Nonlinear Attitude Estimation in SO(3):
Application to a Quadrotor UAV

Miguel Braz Teixeira de Brito Figueirôa
Instituto Superior Técnico
Lisboa, Portugal
miguel.figueiroa@tecnico.ulisboa.pt

Abstract
The objective of this work is to study and compare nonlinear attitude estimation methods to be used for a quadrotor state estimation and control. We begin by introducing the QuavIST quadrotor Unmanned Aerial Vehicle (UAV) platform, alongside with the mathematical equations that govern the aircraft motion in flight. The nonlinear equations of motion are linearized about an operating point, and finally the computational model is presented. We then focus on the problem of attitude estimation, and start by reviewing the current state of the art. Three nonlinear estimators from current literature are selected and presented: one Extended Kalman filter and two formulated in the Special Orthogonal Group SO(3). A fourth, self-developed method in SO(3) is formulated using Lyapunov theory and considerations are made about the impact of gyroscope bias and of using Global Positioning System (GPS) measurements to assist attitude estimation. We then introduce a selected position estimator, to be used as complementary information to the attitude filters. The performance of the presented attitude and position estimators is evaluated in a series of simulation scenarios emulating real life quadrotor applications.

Keywords: Nonlinear Attitude Estimator, Lyapunov Estimator, Extended Kalman Filter, Quadrotor

1 Introduction
A quadrotor consists of a cross-shaped horizontal frame with a rotor and respective propeller assembled on each of the four corner ends, aligned with the vertical axis. The central section is reserved for sensors, microprocessors and other electronic elements and batteries to power the four electric motors. Motion control is achieved by independently controlling the rotation speed of each of the four propellers. In academia, quadrotors have been used as a platform to test a variety of estimation and control algorithms, Simultaneous Location and Mapping (SLAM) methodologies, methods in the field of computer vision and other technologies. It is also increasingly popular among the hobbyist aircraft model community [1]. The private sector continues to find ingenious applications for the quadrotor platform, from retrieving aerial footage of sports events or journalistic footage, to easing in the inspection of civil infrastructure [2]–[4], and is also being considered as a platform capable of doorstep delivery of small but high value items such as urgent medicine [5] or consumer goods [6].

The prime objective of this work is to study and evaluate the performance of state of the art attitude and position estimators. A solid foundation in state estimation is critical for later development of advanced control algorithms, paving the way to expand the scope of possible quadrotor applications.

One of the main contributions brought forth from this work is the development of a novel nonlinear attitude estimation technique in SO(3), and comparison with the previously presented estimation methods. This novel approach and respective comparison resulted in a publication to be presented at an international conference [7].

Following this introductory section, Section 2 introduces the QuavIST platform, alongside with the mathematical formulation required to model the aircraft. Section 3 presents three nonlinear estimators from current literature are a fourth, self-developed method. Considerations are made about the impact of gyroscope bias and of using Global Positioning System (GPS) measurements to assist attitude estimation. Section 4 introduces the selected position estimator. In Section 5 the performance of the presented attitude and position estimators is evaluated in a series of simulation scenarios.
emulating real life quadrotor applications. Section 6 is reserved for concluding remarks and recommendations for future work.

2 THE QUADROTOR UAV

The QuaVIST is a commercial quadrotor manufactured by UAVISION. It features not only the functions required for an aircraft to fly autonomously (attitude stabilization GPS navigation, communication protocols) but also software to be run on a handheld Ground Control Station, making it possible to monitor and even change flight parameters.

The UAV is custom modified to house an on board PC-104 computer that receives information from the onboard sensors and performs high level computational tasks. It is used to test and validate top level control algorithms such as nonlinear attitude control [8] and vision based state estimation [9]. The computer is linked to a webcam that takes pictures of the ground below at regular intervals. Sensor information, such as heading and GPS location is appended to the photos.

The low level control electronics send periodic messages to both the onboard computer and a wireless transmitter that further sends it to the ground control station.

![Figure 1. QuaVIST platform, to the left and Ground Control Station (GCS) in the right, from [10]](image)

In order to model the flight behavior of the UAV consider the quadrotor presented in Figure 2. Two reference frames are defined: \{A\} is the inertial North East Down (NED) frame and \{B\} corresponds to the body fixed frame, centered on the quadrotor center of mass. \(\alpha_i\) and \(f_i\) correspond respectively to the rotation speed and thrust force generated by each rotor.

![Figure 2. Quadrotor frames of reference](image)

The orientation of the aircraft can be described by the Euler angles \(\varepsilon = [\phi, \theta, \psi]^T\): roll (\(\phi\)), pitch (\(\theta\)) and yaw (\(\psi\)). It can also be described by a rotation matrix \(R\) that expresses the rotation from \{B\} to \{A\}. The dynamics of a quadrotor in flight is modeled by the set of equations

\[
\begin{align*}
\dot{\varepsilon} &= R(f_\Sigma + f_a)/m + g \\
\dot{\omega} &= -\omega \times f_\omega + \tau + \tau_a \\
\dot{p} &= v \\
\dot{R} &= R[\omega]_x
\end{align*}
\] (1)

where \(\dot{\varepsilon} = [v_N, v_E, v_D]\) corresponds to the linear velocity vector in \{A\}, \(\omega\) the rotational velocity measured in \{B\} and \(p = [p_N, p_E, p_D]\) the position in \{A\}. \(f_\Sigma \in \mathbb{R}^3\) corresponds the vector of the sum of forces generated by the propellers and \(f_a\) the sum of other aerodynamic effects, both in \{A\}. The term \(g\) is the gravity acceleration vector in \{A\}. \(\tau, \tau_a \in \mathbb{R}^3\) represent the moments acting on the aircraft, generated from the propellers and from other aerodynamics effects, respectively. The rotation kinematics are formulated in the Special Orthogonal Group is denoted by

\[
SO(3) = \{ A \in \mathbb{R}^{3\times3} | A^T A = I, \det(A) = 1 \}
\] (2)

and its Lie algebra is denoted by \(so(3)\), the set of anti-symmetric matrices. The cross map \([\cdot]_\times : \mathbb{R}^{3\times3} \mapsto so(3)\) is a Lie algebra isomorphism, with

\[
[x]_\times = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}
\] (3)
so that \([x]_\times y = x \times y\), \(\forall x, y \in \mathbb{R}^3\). The inverse of the cross map is denoted by the vee map, \([\cdot]_v : so(3) \mapsto \mathbb{R}^3\).

As each propeller rotates with angular velocity \(\alpha_i\) it produces a force \(f_i\) along negative \(\hat{b}_3\) and a counter rotating moment on the aircraft, \(\tau_i\) (Figure 2). The force and moment generated can be calculated related by according to the propeller characteristics.

\[
c_{MT} = \frac{\tau_i}{f_i}
\] (4)

The forces and moments acting of the aircraft generated by the combined effect of the four propellers can be computed according to

\[
\begin{bmatrix} f_{\Sigma}^{\text{ph}} \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -c_{MT} & c_{MT} & -c_{MT} & c_{MT} \end{bmatrix} \begin{bmatrix} f_1^{\text{ph}} \\ f_2^{\text{ph}} \\ f_3^{\text{ph}} \\ f_4^{\text{ph}} \end{bmatrix}
\] (5)

where \(d\) is the distance between the propeller and the quadrotor center of mass.

According to recent literature [11], there are additional aerodynamic effects to be considered for small unmanned quadrotors such as Blade Flapping and Induced Drag.. For the estimation purposes of this work the effect of adding these additional aerodynamic forces is negligible, and do the quadrotor is considered to be subject only to drag forces from wind based on Dryden Wind Turbulence Model [12].
The quadrotor is fitted with several sensors, typically subject to white noise, $\sigma_i$

1) **Rate gyroscope:**
The rate gyroscope measures the angular velocity of \( \{B\} \) relative to \( \{A\} \), expressed in \( \{B\} \). The error model used is

\[
\bar{\omega} = \omega + \sigma_\omega + \mu_\omega
\]  
(6)

with $\omega \in \mathbb{R}^3$ denoting the true value of the angular rate, $\sigma_\omega$ denoting additive white noise and $\mu_\omega$ corresponding to a constant (or slowly time varying) bias.

2) **Accelerometer:**
Let the constant gravitational acceleration vector be $g \in \mathbb{R}^3$ and $a$ denote the linear acceleration of \( \{B\} \) relative to \( \{A\} \) expressed in \( \{B\} \). The accelerometer measures the instantaneous of acceleration of \( \{B\} \), $\bar{a}$, minus the effect of the gravitational acceleration, expressed in \( \{B\} \):

\[
\bar{a} = a - R^T g + \sigma_a + \mu_a
\]  
(7)

3) **Magnetometer:**
The magnetometer measures the specific Earth magnetic field vector, $m \in \mathbb{R}^3$.

\[
\bar{n} = R^T m + d_n + \sigma_n
\]  
(8)

For the purposes of estimation, in this paper we will not consider the local magnetic disturbances $d_n$ as it is customary in the literature [11], [13].

4) **Global Positioning System:**
The positioning information derived from a GPS receiver is assumed to be corrupted by white noise, according to the expected error budget for horizontal positioning using L1 Civilian grade GPS receiver found in [14].

\[
\bar{p} = p + \sigma_p
\]  
(9)

GPS receivers also provide velocity information, and the same error model is used, based on [15].

\[
\bar{v} = v + \sigma_v
\]  
(10)

5) **Barometer:**
The quadrotor is fitted with a digital pressure sensor that measures ambient air pressure in order to obtain a vertical position estimate based on the atmospheric pressure gradient with the following error model

\[
\bar{p}_{\text{baro}} = p + \sigma_{\text{baro}}
\]  
(11)

6) **Sonar**
Used for high precision altitude measurements, unlike GPS and the barometric sensor, the sonar measures a relative height and no absolute height, and typically have limited range ($\approx 5m$).

\[
\bar{p}_{\text{sonar}} = p + \sigma_{\text{sonar}}
\]  
(12)

The mathematical formulation presented above is used in a Simulink (MATLAB®) simulation. It uses the set of equations (1) and the relationship (5) coupled with the linearized model of the motor/propeller assembly developed in [16]. Position and Attitude estimation is the subject of this work. A Linear Quadratic Regulator (LQR) is used for attitude and vertical positioning while a saturated Proportional-Derivative (PD) controller handles high level path following.

![Figure 3. Block diagram of simulation model](image)

### 3 Attitude Estimation

The primary source of information regarding attitude are the rate gyros, but these suffer from the inconvenience of measuring the rate of change of $R$ as opposed to $R$ itself. Both the magnetometer and the accelerometer provide vector measurements: the direction of the magnetic North and the direction of gravity (when the acceleration of \( \{B\} \) is negligible).

\[
\bar{n} \approx R^T m
\]  
(13)

\[
\bar{a} = -R^T g
\]  
(14)

Each of these measurements constrain two of the three degrees of freedom (DOF) of $R$ and if these vectors are not collinear they can be used to obtain a measurement of $R$, $\bar{R}$. In this work we compute $\bar{R}$ using the Single Value Decomposition (SVD) solution to Wahba’s Problem as described in [17], which minimizes the loss function

\[
L(R) = \frac{1}{2} W_{\text{SVD}} [V_{\{A\}} - RV_{\{B\}}]^T [V_{\{A\}} - RV_{\{B\}}]
\]  
(15)

with $V_{\{A\}} = [g \ m]$, $V_{\{B\}} = [-\bar{a} \ \bar{n}]$ and $W_{\text{SVD}}$ a diagonal matrix of positive weights.

A number of attitude estimation methods was studied:

- The Extended Kalman Filter
- Nonlinear Complementary filters in SO3
- Trace Based Filter in SO(3)

#### 3.1 Extended Kalman Filter

We will use the Extended Kalman Filter (EKF) as a performance benchmark as it is a classical attitude estimation method. Also, it is closer to the direct cosine matrix (DCM) formulation used in the following nonlinear filters as opposed to the quaternion form used in the also common Multiplicative Extended Kalman
Filter (MEKF). The EKF approach used here considers the gyroscopic outputs as the observer input, \( u = \vec{a} \), and produces an estimate of the attitude angles \( \varepsilon = [\phi, \theta, \psi]^T \) (corresponding to the estimator state \( x \)), that is later corrected by the measurements of the accelerometer and the magnetometer via the Kalman gain. This algorithm considers only the linearization around the yaw angle \( \psi \), and therefore assumes small (< 20°) roll (\( \phi \)) and pitch (\( \theta \)) angles [16]

\[
y = \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix}
\]

where the indices 1, 2 refer to the respective component of the measurement vector, \( n_{12} = [\vec{n}_1, \vec{n}_2]^T \) and \( \| \cdot \| \) corresponds to the Euclidean norm.

The trim conditions at iteration \( k \) is defined for the state \( x \) and for the output \( y \), respectively, as

\[
x_k^0 = [0, 0, \psi_{k-1}]^T
\]

\[
y_k^0 = \begin{bmatrix} 0 \\ 0 \\ \frac{n_{1,k-1}}{\| n_{12,k-1} \|} \\ \frac{n_{2,k-1}}{\| n_{12,k-1} \|} \end{bmatrix}
\]

The feedforward elements of the EKF (denoted by the superscript \( \sim \)) are obtained from

\[
\begin{align*}
\hat{x}_k &= x_k^0 + A_d(\hat{x}_{k-1} - x_k^0) + B_d u_k \\
P_k &= A_d P_{k-1} A_d^T + Q_k
\end{align*}
\]

where \( A_d \) and \( B_d \) correspond to the dynamic state space discrete matrices, \( P_k \) is the process noise covariance matrix and \( K_k \) is the EKF covariance matrix in iteration \( k \).

The output \( C_k \) matrix of the EKF linearized about the trim condition, is

\[
C_k = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\cos(\hat{\psi}_k) & 0 \\ 0 & 0 & \sin(\hat{\psi}_k) & 0 \end{bmatrix}
\]

The measurement vector \( y \) updates the estimate according to the classical Kalman Filter theory, following equations (22)-(24)

\[
K_k = P_k C_k^T (C_k P_k C_k^T + R_k)^{-1}
\]

\[
\hat{x}_k = \hat{x}_k - x_k^0 + C_k y_k
\]

\[
P_k = (I_3 - K_k C_k) P_k
\]

with \( R_k \) the measurement noise covariance matrix and \( K_k \) the Kalman filter gain matrix.

### 3.2 Passive Complementary Filter in SO(3)

Mahony et al. [13] presents three nonlinear attitude estimators in SO(3), one of which the Passive Complementary Filter (PCF), where the estimator for the rotation matrix is in the form

\[
\hat{R} = R[\hat{\omega} + k_P P_a(\hat{R})]_	imes
\]

with \( k_p \) a positive scalar gain, \( P_a \) the anti-symmetric operator

\[
\Psi_a = \frac{1}{2}(A - A^T)
\]

\( R \) and \( \hat{R} \) the estimation and measurement, respectively, of the rotation matrix and \( \hat{R} = R^T \hat{R} \) the rotation matrix estimation error.

### 3.3 Explicit Complementary Filter in SO(3)

Mahony et al. [13] also presents the explicit version of the complementary filter. Instead of relying on a measured rotation matrix, this formulation directly uses the measured accelerometer and magnetometer directions to correct the estimated attitude. This allows for different scalar weights to be given to the accelerometer and magnetometer measurements, \( k_a \) and \( k_n \) respectively. The Explicit Complementary Filter (ECF) in SO(3) is given by

\[
\hat{R} = R[\hat{\omega} + \alpha]_x
\]

with

\[
\alpha = k_a (\vec{n} \times \hat{R}^T g) + k_n (\vec{n} \times \hat{R}^T m)
\]

### 3.4 Trace Based Filter in SO(3)

This novel attitude estimation method is based on the formulation in [18]

Given attitude and angular velocity measurements \( (\vec{R}, \hat{\omega}) \) and corresponding estimates \( (\hat{R}, \hat{\omega}) \), we consider an attitude error function \( \Psi : SO(3) \times SO(3) \rightarrow \mathbb{R} \), an attitude error vector \( e_R \in \mathbb{R}^3 \), and an angular velocity error vector \( e_\omega \in \mathbb{R}^3 \) as

\[
\Psi(\hat{R}, R) = \frac{1}{2} \text{tr} \left(D(I - \hat{R}^T R)\right)
\]

\[
e_R = \frac{1}{2} \left[D \hat{R}^T R - \hat{R}^T \hat{R} D\right]_v \]

\[
e_\omega = \hat{\omega} - \hat{R}^T \hat{R} \hat{\omega}
\]

where \( D \in \mathbb{R}^{3 \times 3} \) corresponds to the positive definite diagonal weighing matrix and \( \Psi(\hat{R}, R) \) is locally positive-definite about \( \hat{R} = R \).

To obtain the proposed attitude estimator, consider the following Lyapunov function candidate

\[
W = a \Psi(\hat{R}, R) + \frac{1}{2} e_\omega^T e_\omega
\]

with \( a \) a positive scalar parameter, and where \( W > 0 \) for \( \hat{R} \rightarrow R \), for any \( \hat{R}, R \in SO(3) \) The time derivative of the Lyapunov function \( W \) is given by

\[
\dot{W} = a e_\omega^T e_R + e_\omega^T \dot{e}_\omega = e_\omega^T (a e_R + \dot{e}_\omega)
\]

If we choose the input \( \dot{e}_\omega \) such that

\[
ae_R + \dot{e}_\omega = -\Delta e_\omega
\]
with $\Delta$ a positive definite diagonal weighing matrix, we guarantee that $\dot{W} < 0$ and the attitude estimator is almost-globally asymptotically stable.

For known attitude and angular velocity measurements, $\hat{\mathbf{R}}$ and $\hat{\omega}$, the estimator dynamics is then given by

$$
\dot{\mathbf{e}}_\omega = -\mathbf{e}_\omega - \frac{\mu}{2}[D \hat{\mathbf{R}}^T \hat{\mathbf{R}} - \hat{\mathbf{R}}^T D] \mathbf{v}
$$

with

$$
\hat{\mathbf{R}} = \hat{\mathbf{R}}[\hat{\omega}]_x.
$$

(35)  

(36)  

(37)

### 3.5 GPS assisted Attitude Estimation

GPS measurements can be used to improve the attitude estimate by using a single GPS receiver fixed in $\mathcal{B}$ that provides a measurement of the velocity vector $\mathbf{v} \in \mathbb{R}^3$ expressed in the inertial frame of reference $\{I\}$. With this additional measurement it is possible to construct an observer for the linear velocity $\mathbf{v}$ [19], [20]. This is motivated by the fact that the relationship

$$
\hat{\mathbf{a}} \approx -\hat{\mathbf{R}}^T \mathbf{g}
$$

only holds true for small accelerations since

$$
\hat{\mathbf{a}} = \mathbf{a} - \hat{\mathbf{R}}^T \mathbf{g} + \sigma_a + \mu_a
$$

(14)  

(7)

For high horizontal accelerations the term $\mathbf{a} = \hat{\mathbf{R}}^T \mathbf{v}$ becomes large enough to compromise the attitude estimate. A velocity observer can be used to derive a better estimate of $\hat{\mathbf{a}} = \hat{\mathbf{R}}^T \mathbf{v}$. Should such estimate be available one can use the following equation instead of (14) to obtain improved attitude estimation performance in high dynamic maneuvers.

$$
\hat{\mathbf{a}} - \hat{\mathbf{R}}^T \mathbf{v} \approx \hat{\mathbf{R}}^T \mathbf{g}
$$

(14)  

(38)

In [19] two velocity aided attitude estimator are presented. The first, the Velocity-aided Invariant Attitude Observer (VIAO) corresponds to a modified ECF that also includes a linear velocity observer, and a modified function for the correction term $\mathbf{a}$.

$$
\begin{align*}
\hat{\omega} &= \mathbf{e}_\omega + \hat{\mathbf{R}}^T \hat{\mathbf{R}} \dot{\omega} \\
\hat{\mathbf{R}} &= \hat{\mathbf{R}}[\hat{\omega} + \alpha]_x
\end{align*}
$$

(39)

The second observer, the Velocity-aided Cascaded Attitude Observer (VCAO) uses an additional virtual matrix $Q$ is added. This matrix is such that $Q \hat{\mathbf{a}}$ functions as an estimate of $\mathbf{v}$.

$$
\begin{align*}
\dot{\mathbf{v}} &= k_v(\mathbf{v} - \hat{\mathbf{v}}) + \mu \mathbf{g} + \hat{\mathbf{R}} \hat{\mathbf{a}} \\
Q &= Q[\omega]_x + k_q(\mathbf{v} - \hat{\mathbf{v}}) \hat{\mathbf{a}}^T
\end{align*}
$$

(40)

### 3.6 Gyroscope Bias Estimation

A common limitation of current low cost gyroscopes is the presence of a small, slowly varying bias term, $\mu_\omega$. This bias term can significantly limit the usefulness of the gyroscope for determining the quadrotor attitude. For any nonlinear attitude estimator such as the ones presented the estimator dynamics are

$$
\dot{\mathbf{a}} = \mathbf{a} + \alpha
$$

(14)  

(44)

where the calculation of estimated rotation velocity, $\hat{\omega}$, depends on the algorithm used.

In both nonlinear complementary algorithms, PCF and ECF, the estimated rotation velocity $\hat{\omega}$ can be written as

$$
\hat{\omega} = \bar{\omega} + \alpha
$$

(45)

where $\alpha$ is a correction term based on the vectorial measurements, obtained via $k_r \mathbf{p}_a(\hat{\mathbf{R}})$ and as per (28) for the PCF and ECF, respectively. It is this correction term $\alpha$ that drives the bias estimation,

$$
\dot{\hat{\mathbf{a}}} = \hat{\mathbf{R}}[\hat{\omega} + \alpha - \hat{\mu}]_x, \quad \dot{\hat{\mu}} = -k_{\mu} \alpha
$$

(46)  

(47)

where $0 < k_{\mu} < 1$ to capture only the low frequency aspect of the correction term $\alpha$. This modification maintains the original ECF and PCF stability properties [13].

The extension of the Lyapunov formulation presented in [7] to include the gyroscope estimation was attempted, unsuccessfully. Another option considered in the course of this work is to draw inspiration from the nonlinear complementary filter bias formulation, yielding

$$
\begin{align*}
\alpha &= \mathbf{e}_\omega + \hat{\mathbf{R}}^T \hat{\mathbf{R}} \dot{\hat{\omega}} \\
\dot{\hat{\mu}} &= -k_{\mu} (\mathbf{e}_\omega + \hat{\mathbf{R}}^T \hat{\mathbf{R}} \dot{\hat{\mu}})
\end{align*}
$$

(46)  

(47)

for the TBF. Although the stability of this formulation was not proved mathematically, it shows promising experimental results.
4. POSITION ESTIMATION

The problem of estimation position in $\mathbb{R}^3$ can be posed as the aggregation of three problems of position estimation in $\mathbb{R}$. Linear motion is governed by the Newton Equation $m\ddot{x} = \Sigma F_{ext}$, with $m$ the body mass, $\dot{x}$ the body acceleration and $\Sigma F_{ext}$ the sum of external forces acting on the body. For each axis of the Euclidean space $c \in \{1,2,3\}$ we have

$$x_c = \begin{bmatrix} p_{ph} \\ v \end{bmatrix}_c, \quad u = (Ra)_c$$

(48)

and thus the process equation is

$$\begin{bmatrix} p_{ph} \\ v \end{bmatrix}_c = \begin{bmatrix} 1 & t_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{ph} \\ v \end{bmatrix}_{c,k-1} + \begin{bmatrix} 0 & t_s \\ t_s & 0 \end{bmatrix} (Ra)_{c,k-1}$$

(49)

By discretizing for time step $t_s$ we can formulate the Kalman Filter prediction phase, equal for every axis.

$$\begin{bmatrix} \hat{p}_{-ph} \\ \hat{v} \end{bmatrix}_{c,k} = \begin{bmatrix} 1 & t_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{ph} \tilde{v} \\ 0 \end{bmatrix}_{c,k-1} + \begin{bmatrix} t_s^2/2 & t_s \\ t_s & 0 \end{bmatrix} (\tilde{R}a)_{c,k-1}$$

(50)

For the update phase it is necessary to take into account that the vertical axis ($c = 3$) may also take advantage of measurements from the barometric sensor $\tilde{p}_{baro}$ and the sonar sensor $p_{sonar}$. Thus

$$y_{c,k} = \begin{bmatrix} p_{ph} \\ \tilde{p}_{baro} \\ p_{sonar} \end{bmatrix}_{c,k}, \quad c = \{1,2\}$$

(51)

$$y_{c,k} = \begin{bmatrix} \tilde{p} \\ \tilde{v} \end{bmatrix} \begin{bmatrix} \tilde{p}_{baro} \\ \tilde{p}_{sonar} \end{bmatrix}_{c,k}, \quad c = \{3\}$$

(52)

And the correction phase of the filter takes the form

$$\begin{bmatrix} p_{ph} \\ \tilde{v} \end{bmatrix}_{c,k} = \begin{bmatrix} p_{-ph} \\ \tilde{v} \end{bmatrix}_{c,k} + K_{c,k} \left( y_{c,k} - H_{c,k} \begin{bmatrix} p_{-ph} \\ \tilde{v} \end{bmatrix}_{c,k} \right)$$

(53)

In order to cope with the fact that sensor measurements are asynchronous, in each time step the measurement vector $y$ and the output matrix $H_c$ must be populated according to the sensor measurements available at each time.

The discrete implementation for sample time $t_s$ can be written

$$\tilde{v}_{k+1} = (1 - \gamma_d) \tilde{v}_k + \gamma_d t_s (\delta_{k+1} + \tilde{\delta}_k)$$

(54)

with $\gamma_d$, the discrete time gain, dependent on selected sample time, $t_s$, and filter time constant, $t_a$.

$$\gamma_d = \frac{t_s}{t_s + \gamma_a}$$

(55)

5. RESULTS AND DISCUSSION

5.1 SIMULATION SETUP

The performance of discussed estimators was evaluated in computer simulation conditions. Three missions with horizontal trajectories were considered

1. 100m wide square trajectory
2. 300m x 100m oval trajectory
3. 3 loops of 50m radius circle

![Reference Trajectories](image)

Figure 4. Map of the 3 missions considered

Missions 1 and 2 have a prescribed horizontal velocity of 5 m/s and mission 3 features 10 m/s. The yaw angle is held constant at 10°.

Path following is accomplished via LQR control coupled with a PD controller for attitude and altitude stabilization. Simulations consider the presence of wind composed of a constant term of $(2, 3, 0)$ m/s added with 4 m/s gusts from a Dryden Wind Turbulence model [12].

<table>
<thead>
<tr>
<th>Sensor</th>
<th>$\tau_s$ [s]</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accelerometer [m/s²]</td>
<td>0.02</td>
<td>0.1</td>
</tr>
<tr>
<td>Barometric [m]</td>
<td>0.1</td>
<td>0.69</td>
</tr>
<tr>
<td>GPS horizontal Position [m]</td>
<td>0.25</td>
<td>2</td>
</tr>
<tr>
<td>GPS vertical Position [m]</td>
<td>0.25</td>
<td>5.22</td>
</tr>
<tr>
<td>GPS Velocity [m/s]</td>
<td>0.25</td>
<td>0.2</td>
</tr>
<tr>
<td>Gyroscope [rad/s]</td>
<td>0.02</td>
<td>0.0175</td>
</tr>
<tr>
<td>Magnetometer [N/A]</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>Sonar [m]</td>
<td>0.02</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Table 1. Sensor Properties

The sensors are modeled with the properties presented in Table 1. Where sensor specification could not be obtained from experimental measurements due to the proprietary nature of the QuaVIST platform, the specifications present in datasheet was used.

Additionally, selected simulations also feature gyroscope bias of $1°/sec$ on all three axes.

Initially the parameters for the Extended Kalman Filter were derived directly from the sensor characteristics: the process noise covariance matrix $Q$ defined from the gyroscopic noise variance $\sigma_g^2$ and the measurement noise covariance matrix $R$ determined from the accelerometer and magnetometer noise.
characteristics, $\sigma_a$ and $\sigma_n$. However, some manual iteration suggested that better performance could be obtained using $Q_c = \text{diag}(\sigma_a^2, \sigma_a^2, \sigma_a^2)$ and $R_c = \text{diag}(1.26, 1.26, 3.06, 3.06) \times 10^{-2}$.

The parameters for the nonlinear estimators were also iterated until good performance was achieved. The Explicit Complementary Filter (ECF) parameters are set to $k_n = 0.5$ and $k_p = 1$. The Passive Complementary Filter was tuned to $k_p = 0.3$, whereas for the Trace Based Filter (TBF) the parameters were set to $a = 1$, $D = \text{diag}(25, 25, 25)$ and $\Delta = \text{diag}(45, 45, 45)$. The measured rotation matrix $\tilde{R}$ is calculated via SVD algorithm, with $W_{\text{SVD}} = \text{diag}(1, 5)$.

Both the sensors and the estimators are evaluated at a frequency of 50Hz and, unless otherwise stated, discretization is performed via Euler method.

### 5.2 Effect of Gyroscope Bias

The first comparison considers the case of the above mentioned nonlinear estimations and intends to assess the effect of gyroscopic bias and what improvement can be gain from the online estimation.

Three cases are considered:

1. Unbiased gyroscope measurements and the standard estimator formulation, i.e. without bias estimation
2. Biased gyroscope measurements and the standard estimator formulation
3. Biased Gyroscope measurements and online bias estimation

| & Mission 1 & Mission 2 & Mission 3 |
|---|---|---|---|
| RMSE | $\varphi$ | $\theta$ | $\psi$ | $\varphi$ | $\theta$ | $\psi$ | $\varphi$ | $\theta$ | $\psi$ |
| **EKF** | | | | | | | | | |
| Case 1 | 1.51 | 1.05 | 3.99 | 1.08 | 1.31 | 3.07 | 3.65 | 3.32 | 6.49 |
| Case 2 | 6.56 | 6.42 | 12.58 | 6.48 | 6.55 | 12.32 | 7.30 | 7.00 | 13.04 |
| Case 3 | 2.69 | 1.89 | 6.28 | 2.06 | 2.04 | 4.75 | 5.58 | 5.02 | 10.25 |
| **PCF** | | | | | | | | | |
| Case 1 | 1.98 | 0.50 | 2.74 | 1.19 | 0.41 | 1.65 | 5.11 | 1.25 | 7.29 |
| Case 2 | 3.93 | 3.23 | 4.32 | 3.70 | 3.22 | 3.79 | 5.89 | 3.39 | 8.42 |
| Case 3 | 2.49 | 0.92 | 3.39 | 1.62 | 0.73 | 2.18 | 6.73 | 1.99 | 9.60 |
| **ECF** | | | | | | | | | |
| Case 1 | 1.80 | 1.03 | 1.94 | 1.05 | 0.66 | 1.33 | 4.23 | 1.80 | 5.03 |
| Case 2 | 4.28 | 1.10 | 7.94 | 4.11 | 0.73 | 8.02 | 5.56 | 1.69 | 8.68 |
| Case 3 | 2.35 | 1.14 | 3.49 | 1.56 | 0.72 | 2.60 | 5.48 | 1.76 | 8.32 |
| **TBF** | | | | | | | | | |
| Case 1 | 2.11 | 0.61 | 3.45 | 1.23 | 0.37 | 1.82 | 5.34 | 1.28 | 8.33 |
| Case 2 | 3.59 | 3.05 | 4.13 | 3.26 | 2.93 | 2.66 | 5.88 | 3.32 | 9.08 |
| Case 3 | 2.64 | 0.98 | 3.88 | 1.60 | 0.69 | 2.08 | 6.85 | 1.94 | 9.53 |

*Table 2. Simulation Results for Nonlinear Attitude Estimation: Effect of Gyroscope Bias*

For the ECF and the PCF the bias gain $k_\mu$ is set to 0.1, whereas for the TBF the bias estimation is achieved using (47) with a gain of $k_\mu = 0.1$. Each (case, mission, estimator) tuple was evaluated 30 times and Table 2 summarizes the results.

Case 1 is a useful to evaluate the influence of the aggressiveness of the missions on the estimators. It can be seen that the high aggressiveness of the third mission severely hampers the performance of all estimators, particularly the EKF. Case 1 also reveals the common tendency that the yaw angle ($\psi$) estimation is consistently less accurate that the pitch ($\theta$) and roll ($\phi$) angles, even more so in the case of the EKF and the PCF. This is most likely due to the higher noise of the magnetometer when compared to the accelerometer. Still for case 1, one can see that the TBF and the ECF filters share the best performance, particularly in the roll and pitch axes.

As bias is added to the gyroscope, the accuracy of the estimators worsens significantly, as the correcting effect of the vectorial measurements cannot fully cancel the bias error. However, by adding the online estimation of the bias effect in case 3, the performance is improved to figures closer to case 1 for missions 1 and 2, but not for mission 3.

The performance of the estimators deteriorates as mission aggressiveness increases. In mission 2, the one with the smoothest trajectory it is about 1° RMSE on each axis, 2° RMSE in mission 1 and upwards of 3° RMSE in the third, the one with the most aggressive maneuvers.

### 5.3 GPS assisted attitude estimation

This set of simulations is used to evaluate the effect of GPS assistance on the performance of the nonlinear Filters: PCF, ECF and TBF. The simulation setup is the same as before regarding missions and parameters. Again, a total of three cases were considered:

1. Without GPS assistance nor bias in the gyroscope
2. With GPS assistance and without bias in the gyroscope
3. With both GPS assistance and bias in the gyroscope and with bias estimation

Case 1 correspond to case 1 in Table 2, where sensor measurements are only corrupted with white noise. In case 2 the performance of all estimators is improved significantly. In the case of the EKF, the improvement is limited to the pitch ($\theta$) and roll ($\phi$) axes while yaw ($\psi$) angle estimation shows no sign of improvement. The performance of the nonlinear filters is significantly improved, not only in the pitch ($\theta$) and roll ($\phi$) axes but also in the yaw ($\psi$) axis. The difference of effect in the yaw axis can be explained by the fact that in the EKF, only the pitch ($\theta$) and roll ($\phi$) axes depend on the measured acceleration, whereas in the nonlinear filters presented the acceleration can potentially be used to aid in the estimation of all three attitude angles.

Case 3 can be considered the one that better represents realistic conditions. It takes into account the bias dynamics and indicates the performance gains that can be expected by including GPS measurements. When comparing case 1 to case 3, it is clear that the performance of almost every mission/axis pair is improved, even in mission 2, with the smoothest trajectory.

5.4 Temporary GPS Unavailability

Attitude estimation and control is of vital importance and GPS positioning can suffer from reduced accuracy or even signal unavailability indoors or even when satellite line of sight is blocked by dense forest canopy. Taking advantage of GPS measurements to improve attitude estimation raises the question of what effect and eventual temporary GPS unavailability could have on the attitude estimation. Such occurrence was tested using the TBF in mission 3 considering two scenarios.

1. GPS measurement interruption in the interval $t = [42, 47]$, representing a short term unavailability.

Such is the case of the GPS receiver needing to change the set of tracked satellites because on has moved out of sight, either due to its orbit or due to local disturbances such as building blocking the signal.

2. GPS measurement interruption in the interval $t = [40, 80]$, representing a short term unavailability. Such would be the case of a manually piloted quadrotor entering a building.

Studies performed indicate that a short term GPS unavailability has little impact on the attitude estimation. Figure 5 shows that even in mission 3, with the most aggressive maneuvers, the GPS measurement interruption did not affect attitude estimation significantly.

![Figure 5. Impact of short term GPS unavailability in the attitude estimation](image)

However, as Figure 6 shows, a long term GPS unavailability leads to a significant peak in attitude error. Contrary to what was expected, this error spike occurs not upon losing the GPS signal but as it is recovered, at $t = 80$s. A possible explanation is that the sudden recovery of velocity information after a long GPS interruption is interpreted by the filter in (53) and (54) as

<table>
<thead>
<tr>
<th>RMSE</th>
<th>Mission 1</th>
<th>Mission 2</th>
<th>Mission 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi$</td>
<td>$\theta$</td>
<td>$\psi$</td>
</tr>
<tr>
<td>EKF</td>
<td>Case 1</td>
<td>1.52</td>
<td>1.04</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.64</td>
<td>0.69</td>
<td>4.08</td>
</tr>
<tr>
<td>Case 3</td>
<td>2.16</td>
<td>2.09</td>
<td>6.19</td>
</tr>
<tr>
<td>PCF</td>
<td>Case 1</td>
<td>1.96</td>
<td>0.51</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.58</td>
<td>0.35</td>
<td>0.73</td>
</tr>
<tr>
<td>Case 3</td>
<td>1.01</td>
<td>0.79</td>
<td>1.17</td>
</tr>
<tr>
<td>ECF</td>
<td>Case 1</td>
<td>1.81</td>
<td>1.03</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.61</td>
<td>0.48</td>
<td>0.66</td>
</tr>
<tr>
<td>Case 3</td>
<td>1.23</td>
<td>0.52</td>
<td>2.14</td>
</tr>
<tr>
<td>TBF</td>
<td>Case 1</td>
<td>2.12</td>
<td>0.59</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.88</td>
<td>0.59</td>
<td>1.10</td>
</tr>
<tr>
<td>Case 3</td>
<td>1.23</td>
<td>1.06</td>
<td>1.26</td>
</tr>
</tbody>
</table>

Table 3. Simulation Results for Attitude Estimation: Effect of GPS aid
a high acceleration. Refinement of the $\hat{\theta}$ observer to cope with these long term GPS measurement interruptions is still a work in progress.

5.5 **POSITION ESTIMATOR PERFORMANCE**

Filtering the measurements from standalone civilian grade GPS receiver using the filter proposed in section 4 results in an improvement in horizontal accuracy in the order of 80% and vertical position accuracy improves about 90%. The studies did not consider the sonar measurements due to its saturation at an altitude of about 5m.

<table>
<thead>
<tr>
<th>Position</th>
<th>Mission 1</th>
<th>Mission 2</th>
<th>Mission 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE [m]</td>
<td>N/E</td>
<td>D</td>
<td>N/E</td>
</tr>
<tr>
<td>Filtered</td>
<td>0.41</td>
<td>0.70</td>
<td>0.39</td>
</tr>
<tr>
<td>Measured</td>
<td>2.05</td>
<td>5.23</td>
<td>2.05</td>
</tr>
</tbody>
</table>

Table 4. Position Estimator Performance

6 **CONCLUSION**

There are a multitude of attitude and position estimators in current literature. In this work some alternatives were studied, with the following conclusions:

- Despite featuring reasonable performance in the roll ($\phi$) and pitch ($\theta$) axes, the studied EKF formulation provides poor yaw ($\psi$) angle estimates.
- Sharing similar accuracy, all the studied attitude estimator formulations in $SO(3)$ outperform the EKF and present a good approach provided the quadrotor flight is restricted to smooth trajectories, that is, acceleration is kept within a small bound.
- Although lacking mathematical proof, the proposed method for gyroscope bias estimation leads to good TBF performance, at times outperforming both the ECF and the PCF
- Uncompensated gyroscope bias can have significantly detrimental effects on attitude estimation, regardless of method.
- The envelope of operation of attitude estimators can be expanded outside the small acceleration conditions by using complementary velocity information from GPS or other sensors. However, care must be taken regarding long term GPS measurement interruption.

Interesting future developments may include:

- Experimental validation of the studied estimators using the QuaVIST platform.
- Obtain a more refined error model for L1 civilian grade GPS receiver, either based on current literature or experimental tests.
- Lyapunov stability analysis of the TBF with bias estimation, either in the form presented in this work or some other.
- Consolidation of position and attitude estimation integration using Lyapunov stability theory. The proposed methodologies present good simulation results, but a mathematical proof of stability is lacking.
- Studying the possibility of integrating the proposed attitude and position estimators with complementary sensors not considered in the scope in this work, i.e. hybrid approaches taking advantage of computer vision algorithms studies [9] or multi-antenna GPS attitude determination [21].
- Refinement of the acceleration observer to better cope with possible long term GPS measurement interruptions.

**REFERENCES**


