4 we have $Q(\beta_4,\eta_4)=0$, where $Q(.,.)$ is defined in (2.51), and where both of these parameters figure (and where A_4 is not present). However we still need more information to determine their values. At last, we still have to calculate values for $\hat{P}$ and $\hat{C}$. They are both present in equation (2.56), but similarly to what happens with the presence of β_4 and η_4 in the first equation, we

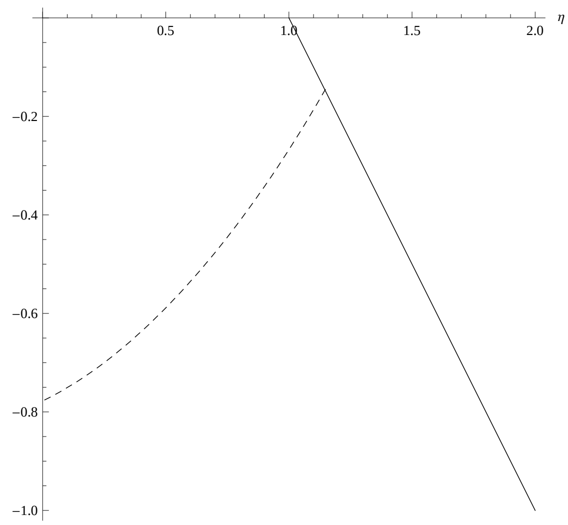

Figure 2.3: Graphic of the possible range of solutions for η and β after considering restrictions, for the numerical illustration proposed in Table 2.2

need more information to determine their values. In this sense, the next and last equation to integrate the model will be dependent on β_4 and η_4, assuming we have a fixed value for $\hat{C}$. This way, following the same procedure used in Adkins and Paxson (2011b) and assuming $\hat{C}$ is fixed initially, we can determine β_4 and η_4 using this new equation together with (2.51), and finally determine $\hat{P}$ using (2.56).

We shall then proceed to the deduction of the last equation. Using $A_4 = \frac{\hat{C}}{(r-\alpha_C)\eta_4 \hat{P}^{\beta_4}\hat{C}^{\eta_4}}$ and $\hat{P} = -\frac{\hat{C}(r-\alpha_P)\beta_4}{(r-\alpha_C)\eta_4}$ in (2.54) we will have the following equivalences,

$$A_4\hat{P}^{\beta_4}\hat{C}^{\eta_4} + \frac{\hat{P}}{r-\alpha_P} - \frac{\hat{C}}{r-\alpha_C} = A_4 {P_I}^{\beta_4}{C_I}^{\eta_4} + \frac{P_I}{r-\alpha_P} - \frac{C_I}{r-\alpha_C} - K \Leftrightarrow$$

$$A_4[\hat{P}^{\beta_4}\hat{C}^{\eta_4} - {P_I}^{\beta_4}{C_I}^{\eta_4}] + \frac{\hat{P}}{r-\alpha_P} - \frac{\hat{C}}{r-\alpha_C} = \frac{P_I}{r-\alpha_P} - \frac{C_I}{r-\alpha_C} - K \overset{def.\,A_4}{\Leftrightarrow}$$

$$\frac{\hat{C}}{(r-\alpha_C)\eta_4\hat{P}^{\beta_4}\hat{C}^{\eta_4}}[\hat{P}^{\beta_4}\hat{C}^{\eta_4} - {P_I}^{\beta_4}{C_I}^{\eta_4}] + \frac{\hat{P}}{r-\alpha_P} - \frac{\hat{C}}{r-\alpha_C} = \frac{P_I}{r-\alpha_P} - \frac{C_I}{r-\alpha_C} - K \Leftrightarrow$$

$$\frac{\hat{C}}{(r-\alpha_C)\eta_4}[1 - \frac{{P_I}^{\beta_4}{C_I}^{\eta_4}}{\hat{P}^{\beta_4}\hat{C}^{\eta_4}}] + \frac{\hat{P}}{r-\alpha_P} - \frac{\hat{C}}{r-\alpha_C} = \frac{P_I}{r-\alpha_P} - \frac{C_I}{r-\alpha_C} - K \overset{def.\,\hat{P}}{\Leftrightarrow}$$

$$\frac{\hat{C}}{(r-\alpha_C)\eta_4}\left[1 - \beta_4 - \eta_4 - \frac{{P_I}^{\beta_4}{C_I}^{\eta_4}}{\hat{C}^{\beta_4+\eta_4}}\left(\frac{-\beta_4(r-\alpha_P)}{\eta_4(r-\alpha_C)}\right)^{-\beta_4}\right] - \frac{P_I}{r-\alpha_P} + \frac{C_I}{r-\alpha_C} + K = 0$$

This condition will now be incorporated in our model in (2.57), leading to:

$$\begin{aligned} H(\beta_4, \eta_4|\hat{C}) &= \frac{\hat{C}}{(r-\alpha_C)\eta_4}\left[1 - \beta_4 - \eta_4 - \frac{{P_I}^{\beta_4}{C_I}^{\eta_4}}{\hat{C}^{\beta_4+\eta_4}}\left(\frac{-\beta_4(r-\alpha_P)}{\eta_4(r-\alpha_C)}\right)^{-\beta_4}\right] \\ &\quad - \frac{P_I}{r-\alpha_P} + \frac{C_I}{r-\alpha_C} + K \\ &= 0 \end{aligned} \tag{2.57}$$

Naturally, using $A_4 = \frac{\hat{P}}{-(r-\alpha_P)\beta_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4}}$ and $\hat{C} = -\frac{\hat{P}(r-\alpha_C)\eta_4}{(r-\alpha_P)\beta_4}$ we could reach an equivalent expression conditioning instead on $\hat{P}$, $H(\beta_4, \eta_4|\hat{P})$.

Equations $Q(\beta_4, \eta_4) = 0$, with Q defined in (2.51), (2.56) and (2.57) form the two-factor model, noting that in expression (2.51) we will now consider β_4 and η_4, instead of β and η. We have 3 equations and 4 unknown variables, and thus the system is indeterminate. However, as we explained before, fixing $\hat{C}$ and using (2.57) will solve us that problem. This last technique is the reason why this method described in Adkins and Paxson (2011b) is called a *quasi analytical* approach. The procedure is all done analytically with the exception of this last step, where a value is fixed for one of the variables. Next we illustrate how we derive the solution, using this procedure.

Obtaining the solution

Below we present the three equations in the two-factor model of this section:

1. $\frac{1}{2}\sigma_P^2\beta_4(\beta_4-1) + \frac{1}{2}\sigma_C^2\eta_4(\eta_4-1) + \rho\sigma_P\sigma_C\beta_4\eta_4 + \alpha_P\beta_4 + \alpha_C\eta_4 - r = 0$
2. $\frac{\hat{P}}{-(r-\alpha_P)\beta_4} = \frac{\hat{C}}{(r-\alpha_C)\eta_4}$
3. $\frac{\hat{C}}{(r-\alpha_C)\eta_4}\left[1 - \beta_4 - \eta_4 - \frac{{P_I}^{\beta_4}{C_I}^{\eta_4}}{\hat{C}^{\beta_4+\eta_4}}\left(\frac{-\beta_4(r-\alpha_P)}{\eta_4(r-\alpha_C)}\right)^{-\beta_4}\right] - \frac{P_I}{r-\alpha_P} + \frac{C_I}{r-\alpha_C} + K = 0$

As we saw before, it is nearly impossible to find a closed-form solution, as the system is indeterminate. So, we choose a fixed value for $\hat{C}$, and then solve a system with (2.51) and (2.57) in order to find β_4 and η_4, and finally, we use (2.56) to identify $\hat{P}$. Alternatively, we could also fix one value for $\hat{P}$ and solve everything else in a similar way. This algorithm is presented below in a more schematic way.

1. Fix a value for $\hat{C}$.
2. Solve the system
$$\begin{cases} Q(\beta_4, \eta_4) = 0 \\ H(\beta_4, \eta_4|\hat{C}) = 0 \end{cases}$$
in order to find β_4 and η_4.
3. Solve
$$\frac{\hat{P}}{-(r-\alpha_P)\beta_4} = \frac{\hat{C}}{(r-\alpha_C)\eta_4}$$
in order to find $\hat{P}$.

This way, we can find β_4, η_4 and $\hat{P}$, using only the three equations we have, as long as we fix some value for either $\hat{C}$ or $\hat{P}$.

Let us consider $\hat{C} = 50$, just like it is done in Adkins and Paxson (2011b), as the fixed value chosen for $\hat{C}$. Also, we will assume the values presented on table 2.3 for the remaining parameters of the problem. Clearly, ranging $\hat{C}$, we can get a discrete version of the true relation between $\hat{P}$ and $\hat{C}$.

Parameters	Symbol	Value
Initial value for the revenues	P_I	80
Initial value for the operating costs	C_I	20
Cost of reinvestment	K	100
Revenue risk-adjusted drift rate	α_P	-0.01
Operating costs risk-adjusted drift rate	α_C	0.04
Risk-free interest rate	r	0.07
Revenue volatility	σ_P	0.3
Operating costs volatility	σ_C	0.3
Correlation between revenues and operating costs	ρ	0

Table 2.3: Parameter Values - Two-factor model

We must just refer that in the two figures (2.2) and (2.3), we assumed $\rho = -0.5$, whereas figure (2.4) assumes $\rho = 0$ instead. The choice of the values for the parameters presented in Table 2.3 is the same as Adkins and Paxson (2011b).

Applying the algorithm described above, leads us to two different solutions, because in the $2nd$ step of the algorithm, $Q(\beta_4, \eta_4) = 0$ and $H(\beta_4, \eta_4|\hat{C}) = 0$ intersect in two distinct points, as we can see in figure 2.4.

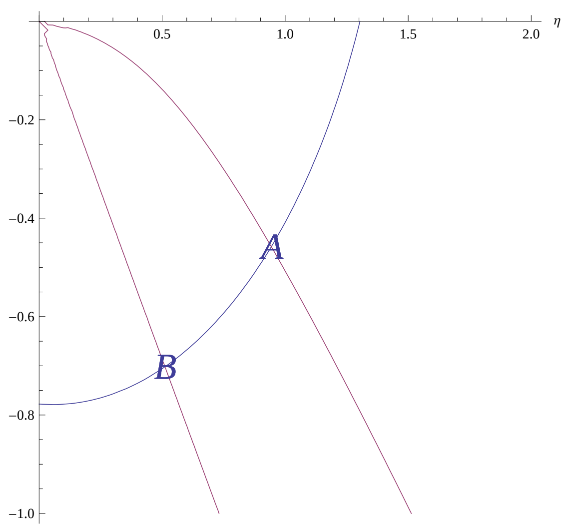

Figure 2.4: Intersection between $Q(\beta_4, \eta_4) = 0$ and $H(\beta_4, \eta_4|\hat{C}) = 0$

Variables	Point A	Point B
η_4	0.9438	0.5114
β_4	-0.4581	-0.7020
$\hat{P}$	64.715	183.041

Table 2.4: Points A and B - Two-factor model solution

Fortunately, we can actually consider only one of these points, since point B violates one of the very first conditions we imposed on the values that the revenues could take. We stated that, in general, $P < P_I$ and what we have in the case of point B is, in fact, $\hat{P} > P_I$, which allows us to eliminate B and select point A as the solution to our problem.

Deterministic Approach

Next, we will explore an attempt to find a solution to the deterministic version of the model we've been working with, as it is also done in Adkins and Paxson (2011b). Apparently, one could ask why should we be worried about a deterministic approach if we already developed a stochastic approach that is more general? The main advantage of this approach is that with it we will be able to determine an expression for the optimal replacement time, which is a big advantage and a really important information to include in our model.

Let P and C be the time-dependent variables previously introduced:

$$dX = \alpha_X X dt + \sigma_X X dZ_X, \quad X \in \{P, C\},\ \alpha_X \in \{\alpha_P, \alpha_C\},\ \sigma_X \in \{\sigma_P, \sigma_C\}$$

Considering a deterministic model implies that both σ_P and σ_C are 0. In this sense, we have that,

$$\begin{cases} dP = \alpha_P P dt \\ \\ dC = \alpha_C C dt \end{cases} \Leftrightarrow \begin{cases} \frac{dP}{dt} = \alpha_P P \\ \\ \frac{dC}{dt} = \alpha_C C \end{cases} \Leftrightarrow \begin{cases} P(t) = P_0 e^{\alpha_P t} \\ \\ C(t) = C_0 e^{\alpha_C t} \end{cases} \Leftrightarrow^{2} \begin{cases} P(t) = P_I e^{\alpha_P t} \\ \\ C(t) = C_I e^{\alpha_C t} \end{cases}$$

According to Dixit and Pindyck (1994), the value of the investment opportunity assuming we invest at some arbitrary future time T is given by:

$$\begin{aligned}(P_I e^{\alpha_P T} - K)e^{-rT} - (C_I e^{\alpha_C T} - K)e^{-rT} &= [P_I e^{\alpha_P T} - K - C_I e^{\alpha_C T} + K]e^{-rT} \\ &= [P_I e^{\alpha_P T} - C_I e^{\alpha_C T}]e^{-rT}\end{aligned}$$

which is precisely the expression that Adkins and Paxson (2011b) present for the deterministic case.

Consequently the present value, N, for such an asset with lifetime T is given by:

$$N = \int_0^T \Big(P_I e^{\alpha_P t} - C_I e^{\alpha_C t}\Big) e^{-rt} dt \tag{2.58}$$

Following Adkins and Paxson (2011b), we also assume that it is a renewal process, in the sense that we consider an infinite chain of replica assets, and T is the (constant and equal) renewal interval. Moreover, we assume that the reinvestment cost is also the same, K. Thus the value of such a renewal process, now denoted by W, is equal to the value of the investment opportunity N plus its value after renewal, discounted back at time T. As we assume equal reinvestment costs and a deterministic situation, it follows that,

[2] $P_0 = P(0) = P_I; C_0 = C(0) = C_I$.

$$
\begin{aligned}
W &= N + (W-K)e^{-rT} \\
&= \int_0^T \Big(P_I e^{\alpha_P t} - C_I e^{\alpha_C t}\Big) e^{-rt} dt + (W-K)e^{-rT} \Leftrightarrow \\
W(1-e^{-rT}) &= \int_0^T \Big(P_I e^{\alpha_P t} - C_I e^{\alpha_C t}\Big) e^{-rt} dt - Ke^{-rT}
\end{aligned}
$$

W can then be expressed as,

$$
W = \frac{1}{1-e^{-rT}} \left(\int_0^T \Big(P_I e^{\alpha_P t} - C_I e^{\alpha_C t}\Big) e^{-rt} dt - Ke^{-rT} \right) \tag{2.59}
$$

As it is stated in Adkins and Paxson (2011b), the consideration of a deterministic approach leads us to some changes in the results we've obtained so far, and will originate some new results. These new results are the following,

$$
\alpha_P \beta_4 + \alpha_C \eta_4 - r = 0 \tag{2.60}
$$

$$
\left(\frac{P_I}{\hat{P}}\right)^{\beta_4} \left(\frac{C_I}{\hat{C}}\right)^{\eta_4} = e^{-r\hat{T}} \tag{2.61}
$$

$$
\hat{T} = \frac{\beta_4}{r} \ln\left(\frac{\hat{P}}{P_I}\right) + \frac{\eta_4}{r} \ln\left(\frac{\hat{C}}{C_I}\right) \tag{2.62}
$$

$$
\hat{P}\left(\frac{1}{r} + \frac{\alpha_P}{r} \times \frac{e^{-r\hat{T}}}{r-\alpha_P}\right) - \hat{C}\left(\frac{1}{r} + \frac{\alpha_C}{r} \times \frac{e^{-r\hat{T}}}{r-\alpha_C}\right) = A_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4} - A_4 P_I^{\beta_4} C_I^{\eta_4} + \frac{\hat{P}}{(r-\alpha_P)} - \frac{\hat{C}}{(r-\alpha_C)} \tag{2.63}
$$

$$
\hat{P}\left(\frac{1}{r} + \frac{\alpha_P}{r} \times \frac{e^{-r\hat{T}}}{r-\alpha_P}\right) - \hat{C}\left(\frac{1}{r} + \frac{\alpha_C}{r} \times \frac{e^{-r\hat{T}}}{r-\alpha_C}\right) = \frac{P_I}{(r-\alpha_P)} - \frac{C_I}{(r-\alpha_C)} - K \tag{2.64}
$$

which now we prove.

Let us start by considering the ellipse equation (2.51), but now implementing the deterministic approach. Then $\sigma_P = \sigma_C = 0$, and thus,

$$
\begin{aligned}
& Q(\beta_4, \eta_4) = \frac{1}{2}\sigma_P^2 \beta_4(\beta_4 - 1) + \frac{1}{2}\sigma_C^2 \eta_4(\eta_4 - 1) + \rho\sigma_P\sigma_C\beta_4\eta_4 + \alpha_P\beta_4 + \alpha_C\eta_4 - r = 0 \\
\Leftrightarrow^{3} \quad & \alpha_P\beta_4 + \alpha_C\eta_4 - r = 0
\end{aligned}
$$

which gives us immediately result (2.60).

[3] $\sigma_P = \sigma_C = 0$

In order to derive result (2.61), we have to take the firm's value F, and consider the component of F that denotes the renewal option value. This is also the component that solves the homogeneous part of (2.45). With that in mind, we will use the notation F_H for this part of F. Considering the fourth quadrant for the parameters β and η we have,

$$F_H(P, C) = A_4 P^{\beta_4} C^{\eta_4} \tag{2.65}$$

It is true that the value of F_H for the initial values of (P_I, C_I) is equal to its value for the limiting values of $(\hat{P}, \hat{C})$, discounted at the present time. This is expressed by the following equations,

$$\begin{aligned} F_H(P_I, C_I) &= F_H(\hat{P}, \hat{C}) e^{-r\hat{T}} \Leftrightarrow \\ A_4 P_I^{\beta_4} C_I^{\eta_4} &= A_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4} e^{-r\hat{T}} \end{aligned}$$

Using this reasoning we arrive at result (2.61).

The expression (2.62) for $\hat{T}$ that is going to be derived next, differs a bit from what is presented in Adkins and Paxson (2011b). Considering the result we've just derived, see (2.61), we believe it is more intuitive to derive $\hat{T}$ directly from this equation. With this in mind, we shall now take (2.61), and obtain $\hat{T}$.

$$\begin{aligned} & \left(\frac{P_I}{\hat{P}}\right)^{\beta_4} \left(\frac{C_I}{\hat{C}}\right)^{\eta_4} = e^{-r\hat{T}} \\ & \Leftrightarrow -r\hat{T} = -\beta_4 \ln\left(\frac{\hat{P}}{P_I}\right) - \eta_4 \ln\left(\frac{\hat{C}}{C_I}\right) \\ & \Leftrightarrow \hat{T} = \frac{\beta_4}{r} \ln\left(\frac{\hat{P}}{P_I}\right) + \frac{\eta_4}{r} \ln\left(\frac{\hat{C}}{C_I}\right) \end{aligned}$$

The derivation of the two remaining equalities is easy but long once it comes down to a lot of computations using the two first results mentioned in this section, (2.60) and (2.61).

In order to prove (2.63), we start with the left side of this equation, and we simplify it, using some of the already proved relationships:

$$\hat{P}\left(\frac{1}{r}+\frac{\alpha_P}{r}\times\frac{e^{-r\hat{T}}}{r-\alpha_P}\right)-\hat{C}\left(\frac{1}{r}+\frac{\alpha_C}{r}\times\frac{e^{-r\hat{T}}}{r-\alpha_C}\right)=$$

$$=\frac{\hat{P}}{r}+\frac{\hat{P}\ \alpha_P\ e^{-r\hat{T}}}{r(r-\alpha_P)}-\frac{\hat{C}}{r}-\frac{\hat{C}\ \alpha_C\ e^{-r\hat{T}}}{r(r-\alpha_C)}$$

$$=^{4}\frac{\hat{P}}{r}+\frac{\hat{P}\ \alpha_P}{r(r-\alpha_P)}\left(\frac{P_I}{\hat{P}}\right)^{\beta_4}\left(\frac{C_I}{\hat{C}}\right)^{\eta_4}-\frac{\hat{C}}{r}-\frac{\hat{C}\ \alpha_C}{r(r-\alpha_C)}\left(\frac{P_I}{\hat{P}}\right)^{\beta_4}\left(\frac{C_I}{\hat{C}}\right)^{\eta_4}$$

$$=\frac{\hat{P}}{r}+\frac{\alpha_P\ (-\beta_4)}{r}\times\underbrace{\frac{\hat{P}}{(-\beta_4)(r-\alpha_P)}\times\frac{1}{\hat{P}^{\beta_4}\hat{C}^{\eta_4}}}_{=A_4}\times P_I^{\beta_4}C_I^{\eta_4}$$

$$-\frac{\hat{C}}{r}-\frac{\alpha_C\ \eta_4}{r}\times\underbrace{\frac{\hat{C}}{\eta_4(r-\alpha_C)}\times\frac{1}{\hat{P}^{\beta_4}\hat{C}^{\eta_4}}}_{=A_4}\times P_I^{\beta_4}C_I^{\eta_4}$$

$$=\frac{\hat{P}}{r}-\frac{\hat{C}}{r}+A_4P_I^{\beta_4}C_I^{\eta_4}\underbrace{\left[\frac{\alpha_P(-\beta_4)}{r}-\frac{\alpha_C\eta_4}{r}\right]}_{=\frac{-r}{r}=-1}$$

$$=\frac{\hat{P}}{r}-\frac{\hat{C}}{r}-A_4P_I^{\beta_4}C_I^{\eta_4}$$

$$=\frac{\hat{P}}{r}\times\frac{(r-\alpha_P)}{(r-\alpha_P)}-\frac{\hat{C}}{r}\times\frac{(r-\alpha_C)}{(r-\alpha_C)}-A_4P_I^{\beta_4}C_I^{\eta_4}$$

$$=\frac{\hat{P}}{(r-\alpha_P)}\times\frac{(r-\alpha_P)}{r}-\frac{\hat{C}}{(r-\alpha_C)}\times\frac{(r-\alpha_C)}{r}-A_4P_I^{\beta_4}C_I^{\eta_4}$$

$$=\frac{\hat{P}}{(r-\alpha_P)}\times\left[1-\frac{\alpha_P}{r}\right]-\frac{\hat{C}}{(r-\alpha_C)}\times\left[1-\frac{\alpha_C}{r}\right]-A_4P_I^{\beta_4}C_I^{\eta_4}$$

$$=-\frac{\hat{P}\ \alpha_P}{r(r-\alpha_P)}+\frac{\hat{C}\ \alpha_C}{r(r-\alpha_C)}-A_4P_I^{\beta_4}C_I^{\eta_4}+\frac{\hat{P}}{(r-\alpha_P)}-\frac{\hat{C}}{(r-\alpha_C)}$$

$$=-\frac{\hat{P}}{(r-\alpha_P)\beta_4}\frac{\alpha_P\ \beta_4}{r}+\frac{\hat{C}}{(r-\alpha_C)\eta_4}\frac{\alpha_C\ \eta_4}{r}-A_4P_I^{\beta_4}C_I^{\eta_4}+\frac{\hat{P}}{(r-\alpha_P)}-\frac{\hat{C}}{(r-\alpha_C)}$$

$$=\underbrace{-\frac{\hat{P}}{(r-\alpha_P)\beta_4}\ \frac{1}{\hat{P}^{\beta_4}\hat{C}^{\eta_4}}}_{=A_4}\frac{\alpha_P\ \beta_4}{r}\ \hat{P}^{\beta_4}\hat{C}^{\eta_4}+\underbrace{\frac{\hat{C}}{(r-\alpha_C)\eta_4}\ \frac{1}{\hat{P}^{\beta_4}\hat{C}^{\eta_4}}}_{=A_4}\frac{\alpha_C\ \eta_4}{r}\ \hat{P}^{\beta_4}\hat{C}^{\eta_4}$$

$$-A_4P_I^{\beta_4}C_I^{\eta_4}+\frac{\hat{P}}{(r-\alpha_P)}-\frac{\hat{C}}{(r-\alpha_C)}$$

$$=A_4\hat{P}^{\beta_4}\hat{C}^{\eta_4}\underbrace{\left[\frac{\alpha_P\beta_4}{r}+\frac{\alpha_C\eta_4}{r}\right]}_{=\frac{r}{r}=1}-A_4P_I^{\beta_4}C_I^{\eta_4}+\frac{\hat{P}}{(r-\alpha_P)}-\frac{\hat{C}}{(r-\alpha_C)}$$

$$=A_4\hat{P}^{\beta_4}\hat{C}^{\eta_4}-A_4P_I^{\beta_4}C_I^{\eta_4}+\frac{\hat{P}}{(r-\alpha_P)}-\frac{\hat{C}}{(r-\alpha_C)}$$

and thus we end up with the right-hand side of equation (2.63), and so the result is proved.

[4] $e^{-r\hat{T}}=\left(\frac{P_I}{\hat{P}}\right)^{\beta_4}\left(\frac{C_I}{\hat{C}}\right)^{\eta_4}$

Once (2.63) is proved, in order to prove expression (2.64) we consider the **value-matching relationship** given by (2.54), and from there we have that,

$$A_4 \hat{P}^{\beta_4} \hat{C}^{\eta_4} - A_4 P_I^{\beta_4} C_I^{\eta_4} + \frac{\hat{P}}{(r-\alpha_P)} - \frac{\hat{C}}{(r-\alpha_C)} = \frac{P_I}{(r-\alpha_P)} - \frac{C_I}{(r-\alpha_C)} - K$$

Thus, using (2.63) it follows that,

$$\hat{P}\left(\frac{1}{r} + \frac{\alpha_P}{r} \times \frac{e^{-r\hat{T}}}{r-\alpha_P}\right) - \hat{C}\left(\frac{1}{r} + \frac{\alpha_C}{r} \times \frac{e^{-r\hat{T}}}{r-\alpha_C}\right) = \frac{P_I}{(r-\alpha_P)} - \frac{C_I}{(r-\alpha_C)} - K$$

This concludes our section on investment under uncertainty, as well as our chapter on introductory theoretical notions. In the next chapter we will introduce the reciprocal switching models on energy markets.

Chapter 3

Energy-Switching Options

In this chapter we will continue studying investment under uncertainty but now from a different perspective. In the first section, in a similar way as considered in Adkins and Paxson (2011a) we will start by considering the production of some saleable output. Later on we will look at a specific application considering energy production. The uncertainty will then be related to the feedstock source that one should choose. The optimality of the feedstock's source in use might change with several factors. It can, for example, vary with cost, season, etc. In this sense, our main goal is to decide which feedstock is optimal to use in each moment, and make a switch, if necessary, according to an optimal criteria to be defined later, following the same procedure used in Adkins and Paxson (2011a).

This approach assumes the behaviour of the feedstock prices to follow a geometric Brownian motion dynamic, and it follows, in a lot of aspects, the reasoning presented in Adkins and Paxson (2011a).

Posteriorly, in a second section, we make a few changes to our initial assumptions. First of all, a change of the dynamic for the feedstock prices is considered. Instead of the geometric Brownian motion, we assume that the prices behave according to an arithmetic Brownian motion. The choice for the new dynamic is inspired in Hem and Svendsen (2010), where such a behaviour is considered for the long-term feedstock prices.

Also based on Hem and Svendsen (2010), we will adopt the position of a company responsible for the extraction of both oil and natural gas, which will now become our two possible feedstocks, and the development of this next model shall be based on their perspective. The main idea of this new approach is to consider a concrete scenario, based on the one presented in Hem and Svendsen (2010) and adapt the first established model to it, that has been developed following the reasoning presented in Adkins and Paxson (2011a), considering a different price dynamic.

3.1 Geometric Brownian Motion Dynamic

We shall now consider the production line of a certain item, and the existence of two possible sources for the feedstock in use, as described in Adkins and Paxson (2011a). We are going to denote the feedstock coming from the first source as feedstock 1 and the one coming from the second source as feedstock 2, and we will refer to the one in use as the incumbent and to the the one that is not being used at the moment as the substitute. We will also fix switching costs, for whenever we consider the switch between the incumbent feedstock and the substitute.

Let X_1 denote the price of feedstock 1 and X_2 the price of feedstock 2. In this first approach we will assume both variables to follow a geometric Brownian motion as we can see below in (3.1) and once again as it is done in Adkins and Paxson (2011a). Thus the dynamic for the price of feedstock I, for $I \in \{1, 2\}$ is given by,

$$dX_I = \alpha_I X_I dt + \sigma_I X_I dZ_I \tag{3.1}$$

where α_I is the risk-adjusted drift rate for feedstock I, σ_I the volatility rate and dZ_I the increment of a standard Wiener process. Also, we have that $Cov[dX_1, dX_2] = \rho\sigma_1\sigma_2 dt$, where $|\rho| \leq 1$, and thus similarly to what was stated in the last chapter, $\rho\sigma_1\sigma_2$ will denote the covariance term.

We must now define F_I, as the firm's value when feedstock I is the incumbent. The firm's value is composed exactly by the same elements we described in the previous chapter: one term representing the option value, now called switching option value, and one term denoting the incumbent asset value. We must also define D_0 as the output or selling price of our final product. In this sense, and according to Adkins and Paxson (2011a), the net cash flow for this case scenario is going to be given by $D_0 - X_I$, meaning that we have the price at which the final product is being sold minus the price paid for the feedstock used. So, once more combining the dynamic programming principle with Ito's lemma and applying them to $e^{-rt}F_I(X_1, X_2)$, leads us to the following dynamic programming equation,

$$\frac{1}{2}\sigma_1^2 X_1^2 \frac{\partial^2 F_I}{\partial X_1^2} + \frac{1}{2}\sigma_2^2 X_2^2 \frac{\partial^2 F_I}{\partial X_2^2} + \rho\sigma_1\sigma_2 X_1 X_2 \frac{\partial^2 F_I}{\partial X_1 \partial X_2} + \alpha_1 X_1 \frac{\partial F_I}{\partial X_1} + \alpha_2 X_2 \frac{\partial F_I}{\partial X_2} - rF_I + (D_0 - X_I) = 0 \tag{3.2}$$

where r denotes the risk-free interest rate, and F_I is always a short term for $F_I(X_1, X_2)$.

Moreover, we remark that, similarly to what we have already in section 2.2. of chapter 2, we use the Ito's formula for a two dimensional process (although in section 2.1 we derived the dynamic programming principle and equation for the one dimensional case, for simplicity of notation), that can be adapted to this case. We also refer Björk (2004) for the Ito's formula in the multidimensional case.

Characteristic Root Equation

Similarly to what was done before, and following the same course of action used in Adkins and Paxson (2011a), next we propose a solution to equation (3.2), compute its derivatives and replace them in (3.2). At last, we must analyse the result and see what sort of information we can outdraw from it.

According to Adkins and Paxson (2011a), the proposed solution, that we will denote by G_I, is going to be given by,

$$G_I(X_1, X_2) = A_I X_1^{\beta_I} X_2^{\eta_I} + \frac{D_0}{r} - \frac{X_I}{r - \alpha_I} \tag{3.3}$$

for $I \in \{1, 2\}$, and where A_I, β_I and η_I are parameters still to be determined. It is worth mentioning that the term $A_I X_1^{\beta_I} X_2^{\eta_I}$ in (3.3) corresponds to the switching option value and is also the solution for the homogeneous part of (3.2), whereas the term $\frac{D_0}{r} - \frac{X_I}{r-\alpha_I}$ corresponds to the incumbent value in the absence of a switching option. The derivatives of G_I are given by,

$$\frac{\partial G_I}{\partial X_1} = \begin{cases} A_I \beta_I X_1^{\beta_I - 1} X_2^{\eta_I} - \frac{1}{r-\alpha_1} & \text{if } I = 1 \\ A_I \beta_I X_1^{\beta_I - 1} X_2^{\eta_I} & \text{if } I = 2 \end{cases} \tag{3.4}$$

$$\frac{\partial G_I}{\partial X_2} = \begin{cases} A_I \eta_I X_1^{\beta_I} X_2^{\eta_I - 1} & \text{if } I = 1 \\ A_I \eta_I X_1^{\beta_I} X_2^{\eta_I - 1} - \frac{1}{r-\alpha_2} & \text{if } I = 2 \end{cases} \tag{3.5}$$

$$\frac{\partial^2 G_I}{\partial X_1^2} = A_I \beta_I (\beta_I - 1) X_1^{\beta_I - 2} X_2^{\eta_I} \tag{3.6}$$

$$\frac{\partial^2 G_I}{\partial X_2^2} = A_I \eta_I (\eta_I - 1) X_1^{\beta_I} X_2^{\eta_I - 2} \tag{3.7}$$

$$\frac{\partial^2 G_I}{\partial X_1 \partial X_2} = A_I \beta_I \eta_I X_1^{\beta_I - 1} X_2^{\eta_I - 1} \tag{3.8}$$

Replacing all of these derivatives in equation (3.2) we obtain the following,

$$\begin{aligned} &\frac{1}{2}\sigma_1^2 X_1^2 A_I \beta_I (\beta_I - 1) X_1^{\beta_I - 2} X_2^{\eta_I} + \frac{1}{2}\sigma_2^2 X_2^2 A_I \eta_I (\eta_I - 1) X_1^{\beta_I} X_2^{\eta_I - 2} + \rho\sigma_1\sigma_2 X_1 X_2 A_I \beta_I \eta_I X_1^{\beta_I - 1} X_2^{\eta_I - 1} \\ &+\alpha_1 X_1 A_I \beta_I X_1^{\beta_I - 1} X_2^{\eta_I} + \alpha_2 X_2 A_I \eta_I X_1^{\beta_I} X_2^{\eta_I - 1} - \alpha_I X_I \frac{1}{r - \alpha_I} - r\left(A_I X_1^{\beta_I} X_2^{\eta_I} + \frac{D_0}{r} - \frac{X_I}{r - \alpha_I} \right) \\ &+(D_0 - X_I) = 0 \Leftrightarrow \\ &A_I X_1^{\beta_I} X_2^{\eta_I} [\frac{1}{2}\sigma_1^2 \beta_I (\beta_I - 1) + \frac{1}{2}\sigma_2^2 \eta_I (\eta_I - 1) + \rho\sigma_1\sigma_2 \beta_I \eta_I + \alpha_1 \beta_I + \alpha_2 \eta_I - r] \\ &-\frac{\alpha_I X_I}{r - \alpha_I} - D_0 + \frac{r X_I}{r - \alpha_I} + D_0 - X_I = 0 \end{aligned} \tag{3.9}$$

This originates what is denoted in Adkins and Paxson (2011a) as the *characteristic root equation* given by $\frac{1}{2}\sigma_1^2 \beta_I (\beta_I - 1) + \frac{1}{2}\sigma_2^2 \eta_I (\eta_I - 1) + \rho\sigma_1\sigma_2 \beta_I \eta_I + \alpha_1 \beta_I + \alpha_2 \eta_I - r = 0$, and we can define a function Q_I, for each $I \in \{1, 2\}$, of β_I and η_I, as,

$$Q_I(\beta_I, \eta_I) = \frac{1}{2}\sigma_1^2 \beta_I (\beta_I - 1) + \frac{1}{2}\sigma_2^2 \eta_I (\eta_I - 1) + \rho\sigma_1\sigma_2 \beta_I \eta_I + \alpha_1 \beta_I + \alpha_2 \eta_I - r = 0 \tag{3.10}$$

The graphic of this function originates an ellipse. Similarly to what we did in the last chapter, we will present the graphic of $Q_I(\beta_I, \eta_I) = 0$ in figure 3.1, where the values for the fixed constants are presented in table 3.1, and are the same ones used in Adkins and Paxson (2011a).

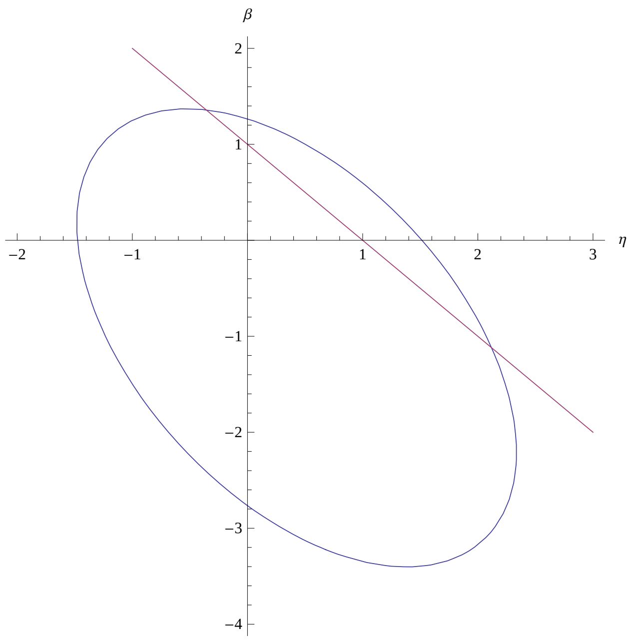

Figure 3.1: Graphic of the ellipse defined by $Q_I(\beta_I, \eta_I) = 0$ and the line $\beta_I + \eta_I = 1$

Parameters	Symbol	Value
Feedstock 1 volatility	σ_1	0.2
Feedstock 2 volatility	σ_2	0.25
Feedstock 1 risk-adjusted drift rate	α_1	0.05
Feedstock 2 risk-adjusted drift rate	α_2	0.03
Risk-free interest rate	r	0.07
Correlation between X_1 and X_2	ρ	0.5

Table 3.1: Parameter Values - Ellipse equation - Energy switch model considering a geometric Brownian motion dynamic

Once more the homogeneity-degree-1 condition, given by $\beta_I + \eta_I = 1$, is represented in figure 3.1, even though it won't be used for now. However for some simplifications presented in Adkins and Paxson (2011a) such an assumption is, in fact, made.

The procedure used from now on is going to be to divide the term of G_I correspondent to the option value into four quadrants and afterwards identify which quadrant will be relevant to use in the future. As in Adkins and Paxson (2011a), the four quadrants are going to be,

$$
\begin{aligned}
I &: \{\beta_I, \eta_I\} \quad \beta_I \geq 0; \eta_I \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I1}, \eta_{I1}\} \\
II &: \{\beta_I, \eta_I\} \quad \beta_I \geq 0; \eta_I \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I2}, \eta_{I2}\} \\
III &: \{\beta_I, \eta_I\} \quad \beta_I \leq 0; \eta_I \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I3}, \eta_{I3}\} \\
IV &: \{\beta_I, \eta_I\} \quad \beta_I \leq 0; \eta_I \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I4}, \eta_{I4}\}
\end{aligned}
$$

Thus, let us rewrite G_I as,

$$G_I(X_1, X_2) = A_{I1}X_1^{\beta_{I1}}X_2^{\eta_{I1}} + A_{I2}X_1^{\beta_{I2}}X_2^{\eta_{I2}} + A_{I3}X_1^{\beta_{I3}}X_2^{\eta_{I3}} + A_{I4}X_1^{\beta_{I4}}X_2^{\eta_{I4}} + \frac{D_0}{r} - \frac{X_I}{r - \alpha_I} \tag{3.11}$$

where we apply similar arguments to the ones that we presented in chapter 2 to justify this quadrant division.

We must now proceed with the choice of the right quadrant to appear in G_I, for each $I \in \{1, 2\}$. In order to do this, some economical arguments will be presented according to what is stated in Adkins and Paxson (2011a), that leave no doubts of what is the most suitable quadrant to select in each case.

Let us start by supposing that the incumbent feedstock is feedstock 1, and thus we shall treat G_1. So, we know that whenever the price of feedstock 1 tends to zero, the correspondent option value present in the firm's value, let's denote it as G_{H1}, should do so as well. Similarly, whenever the price for feedstock 2 is infinitely large, the correspondent option value in G_{H1}, will tend to zero. Summarizing this two last arguments we have the following,

1. $\lim_{X_1 \to 0} G_{H1} = 0 \Leftrightarrow \lim_{X_1 \to 0} A_1X_1^{\beta_1}X_2^{\eta_1} = 0 \Rightarrow \beta_1 > 0$
2. $\lim_{X_2 \to \infty} G_{H1} = 0 \Leftrightarrow \lim_{X_2 \to \infty} A_1X_1^{\beta_1}X_2^{\eta_1} = 0 \Rightarrow \eta_1 \leq 0$

This implies that whenever feedstock 1 is the incumbent we should choose the second quadrant for G_1. In this sense, we will have,

$$G_1(X_1, X_2) = A_{12}X_1^{\beta_{12}}X_2^{\eta_{12}} + \frac{D_0}{r} - \frac{X_1}{r - \alpha_1} \tag{3.12}$$

Now we ought to focus our attention on the scenario where feedstock 2 is the incumbent, and so we will be dealing with G_2. Similarly to what happened in the previous case, we can state that whenever the price of feedstock 1 tends to infinity the option value term of the firm's value when feedstock 2 is the incumbent, let us denote it by G_{H2}, will tend to zero. Also, when the price of feedstock 2 tends to zero, G_{H2} will tend to zero as well. Considering all this, we have that,

1. $\lim_{X_1 \to \infty} G_{H2} = 0 \Leftrightarrow \lim_{X_1 \to \infty} A_2X_1^{\beta_2}X_2^{\eta_2} = 0 \Rightarrow \beta_2 \leq 0$
2. $\lim_{X_2 \to 0} G_{H2} = 0 \Leftrightarrow \lim_{X_2 \to 0} A_2X_1^{\beta_2}X_2^{\eta_2} = 0 \Rightarrow \eta_2 > 0$

This results imply that we should choose the fourth quadrant whenever feedstock 2 is the incumbent feedstock. Thus,

$$G_2(X_1, X_2) = A_{24}X_1^{\beta_{24}}X_2^{\eta_{24}} + \frac{D_0}{r} - \frac{X_2}{r - \alpha_2} \tag{3.13}$$

Before we move any further we must note three aspects that the reader should take into account. Firstly, the notation G_{HI}, $I \in \{1, 2\}$ for the option value term figuring in the firm's value expression whenever feedstock I is the incumbent, was adopted because the option value term in G_I also corresponds to the solution of the homogeneous part of (3.2), as we mentioned above. Secondly, it must be taken into account that all these arguments, and the consequent choice of the quadrants for each G_I, follow the perspective of someone interested in acquiring the final product. Clearly, if we were considering the perspective of the company responsible for the production the whole reasoning would be opposite. We choose to take this point of view to follow the reasoning presented in Adkins and Paxson (2011a). Later on, this perspective will be reversed, and we will analyse the company's side as well. At last, we must observe that the quadrant choice that was made above will influence directly the definition of the *characteristic root equation*, presented in (3.10). If we are going to consider from now on, that when the incumbent feedstock is 1, we'll have the values of β_1 and η_1 belonging to the second quadrant, and when the incumbent feedstock is 2, we'll have those values belonging to the fourth quadrant, then $Q_I(\beta_I, \eta_I)$, can be rewritten for each $I \in \{1, 2\}$, as follows,

$$Q_I(\beta_I, \eta_I) = 0 \Leftrightarrow \begin{cases} Q_1(\beta_{12}, \eta_{12}) = 0 & \text{if } I = 1 \\ \\ Q_2(\beta_{24}, \eta_{24}) = 0 & \text{if } I = 2 \end{cases} \tag{3.14}$$

Value-Matching Relationship and Smooth Pasting Conditions

Once more following the same course of action we adopted before, we shall now establish the value-matching relationships and smooth pasting conditions for this case. However, we should first define $\hat{X}_{IJ}$ and K_{IJ}.

$\hat{X}_{IJ}$, when $I \neq J$, denotes the threshold price for feedstock I, whenever I is the incumbent and J is the substitute. However, when $I = J$, we get the case where there is only an incumbent feedstock and no substitute. This means that when we refer to a certain price as a threshold price, we are evaluating it in the period of time immediately before a switch is considered. In this sense, for $I, J \in \{1, 2\}$ we have the following,

- $\hat{X}_{12}$ is the threshold price for feedstock 1, being it the incumbent, when 2 is the substitute;
- $\hat{X}_{21}$ is precisely the opposite of the previous one, thus the threshold price for feedstock 2 when it is the incumbent and 1 is the substitute;
- $\hat{X}_{11}$ brings up the case where there is no other substitute feedstock and thus, in this case, it is the threshold price for feedstock 1 whenever the substitute feedstock is also 1;
- $\hat{X}_{22}$ similarly, is the threshold price for feedstock 2 whenever the substitute feedstock is also 2.

K_{IJ} is more simple to define, as it denotes the cost of switching from feedstock I to feedstock J, whenever I is the incumbent and J is the substitute. K_{IJ} can only be defined when we have $I \neq J$.

We are now in conditions of presenting the value-matching relationships and posteriorly the smooth pasting conditions.

As the function G_I originates two different functions, one for each value of $I \in \{1, 2\}$, this will produce two value-matching relationships. According to Adkins and Paxson (2011a), they state that

1. for the case where feedstock 1 is the incumbent and feedstock 2 is the substitute, the difference between the value of the substitute $G_2(\hat{X}_{12}, \hat{X}_{22})$ and the value of the incumbent $G_1(\hat{X}_{12}, \hat{X}_{22})$, immediately before a switch, is equal to the switching cost K_{12}.

2. for the case where feedstock 2 is the incumbent and feedstock 1 is the substitute, the difference between the value of the substitute $G_1(\hat{X}_{11}, \hat{X}_{21})$ and the value of the incumbent $G_2(\hat{X}_{11}, \hat{X}_{21})$, immediately before a switch, equals the switching cost K_{21}.

These last two statements can be expressed by equations (3.15) and (3.16) presented below.

$$G_1(\hat{X}_{12}, \hat{X}_{22}) = G_2(\hat{X}_{12}, \hat{X}_{22}) - K_{12} \tag{3.15}$$

$$G_2(\hat{X}_{11}, \hat{X}_{21}) = G_1(\hat{X}_{11}, \hat{X}_{21}) - K_{21} \tag{3.16}$$

If we take now the expressions for G_1 and G_2 from (3.12) and (3.13) and substitute them in (3.15) and (3.16), respectively, we obtain the following,

$$A_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}} + \frac{D_0}{r} - \frac{\hat{X}_{12}}{r-\alpha_1} = A_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}} + \frac{D_0}{r} - \frac{\hat{X}_{22}}{r-\alpha_2} - K_{12} \tag{3.17}$$

$$A_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}} + \frac{D_0}{r} - \frac{\hat{X}_{21}}{r-\alpha_2} = A_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}} + \frac{D_0}{r} - \frac{\hat{X}_{11}}{r-\alpha_1} - K_{21} \tag{3.18}$$

Once the value-matching relationships were already presented, we should now proceed in order to obtain the smooth pasting conditions associated with (3.15) and (3.16). As we have two variables considered in each value-matching relationship, (3.15) and (3.16) will originate two smooth pasting conditions each, one concerning each variable, which makes a total of four smooth pasting conditions. These conditions essentially guarantee the smoothness of G_1 and G_2 in the threshold points. They also make the problem in question a free boundary problem. Let us then present them below,

$$\left.\frac{\partial G_1}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12}\\X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12}\\X_2=\hat{X}_{22}}} \Leftrightarrow A_{12}\beta_{12}\hat{X}_{12}^{\beta_{12}-1}\hat{X}_{22}^{\eta_{12}} - \frac{1}{r-\alpha_1} = A_{24}\beta_{24}\hat{X}_{12}^{\beta_{24}-1}\hat{X}_{22}^{\eta_{24}} \tag{3.19}$$

$$\left.\frac{\partial G_1}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12}\\X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12}\\X_2=\hat{X}_{22}}} \Leftrightarrow A_{12}\eta_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}-1} = A_{24}\eta_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}-1} - \frac{1}{r-\alpha_2} \tag{3.20}$$

$$\left.\frac{\partial G_2}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{11}\\X_2=\hat{X}_{21}}} = \left.\frac{\partial G_1}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{11}\\X_2=\hat{X}_{21}}} \Leftrightarrow A_{24}\beta_{24}\hat{X}_{11}^{\beta_{24}-1}\hat{X}_{21}^{\eta_{24}} = A_{12}\beta_{12}\hat{X}_{11}^{\beta_{12}-1}\hat{X}_{21}^{\eta_{12}} - \frac{1}{r-\alpha_1} \tag{3.21}$$

$$\left.\frac{\partial G_2}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{11}\\X_2=\hat{X}_{21}}} = \left.\frac{\partial G_1}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{11}\\X_2=\hat{X}_{21}}} \Leftrightarrow A_{24}\eta_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}-1} - \frac{1}{r-\alpha_2} = A_{12}\eta_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}-1} \tag{3.22}$$

At the light of what is done in Adkins and Paxson (2011a), working with the smooth pasting conditions in sets of two equations (the first two conditions and the last two), we are able to isolate the terms $A_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}}$, $A_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}}$, $A_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}}$ and $A_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}}$, as it is presented in (3.23), (3.24), (3.25) and (3.26).

$$A_{12}\hat{X}_{12}^{\beta_{12}}\hat{X}_{22}^{\eta_{12}} = \frac{1}{\Delta}\left[\frac{\eta_{24}\hat{X}_{12}}{r-\alpha_1} + \frac{\beta_{24}\hat{X}_{22}}{r-\alpha_2}\right] \tag{3.23}$$

$$A_{24}\hat{X}_{12}^{\beta_{24}}\hat{X}_{22}^{\eta_{24}} = \frac{1}{\Delta}\left[\frac{\eta_{12}\hat{X}_{12}}{r-\alpha_1} + \frac{\beta_{12}\hat{X}_{22}}{r-\alpha_2}\right] \tag{3.24}$$

$$A_{12}\hat{X}_{11}^{\beta_{12}}\hat{X}_{21}^{\eta_{12}} = \frac{1}{\Delta}\left[\frac{\eta_{24}\hat{X}_{11}}{r-\alpha_1} + \frac{\beta_{24}\hat{X}_{21}}{r-\alpha_2}\right] \tag{3.25}$$

$$A_{24}\hat{X}_{11}^{\beta_{24}}\hat{X}_{21}^{\eta_{24}} = \frac{1}{\Delta}\left[\frac{\eta_{12}\hat{X}_{11}}{r-\alpha_1} + \frac{\beta_{12}\hat{X}_{21}}{r-\alpha_2}\right] \tag{3.26}$$

where $\Delta = \beta_{14}\eta_{22} - \beta_{22}\eta_{14}$.

If we substitute now the last four results in the original value-matching relationships, given by (3.17) and (3.18), we obtain the following equations, as in Adkins and Paxson (2011a).

$$\frac{\hat{X}_{12}}{r-\alpha_1}\left[1 - \frac{\eta_{24}-\eta_{12}}{\Delta}\right] - \frac{\hat{X}_{22}}{r-\alpha_2}\left[1 - \frac{\beta_{12}-\beta_{24}}{\Delta}\right] = K_{12} \tag{3.27}$$

$$\frac{\hat{X}_{21}}{r-\alpha_2}\left[1 - \frac{\beta_{12}-\beta_{24}}{\Delta}\right] - \frac{\hat{X}_{11}}{r-\alpha_1}\left[1 - \frac{\eta_{24}-\eta_{12}}{\Delta}\right] = K_{21} \tag{3.28}$$

It is worth mentioning that these two last equations are really useful simplifications of the value-matching relationships, once we have eliminated two unknown parameters A_{12} and A_{24}. Also, as G_I verified all boundary conditions we imposed, we can now assume that it truly represents the firm's value given by F_I.

The model constituted by the two *characteristic root equations* $Q_1(\beta_{12}, \eta_{12}) = 0$ and $Q_2(\beta_{24}, \eta_{24}) = 0$, the two value-matching relationships (3.17) and (3.18) and the four smooth pasting conditions (3.19) - (3.22), constitute our general model for investments concerning switching options. One can observe that this model contains eight equations, four unknown variables $\hat{X}_{12}$, $\hat{X}_{22}$, $\hat{X}_{21}$ and $\hat{X}_{11}$, and six unknown parameters A_{12}, A_{24}, β_{12}, β_{24}, η_{12} and η_{24}. Clearly, a solution to this model cannot be derived analytically. One could always try to reach a solution numerically, however we adopted another approach. We will consider a simplification to this general model, also mentioned in Adkins and Paxson (2011a), that is going to be presented next, and where an analytical solution will be easier to reach.

Single Opportunity Switch Model

According to Adkins and Paxson (2011a), this simplified model considers exclusively scenarios where one has only one opportunity to switch from feedstock 1 to feedstock 2, or from feedstock 2 to feedstock 1.

Naturally, this new feature will induce some changes in both the value-matching relationships and smooth pasting conditions. These changes will be quite profitable to us, once the complexity of the model will decrease tremendously, allowing us to solve it using a method we are already familiar with. Similarly to Adkins and Paxson (2011a), we will use the index s in all the computations from now on, to emphasize the fact that we are dealing with a single opportunity switch model.

Let us start by evaluating the effects on the first value-matching relationship. We should recall that we have feedstock 1 as the incumbent and feedstock 2 as the substitute. What will happen now is that once we switch from feedstock 1 to 2, there will no longer be the possibility of switching back to feedstock 1. In this sense, we need to select the term in the value-matching relationship that represents this possibility of change and eliminate it from the equation. The equation in question will be given by (3.29) and it is presented below.

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} - \frac{\hat{X}_{12s}}{r-\alpha_1} = A_{24s}\hat{X}_{12s}^{\beta_{24s}}\hat{X}_{22s}^{\eta_{24s}} - \frac{\hat{X}_{22s}}{r-\alpha_2} - K_{12} \tag{3.29}$$

The term in this equation that corresponds to the possible switch back from feedstock 2 to feedstock 1 is $A_{24s}\hat{X}_{12s}^{\beta_{24s}}\hat{X}_{22s}^{\eta_{24s}}$, as it is described in Adkins and Paxson (2011a). This term represents precisely the switching option value element in the composition of the firm's value when the incumbent feedstock is 2, G_2, in the points $\hat{X}_{12}$ and $\hat{X}_{22}$. Eliminating this term from (3.29) we get the "new" value-matching relationship for this specific model:

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} - \frac{\hat{X}_{12s}}{r-\alpha_1} = -\frac{\hat{X}_{22s}}{r-\alpha_2} - K_{12} \tag{3.30}$$

Consequently, we can now obtain two new associated smooth pasting conditions. They are the following,

$$A_{12s}\beta_{12s}\hat{X}_{12s}^{\beta_{12s}-1}\hat{X}_{22s}^{\eta_{12s}} - \tfrac{1}{r-\alpha_1} = 0 \Leftrightarrow A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} = \tfrac{\hat{X}_{12s}}{\beta_{12s}(r-\alpha_1)} \tag{3.31}$$

$$A_{12s}\eta_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}-1} = -\tfrac{1}{r-\alpha_2} \Leftrightarrow A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} = -\tfrac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} \tag{3.32}$$

This implies that,

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} = \frac{\hat{X}_{12s}}{\beta_{12s}(r-\alpha_1)} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} \tag{3.33}$$

Taking a step back and looking at chapter 2, section 2.3., we can see that the second equation figuring in the final model presented there has exactly the same form as (3.33). In this sense, we can follow the same reasoning here and derive a function H, as it is done in Adkins and Paxson (2011b), consistent with our present case scenario. So, considering that from (3.33) we have,

$$A_{12s} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}}} \tag{3.34}$$

$$\hat{X}_{12s} = -\frac{\hat{X}_{22s}\beta_{12s}(r-\alpha_1)}{\eta_{12s}(r-\alpha_2)} \tag{3.35}$$

Now, taking (3.30), and using results (3.34) and (3.35), it follows that,

$$A_{12s}\hat{X}_{12s}^{\beta_{12s}}\hat{X}_{22s}^{\eta_{12s}} - \frac{\hat{X}_{12s}}{r-\alpha_1} = -\frac{\hat{X}_{22s}}{r-\alpha_2} - K_{12} \quad \Leftrightarrow \tag{3.36}$$

$$-\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} - \frac{\hat{X}_{12s}}{r-\alpha_1} = -\frac{\hat{X}_{22s}}{r-\alpha_2} - K_{12} \quad \Leftrightarrow \tag{3.37}$$

$$-\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} - \frac{-\frac{\hat{X}_{22s}\beta_{12s}(r-\alpha_1)}{\eta_{12s}(r-\alpha_2)}}{r-\alpha_1} = -\frac{\hat{X}_{22s}}{r-\alpha_2} - K_{12} \quad \Leftrightarrow \tag{3.38}$$

$$\frac{\hat{X}_{22s}}{(r-\alpha_2)}\left[-\frac{1}{\eta_{12s}} + \frac{\beta_{12s}}{\eta_{12s}} + 1\right] + K_{12} = 0 \quad \Leftrightarrow \tag{3.39}$$

$$\frac{\hat{X}_{22s}}{(r-\alpha_2)}\left[\frac{\eta_{12s}+\beta_{12s}-1}{\eta_{12s}}\right] + K_{12} = 0 \tag{3.40}$$

Thus, if we fix a value for $\hat{X}_{22s}$ we can define a function of β_{12s} and η_{12s} that together with the *characteristic root equation* will allow us to determine values for these two parameters, following the same methodology used in chapter 2. Such a function. H, is defined in (3.41):

$$H(\beta_{12s}, \eta_{12s}|\hat{X}_{22s}) = \frac{\hat{X}_{22s}}{(r-\alpha_2)}\left[\frac{\eta_{12s}+\beta_{12s}-1}{\eta_{12s}}\right] + K_{12} \tag{3.41}$$

where we use the notation $H(.|x)$ to denote that x is fixed.

Now, we ought to proceed exactly the same way for the second value-matching relationship. Following the same course of action used in the first case, we have now the scenario where feedstock 2 is the incumbent and feedstock 1 is the substitute, and there is only one opportunity to switch from 2 to 1. This is given by,

$$A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} - \frac{\hat{X}_{21s}}{r-\alpha_2} = A_{12s}\hat{X}_{11s}^{\beta_{12s}}\hat{X}_{21s}^{\eta_{12s}} - \frac{\hat{X}_{11s}}{r-\alpha_1} - K_{21} \tag{3.42}$$

We need to select the term in (3.42) that corresponds to the possibility of changing back from feedstock 1 to 2, and eliminate it from the equation. Once more according to Adkins and Paxson (2011a), the term in question is $A_{12s}\hat{X}_{11s}^{\beta_{12s}}\hat{X}_{21s}^{\eta_{12s}}$, which represents also the switching option value element of the firm's value when the incumbent feedstock is 1, evaluated in the points $\hat{X}_{11s}$ and $\hat{X}_{21s}$. Eliminating then the term, we obtain the following,

$$A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} - \frac{\hat{X}_{21s}}{r-\alpha_2} = -\frac{\hat{X}_{11s}}{r-\alpha_1} - K_{21} \tag{3.43}$$

The two smooth pasting conditions associated with (3.43) are given by,

$$A_{24s}\beta_{24s}\hat{X}_{11s}^{\beta_{24s}-1}\hat{X}_{21s}^{\eta_{24s}} = -\tfrac{1}{r-\alpha_1} \Leftrightarrow A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} = -\tfrac{\hat{X}_{11s}}{\beta_{24s}(r-\alpha_1)} \tag{3.44}$$

$$A_{24s}\eta_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}-1} - \tfrac{1}{r-\alpha_2} = 0 \Leftrightarrow A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} = \tfrac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} \tag{3.45}$$

Results (3.44) and (3.45) imply that,

$$A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} = -\frac{\hat{X}_{11s}}{\beta_{24s}(r-\alpha_1)} = \frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} \tag{3.46}$$

From (3.46) we can outdraw the next two results, that will be posteriorly applied in the derivation of an equivalent H function in this case.

$$A_{24s} = \frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}}} \tag{3.47}$$

$$\hat{X}_{11s} = -\frac{\hat{X}_{21s}\beta_{24s}(r-\alpha_1)}{\eta_{24s}(r-\alpha_2)} \tag{3.48}$$

Finally, taking (3.47) and (3.48), and using them in (3.43) leads us to,

$$A_{24s}\hat{X}_{11s}^{\beta_{24s}}\hat{X}_{21s}^{\eta_{24s}} - \frac{\hat{X}_{21s}}{r-\alpha_2} = -\frac{\hat{X}_{11s}}{r-\alpha_1} - K_{21} \quad \Leftrightarrow \tag{3.49}$$

$$\frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} - \frac{\hat{X}_{21s}}{r-\alpha_2} = -\frac{\hat{X}_{11s}}{r-\alpha_1} - K_{21} \quad \Leftrightarrow \tag{3.50}$$

$$\frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} - \frac{\hat{X}_{21s}}{r-\alpha_2} = -\frac{-\frac{\hat{X}_{21s}\beta_{24s}(r-\alpha_1)}{\eta_{24s}(r-\alpha_2)}}{r-\alpha_1} - K_{21} \quad \Leftrightarrow \tag{3.51}$$

$$\frac{\hat{X}_{21s}}{(r-\alpha_2)}\left[\frac{1}{\eta_{24s}} - 1 - \frac{\beta_{24s}}{\eta_{24s}}\right] + K_{21} = 0 \quad \Leftrightarrow \tag{3.52}$$

$$\frac{\hat{X}_{21s}}{(r-\alpha_2)}\left[\frac{1-\eta_{24s}-\beta_{24s}}{\eta_{24s}}\right] + K_{21} = 0 \tag{3.53}$$

Thus, we just reached a function H, specified in (3.54), that for a fixed value of $\hat{X}_{21s}$, gives us an expression of β_{24s} and η_{24s}. This expression together with the corresponding *characteristic root equation* will allow us to determine values for β_{24s} and η_{24s}.

$$H(\beta_{24s}, \eta_{24s}|\hat{X}_{21s}) = \frac{\hat{X}_{21s}}{(r-\alpha_2)}\left[\frac{1-\eta_{24s}-\beta_{24s}}{\eta_{24s}}\right] + K_{21} \tag{3.54}$$

Summarizing, the single opportunity switch model is constituted by the following elements:

1. Two *characteristic root equations*, that were also present in the more general model and that did not suffer any alteration when we consider a single opportunity to switch, besides the presence of an index s, whose purpose was already referred above. They are $Q_1(\beta_{12s}, \eta_{12s}) = 0$ and $Q_2(\beta_{24s}, \eta_{24s}) = 0$;

2. Two relationships given by (3.33) and (3.46), derived from the corresponding smooth pasting conditions;

3. Two H functions, one given by (3.41), that we denote by H_{22}, and the other one given by (3.54), that we denote by H_{21}.

This is exactly the same composition of the model presented in chapter 2, section 2.2.3. and also described in Adkins and Paxson (2011b). The unknowns we wish to determine with this model are β_{12s}, η_{12s}, β_{24s}, η_{24s}, $\hat{X}_{12s}$ and $\hat{X}_{11s}$, once $\hat{X}_{21s}$ and $\hat{X}_{22s}$ are being considered as fixed. Once again, the act of fixing the values for these two variables is what makes this methodology a *quasi analytical* approach.

As in Adkins and Paxson (2011a), we can also observe that from the moment we eliminated the terms in the value-matching relationships, corresponding to either the switch back from feedstock 2 to 1 or from 1 to 2, and as soon as the values for $\hat{X}_{21s}$ and $\hat{X}_{22s}$ were fixed, the problem concerning the determination of β_{12s}, η_{12s}, $\hat{X}_{12s}$ (see equations (3.35) and (3.41)), and the problem of determining β_{24s}, η_{24s}, $\hat{X}_{11s}$ (see equations (3.48) and (3.54)), became completely separated. In this sense, and with all the tools derived so far, we can now present an algorithm that one can use to determine each set of (three) values, separately. Such an algorithm is going to be given by,

1. Start by fixing $\hat{X}_{22s}$ ($\hat{X}_{21s}$).

2. Solve the system
$$\begin{cases} Q_1(\beta_{12s}, \eta_{12s}) = 0 \\ H(\beta_{12s}, \eta_{12s} | \hat{X}_{22s}) = 0 \end{cases}$$
$$\left(\begin{cases} Q_2(\beta_{24s}, \eta_{24s}) = 0 \\ H(\beta_{24s}, \eta_{24s} | \hat{X}_{21s}) = 0 \end{cases} \right)$$
in order to find β_{12s}, η_{12s} (β_{24s}, η_{24s}).

3. Solve
$$\frac{\hat{X}_{12s}}{\beta_{12s}(r-\alpha_1)} = -\frac{\hat{X}_{22s}}{\eta_{12s}(r-\alpha_2)} \quad \left(-\frac{\hat{X}_{11s}}{\beta_{24s}(r-\alpha_1)} = \frac{\hat{X}_{21s}}{\eta_{24s}(r-\alpha_2)} \right)$$
in order to find $\hat{X}_{12s}$ ($\hat{X}_{11s}$).

The assumption of a single switch opportunity was made because such a scenario is observed quite often, as it is stated in Adkins and Paxson (2011a). Also, later on when we consider different dynamics for the feedstock prices, and apply this model to a more concrete scenario, this single switch assumption will be required, and so all this reasoning will be quite useful then.

The next chapter will present a specific situation of an energy switching market, and a model to describe that scenario will be developed. Also, an attempt of developing this kind of model considering other dynamics to the feedstock prices will be made.

3.2 Arithmetic Brownian Motion Dynamic

As we mentioned in the beginning of the current chapter, in the approach that follows next, we will adopt a few changes in our initial assumptions.

First of all, the dynamic used to describe the prices behaviour will be the arithmetic Brownian motion, instead of the geometric Brownian motion considered previously.

Also, we will assume a particular scenario, where the single switch model developed before and presented in Adkins and Paxson (2011a) will be applied. This new scenario, that is going to be assumed from now on, is the production of oil and natural gas. Based on the problem described in Hem and Svendsen (2010), we will adopt the position of a company responsible for the extraction of both oil and gas, and we shall adapt the next approach to this perspective.

Before we move on to the description of the model itself, we must familiarize ourselves with the process behind the extraction of these two feedstocks. For obvious reasons, we do not provide a full description of such production process, and we just explain briefly and in a quite informal way how it is done. According to Hem and Svendsen (2010), in order to extract oil, one should first inject natural gas, and once all the oil has been extracted we then proceed to the natural gas extraction. In this sense, there is only one switch available for us to make, and so the single opportunity switch model described in Adkins and Paxson (2011a) will be the one that suits our interest best. Nevertheless, the general model should always be derived first, before developing the specific cases, which we present next.

At last, we must also evaluate the impact of these new assumptions in the saleable output considered. In these scenario, considering this kind of energy markets, we will no longer have two feedstocks for one outcoming product. We will now have two feedstocks that are being extracted, and that will be sold as two distinct energy resources. So, instead of only one output price given by D_0, we will now be interested in two extraction prices, E_{oil} and E_{gas}, denoting the extraction prices of oil and gas, respectively.

We are now in conditions to start dealing with the model itself. So, let X_I, $I \in \{1, 2\}$, denote the selling price for feedstock I, where now we actually know what will each value of I refer to. We shall consider oil to be feedstock 1 and natural gas to be feedstock 2. Assuming that both of these prices follow an arithmetic Brownian motion, for $I \in \{1, 2\}$, we have the following,

$$dX_I = \alpha_I dt + \sigma_I dZ_I \tag{3.55}$$

Naturally, α_I is the risk-adjusted drift rate, σ_I is the the volatility rate, and dZ_I is the increment of a standard Wiener process. Additionally, we can describe the dependence between the two variables by its covariance term $\rho\sigma_1\sigma_2$ where $Cov[dX_1, dX_2] = \rho\sigma_1\sigma_2 dt$, with $|\rho| \leq 1$.

Once again, let $F_I = F_I(X_1, X_2)$ be the function that represents the firm's value when feedstock I is the incumbent feedstock. The next step, similarly to what has been done so far, is to take $e^{-rt}F_I(X_1, X_2)$ and apply the dynamic programming principle together with Ito's lemma and obtain the correspondent dynamic programming equation. However we need to define first the net cash flow for this case.

On the last chapter we considered the production of a specific product, where one could use two different feedstock prices. So, the output price given by D_0 was referring to one product only. In this case, we are extracting two different feedstocks, oil and gas, that will originate two different saleable outputs. In this sense, instead of having one output price D_0, we will define extraction prices instead, for oil as E_1 and for natural gas as E_2.

One must also note that in the previous case, we were buying the feedstock at a price X_I, and now once we are the producers, we are selling it, also at a price X_I. Thus, the net cash flow, when feedstock I is the incumbent is going to be given by $X_I - E_I$. We are then in conditions to obtain the partial differential equation considered in the dynamic programming equation, that shall be given by,

$$\frac{1}{2}\sigma_1^2\frac{\partial^2 F_I}{\partial X_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 F_I}{\partial X_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 F_I}{\partial X_1 \partial X_2} + \alpha_1\frac{\partial F_I}{\partial X_1} + \alpha_2\frac{\partial F_I}{\partial X_2} - rF_I + (X_I - E_I) = 0 \tag{3.56}$$

where r denotes the risk-free interest rate, as usual.

Characteristic Root Equation

Now, we shall propose a solution to this equation, meaning that a concrete function denoted by $G_I(X_1, X_2)$ that satisfies (3.56) will be suggested. Then, this function is going to be placed in the equation, and the required computations will be made, in order to obtain a condition that will later integrate our final model.

The function G_I that we propose is the following,

$$G_I(X_1, X_2) = A_I e^{X_1\beta_I + X_2\eta_I} + \frac{X_I}{r} + \frac{\alpha_I}{r^2} - \frac{E_I}{r} \tag{3.57}$$

We should once more notice that in this function the term $A_I e^{X_1\beta_I + X_2\eta_I}$ corresponds to the solution of the homogeneous part of (3.56) and it represents the switching option value. The term $\frac{X_I}{r} + \frac{\alpha_I}{r^2} - \frac{E_I}{r}$ denotes the incumbent asset value associated with the investment.

The first and second order derivatives are given by,

$$\frac{\partial G_I}{\partial X_1} = \begin{cases} A_I\beta_I e^{X_1\beta_I + X_2\eta_I} + \frac{1}{r} & \text{if } I = 1 \\ A_I\beta_I e^{X_1\beta_I + X_2\eta_I} & \text{if } I = 2 \end{cases} \tag{3.58}$$

$$\frac{\partial G_I}{\partial X_2} = \begin{cases} A_I\eta_I e^{X_1\beta_I + X_2\eta_I} & \text{if } I = 1 \\ A_I\eta_I e^{X_1\beta_I + X_2\eta_I} + \frac{1}{r} & \text{if } I = 2 \end{cases} \tag{3.59}$$

$$
\begin{aligned}
\frac{\partial^2 G_I}{\partial X_1^2} &= A_I \beta_I^2 e^{X_1\beta_I + X_2\eta_I} && (3.60)\\
\frac{\partial^2 G_I}{\partial X_2^2} &= A_I \eta_I^2 e^{X_1\beta_I + X_2\eta_I} && (3.61)\\
\frac{\partial^2 G_I}{\partial X_1 \partial X_2} &= A_I \beta_I \eta_I e^{X_1\beta_I + X_2\eta_I} && (3.62)
\end{aligned}
$$

Now substituting them in (3.56), we get:

$$
\begin{aligned}
&\frac{1}{2}\sigma_1^2 A_I \beta_I^2 e^{X_1\beta_I + X_2\eta_I} + \frac{1}{2}\sigma_2^2 A_I \eta_I^2 e^{X_1\beta_I + X_2\eta_I} + \rho\sigma_1\sigma_2 A_I \beta_I \eta_I e^{X_1\beta_I + X_2\eta_I} + \alpha_1 A_I \beta_I e^{X_1\beta_I + X_2\eta_I}\\
&+\alpha_2 A_I \eta_I e^{X_1\beta_I + X_2\eta_I} - r A_I e^{X_1\beta_I + X_2\eta_I} + \frac{\alpha_I}{r} + E_I - X_I - \frac{\alpha_I}{r} + (X_I - E_I) = 0 \Leftrightarrow\\
&A_I e^{X_1\beta_I + X_2\eta_I}\left(\frac{1}{2}\sigma_1^2\beta_I^2 + \frac{1}{2}\sigma_1^2\eta_I^2 + \rho\sigma_1\sigma_2\beta_I\eta_I + \alpha_1\beta_I + \alpha_2\eta_I - r\right) = 0 \qquad (3.63)
\end{aligned}
$$

This leads us to the condition presented in (3.64), which, similarly to what happened in the previous chapters and to what is presented in Adkins and Paxson (2011b) and Adkins and Paxson (2011a), can also be called *characteristic root equation*:

$$
Q_I(\beta_I, \eta_I) = \frac{1}{2}\sigma_1^2\beta_I^2 + \frac{1}{2}\sigma_2^2\eta_I^2 + \rho\sigma_1\sigma_2\beta_I\eta_I + \alpha_1\beta_I + \alpha_2\eta_I - r = 0 \qquad (3.64)
$$

The plot of $Q_I(\beta_I, \eta_I) = 0$ originates an ellipse that can be seen in figure 3.2, where the values for the fixed constants are presented in table 3.2, and are the same ones considered in the first section, and thus in Adkins and Paxson (2011a).

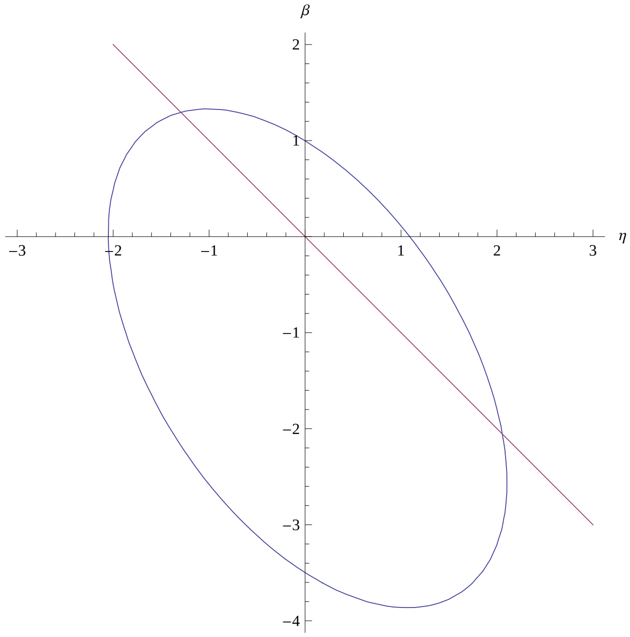

Figure 3.2: Graphic of the ellipse defined by $Q_I(\beta_I, \eta_I) = 0$ and the line $\beta_I + \eta_I = 0$

Parameters	Symbol	Value
Feedstock 1 volatility	σ_1	0.2
Feedstock 2 volatility	σ_2	0.25
Feedstock 1 risk-adjusted drift rate	α_1	0.05
Feedstock 2 risk-adjusted drift rate	α_2	0.03
Risk-free interest rate	r	0.07
Correlation between X_1 and X_2	ρ	0.5

Table 3.2: Parameter Values - Ellipse equation - Energy switch model considering an arithmetic Brownian motion dynamic

In this case the homogeneity-degree-1 condition, given by $\beta_I + \eta_I = 1$, does not appear in figure 3.2. However, the condition $\beta_I + \eta_I = 0$ does, as it will appear later on when we consider the single opportunity switch simplification.

The next step will be to take the term in G_I, that corresponds to the solution of the homogeneous part of (3.56), and divide it in four different quadrants. The second step will be to use economical arguments in order to choose which quadrant is relevant to integrate the structure of each G_I, for $I \in \{1, 2\}$.

The four quadrants and the correspondent division in the function are presented below.

$$
\begin{aligned}
I &: \{\beta_I, \eta_I\} \quad \beta_I \geq 0; \eta_I \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I1}, \eta_{I1}\} \\
II &: \{\beta_I, \eta_I\} \quad \beta_I \geq 0; \eta_I \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I2}, \eta_{I2}\} \\
III &: \{\beta_I, \eta_I\} \quad \beta_I \leq 0; \eta_I \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I3}, \eta_{I3}\} \\
IV &: \{\beta_I, \eta_I\} \quad \beta_I \leq 0; \eta_I \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I4}, \eta_{I4}\}
\end{aligned}
$$

and therefore,

$$G_I(X_1,X_2) = A_{I1}e^{X_1\beta_{I1}+X_2\eta_{I1}} + A_{I2}e^{X_1\beta_{I2}+X_2\eta_{I2}} + A_{I3}e^{X_1\beta_{I3}+X_2\eta_{I3}} + A_{I4}e^{X_1\beta_{I4}+X_2\eta_{I4}} + \frac{X_I}{r} + \frac{\alpha_I}{r^2} - \frac{E_I}{r} \tag{3.65}$$

where we use in fact similar arguments as the ones that we presented in chapter 2 to justify this quadrant division.

The economical arguments that will lead us to the choice of the right quadrant in each case are presented below. We should take into account that now our perspective has changed. We recall that in the previous section we assumed the perspective from the final consumer of the product, as happens in Adkins and Paxson (2011a), which we no longer assume in the present problem. Once we are now assuming the position of a company responsible for the extraction of these two feedstocks, we are no longer thinking as the buyer of the final product, which in this case would be an energy resource.

Let us start with the scenario where the incumbent feedstock is oil, thus feedstock 1. Whenever the oil price tends to zero, we are interested in switching from 1 to 2, as it will no longer be profitable for us to keep extracting, and posteriorly selling oil. So, the price of the segment in the firm's value that corresponds to the option value, denoted by G_{H1}, will increase infinitely, meaning that $\lim_{X_1\to 0} G_{H1} = \infty$. This doesn't actually impose any restrction regarding the value of β_1, but once we consider the condition presented in figure 3.2, we can observe that $\beta_1 \leq 0$. On the other hand, whenever we observe that the price at which we sell the natural gas is very high, we also want to make the switch from 1 to 2, once we will have more profit if we start extracting gas. So, G_{H1} will assume extremely high values. This means that $\lim_{X_2\to\infty} G_{H1} = \infty$, which implies that η_1 has to be non-negative and thus, $\eta_1 \geq 0$. Then, we can conclude that the right quadrant to choose when feedstock 1 is the incumbent is the fourth. Thus, G_1 can be written as follows,

$$G_1(X_1,X_2) = A_{14}e^{X_1\beta_{14}+X_2\eta_{14}} + \frac{X_1}{r} + \frac{\alpha_1}{r^2} - \frac{E_1}{r} \tag{3.66}$$

Below this, the correspondent *characteristic root equation* for $I = 1$ will be given by $Q_1(\beta_{14}, \eta_{14}) = 0$.

Now we should consider the case where the incumbent feedstock is gas, meaning feedstock 2. The reasoning here is very similar to what was used in the previous case. So, whenever the gas price tends to 0, the option value must tend to infinity, $\lim_{X_2\to 0} G_{H2} = \infty$. This happens because as we are selling the feedstock at a really low price, we will not be making a lot of profit from its extraction and so the best decision for us is to switch. Again this doesn't necessarily imply any restrictions regarding the values that η_2 can take, however when the restriction $\beta_I + \eta_I = 0$ is considered, it follows that $\eta_2 \leq 0$. Similarly, whenever the oil price increases infinitely, we will be really interested in switching from 2 to 1 as it will be more profitable for us to start extracting oil. In this sense, the option value must tend to infinity and thus, $\lim_{X_1\to\infty} G_{H2} = \infty$. This implies that $\beta_2 \geq 0$. Therefore the right quadrant is going to be the second one,

$$G_2(X_1,X_2) = A_{22}e^{X_1\beta_{22}+X_2\eta_{22}} + \frac{X_2}{r} + \frac{\alpha_2}{r^2} - \frac{E_2}{r} \tag{3.67}$$

With this result in mind, we can now obtain the *characteristic root equation* when $I = 2$, $Q_2(\beta_{22}, \eta_{22}) = 0$.

Next we present the corresponding value-matching and smooth pasting conditions.

Value-Matching Relationships and Smooth Pasting Conditions

Before we move on and present the value-matching relationships and the smooth pasting conditions, there is the need to introduce some new variables and constants to the problem. This might seem a bit repetitive, once we did the same thing in previous section. However, one should always keep in mind that we are now applying this whole reasoning to oil an gas extraction, instead of two unknown feedstocks, and thus all of the variables and constants should be redefined accordingly.

Let's start by defining K_{12} and K_{21}, which represent, respectively, the cost of changing to feedstock 2 when 1 is the incumbent, and the cost of changing to feedstock 1 when 2 is the incumbent. In other words, K_{12} is the cost of switching from oil to gas, and K_{21} is the cost of switching from gas to oil. They are both fixed constants, and this definition is coherent with the definition given in the previous chapter.

Now we need to introduce $\hat{X}_{IJ}$, where I denotes the incumbent feedstock and J the substitute, and which, as $I, J \in \{1, 2\}$, generates four different variables. Thus, $\hat{X}_{IJ}$ denotes the threshold value for feedstock I whenever J is the substitute. Again, this definition is coherent, and quite similar, to the one presented in the last chapter. The four variables that will be originated here are:

- $\hat{X}_{12}$, which is the threshold price for oil when gas is the substitute;
- $\hat{X}_{21}$, which precisely the opposite of the previous one, thus the threshold price for gas when oil is the substitute;
- $\hat{X}_{11}$, which brings up the case where there is no other substitute feedstock and thus, in this case, it is the threshold price for oil whenever the substitute feedstock is also oil;
- $\hat{X}_{22}$, similarly, is the threshold price for gas whenever the substitute feedstock is also gas.

Regarding the value-matching relationships, as it is stated in Adkins and Paxson (2011a):

1. for the case where feedstock 1 is the incumbent and feedstock 2 is the substitute, the difference between the value of the substitute $G_2(\hat{X}_{12}, \hat{X}_{22})$ and the value of the incumbent $G_1(\hat{X}_{12}, \hat{X}_{22})$, immediately before a switch, is equal to the switching cost K_{12}.
2. for the case where feedstock 2 is the incumbent and feedstock 1 is the substitute, the difference between the value of the substitute $G_1(\hat{X}_{11}, \hat{X}_{21})$ and the value of the incumbent $G_2(\hat{X}_{11}, \hat{X}_{21})$, immediately before a switch, equals the switching cost K_{21}.

Thus, we end up with the following two equations:

$$G_1(\hat{X}_{12}, \hat{X}_{22}) = G_2(\hat{X}_{12}, \hat{X}_{22}) - K_{12} \tag{3.68}$$

$$G_2(\hat{X}_{11}, \hat{X}_{21}) = G_1(\hat{X}_{11}, \hat{X}_{21}) - K_{21} \tag{3.69}$$

According to Hem and Svendsen (2010), one has to start with oil extraction and only afterwards we can proceed to the gas extraction. In this sense, we don't think it is necessary to consider the case where we start with gas as the incumbent feedstock. Thus, from now on, we will only consider the first value-matching relationship presented.

Substituting then (3.66) in the last equation we obtain the following,

$$A_{14}e^{\hat{X}_{12}\beta_{14}+\hat{X}_{22}\eta_{14}} + \frac{\hat{X}_{12}}{r} + \frac{\alpha_1}{r^2} - \frac{E_1}{r} = A_{22}e^{\hat{X}_{12}\beta_{22}+\hat{X}_{22}\eta_{22}} + \frac{\hat{X}_{22}}{r} + \frac{\alpha_2}{r^2} - \frac{E_2}{r} - K_{12} \tag{3.70}$$

Also, it should be noticed that once we consider this restriction, and consequently we ignore the second value-matching relationship, presented in (3.69), we are also eliminating the variables $\hat{X}_{11}$ and $\hat{X}_{21}$ from the problem.

Let us proceed and derive the smooth-pasting conditions. There will be two smooth pasting conditions associated with this value-matching relationship. They are presented below.

$$\left.\frac{\partial G_1}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_1}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} \Leftrightarrow A_{14}\beta_{14}e^{\hat{X}_{12}\beta_{14}+\hat{X}_{22}\eta_{14}} + \tfrac{1}{r} = A_{22}\beta_{22}e^{\hat{X}_{12}\beta_{22}+\hat{X}_{22}\eta_{22}} \tag{3.71}$$

$$\left.\frac{\partial G_1}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} = \left.\frac{\partial G_2}{\partial X_2}\right|_{\substack{X_1=\hat{X}_{12}\\ X_2=\hat{X}_{22}}} \Leftrightarrow A_{14}\eta_{14}e^{\hat{X}_{12}\beta_{14}+\hat{X}_{22}\eta_{14}} = A_{22}\eta_{22}e^{\hat{X}_{12}\beta_{22}+\hat{X}_{22}\eta_{22}} + \tfrac{1}{r} \tag{3.72}$$

In the smooth-pasting conditions, the terms $A_{14}e^{\hat{X}_{12}\beta_{14}+\hat{X}_{22}\eta_{14}}$ and $A_{22}e^{\hat{X}_{12}\beta_{22}+\hat{X}_{22}\eta_{22}}$ can be isolated so that expressions (3.73) and (3.74) can emerge, and posteriorly be replaced in the value-matching relationship, in order to achieve a more simplified equality. These two expressions are presented below.

$$A_{14}e^{\hat{X}_{12}\beta_{14}+\hat{X}_{22}\eta_{14}} = \frac{1}{\Delta}\left(-\frac{\eta_{22}+\beta_{22}}{r}\right) \tag{3.73}$$

$$A_{22}e^{\hat{X}_{12}\beta_{22}+\hat{X}_{22}\eta_{22}} = \frac{1}{\Delta}\left(-\frac{\eta_{14}+\beta_{14}}{r}\right) \tag{3.74}$$

where $\Delta = \beta_{14}\eta_{22} - \beta_{22}\eta_{14}$.

Finally, taking these expressions and using them in the value-matching relationship, in order to simplify it, will lead us to the following result:

$$\hat{X}_{12} - \hat{X}_{22} + \frac{\beta_{14}+\eta_{14}-\beta_{22}-\eta_{22}}{\Delta} = E_1 - E_2 - \frac{1}{r}(\alpha_1-\alpha_2) - rK_{12} \tag{3.75}$$

This is, in fact, a simplified version of the value-matching relationship, as the dependence on A_{14} and A_{22} was eliminated. Such a result will be quite valuable in the next section. Also, and similarly to what happened in the previous section, we can observe that G_I verified the boundary conditions we imposed and thus it can be considered from now on as the true firm's value, initially represented by F_I.

Following the same reasoning used in the previous chapter we can conclude that the final model will be constituted by five equations,

- two *characteristic root equations*: $Q_1(\beta_{14}, \eta_{14}) = 0$ and $Q_2(\beta_{22}, \eta_{22}) = 0$
- one value-matching relationship given by (3.70)
- two smooth-pasting conditions given by (3.71) and (3.72)

where six unknown parameters shall be determined: $A_{14}, A_{22}, \beta_{14}, \eta_{14}, \beta_{22}, \eta_{22}$, and the value of the two remaining unknown variables $\hat{X}_{12}$ and $\hat{X}_{22}$ should be found.

Once again, we have more unknown variables and parameters than equations, which constitutes an indetermination. Also, the model we have developed so far doesn't consider the restriction of a single switch opportunity, and that is a very important assumption to make when dealing with oil and gas extraction. With this in mind, we will consider now the simplification of this general model, where the single opportunity switch is considered, as well as the admission of fixed values for one of the two unknown variables, again similarly to what happened in previous chapters and in Adkins and Paxson (2011b) and Adkins and Paxson (2011a), so that we can determine the values of the unknown parameters.

Single Switch Opportunity Model

Let us now proceed to the implementation of the single opportunity switch model for the arithmetic Brownian motion. As we mentioned before, according to Hem and Svendsen (2010), when we are extracting oil and natural gas, we start with oil extraction and then we switch to natural gas. In this process, there is only one possible change and it has to be from oil to gas. In this sense, we only have one opportunity to change, and the decision to switch is irreversible. So, we must realize that this scenario will induce quite some changes in all the components of our model, from the *characteristic root equations* to the value-matching and smooth pasting conditions.

We shall then take the first value-matching relationship, given by (3.68) and (3.70) and start implementing this single opportunity feature in it, similarly to what we did in the last section. Again we will use an s in the index to point out that we are now developing the single opportunity switch model, as it's done in Adkins and Paxson (2011a). Thus, taking (3.70), we need to select the term that is allowing to switch back from gas to oil, once the first switch from oil to gas is made. This term is $A_{22s}e^{\hat{X}_{12s}\beta_{22s}+\hat{X}_{22s}\eta_{22s}}$, and as we identified it, we must now eliminate it. We will obtain the following "rewritten" value-matching relationship,

$$A_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}} + \frac{\hat{X}_{12s}}{r} + \frac{\alpha_1}{r^2} - \frac{E_1}{r} = \frac{\hat{X}_{22s}}{r} + \frac{\alpha_2}{r^2} - \frac{E_2}{r} - K_{12} \tag{3.76}$$

By ignoring the second value-matching relationship and its correspondent smooth pasting conditions, we actually eliminated from the problem the variables $\hat{X}_{11s}$ and $\hat{X}_{21s}$. Also, once we removed the term $A_{22s}e^{\hat{X}_{12s}\beta_{22s}+\hat{X}_{22s}\eta_{22s}}$ from the first value-matching relationship we also eliminated from the problem the following unknown parameters: A_{22}, β_{22} and η_{22}. In this sense, it would no longer be reasonable to consider the second *characteristic root equation* $Q_2(\beta_{22s}, \eta_{22s}) = 0$, and so this equation shall be eliminated as well.

Summarizing, we now have the unknown parameters A_{14s}, β_{14s} and η_{14s}, and the variables $\hat{X}_{12s}$ and $\hat{X}_{22s}$. We should then proceed and take (3.76) in order to derive the smooth pasting conditions associated with it. They are the following,

$$A_{14s}\beta_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}}+\frac{1}{r}=0 \quad \Leftrightarrow \quad A_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}}=-\frac{1}{\beta_{14s}r} \tag{3.77}$$

$$A_{14s}\eta_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}}=\frac{1}{r} \quad \Leftrightarrow \quad A_{14s}e^{\hat{X}_{12s}\beta_{14s}+\hat{X}_{22s}\eta_{14s}}=\frac{1}{\eta_{14s}r} \tag{3.78}$$

This implies that,

$$-\frac{1}{\beta_{14s}r}=\frac{1}{\eta_{14s}r}\Leftrightarrow -\beta_{14s}=\eta_{14s}\Leftrightarrow \beta_{14s}+\eta_{14s}=0 \tag{3.79}$$

Remark that this is precisely the condition appearing in figure 3.2. If we observe now this plot once again, taking into account that β_{14s} and η_{14s} are in the fourth quadrant and that $\beta_{14s}+\eta_{14s}=0$, we can easily determine the values for this two parameters. Naturally, we will replace $\beta_{14s}=-\eta_{14s}$ in the *characteristic root equation*.

$$\begin{aligned}
&Q_1(\beta_{14s},\eta_{14s})=\frac{1}{2}\sigma_1^2\beta_{14s}^2+\frac{1}{2}\sigma_2^2\eta_{14s}^2+\rho\sigma_1\sigma_2\beta_{14s}\eta_{14s}+\alpha_1\beta_{14s}+\alpha_2\eta_{14s}-r=0\Leftrightarrow\\
&\frac{1}{2}\sigma_1^2\beta_{14s}^2+\frac{1}{2}\sigma_2^2(-\beta_{14s})^2-\rho\sigma_1\sigma_2\beta_{14s}^2+\alpha_1\beta_{14s}-\alpha_2\beta_{14s}-r=0\Leftrightarrow\\
&\beta_{14s}^2\left[\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_2^2-\rho\sigma_1\sigma_2\right]+\beta_{14s}(\alpha_1-\alpha_2)-r=0\Leftrightarrow\\
&\beta_{14s}=\frac{-(\alpha_1-\alpha_2)\pm\sqrt{(\alpha_1-\alpha_2)^2+(2r\sigma_1^2+2r\sigma_2^2-4r\rho\sigma_1\sigma_2)}}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}
\end{aligned}$$

As we are in the fourth quadrant and $\beta_{14s}\leq 0$, we must have,

$$\beta_{14s}=\frac{-(\alpha_1-\alpha_2)-\sqrt{(\alpha_1-\alpha_2)^2+(2r\sigma_1^2+2r\sigma_2^2-4r\rho\sigma_1\sigma_2)}}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} \tag{3.80}$$

Now, the next step would be to proceed to the value-matching relationship (3.76) and determine the unknown variables. However, and similarly to what happened before, we now have two unknown variables and only one value-matching relationship to use. Once more, we will fix a value for one of the variables, which is going to be again $\hat{X}_{22s}$.

As (3.76) also depends on A_{14s} and we cannot determine its value yet with the information we've got so far. We have to take an expression for the value matching relationship that doesn't depend on this parameters. In order to do this we need to go back to equation (3.75), using the fact that $\beta_{22}=\eta_{22}=0$:

$$\hat{X}_{12s}-\hat{X}_{22s}+\frac{\beta_{14s}+\eta_{14s}}{\Delta}=E_1-E_2-\frac{1}{r}(\alpha_1-\alpha_2)-rK_{12} \tag{3.81}$$

Moreover, as $\beta_{14s} + \eta_{14s} = 0$ then we set that:

$$\hat{X}_{12s} - \hat{X}_{22s} = E_1 - E_2 - \frac{1}{r}(\alpha_1 - \alpha_2) - rK_{12} \tag{3.82}$$

So, fixing the value for $\hat{X}_{22s}$, we may derive $\hat{X}_{12s}$ using (3.82).

Summarizing the whole procedure we can now establish an algorithm to solve the single opportunity switch model.

1. Obain the value for β_{14s} using (3.80).
2. Obtain the value for η_{14s} using the equality $\beta_{14s} = -\eta_{14s}$.
3. Fix a value for $\hat{X}_{22s}$.
4. Obtain the value for $\hat{X}_{12s}$ using (3.82).

Additionally, we could obtain the value for A_{14s} using one of the smooth pasting conditions, (3.77) or (3.78), and then obtain the firm's value using (3.66).

Next, we will consider a mean reverting process to describe the prices and we will make an attempt, although slightly more incomplete, to develop the same model we just did.

Chapter 4

Mean Reverting Process

This chapter follows partially the idea presented in the second section of chapter 3, where we study the switching option to change from oil production to gas production, assuming that the underlying prices follow an arithmetic Brownian motion. When we assumed a single switch, we were able to present a quasi-analytical method.

In this chapter we follow the same line of reasoning for the case where the prices are modelled by a mean-reverting process, in a two dimensional approach. We follow also closely Dixit and Pindyck (1994), as they also study an investment problem under the mean-reverting assumption, however in a one dimensional setting.

When compared to either the geometric Brownian motion, or the arithmetic Brownian motion, this process is quite more difficult to handle, and so the model that is going to be presented in this chapter, will not be as complete or as general as the ones presented in chapter 3.

One new assumption that is done in this model is that we will consider no correlation between the two uncertain factors. This is not a very realistic consideration, but we believe that as a first approach using this process, it is reasonable to make such an assumption. In a posterior approach, one could try to eliminate this assumption.

Also, another important restriction of this approach, is that we only present a solution to the homogeneous part of the dynamic programming equation. Once more, due to the difficulty in handling this process, we decided that in a first approach, presenting a solution of the homogeneous part of the problem exclusively could be already quite satisfactory. In this sense, the solution for the non homogeneous part of this problem will be left open to solve in a posterior approach.

Let us then consider this (geometric version) of the Ornstein Uhlenbeck process, which constitutes a mean reverting process, also presented in Dixit and Pindyck (1994) in a unidimensional way. This process will then describe the behaviour of the selling price of feedstock I, given by X_I, for $I \in \{1, 2\}$. Once again we should refer to oil as feedstock 1 and to natural gas as feedstock 2. Thus,

$$dX_I = \theta_I(\mu_I - X_I)X_I dt + \sigma_I X_I dz_I \tag{4.1}$$

where θ_I will be the risk-adjusted rate of the reversion towards the mean, $\mu_I = \mathbb{E}[X_I]$ will denote precisely the mean of X_I, and σ_I is going to denote the volatility rate, as usual. Additionally, as we already stated, we will assume $\rho = 0$, in $cov[dX_1, dX_2] = \rho\sigma_1\sigma_2 dt$, which implies that the prices for oil and natural gas are assumed to be uncorrelated.

We shall now consider $F_I = F_I(X_1, X_2)$ to be the firm's value when feedstock I is the incumbent. Similarly to the previous approaches we will apply the dynamic programming principle together with Ito's lemma to $e^{-rt}F_I(X_1, X_2)$, which will lead us to the following partial differential equation associated with the dynamic programming equation:

$$\frac{1}{2}\sigma_1^2 X_1^2 \frac{\partial^2 F_I}{\partial X_1^2} + \frac{1}{2}\sigma_2^2 X_2^2 \frac{\partial^2 F_I}{\partial X_2^2} + \theta_1(\mu_1 - X_1)X_1 \frac{\partial F_I}{\partial X_1} + \theta_2(\mu_2 - X_2)X_2 \frac{\partial F_I}{\partial X_2} - rF + (X_I - E_I) = 0 \tag{4.2}$$

The net cash flow is still defined in the same way as before. It is given by $(X_I - E_I)$, the selling price for feedstock I minus the corresponding extraction price. Also, one should notice that the term in (4.2) where ρ was present has disappeared. This actually reduces the complexity of the problem quite significantly.

Following the same methodology that has been used in the previous models, we must now propose a function which we will denote by G_I that solves (4.2). After this, we should compute its derivatives, replace them in (4.2), and then obtain what we have been calling the *characteristic root equation*.

However, in this case, and as we stated initially, we will only solve the homogeneous part of the dynamic programming equation. In this sense, we are not actually going to propose a function G_I, but only its component that corresponds to the solution of the homogeneous part of (4.2). Such a solution denotes also the option value part of what we define as the firm's value, and as we will not obtain the other term corresponding to the incumbent asset value, it is more accurate to state that we will propose an expression for G_{HI}, and not for G_I globally. This also implies that in this chapter, we will never deal with the firm's value in its total, but we will be dealing exclusively with the switching option value, G_{HI}. This notation for G_{HI} was already used beforehand.

The solution of the homogeneous part of (4.2) is the following:

$$G_{HI}(X_1, X_2) = A_I X_1^{\beta_I} X_2^{\eta_I} h_1(X_1) h_2(X_2) \tag{4.3}$$

where h_1 and h_2 are functions still to be derived. Therefore, we compute the correspondent first and second partial derivatives:

$$\frac{\partial G_I}{\partial X_1} = A_I \beta_I X_1^{\beta_I - 1} X_2^{\eta_I} h_1(X_1) h_2(X_2) + A_I X_1^{\beta_I} X_2^{\eta_I} h_1'(X_1) h_2(X_2) \tag{4.4}$$

$$\frac{\partial G_I}{\partial X_2} = A_I\eta_I X_1^{\beta_I} X_2^{\eta_I - 1} h_1(X_1)h_2(X_2) + A_I X_1^{\beta_I} X_2^{\eta_I} h_1(X_1)h_2'(X_2) \tag{4.5}$$

$$\frac{\partial^2 G_I}{\partial X_1^2} = A_I\beta_I(\beta_I - 1)X_1^{\beta_I - 2}X_2^{\eta_I} h_1(X_1)h_2(X_2) + 2A_I\beta_I X_1^{\beta_I - 1}X_2^{\eta_I} h_1'(X_1)h_2(X_2) + A_I X_1^{\beta_I} X_2^{\eta_I} h_1''(X_1)h_2(X_2) \tag{4.6}$$

$$\frac{\partial^2 G_I}{\partial X_2^2} = A_I\eta_I(\eta_I - 1)X_1^{\beta_I}X_2^{\eta_I - 2} h_1(X_1)h_2(X_2) + 2A_I\eta_I X_1^{\beta_I}X_2^{\eta_I - 1} h_1(X_1)h_2'(X_2) + A_I X_1^{\beta_I} X_2^{\eta_I} h_1(X_1)h_2''(X_2) \tag{4.7}$$

After replacing the derivatives in (4.2), and making some necessary computations to simplify the result, we obtain the following equation:

$$\begin{aligned}
&A_I X_1^{\beta_I} X_2^{\eta_I} h_1(X_1)h_2(X_2)\left[\frac{1}{2}\sigma_1^2\beta_I(\beta_I - 1) + \frac{1}{2}\sigma_2^2\eta_I(\eta_I - 1) + \theta_1\beta_I\mu_1 + \theta_2\eta_I\mu_2 - r\right] + \\
&A_I X_1^{\beta_I + 1} X_2^{\eta_I} h_2(X_2)\left[\beta_I\sigma_1^2 h_1'(X_1) + \frac{1}{2}\sigma_1^2 X_1 h_1''(X_1) + \theta_1\beta_I h_1(X_1) + \theta_1(\mu_1 - X_1)h_1'(X_1)\right] + \\
&A_I X_1^{\beta_I} X_2^{\eta_I + 1} h_1(X_1)\left[\eta_I\sigma_2^2 h_2'(X_2) + \frac{1}{2}\sigma_2^2 X_2 h_2''(X_2) + \theta_2\eta_I h_2(X_2) + \theta_2(\mu_2 - X_2)h_2'(X_2)\right] = 0
\end{aligned}$$

We can observe in the result presented above that three distinct terms can be identified, and its sum must equal zero. As, neither the variables X_1 and X_2, nor the parameter A_I or the functions h_1 and h_2 can be zero, we are forced to conclude that the remaining expressions included in each term must be zero. This is more formally stated below in (4.8), (4.9) and (4.10).

$$\frac{1}{2}\sigma_1^2\beta_I(\beta_I - 1) + \frac{1}{2}\sigma_2^2\eta_I(\eta_I - 1) + \theta_1\beta_I\mu_1 + \theta_2\eta_I\mu_2 - r = 0 \tag{4.8}$$

$$\frac{1}{2}\sigma_1^2 X_1 h_1''(X_1) + (\beta_I\sigma_1^2 + \theta_1(\mu_1 - X_1))h_1'(X_1) + \theta_1\beta_I h_1(X_1) = 0 \tag{4.9}$$

$$\frac{1}{2}\sigma_2^2 X_2 h_2''(X_2) + (\eta_I\sigma_2^2 + \theta_2(\mu_2 - X_2))h_2'(X_2) + \theta_2\eta_I h_2(X_2) = 0 \tag{4.10}$$

The first result is, in fact, the *characteristic root equation*. Thus, we can once more define a function $Q_I(\beta_I, \eta_I)$ (see (4.11)), to help us handling it. As a new feature in this chapter, we have to deal with two more equations (i.e. (4.9) and (4.10)), besides the one that leads us to the *characteristic root equation*:

$$Q_I(\beta_I, \eta_I) = \frac{1}{2}\sigma_1^2\beta_I(\beta_I - 1) + \frac{1}{2}\sigma_2^2\eta_I(\eta_I - 1) + \theta_1\beta_I\mu_1 + \theta_2\eta_I\mu_2 - r \tag{4.11}$$

Let us start with the second result, (4.9). We shall consider the following variable change, as it is done in Dixit and Pindyck (1994), presented in (4.12):

$$Y_1 = \frac{X_1 2\theta_1}{\sigma_1^2} \Leftrightarrow X_1 = \frac{Y_1\sigma_1^2}{2\theta_1} \tag{4.12}$$

Associated with it, we must introduce function g_1, that will be such that $h_1(X_1) = g_1(Y_1)$. Consequently, due to the variable change and the definition of Y_1, we will have $h_1'(X_1) = g_1'(Y_1)\frac{2\theta_1}{\sigma_1^2}$ and $h_1''(X_1) = g_1''(Y_1)\frac{4\theta_1^2}{\sigma_1^4}$. Applying this variable change to (4.9), we will obtain the following,

$$\begin{aligned}
&\frac{1}{2}\sigma_1^2\frac{Y_1\sigma_1^2}{2\theta_1}g_1''(Y_1)\frac{4\theta_1^2}{\sigma_1^4} + (\beta_I\sigma_1^2 + \theta_1(\mu_1 - \frac{Y_1\sigma_1^2}{2\theta_1}))g_1'(Y_1)\frac{2\theta_1}{\sigma_1^2} + \theta_1\beta_I g_1(Y_1) = 0 \Leftrightarrow \\
&Y_1\theta_1 g_1''(Y_1) + (2\theta_1\beta_I + \frac{2\theta_1^2\mu_1}{\sigma_1^2} - \theta_1 Y_1)g_1'(Y_1) + \theta_1\beta_I g_1(Y_1) = 0 \Leftrightarrow \\
&Y_1 g_1''(Y_1) + (\underbrace{2\beta_I + \frac{2\theta_1\mu_1}{\sigma_1^2}}_{=b_1} - Y_1)g_1'(Y_1) + \beta_I g_1(Y_1) = 0
\end{aligned} \tag{4.13}$$

Let us define b_1 as it is described in (4.13) (similar to Dixit and Pindyck (1994)). This equation is known as the *Kummer* equation. In general, *Kummer* equations are of the form:

$$zk''(z) + (b - z)k'(z) - ak(z) = 0 \tag{4.14}$$

There are two possible solutions to this equation, one given by

$$H(z, a, b) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots \tag{4.15}$$

that is designated as the *Kummer function*. The second one is given by

$$U(z, a, b) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}H(z, a, b) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}H(z, a-b+1, 2-b) \tag{4.16}$$

and it is called a *Tricomi* function. Of course, we have that a and b are just constant parameters.

The choice regarding which function to use should be made taking into account the limiting behaviour we wish to observe in the considered scenario. As the two solutions present different limiting behaviours, one should choose the one whose behaviour is more in agreement with the scenario in question. Since Dixit and Pindyck (1994) consider the first solution, we will have $b = b_1$, $a = -\beta_I$ and $Y_1 = \frac{X_1 2\theta_1}{\sigma_1^2}$ meaning that the presented solution for (4.13) is given by $H(\frac{X_1 2\theta_1}{\sigma_1^2}, -\beta_I, b_1) = g_1(Y_1) = h(X_1)$. Using the same reasoning for g_2, with the obvious change of variables and parameters, we end up with the following function for $h_2(X_2) = g_2(Y_2) = H(\frac{X_2 2\theta_2}{\sigma_2^2}, -\eta_I, b_2)$.

Thus, in the view of these results and once more according to Dixit and Pindyck (1994), it follows that G_{HI} is given by:

$$G_{HI}(X_1, X_2) = A_I X_1^{\beta_I} X_2^{\eta_I} H\left(\frac{X_1 2\theta_1}{\sigma_1^2}, -\beta_I, b_1\right) H\left(\frac{X_2 2\theta_2}{\sigma_2^2}, -\eta_I, b_2\right) \tag{4.17}$$

Proceeding now to the next step of the methodology we've been following so far, we shall divide G_{HI} in four quadrants for the values of β_I and η_I. Based on economical arguments we can choose the relevant quadrant to integrate each G_{HI}, for $I \in \{1, 2\}$. The quadrant division is presented below, and it is the same as considered in the previous cases treated in the last chapters.

$$\begin{aligned}
I &: \{\beta_I, \eta_I\} \quad \beta_I \geq 0; \eta_I \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I1}, \eta_{I1}\} \\
II &: \{\beta_I, \eta_I\} \quad \beta_I \geq 0; \eta_I \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I2}, \eta_{I2}\} \\
III &: \{\beta_I, \eta_I\} \quad \beta_I \leq 0; \eta_I \leq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I3}, \eta_{I3}\} \\
IV &: \{\beta_I, \eta_I\} \quad \beta_I \leq 0; \eta_I \geq 0, \text{ where the points in these conditions will be denoted by } \{\beta_{I4}, \eta_{I4}\}
\end{aligned}$$

As we are considering the exact same case scenario as in the second section of chapter 3, we can avoid repeating the same arguments, that are similar to the ones used in Adkins and Paxson (2011a), and conclude that whenever feedstock 1 is the incumbent, the selected quadrant is going to be the fourth, and whenever the incumbent feedstock is 2, the considered quadrant is the second. Thus, we can write now G_{H1} and G_{H2} in their final form:

$$\begin{gathered}
G_{H1}(X_1, X_2) = A_{14} X_1^{\beta_{14}} X_2^{\eta_{14}} H(\tfrac{X_1 2\theta_1}{\sigma_1^2}, -\beta_{14}, b_{1(14)}) H(\tfrac{X_2 2\theta_2}{\sigma_2^2}, -\eta_{14}, b_{2(14)}), \\
\text{where } b_{1(14)} = 2\beta_{14} + \tfrac{2\theta_1\mu_1}{\sigma_1^2} \text{ and } b_{2(14)} = 2\eta_{14} + \tfrac{2\theta_2\mu_2}{\sigma_2^2}
\end{gathered} \tag{4.18}$$

$$\begin{gathered}
G_{H2}(X_1, X_2) = A_{22} X_1^{\beta_{22}} X_2^{\eta_{22}} H(\tfrac{X_1 2\theta_1}{\sigma_1^2}, -\beta_{22}, b_{1(22)}) H(\tfrac{X_2 2\theta_2}{\sigma_2^2}, -\eta_{22}, b_{2(22)}), \\
\text{where } b_{1(22)} = 2\beta_{22} + \tfrac{2\theta_1\mu_1}{\sigma_1^2} \text{ and } b_{2(22)} = 2\eta_{22} + \tfrac{2\theta_2\mu_2}{\sigma_2^2}
\end{gathered} \tag{4.19}$$

Once we only have one part of the function G, that corresponds to option value, we consider that it would make no sense to establish value-matching and smooth pasting conditions, for an incomplete function G. So, we end our presentation at this point.

In order to obtain the remaining element of the firm's value, we suggest that a numerical approach is performed, as we do not believe that such a solution can be found analytically.

Chapter 5

Conclusion

In this section we would like to make a few final considerations to what has been stated before. We shall present the main conclusions one can take from the work presented, as well as a selection of the aspects that should be improved in a future approach.

Regarding chapter 2, we can say that the models concerning investment under two sources of uncertainty originally developed in Dixit and Pindyck (1994) and Adkins and Paxson (2011b), were successfully presented and adapted to our point of view.

In the first one presented, we consider that the reduction from the two factors to one (as in Dixit and Pindyck (1994)), accomplishes a very significant reduction in the complexity of the model. In cases where the problem can be reduced to one factor, the approach presented by Dixit and Pindyck (1994) is very useful. However, the second method presented, authored by Adkins and Paxson (2011b), allows to handle two uncertainty factors, and it is solved using a *quasi analytical* approach, which involves fixing a value for one the variables in question. This last approach of Adkins and Paxson (2011b) is more general and consequently more interesting for a wider range of applications.

Considering the switching model presented in chapter 3, we find that, independently of the dynamics considered, treating the switching model without any extra considerations leads us to a classic case of indeterminacy. However, we managed to solve the single opportunity switching model for the geometric Brownian motion (as in Adkins and Paxson (2011a)) as well as for the arithmetic Brownian motion. For the case of the arithmetic Brownian motion, we consider an application of a fossil fuel company that has to decide when to stop extracting oil and start extracting natural gas.

Finally, when we attempted to consider the dynamics of the mean reverting process, but keeping the same scenario of oil and gas extraction considered before, we could only treat the homogeneous part of the problem. This means that a solution was successfully found, however it only solves part of the problem. We believe that the remaining part of the solution that could not be found yet, isn't easily achieved analytically. So, what we propose as future work in this problem is the application of numerical methods that will hopefully lead us to reach such a solution.

Additionally, we consider that a practical application of this models can be very interesting to observe. Thus, in this sense, we hope to apply actual real data to this problem and analyse the behaviour of each of the presented models.

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