Analytical and Numerical Techniques For Pulse Propagation In Optical Fibers: Linear And Nonlinear Regimes

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And last but not least to my family for providing me the best education and for believing in me.
Abstract

The main topic of this dissertation is the study of the linear and nonlinear regimes in optical fibers. In both cases numerical simulations were developed based on the FFT (fast Fourier transform) and SSFM (split-step Fourier method) algorithms.

Taking into account the path followed by the optical signal, from the emitter to the detector, the first step is the analysis of the semiconductor laser - namely, the rate equations corresponding to the direct modulation of the injection current.

The second step, after a brief introduction to the optical fiber structure and its modal analysis, focuses on the effect of the group velocity dispersion (GVD) on pulse propagation. Special emphasis is given to the chirp that results from the direct modulation of the laser.

In the next step the differential equation that governs the pulse propagation in the linear regime was presented. Using this equation, dispersion management was studied by introducing DCFs (dispersion compensating fibers). For this technique different kinds of pulses were used in order to simulate the transmission along the optical fiber.

Finally, the influence of nonlinear effects on pulse propagation was analyzed. In this regime, solitons can propagate as long as special circumstances are met, namely a perfect balance between GVD and SPM (self-phase modulation). The interaction between solitons and its influence on pulse propagation was also studied using dispersion management based either on periodic dispersion maps or on DDFs (dispersion decreasing fibers).

Keywords:
Linear regime; nonlinear regime; semiconductor lasers; modal analysis; dispersion; pulse broadening; soliton; chirp
Resumo

O tema principal desta dissertação é o estudo dos regimes linear e não linear nas fibras ópticas. Em ambos os casos simulações numéricas foram desenvolvidas com base nos algoritmos de FFT (fast Fourier transform) e SSFM (split-step Fourier method).

Tendo em consideração o trajeto seguido pelo sinal óptico, desde do emissor até ao detector, o primeiro passo é a análise do laser semiconductor – nomeadamente, as equações das taxas que correspondem à modulação directa da corrente de injeção.

O segundo passo, após uma breve introdução da estrutura da fibra óptica e da sua análise modal, foca no efeito da dispersão da velocidade de grupo (DVG) na propagação de impulsos. Especial enfase é dado ao Chirp que resulta da modulação directa do laser.

No passo seguinte foi apresentada a equação diferencial que governa a propagação de impulsos em regime linear. Com recurso a esta equação, a gestão da dispersão foi estudada introduzindo as fibras compensadores de dispersão. Para esta técnica diferentes tipos de impulsos foram usados de modo a simular a transmissão ao longo da fibra óptica.

Por fim, a influência dos efeitos não lineares na propagação de impulsos foi analisada. Neste regime, os solitões podem propagar-se desde que circunstâncias especiais são verificadas, nomeadamente um equilíbrio perfeito entre a DVG e a AMF (auto-modulação de fase). A interacção entre solitões e a sua influência na propagação de impulsos foi também estudada recorrendo à gestão de dispersão baseada quer em mapas de dispersão periódicos quer em fibras de dispersão decrescente.

Palavras chave:
Regime linear; regime não linear; lasers semicondutores; análise modal; dispersão; alargamento de impulsos; solitão; chirp
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<td>Effective area</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Attenuation constant</td>
</tr>
<tr>
<td>$B(z,t)$</td>
<td>Longitudinal distribution of the electrical field in function of $z$ and $t$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Longitudinal propagation constant</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>Group velocity dispersion (GVD) dispersion</td>
</tr>
<tr>
<td>$c$</td>
<td>Propagation velocity of light in vacuum</td>
</tr>
<tr>
<td>$C$</td>
<td>Chirp parameter</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Dielectric contrast</td>
</tr>
<tr>
<td>$d$</td>
<td>Laser width</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Gain compression coefficient</td>
</tr>
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<td>$E(x, y, z, t)$</td>
<td>Electric field</td>
</tr>
<tr>
<td>$F(x, y)$</td>
<td>Transversal distribution of the electric field in function of $x$ and $y$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Normalized frequency</td>
</tr>
<tr>
<td>$G$</td>
<td>Gain of an optical amplifier</td>
</tr>
<tr>
<td>$G_0$</td>
<td>Elementary rate of stimulated transmission</td>
</tr>
<tr>
<td>$I_0$</td>
<td>Injection current</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Modulation depth of the injection current</td>
</tr>
<tr>
<td>$I_{\text{th}}$</td>
<td>Threshold current</td>
</tr>
<tr>
<td>$L$</td>
<td>Length</td>
</tr>
<tr>
<td>$L_D$</td>
<td>Dispersion length</td>
</tr>
<tr>
<td>$L_{\text{NL}}$</td>
<td>Nonlinear length</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wavelength</td>
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<tr>
<td>$N_0$</td>
<td>Population of electrons</td>
</tr>
<tr>
<td>$N_a$</td>
<td>Average density of electrons in the active zone</td>
</tr>
<tr>
<td>$N_{\text{th}}$</td>
<td>Number of electrons in the oscillation threshold</td>
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<td>$n_1$</td>
<td>Refractive index in the core of the optical fiber</td>
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<tr>
<td>$n_2$</td>
<td>Refractive index in the cladding of the optical fiber</td>
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<tr>
<td>$\sigma(\xi)$</td>
<td>Effective length of the pulse</td>
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<td>$\Omega$</td>
<td>Frequency shift</td>
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<tr>
<td>$q$</td>
<td>Electron charge</td>
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<tr>
<td>$q_0$</td>
<td>Half of the initial temporal separation between solitons (normalized)</td>
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<td>$S_0$</td>
<td>Population of photons</td>
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<td>$t$</td>
<td>Temporal variable</td>
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<td>$T$</td>
<td>Temporal pulse</td>
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<tr>
<td>$\tau$</td>
<td>Normalized time variable</td>
</tr>
<tr>
<td>$\tau_e$</td>
<td>Average life time of electrons</td>
</tr>
<tr>
<td>$\tau_p$</td>
<td>Average life time of protons</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Phase difference between solitons</td>
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</table>
\( u(\zeta, \tau) \) Normalized amplitude of a soliton
\( v_g \) Group velocity
\( V_a \) Laser cavity volume in the active zone
\( w \) Laser length
\( \omega \) Angular frequency
\( \omega_0 \) Angular frequency of the carrier
\( z \) Spatial variable
\( \zeta \) Normalized spatial variable
\( k_0 \) Wavenumber in vacuum
\( \tilde{A}(0, \omega) \) Normalized function
\( F(x) \) Modal function
\( J_m \) Bessel function of the first specie
\( K_m \) Bessel function of the second specie
\( u \) Normalized refractive index in the core
\( w \) Normalized refractive index in the cladding
\( b \) Normalized propagation constant
\( \tilde{n} \) Modal refractive index
\( \nu \) Normalized frequency
\( P_{in} \) Input power
\( \tilde{n}_g \) Group index
\( D \) Dispersion coefficient
\( S \) Dispersion slope
\( D_M \) Material dispersion coefficient
\( D_W \) Waveguide dispersion coefficient
\( B \) Bitrate
\( T_B \) Period of a bit
\( \lambda_{2D} \) Zero dispersion waveguide
\( \Delta \lambda \) Spectral width of the source
\( F(x, y) \) Transversal variation of the mode \( LP_{01} \)
\( B(0, t) \) Longitudinal variation of the mode \( LP_{01} \)
\( A(0, t) \) Pulse amplitude at the fiber entrance
\( A(z, t) \) Pulse amplitude at the fiber exit
\( \tau(z) \) Pulse width at distance \( z \)
\( \Phi(t) \) Phase variation with time
\( \delta\omega(t) \) Chirp
\( \gamma \) Nonlinear coefficient
\( \Phi_{NL} \) Nonlinear phase
\( \beta_{21} \) Dispersion coefficient in the optical fiber with length \( L_1 \)
\( \beta_{22} \) Dispersion coefficient in the optical fiber with length \( L_2 \)
\( \theta_{0\text{max}} \) Maximum acceptance angle
## List of Acronyms

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<th>Acronym</th>
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<td>Fast Fourier Transform</td>
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<td>SSFM</td>
<td>Split-Step Fourier Method</td>
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<td>GVD</td>
<td>Group Velocity Dispersion</td>
</tr>
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<td>DCF</td>
<td>Dispersion-Compensating Fiber</td>
</tr>
<tr>
<td>SPM</td>
<td>Self-Phase Modulation</td>
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<tr>
<td>NLS</td>
<td>Nonlinear Schroedinger Equation</td>
</tr>
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<td>LP</td>
<td>Linearly Polarized</td>
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<td>SMF</td>
<td>Single Mode Fiber</td>
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<tr>
<td>ISI</td>
<td>Inter-Symbol Interference</td>
</tr>
<tr>
<td>DDF</td>
<td>Dispersion Decreasing Fiber</td>
</tr>
<tr>
<td>IFFT</td>
<td>Inverse Fast Fourier Transform</td>
</tr>
<tr>
<td>DLR</td>
<td>Dispersive Linear Regime</td>
</tr>
<tr>
<td>NDLR</td>
<td>Non-Dispersive Linear Regime</td>
</tr>
<tr>
<td>NLDR</td>
<td>Nonlinear Dispersive Regime</td>
</tr>
<tr>
<td>NLNDR</td>
<td>Nonlinear Non-Dispersive Regime</td>
</tr>
<tr>
<td>NA</td>
<td>Numerical Aperture</td>
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</table>
Chapter 1

Introduction

In this chapter a brief description and overview of the work is presented. The basic structure of a fiber-optic communication system is presented. The main technological breakthroughs of optical fibers that contributed for their use communication systems were addressed. A survey, main objectives, state of the art, framework and original contributions are the sections in this chapter.
1.1. Survey

In the modern age, the demand for networks of higher capacities at lower costs is increasing. Optical communication technology has been developing at a rapid speed in order to achieve larger transmission capacities and longer distances.

Global telecommunication networks are continuously being challenged with the increasing demand for higher data rate transmission. Early communication systems based on axial cables, radio and microwave have stricter bandwidth constraints and can witness high losses being unable to meet the demanded bandwidths.

The optical fiber, thanks to its low loss and large bandwidth represents an obvious candidate to fulfill the present and future demand for audio, video and data transmission. All factors combined make the optical fiber a very desirable and suitable communication media in global telecommunication and has been growing as an innovative technology over the past few decades.

In the optical communication systems, the optical fiber is the main element of the communication channel. Typically, the channel is composed with many sections of fiber along with amplifiers and filters capable of amplifying and conditioning the optical signal.

1.2. Motivation and objectives

This dissertation has the main objective to study the pulse propagation in both linear and nonlinear regimes for certain pulses. In addition it will address the dispersion effect. A dispersion compensation technique will be applied in the linear regime and the soliton propagation will be studied in the nonlinear case.

Starting of by analyzing the signal generation semiconductor lasers will be studied based on the rate equations model. Important notions will be defined such as the threshold current from which the laser starts emitting. Next the behavior for different type of injection currents will be examined taking some conclusions regarding its influence through time and with the number of electrons and photons in the cavity.

It is commonly known that when a pulse is propagating in an optical fiber, it can suffer a dispersion effect. The dispersion manifested is mainly in the time domain and can affect the pulse shape.
In real situations a train of pulses is transmitted and when this train arrives at the receiver, the pulses shapes are not the same.

In some situations, when strong dispersion is manifested, the pulse invades the bit slot of another pulse being this effect called inter-symbolic interference (ISI). The main problem with this type of interference is that the receiver may not be able to distinguish the initial message so that it could change the information sent by the emitter. For example, with ISI, the receiver might detect a 0 instead of a 1 or vice-versa.

The pulse can also be affected by dispersion in the spectral domain. When direct modulation occurs the pulse suffers a chirp effect, which basically consists in a broadening of the spectrum of the pulse. This effect will be studied in the dissertation.

Some examples on how to avoid the interferences will be presented. The technique in which we will focus is the DCF (dispersion compensating fiber).

At last, the importance of the nonlinear regime and their advantages comparing with the linear case will be discussed and put in evidence given some importance to the special situation when solitons appear.

1.3. State of the Art

The history of long distance communications using optics dates back probably to the early times as the human species origin. We can easily recall the example of fire signals to visually telecommunicate and the communications between ships using flags as signals or more recently using powerful light sources.

Naturally, the references mentioned above represent optical communications, but are not example of guided optical communication. Only in 1870 we have perhaps the first approach to the principle of optical fiber when in England light was transmitted inside a thin water jet.

The first experiments with dielectric waveguides were performed only after a few decades in 1910. However the extremely high attenuation levels registered back then reduced any possible interest on the waveguide solution applied to telecommunications. Until the end of the end of the 60's the dielectrics used have attenuations levels typically in the order of thousands of dB km\(^{-1}\).

In the beginning of the 70's it was possible to synthetize dielectrics with attenuations levels typically of 20 dB km\(^{-1}\) and nowadays the most common values are around 0.2 dB km\(^{-1}\). It is
thought that in a near future it may be possible to reduce these values at least in magnitude order.

Using a large spectral range that starts in the far infrared region, passing through the visible spectrum and ending in the ultraviolet, optical fibers offer fantastic possibilities regarding the transmission capacity, and can be considered theoretical values $10^4$ superior to the actual systems.

Nowadays optical fibers are operated in the infrared band, in wavelengths that vary between 800 to 1800 nm, i.e., with frequencies in the order of hundreds of THz.

Having carriers with such high frequencies it is possible to achieve really high bandwidths, in the order of hundreds of GHz, which correspond to highly transmission rates.

Almost after a century since the first works, optical fibers, associated with extremely low attenuation and high transmission rates, have reached a technological state that empowers them as the privileged communication channel at short and long distances in the present days and also in the near future, being in the base of a real revolution in the telecommunications world.

1.4. Framework

A communication system is a link connecting two points through which the information is conveyed from one end to the other. With the increasing number of the available telecommunication services and the generalization of communication systems their respective mass adoption was possible.

Thus it becomes necessary to increase the capacity of the systems to meet the existing information traffic. The use of optical fibers came to meet this demand since their discovery enabled the revolution of the telecommunication business making it possible to establish high quality links and high capacity over very long distances.

Looking at the past three decades, the advances made in this device have enhanced and redesigned the technology used in fibers. In addition to the already existing application in telecommunications, optical fibers are also widely used in the other growing fields such as sensors, lasers and fiber amplifiers [11].

It is possible to define an optical communication system as the one that has a carrier with electromagnetic wave signals in the optical spectrum. This signal is contained in a frequency range that goes from the region of the infrared passing through the visible spectrum and ending in the UV domain. The system itself is characterized by core elements: optical transmitter:
converts the electric signals into optical signals sending them to a physical transmission medium (receiver in the case of optical fibers)

**Figure 1- Optical communication system [17]**

The optical communications represent the long distance systems. And long connection distances can compromise the transmitted signal quality due to losses and other degrading effects inherent to transmission. To solve this problem optical amplifiers are used, usually in each branch of fiber that help to compensate the losses by amplifying the signal so that it can reach the destination with the highest quality possible.

### 1.5. Original contributions

In terms of contributions, this dissertation represents an integrated view of the pulse propagation in optical fiber systems. In particular the pulse propagation characterization in linear regime will be made by analyzing the temporal dispersion phenomenon. In nonlinear regime a similar approach studying the joint effect of the self-phase modulation and the temporal dispersion.

The numerical simulation of pulse propagation in both regimes and also the dispersion compensation techniques in the linear regime (DCF) and nonlinear regime (soliton transmission) are also important aspects to be pointed.

Every aspect illustrated in this dissertation will be based in Matlab scripts in order to acknowledge the fundamental conclusions of the topics discussed.
Chapter 2

Semiconductor lasers

The semiconductor lasers use the electric current to produce light. Amongst its advantages we have the fact that they produce a pure spectrum, along with the low volume, reduced cost and high durability. All combined make the laser one of the most suitable light sources to be used in medium and high performance applications. A laser which stands for Light Amplification by Stimulated Emission of Radiation, has the capability of producing electromagnetic radiation which some special characteristics: the light produced is monochromatic (it is centered in specified wavelength), coherent (has well defined phase relations) and collimated (it spreads throughout a beam) [5],[7].

The development of the computer era, made capable the processing of complex numerical calculus methods, which allowed the development of mathematical models that reproduce with high accuracy the behavior of semiconductor lasers.
2.1. Laser model

The models develop are based on the simplification of Maxwell laws, which led to the differential equations denominated rate equations, that in steady state:

\[
\frac{dS}{dt} = \frac{dN}{dt} = 0
\]  

(2.1)

Considering this situation all the zero sub indexes are referred to the steady state from now on. Therefore we can write

\[
\begin{align*}
G_0 S_0 + \beta_{sp} R_{sp}^{(0)} &= \frac{S_0}{\tau_p} \\
G_0 S_0 + \frac{N_0}{\tau_e} &= \frac{I_0}{q}
\end{align*}
\]  

(2.2)

Were we considered, as it is usual, that \( \beta_{sp} \) and \( \tau_e \) are constant and that \( R_{sp}^{(0)} \) represents the spontaneous recombination rate in steady state regime. The gain relates the gain coefficient in the active zone with the gain of the device and it is given by

\[
G_0 = \frac{G_0(N_0 - N_s)}{1 + \varepsilon S_0}
\]  

(2.3)

\( N_0 \) and \( S_0 \) represent the population of electrons and photons respectively.

Generally the function \( G = G(N, S) \) is a non-linear relationship. However, we are going only to analyze the simple case of the linear model where it is admitted that \( G \) only varies with \( N \), in other words, it does not depend on the number of photons.

Due to the reduced value of \( \beta_{sp} \) the effect of spontaneous radiation is minimal when compared to the stimulated radiation and therefore the term \( \beta_{sp}R_{sp} \) may neglected. Taking this into consideration, we obtain the following simplification of equation(2.2):

\[
S_0 \left(G_0 - \frac{1}{\tau_p}\right) = 0
\]  

(2.4)
When the laser is emitting \( S_0 > 0 \), equation (2.4) reveals the following oscillation condition:

\[
G_0 = \frac{1}{\tau_p} \tag{2.5}
\]

In the oscillation threshold we have \( S_0 = 0 \), \( N_0 = N_{th} \) and \( I_0 = I_{th} \). Trying to rewrite equation (2.2) with these new conditions we obtain the equation that gives us the threshold current:

\[
\frac{N_{th}}{\tau_c} = \frac{I_{th}}{q} \Leftrightarrow I_{th} = q \cdot \frac{N_{th}}{\tau_c} \tag{2.6}
\]

Starting from equation (2.3) and replacing \( S_0 = 0 \) as mentioned above we can calculate the population of electrons at the oscillation threshold. For this purpose we also take in account equation(2.5):

\[
G_0 = \frac{G_N (N_{th} - N_i)}{1 + \varepsilon.0} \Leftrightarrow N_{th} = N_i + \frac{G_0}{G_N} \tag{2.7}
\]

\[
\Leftrightarrow N_{th} = N_i + \frac{1}{G_N \tau_p}
\]

This way, observing equation (2.7) we can conclude that in this model, the population of electrons is constant and does not depend on the injection current.

To characterize a semiconductor laser in order to simulate its behavior a set of numerical values is chosen [2]:

\( \lambda = 1.55 \mu m \) : wavelength

\( G_N = 10^4 s^{-1} \) : elementary rate of stimulated transmission

\( N_i = 10^8 \) : number of electrons

\( n_{sp} = 2 \) : spontaneous emission factor

\( \varepsilon = 10^{-7} \) : gain compression coefficient

\( \tau_p = 3 ps \) : average life time of photons

\( q = 1.601 \times 10^{-19} C \) : electron charge
2.2. Oscillation threshold

It is necessary that the injected current is higher than the threshold current $I_{th}$ for the laser to emit light. Otherwise the laser will not emit.

The values of $N$ and $S$ in the steady state regime, $N_0$ and $S_0$, represent the number of electrons and photons respectively which the laser will present after the injection current remains invariable with time. These values will not depend if the current in steady state regime $I_0$ is higher or lower than the threshold current.

Taking in account equation (2.7) it is possible to determine for this situation the numerical value of $N_{th}$:

$$N_{th} = N_t + \frac{1}{G_N \tau_p} = 10^8 + \frac{1}{10^4 \times 3 \times 10^{-12}} = 1.333 \times 10^8$$

This way, it can be obtained the threshold current through equation (2.6):

$$I_{th} = q \frac{N_{th} \tau_c}{\tau_c} = 1.6 \times 10^{-19} \times \frac{1.333 \times 10^8}{2 \times 10^{-9}} = 10.67 mA$$ (2.8)

Assuming that the electrons life time $\tau_c \equiv \tau_{c|N=N_a}$ we have that:

$$\tau_{c|N=N_a} = \frac{1}{A + b_a N_a + c_a N_a^2}$$ (2.9)

Where $N = N_a V_a$ represents the total number of electrons, $N_a$ is the average electron density in the active zone and $V_a$ is the laser cavity volume in the active zone and given by:

$$V_a = wdL$$ (2.10)

The laser cavity dimensions used for the numerical simulation are the following:

**Width**: $w = 2 \times 10^{-6} m$

**Depth**: $d = 0.2 \times 10^{-6} m$

**Length**: $L = 250 \times 10^{-6} m$
From equation (2.10), and replacing the values of the laser cavity mentioned we have:

\[ V_a = w d L = 2 \times 10^{-6} \cdot 0.2 \times 10^{-6} \cdot 250 \times 10^{-6} = 10^{-16} m \]

Therefore in equation (2.9) we can obtain the life time of the electrons:

\[ \frac{1}{\tau_c} = A + b_d \frac{N_{sh}}{V_a} + c_d \left( \frac{N_{sh}}{V_a} \right)^2 \iff \]

\[ \frac{1}{\tau_c} = 10^8 + 10^{-16} \cdot \frac{1.333 \times 10^8}{10^{16}} + 3 \times 10^{-41} \left( \frac{1.333 \times 10^8}{10^{16}} \right)^2 \iff \]

\[ \iff \frac{1}{\tau_c} = 2.866 \times 10^8 s^{-1} \]

\[ \iff \tau_c = 3.489 ns \]

This way, the threshold current \( I_{th} \) is calculated through:

\[ I_{th} = q \frac{N_{sh}}{\tau_c} \quad (2.11) \]

Which replacing the values above gives us:

\[ I_{th} = 1.6 \times 10^{-9} \cdot \frac{1.333 \times 10^8}{3.489 \times 10^{19}} = 6.12 mA \]

Let us consider now the number of electrons and photons in steady state, \( N_0 \) and \( S_0 \) respectively, for two different injection currents. One above the threshold current \( (I_0 = 1.1I_{th}) \) and the other below \( (I_0 = 0.8I_{th}) \)

2.3. Emitting Laser

\[ I_0 = 1.1I_{th} \text{ and } I_p = I_{th} \]
As \( I_0 > I_{th} \) the first condition for a laser in emission is verified \((S_0 > 0)\).

With some algebraic manipulation between equations (2.2) and (2.5) in order to the number of electrons \( N_0 \) and number of photons \( S_0 \) we obtain the following equations:

\[
N_0 = N_t + \frac{1 + \varepsilon S_0}{\tau_p G_N} \tag{2.12}
\]

\[
S_0 = \tau_p \left( \frac{I_0}{q} - \frac{N_0}{\tau_c} \right) \tag{2.13}
\]

Equations (2.12) and (2.13) together represent a 2 equations system with 2 unknown variables and replacing the considered values for the numerical simulation we can obtain \( N_0 \) and \( S_0 \)

From the results obtained it is possible to prove that actually \( \varepsilon S_0 \approx 25 \times 10^{-4} \ll 1 \) and so this parameter reflected in equation (2.12) can be negligible. Therefore, when the laser is emitting \( (I_0 = 1.1 I_{th}) \), equations (2.12) and (2.13) verify approximately:

\[
N_0 \approx N_t + \frac{1}{\tau_p G_N} \tag{2.14}
\]

\[
S_0 = \tau_p \left( \frac{I_0 - q}{q} \frac{N_0}{\tau_c} \right) = \tau_p \left( \frac{I_0}{q} - I_{th} \right) \tag{2.15}
\]

This means, when the laser is emitting, equation (2.3) which reflects the equation to calculate \( G_0 \) can be approximated by the linear model:

\[
G_0 = G_N (N_0 - N_t) \tag{2.16}
\]

Meaning this that \( G_0 \) varies only with \( N_0 \), or in other words, it does not depend on the number of photons involved. [8]

In the following pages there will be presented some interesting results based on the values assumed for the emitting laser mode.
Figure 2 - Injection current for T=0.2ns

Figure 3 - Evolution of the total number of photons and electrons for T=0.2ns
T=0.5 ns

![Graph of Injection current for T=0.5ns](image1)

**Figure 4 - Injection current for T=0.5ns**

![Graph of Evolution of the total number of photons and electrons for T=0.5ns](image2)

**Figure 5 - Evolution of the total number of photons and electrons for T=0.5ns**
In the case illustrated above, the injection current is higher than the threshold value, occurring the population inversion, which means that there is only emission of photons after the threshold current, necessary to the stimulated emission. In the figures above that represent the evolution of the number of electrons and photons in the laser cavity we can verify that for both periods when applying an injection current $I_0$, the population of electrons increases rapidly relative to its initial value in opposition to what we see in the photons population. The increasing number of electrons leads to increase the stimulated emission rate making the number of photons to also increase rapidly. This increase will origin radiative recombination characterized by the decreasing number of electrons in the cavity. The described process will again origin a new increase in the number of electrons in pair with the decrease of the photons already mentioned earlier in this cycle. The oscillatory character between electrons and photons inside the cavity of the laser is then comprehended with this process.

2.4. Laser emitting in a short time timespan

$I_0 = 0.8I_{th}$ and $I_p = I_{th}$

As $I_0 < I_{th}$ the emitting laser condition isn’t verified. The laser is not in action ($S_0 = 0$ ).

To calculate the number of electrons we cannot use the same equation (2.12) since this is only valid for the case in which the laser is enable to emit.

This way and starting with equation (2.2) and knowing $S_0 = 0$ we can obtain a simpler equation that allows us to define the number of electrons in the laser:

$$N_0 = \tau_e \cdot \frac{I_0}{q}$$  \hspace{1cm} (2.17)

The same way as done before, replacing by the numerical values:

$$N_0 = 1.067 \times 10^8$$
**T = 0.2 ns**

*Figure 6 - Injection current for T=0.2 ns*

*Figure 7 - Evolution of the total number of photons and electrons for T=0.2ns*
Figure 8 - Injection current for T=0.5ns

Figure 9 - Evolution of the total number of photons and electrons for T=0.5ns
With this new environment in analysis it is clear that the number of photons involved is almost incomparable to the environment presented in the previous section. In the instant immediately before \( t = 0 \) we only have the injection current \( I_0 = 0.8I_{th} \) which if lower than the threshold, it is insufficient to enable the population inversion. When we reach \( t = 0 \), the current considered at that moment is \( I_0 + I_p = 0.8I_{th} + I_{th} = 1.8I_{th} \), which enables the stimulated emission since \( I_0 + I_p > I_{th} \). The difference comparing to the previous situation is in the initial condition \( S_0 = 0 \) that causes a delay in the laser reaction. After \( t=0 \), \( I_p \) current effect disappears having \( I_0 = 0.8I_{th} \), verifying that the current isn’t enough to enable emission which makes the number of photons to decrease to zero.

2.5. Conclusions

In the first case (\( I_0 = 1.1I_{th} \) and \( I_p = I_{th} \)) the evolution of the photon and electron number in the laser cavity when influenced by the injection current was studied.

Due to the fact that the injection current is always higher than the threshold current the laser works without any interruptions. It is also noticeable that the oscillatory character that the number of photons has, which after the pulse durations tends to stabilize.

In the second situation (\( I_0 = 0.8I_{th} \) and \( I_p = I_{th} \)) the laser only has an injection current higher than the threshold in a really short time span, more specifically for \( t = 0 \) when \( (I_0 + I_p > I_{th}) \) which leads to a delay in the answer given by the laser and consequent increase in the photon number. This situation is such that the number of photons generated becomes insufficient in order to trigger the emission. The device enters a transient regime which will rapidly lower the photon number to zero.
Chapter 3

Basic concepts: Optical Fibers

Whenever a pulse propagates along a fiber, one or more modes can manifest. This phenomenon depends on the design of the optical fiber. The transmission of light is only possible thanks to the difference between the refractive index in the core and the clad having the cladding a lower refractive index, being this characteristic associated to a certain incident angle of the light allowing the total internal reflection phenomenon. The modal theory, discussed in the following sections, explains the transmission of light inside a fiber [3], [13].
3.1. Snell law

Snell law explains that refraction is not possible when the incident angle is high. In other words, there is a critical value for the angle which allows the light to travel through a surface. The total internal reflection phenomenon is what maintains and sustains the light confined inside the fiber. This is one of the simplest processes that occur with light travelling from one media to another. Being the basis of optics and optical fibers it is of keen importance to understand it:

\[ n_1 \sin \theta_i = n_2 \sin \theta_t \]  \hspace{1cm} (3.1)

To understand this law clearly are represented next 3 situations of the reflection phenomenon and a special case where we have total internal reflection:

Through trigonometry it is possible to define a maximum value for the incident angle in relation to the fiber axis and not to the normal referred in the figure above which is called acceptance angle \( \theta_{\text{max}} \). Only the rays that enter with an angle inferior to the acceptance angle propagate along the fiber [16],[21].

In fiber optics, let \( n_1 \) be the core refractive index and \( n_2 \) be the clad refractive index, there is a very useful parameter to describe the capacity of collecting light which is related to the acceptance angle, This parameter is mentioned as Numerical Aperture (NA):

\[ NA = n_0 \sin \theta_{\text{max}} = \sqrt{n_1^2 - n_2^2} \]  \hspace{1cm} (3.2)
Where \( n_0 \) is the refractive index of the medium exterior to the fiber (generally air so that \( n_0 = 1 \)). Most of the times, since the difference between the refractive index of the core and cladding is really small given, the numerical aperture (NA) can be approximated, if \( \Delta \ll 1 \), by:

\[
NA = n_i \sqrt{2\Delta}
\]

(3.3)

Where \( \Delta \) (difference between the refractive index of the core and cladding) represents the dielectric contrast and will be discussed with more detail in the next section.

### 3.2. Optical fibers operated in linear regime

Optical fibers operating in the linear regime, commonly mentioned as optical fibers in single mode regime, are designed to carry only a single ray of light or mode, have smaller dimensions and a higher bandwidth since they have less dispersion.

Being the difference between the refractive index of the core and cladding given by the dielectric contrast which is defined by \( \Delta \) such as:

\[
\Delta = \frac{n_i^2 - n_e^2}{2n_i^2}
\]

(3.4)

This parameter is associated with the dispersion in the fiber. When \( \Delta \to 0 \), the group velocity dispersion will tend to vanish. The only problem with this parameter being low is that in such case the fiber won’t be allowed to confine more light inside the core.

To better study the pulse propagation in optical fibers, the step-index profile will be considered.

By admitting a referential in which the propagation direction of the pulse is in the \( z \) axis direction, it is possible to consider a longitudinal component of the electric field in cylindrical coordinates. Therefore the electrical field’s axial component can be considered to be:

\[
E_z(r, \phi, z, t) = E_0 F(r) \exp(i m \phi) \exp[i (\beta z - \omega t)]
\]

(3.5)

where \( E_0 \) is the amplitude’s field, \( F(r) \) is the modal function, \( m \) is a constant that accepts values for field’s periodicity, \( \beta \) is the longitudinal propagation constant and lastly \( \omega \) stands for frequency.

Taking in consideration the step-index fiber, the refractive index function for this profile will be as follows:
\[ n(r) = \begin{cases} 
 n_1, & r \leq a \\
 n_2, & r > a 
\end{cases} \]  
(3.6)

Where \( a \) is the core radius.

Being \( k \) a generic propagation constant and \( k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda} \) the propagation constant in vacuum, using the ray theory the transverse propagation constant can be obtained:

\[ k^2(r) = n^2(r)k_0^2 - \beta^2 \]  
(3.7)

And if \( k = q \) at the cladding and \( k = h \) at the core, equation (3.7) can be rewritten as:

\[ h^2 = n_1^2k_0^2 - \beta^2 \]  
(3.8)

\[ q^2 = n_2^2k_0^2 - \beta^2 \]  
(3.9)

Since the objective is to have a guided superficial wave, the internal reflection needs to occur between the core and the cladding, for which we need to have \( q = i\alpha \) in order to verify the total internal reflection, \( i.e., \)

\[ \alpha^2 = \beta^2 - n_2k_0^2 \]  
(3.10)

### 3.3. Wavenumber and normalized frequency

Considering the subject being discussed in this chapter, it is common to normalize some constants [1].

This proves to be very convenient when the objective is to obtain simpler expressions. This section has the intention to explain the methodology of these simplifications.

Considering the variable transformations, known as core and cladding’s wavenumber, accordingly to the following equations:

\[ u = ha \]  
(3.11)

\[ v = \alpha a \]  
(3.12)

where \( h \) is the transversal propagation constant and \( \alpha \) is the parameter that measures the attenuation (attenuation constant).
Introducing a new variable, $\bar{n}$, and knowing that the longitudinal propagation constant is defined as $\beta$:

$$\beta = \bar{n}k_0$$  \hspace{1cm} (3.13)

And based on equation (3.8) and (3.9) the result obtained is as follows:

$$u^2 = (n_i - \bar{n}^2)(k_0a)^2$$  \hspace{1cm} (3.14)

$$w^2 = (\bar{n}^2 - n_x)(k_0a)^2$$  \hspace{1cm} (3.15)

Introducing the normalized frequency of the fiber, $\nu$, which is given by:

$$\nu^2 = u^2 + w^2$$  \hspace{1cm} (3.16)

Which leads to:

$$\nu^2 = k_0^2a^2(n_i^2 - n_x^2) = k_0^2a^2n_i^2(2\Delta)$$  \hspace{1cm} (3.17)

Knowing that $\nu^2 = u^2 + a^2(\beta^2 - k_0^2n_x^2)$, $n_x^2 = n_i^2(1 - 2\Delta)$, using equation (3.17), it is possible to define the longitudinal propagation constant $\beta$ as function of the parameters $\nu$, $u$ and $\Delta$:

$$\beta^2 = \frac{1}{a^2}\left(\frac{\nu^2}{2\Delta} - u^2\right)$$  \hspace{1cm} (3.18)

Having equation (3.18) it is possible to define the normalized modal refractive index as:

$$b = 1 - \frac{u^2}{\nu^2} = \frac{w^2}{\nu^2} = \frac{\bar{n}^2 - n_x^2}{n_i^2 - n_x^2}$$  \hspace{1cm} (3.19)

With $\bar{n}$ depending on the longitudinal propagation constant as follows:

$$\bar{n} = \frac{\beta}{k_0}$$  \hspace{1cm} (3.20)

We have $\bar{n}$ varying $n_x \leq \bar{n} \leq n_i$ or $n_xk_0 \leq \beta \leq n_ik_0$ which implies $0 \leq b \leq 1$

The cut-off occurs when total internal reflection is verified, i.e., $\beta = n_xk_0$.
At the limit of high frequencies the following conclusion can be made: \( \lim_{v \to \infty} \frac{\pi}{\lambda} = n_1 \) \( \leftrightarrow \) \( \lim_{v \to \infty} b = 1 \)

The physical understanding of these limits is really important. When \( \beta = n_2k_0 \) it is being considered that the total internal reflection will begin at this stage.

Taking in account that:

\[
\begin{align*}
    w &= v\sqrt{b} \\
    u &= v\sqrt{1-b}
\end{align*}
\]

and defining \( v_c \) as the cut-off normalized frequency, we end up with the following result:

\[
v_c = 2\pi n_1 \frac{a}{\lambda_c} \sqrt{2\Delta}
\]

and being \( \lambda_c \) the cut-off wavelength, it can be concluded that

\[
\Delta = \frac{1}{2} \left( \frac{v_c \lambda_c}{2\pi n_1 a} \right)^2
\]

Since a single-mode fiber is being considered, \( \nu \leq \nu_c \), with \( \nu_c = 2.4084 \). This value (single-mode condition) is obtained by solving two equations of modes with respect to the cut-off frequency [7].

### 3.4. Fiber modes

The propagation of light in a fiber is ruled by an electromagnetic phenomenon and so optical fields are associated which are ruled by the Maxwell laws. An optical mode is a specific solution of the wave equation that satisfies the appropriate boundary and which spatial distribution does not change with propagation. A mode is determined by its propagation constant. Taking into account equation (3.13), each fiber mode propagates with a refractive index with values between \( n_2 \leq \bar{n} < n_1 \).

In an optical fiber, the surface guided waves are generally hybrid modes whose modal equation is given by: (equation demonstrated in [3])
\[ R_m(u)S_m(u) = m^2\left(1 - 2\Delta \frac{u^2}{v^2}\right)^2 \left(\frac{v}{uv}\right)^4 \]  

(3.25)

where the parameters \( R_m(u) \) and \( S_m(u) \) are expressed as:

\[
R_m(u) = \frac{J'_m(u)}{uJ_m(u)} + \frac{K'_m(w)}{wK_m(w)}
\]

(3.26)

\[
S_m(u) = \frac{J'_m(u)}{uJ_m(u)} + (1 - 2\Delta) \frac{K'_m(w)}{wK_m(w)}
\]

(3.27)

where \( J_m(u) \) is the Bessel function of the first kind and \( J'_m(u) \) its derivative. \( K_m(w) \) is the Bessel function of second kind and \( K'_m(w) \) its derivative. Regarding the derivatives they are respectively given by:

\[
J'_m(u) = \frac{1}{2} \left[ J_{m-1}(u) - J_{m+1}(u) \right]
\]

(3.28)

\[
K'_m(w) = \frac{1}{2} \left[ K_{m-1}(w) + J_{m+1}(w) \right]
\]

(3.29)

Hybrid modes are divided into two categories: \( HE_{mn} \) and \( EH_{mn} \). This notation is used to represent whether \( H_z \) or \( E_z \) dominates, respectively. The parameter \( m \) refers to an azimuthal variation while the parameter \( n \) refers to a radial variation. When \( m = 0 \), two modes degenerate: \( TE_{0n} \) and \( TM_{0n} \), corresponding to transverse electric (\( E_z = 0 \)) and transverse magnetic (\( H_z = 0 \)). To describe this special case we can write the following pair of equations:

\[
TE_{0n} \Rightarrow R_0(u) = 0
\]

(3.30)

\[
TM_{0n} \Rightarrow S_0(u) = 0
\]

(3.31)

3.5. LP modes

Being there other types of fiber modes, this section will focus on LP modes or linearly polarized modes. In a fiber with a low dielectric contrast, i.e., \( \Delta \ll 1 \), the LP mode come from the
approximation with \( \frac{n_1}{n_2} \approx 1 \). These fibers are called weakly-guided-fiber. In this particular situation and looking at equation (3.4) the following approximation can be made:

\[
\Delta \approx \frac{n_1 - n_2}{n_1}
\]  

(3.32)

Considering low contrast fibers, the Gloge approximation is valid for the modal equation which allows the use of the condition:

\[
R_m(u) = S_m(u)
\]  

(3.33)

And then equation (3.26) becomes:

\[
R_m(u) = \pm \frac{mv^2}{u^2w^2}
\]  

(3.34)

Where (+) stands for \( \text{EH}_{mn} \) and (-) stands for \( \text{HE}_{mn} \).

For \( \text{EH}_{mn} \) mode, the following modal equation is obtained:

\[
\frac{J_{m+1}(u)}{uJ_m(u)} + \frac{K_{m+1}(w)}{wK_m(w)} = 0
\]  

(3.35)

where

\[
J'_m(u) = -J_{m+1}(u) + \frac{m}{u} J_m(u)
\]  

(3.36)

\[
K'_m(w) = -K_{m+1}(w) + \frac{m}{w} K_m(w)
\]  

(3.37)

The same methodology can be applied to \( \text{HE}_{mn} \) mode, for which we obtain

\[
\frac{J_{m-1}(u)}{uJ_m(u)} - \frac{K_{m-1}(w)}{wK_m(w)} = 0
\]  

(3.38)

where, it has been considered that:

\[
J'_m(u) = -J_{m-1}(u) - \frac{m}{u} J_m(u)
\]  

(3.39)
\[ K'_m(w) = -\frac{K_{m-1}(w)}{w} - \frac{m}{w} K_m(w) \]  \hspace{1cm} (3.40)

Analyzing carefully equations (3.35) and (3.38), and if \( m=0 \), both equations will result in:

\[ \frac{J_1(u)}{uJ_0(u)} - \frac{K_1(w)}{wK_0(w)} = 0 \]  \hspace{1cm} (3.41)

Which corresponds to \( \text{TE}_{0n} \) and \( \text{TM}_{0n} \) modes. Using Gloge’s approximation two modes are degenerated. Taking that in account there are two conditions to be obtained when these two modes are used:

\[ J_{-m}(u) = (-1)^m J_m(u) \]  \hspace{1cm} (3.42)

\[ K_{-m}(w) = K_m(w) \]  \hspace{1cm} (3.43)

Consequently, the \( \text{EH}_{0n} \) and the \( \text{HE}_{0n} \) modes are, respectively, \( \text{TM}_{0n} \) and \( \text{TE}_{0n} \) modes. When considering low dielectric contrast, all modes are linearly polarized and are referred to as \( \text{LP}_{pn} \) modes. When \( \text{EH}_{mn} \) mode is propagating the correspondent \( \text{LP}_{pn} \) mode has \( p = m + 1 \). On the contrary when the propagated mode is \( \text{HE}_{mn} \), the correspondent \( \text{LP}_{pn} \) mode has \( p = m - 1 \).

And the modal equation that relates to the previous can be

\[ u \frac{J_{p-1}(u)}{J_p(u)} + w \frac{K_{p-1}(w)}{K_p(w)} = 0 \]  \hspace{1cm} (3.44)

\[ u \frac{J_{p+1}(u)}{J_p(u)} - w \frac{K_{p+1}(w)}{K_p(w)} = 0 \]  \hspace{1cm} (3.45)

The equations above are equivalent and according to them, \( w \) and \( u \) parameters need to be transformed in terms of \( b \) and \( v \) in order to have a \( b(v) \) function. Next is presented a figure that shows the first LP modes of an optical fiber. This result comes straight from equations (3.44) and (3.45).
Figure 11 - Normalized propagation constants as a function of the normalized frequency

Also from the figure it is noticeable that for a certain value of \( \nu \), only \( LP_{01} \) mode is propagated. This value is the cutoff frequency \( \nu_c = 2.4048 \) which enables the \( LP_{01} \) mode. The expression that determines the maximum core radius is [3]:

\[
a \leq a_{\text{max}} = \frac{\lambda}{2\pi n_1 \sqrt{2\Delta}} \tag{3.46}
\]

This radius influences the number of modes that can be propagated. The fewer modes the lower the core radius is. For a single mode fiber the effective area is lower than a multimode fiber. Being this the reason why non-linear effects will be more accentuated in single mode fibers than in the multimodal.
Chapter 4

Optical Fiber: Linear regime

It is impossible to address the pulse propagation in linear regime theme without mentioning the dispersion in optical fibers, since light propagating in a fiber is submitted to the dispersion effect due to the material and structural characteristics of fibers. It is possible to define generically the dispersion as being the distortion and enlargement that transmitted pulses suffer when propagating through the fiber, which more than limiting the bandwidth of the transmitted single can cause interference, most commonly known as inter-symbolic interference (ISI).
4.1. Dispersion in optical fibers

Dispersion in an optical fiber is originated in the material and structural characteristics and results in the fact that the propagation velocity of light is a function of the wavelength. When the beam with finite spectral width is sent to the fiber, each spectral component of the pulse “travels” throughout with a velocity that depends of its respective wavelength.

The difference between propagation velocities of the different spectral components is known as group velocity dispersion (GVD).

In practical terms, dispersion is the spreading out of a light pulse in time as it propagates down the fiber from which it possible to conclude that the inter-symbolic interference or pulse dispersion is the difference between the bandwidth and the entrance and the correspondent bandwidth of the pulse at the exit of the fiber.

There are two types of dispersion in optical fibers: intramodal and intermodal dispersion. Intramodal, or chromatic dispersion affects all types of fibers. Intermodal, or modal, dispersion occurs in multimode fibers. Each type of dispersion mechanism is responsible for the bandwidth limitation and also to pulse spreading.

Intramodal, or chromatic, dispersion depends mainly on fiber materials being there two types of intramodal dispersion: material and waveguide dispersion.

By working in single-mode regime the main cause of dispersion in optical fibers is eliminated, intermodal dispersion. But as mentioned, there is another important factor when discussing dispersion in a fiber which is the GVD. It may also exist high order dispersion. Considering a single-mode fiber with length $L$. After determining the spectral component, at frequency $\omega$, and arriving at the end of the fiber with a delay $T = \frac{L}{v_g}$, where $v_g$ is the group velocity, defined by:

$$v_g = \left( \frac{d \beta}{d \omega} \right)^{-1}$$  \hspace{1cm} (4.1)

Applying the modal refractive index expression to equation (4.1) it’s shown that:

$$v_g = \frac{c}{\tilde{n}_g}$$  \hspace{1cm} (4.2)

where $\tilde{n}_g$ is the group index, given by:
\[ \bar{n}_g = \bar{n} + \omega \left( \frac{d\bar{n}}{d\omega} \right) \]  

(4.3)

As already mentioned, the dependency on the frequency by the group velocity leads to the pulse broadening. This means that they don’t arrive at the end of the fiber at the same instant.

\[ \Delta T = \frac{dT}{d\omega} \Delta \omega = \frac{d}{d\omega} \left( \frac{L}{v_g} \right) \Delta \omega = L \frac{d^2 \beta}{d\omega^2} \Delta \omega = L \beta_2 \Delta \omega, \quad \beta_2 = \frac{d^2 \beta}{d\omega^2} \]  

(4.4)

where the parameter \( \beta_2 \) is denominated as group velocity dispersion, GVD, and defines how much a certain pulse broadens.

Making \( \omega = 2\pi c/\lambda \) and \( \Delta \omega = (-2\pi c/\lambda^2) \Delta \lambda \) it is possible to re-write (4.4) as:

\[ \Delta T = \frac{d}{d\lambda} \left( \frac{L}{v_g} \right) \Delta \lambda = DL \Delta \lambda, \quad D = \frac{d}{d\lambda} \left( \frac{1}{v_g} \right) = -\frac{2\pi c}{\lambda^2} \beta_2 \]  

(4.5)

in which \( D \) is the dispersion coefficient with the following units: ps/(km·nm). This parameter is a sum of two terms:

\[ D = D_M + D_w \]  

(4.6)

Representing \( D_M \) the material dispersion and \( D_w \) the waveguide dispersion which expressions are:

\[ D_M = -\frac{2\pi c}{\lambda^2} \frac{dn_{2g}}{d\omega} = \frac{1}{c} \frac{dn_{2g}}{d\lambda} \]  

(4.7)

\[ D_W = -\frac{2\pi \Delta}{\lambda^2} \left[ \frac{n_{2g}^2 Vd^2(Vb)}{n_2 \omega V^2} + \frac{dn_{2g}}{d\omega} \frac{d(Vb)}{dV} \right] \]  

(4.8)

where \( n_{2g} \) is the group index of the cladding, and \( V \) and \( b \) are, respectively, the normalized frequency and the normalized propagation constant.

It is possible, with an empirical approach, obtain the total dispersion as function of the wavelength, given by [3]

\[ D(\lambda) = \frac{\lambda S_D}{4} \left[ 1 - \left( \frac{\lambda_{2g}}{\lambda} \right)^4 \right] \]  

(4.9)
where $S_D$ is the dispersion slope where the null value of the GVD is verified.

It is demanded for optical communication systems to be capable of transporting information through long distances and at high rates. It has been verified that both are limited by dispersion, being difficult to achieve considerably high bandwidths and connection lengths. It is possible to estimate the effect of dispersion in bit rate.

For that, and considering, the bit rate is given by $D_p = 1/T_b$ the condition $D_pT_b < 1$ must be satisfied. Applying equation (4.5) we obtain [1]

$$D_pL|D|\Delta \lambda < 1 \quad (4.10)$$

Although the effect of GVD is predominant, when the carrier is near the zero dispersion wavelength $\lambda_{zd}$, where $D = \beta_2 = 0$, it becomes necessary to consider the high order dispersion terms. The high order dispersion is determined by the slope $S$

$$S = \frac{\partial D}{\partial \lambda} \quad (4.11)$$

Applying equations (4.4) and (4.9) in (4.11) it comes [3]

$$S = \frac{4\pi c}{\lambda^3} \beta_2 + \left( \frac{2\pi c}{\lambda^2} \right)^2 \beta_3 = \frac{S_D}{4} \left[ 1 + 3 \left( \frac{\lambda_{zd}}{\lambda} \right)^4 \right] \quad (4.12)$$

Being $\beta_3 = d\beta_2/d\omega$ known as high order dispersion coefficient. This parameter, in WDW system applications, is desirable to have a small value, in order to reduce the accumulated dispersion variation in the different wavelengths.
By showing graphically equations (4.9) and (4.12) for the following values:

\[
\begin{align*}
\lambda_{ZD} &= 1312 \text{nm} \\
S_D &= 0.090 \text{ ps/(km nm}^2)\end{align*}
\]

Figure 12 - Dispersion and dispersion slope as a function of the wavelength
4.2. Linear regime propagation equation

For \( z = 0 \) (entrance of the optical fiber), let us consider a pulse \( A(0,t) \) with an angular frequency carrier \( \omega_0 \). Supposing the electric field polarization is linear on the \( x \) axis, we have that:

\[
E(x, y, 0, t) = \hat{x}E(x, y, 0, t)
\]  
(4.13)

Being:

\[
E(x, y, 0, t) = E_0F(x, y)B(0, t)
\]  
(4.14)

\[
B(0, t) = A(0, t)\exp(-i\omega t)
\]  
(4.15)

In a single-mode fiber, \( F(x, y) \) represents the transversal variation of the fundamental \( LP_{01} \) mode. \( J_0 \) is the Bessel function in cylindrical coordinates system and the transversal coordinate \( r \) corresponds to \( r = \sqrt{x^2 + y^2} \). Having finally:

\[
F(r) = \begin{cases} 
J_0 \left( \frac{ru}{a} \right) & r \leq a \\
\frac{J_0(u)}{k_0(w)} \frac{1}{a} \frac{w}{w} & r \geq a
\end{cases}
\]  
(4.16)

in which \( a \) is the optical fiber core radius, \( u \) the transversal propagation constant the core and \( w \) the attenuation constant in the cladding. These two constants are normalized and, therefore, a dimensional:

Because in \( F(0) = 1 \) and \( F(a) = J_0(u) \), in \( r = 0 \) the electric field amplitude is \( E_0 \). Still in the low contrast fibers domain with a very low \( \Delta \) parameter the LP mode approximation is acceptable.

In order to determine the electric field un any point \( z > 0 \), it is used the Fourier transform of the field in \( z = 0 \). Introducing:

\[
\tilde{A}(z, \omega) = \int_{-\infty}^{\infty} A(z, t)\exp(i\omega t)dt
\]  
(4.17)
\[ \tilde{B}(z, \omega) = \int_{-\infty}^{\infty} B(z, t) \exp(i\omega t) dt \]  

(4.18)

And in the respective inverse transforms:

\[ A(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(z, \omega) \exp(-i\omega t) d\omega \]  

(4.19)

\[ B(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{B}(z, \omega) \exp(-i\omega t) d\omega \]  

(4.20)

From equations (4.14) and (4.15) we take that:

\[ \tilde{E}(x, y, 0, \omega) = E_0 F(x, y) \tilde{B}(0, \omega) \]  

(4.21)

\[ \tilde{B}(0, \omega) = \tilde{A}(0, \omega - \omega_0) \]  

(4.22)

Being \( \beta = \beta(\omega) \) in the fundamental mode:

\[ \tilde{E}(x, y, z, \omega) = E_0 F(x, y) \tilde{B}(z, \omega) \]  

(4.23)

\[ \tilde{B}(z, \omega) = \tilde{B}(0, \omega) \exp[i\beta(\omega)z] \]  

(4.24)

Assuming that the polarization doesn’t changed, it is possible to write equation (4.25) which represents the existing electric field at any point of the fiber \( z > 0 \)

\[ E(x, y, z, t) = \hat{x}E(x, y, z, t) \]  

(4.25)

with

\[ E(x, y, z, t) = E_0 F(x, y) B(z, t) \]  

(4.26)

Taking in account equation (4.24):

\[ B(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0, \omega - \omega_0) \exp[i\beta(\omega)z - \omega t] d\omega \]  

(4.27)

Introducing the frequency shift \( \Omega \) in respect to the carrier:

\[ \Omega = \omega - \omega_0 \]  

(4.28)

\[ B(z, t) = \frac{1}{2\pi} \exp(-i\omega_0 t) \int_{-\infty}^{\infty} \tilde{A}(0, \Omega) \exp[i\beta(\omega_0 + \Omega)z - \Omega t] d\Omega \]  

(4.29)
Developing the Taylor series for $\beta(\omega_0 + \Omega)$ reduces the complexity of calculating the inter
present in equation (4.29). With this technique:

$$\beta(\omega_0 + \Omega) = \beta_0 + \Phi(\Omega)$$  \hspace{1cm} (4.30)

$$\Phi(\Omega) = \sum_{m=1}^{\infty} \frac{\beta_m}{m!} \Omega^m$$  \hspace{1cm} (4.31)

It is now possible to write:

$$B(z,t) = A(z,t)\exp[i(\beta_0 z - \omega_0 t)]$$  \hspace{1cm} (4.32)

$$A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0,\Omega)\exp[i\Phi(\Omega)z - \Omega t]$$  \hspace{1cm} (4.33)

Observing equations (4.32) and (4.33) we can verify that $A(z,t)$ varies slower in time
compared to $B(z,t)$. This is verified because $|\Omega| < \omega_0$, and therefore $\exp(-i\Omega t)$ suffers
oscillations with a lower frequency comparing to $\exp(-i\omega_0 t)$.

Combining equations (4.26) and (4.33) it is possible to write:

$$E(x,y,z,t) = E_0 F(x,y)A(z,t)\exp[i(\beta_0 z - \omega_0 t)]$$  \hspace{1cm} (4.34)

It becomes necessary to calculate $A(z,t)$ starting from $A(0,t)$. For that purpose equation (4.33)
needs to be solved:

For $m = 1,2,3...$

$$A_m(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega^m \tilde{A}(0,\Omega)Q(z,t;\Omega)d\Omega$$  \hspace{1cm} (4.35)

where

$$Q(z,t;\Omega) = \exp[i\Phi(\Omega)z]\exp(-i\Omega t)$$  \hspace{1cm} (4.36)

This way, from equation (4.33)

$$\frac{\partial A}{\partial z} = i\sum_{m=1}^{\infty} \frac{\beta_m}{m!} A_m(z,t)$$  \hspace{1cm} (4.37)
Adding losses to equation (4.37) it can be rewritten as follows where $\alpha$ represents the power attenuation coefficient.

$$\frac{\partial A}{\partial z} = i \sum_{m=1}^{\infty} \frac{\beta_m}{m!} A_m(z,t) - \frac{\alpha}{2} A(z,t)$$  \hspace{1cm} (4.38)

It is known by generalization that:

$$\frac{\partial^m A}{\partial t^m} = -i^{2^m} A_m(z,t)$$  \hspace{1cm} (4.39)

Thus, from equations (4.38) and (4.39) we obtain:

$$\frac{\partial A}{\partial z} + \sum_{m=1}^{\infty} \frac{i^{m-1}}{m!} \frac{\partial^m A}{\partial t^m} + \frac{\alpha}{2} A = 0 $$  \hspace{1cm} (4.40)

Which represents the differential equations that allows to calculate $A(z,t)$ from $A(0,t)$.

Generally pulses are narrow banded, which means $|\Omega| \ll \omega_0$, therefore it is possible to consider the truncation given by the following equations (neglecting the remaining high order terms):

$$\Phi(\Omega) = \beta_1 \Omega + \frac{1}{2} \beta_2 \Omega^2 + \frac{1}{6} \beta_3 \Omega^3$$  \hspace{1cm} (4.41)

In this case, equation (4.40) becomes:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{1}{6} \beta_3 \frac{\partial^3 A}{\partial t^3} + \frac{\alpha}{2} A = 0$$  \hspace{1cm} (4.42)

The propagation equation in the linear regime is given by:

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{1}{6} \beta_3 \frac{\partial^3 A}{\partial t^3} + \alpha A = 0$$  \hspace{1cm} (4.43)

For a situation where the attenuation and high order dispersion is negligible the pulse propagation equation in an optical fiber is given by:

$$A(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0,\omega) \exp(\frac{i}{2} \beta_2 \omega^2 z - i\omega t) d\omega$$  \hspace{1cm} (4.44)
4.3. RMS broadening of a Gaussian pulse with Chirp

This section has the objective to come up with an expression that can relate the pulse width at any point of the fiber and the pulse width and the entrance of the fiber, $\sigma / \sigma_0$ (see appendix B for expression deduction). The importance of this equation is considerable since it gives an estimation on how the pulse broadens along the optical fiber. In order to study the pulse broadening a Gaussian pulse was used with the following equation:

$$A(\tau, t) = A_0 \exp \left[ -\frac{1 + iC}{4} \left( \frac{t}{\sigma_0} \right)^2 \right] \quad (4.45)$$

where $A_0$ represents the amplitude of the pulse, $C$ stands for the Chirp’s parameter (which will be later discussed with more detail) and $\sigma_0$ stands for the pulse width at the entrance of the fiber.

Making use of the expression deduced in appendix B, we obtain:

$$\frac{\sigma^2}{\sigma_0^2} = \left( 1 + \frac{C \beta_2 L}{2\sigma_0^2} \right)^2 + \left( \frac{\beta_2 L}{2\sigma_0^2} \right)^2 + (1 + C^2)^2 \left( \frac{\beta_2 L}{4\sqrt{2}\sigma_0^3} \right)^2 \quad (4.46)$$

Equation (4.46) gives the ration between the pulse width at some point and the entrance of the fiber but it is important to notice that it depends on the value of the Chirp parameter, the third order dispersion and the second order dispersion. For now and to simplify only the second order dispersion influence will be taking in account. By changing some variables as:

$$\zeta = \frac{L}{L_D} \quad (4.47)$$

$$L_D = \frac{2\sigma_0^2}{|\beta_2|} \quad (4.48)$$

$$\beta_2 = |\beta_2| \text{sgn}(\beta_2) = -|\beta_2| \quad (4.49)$$

and by substituting these new variables in equation (4.46) the ration equation is:

$$\frac{\sigma}{\sigma_0} = \sqrt{(1 + C\zeta)^2 + \zeta^2} \quad (4.50)$$
Equation (4.50) was simulated for three different values of Chirp and the outcome is shown as follows:

![Pulse width evolution with the distance: sgn(β₂)=-1](image)

The figure above gives a better understanding of how the Chirp parameter influences pulse broadening. According to the reference [7], an unchirped pulse has its width broaden by a factor of $\sqrt{2}$ at $z = L_0$. On the contrary, chirped pulses may broaden or compress depending on the result of the product $\beta_2 C$ being positive or negative. For the case where $\beta_2 C > 0$, the pulse broadens with a faster rate than an unchirped pulse. However, when $\beta_2 C < 0$, and interesting thing is observed: at first the pulse will compress but then the GVD influence will become stronger comparing to the Chirp parameter which will eventually lead to the broadening of the pulse.

One more conclusion can be taken from the figure and also from equation (4.46): pulse broadening will also be influenced by the increasing of the length of the fiber.
4.4. Pulse propagation

In this section the simulation analysis will be discussed. Two different pulses were used in the simulations: hyperbolic secant (‘sech’) pulse and super Gaussian pulse (with and without Chirp). To obtain the results there were written Matlab scripts applied to the linear regime case and admitting a single-mode fiber.

The numerical simulation was obtained based on the Fast Fourier Transform (FFT) method, about which a detailed explanation can be found in appendix C.

The simulation depends on the following propagation equation:

\[ i \frac{\partial u}{\partial \zeta} - \frac{1}{2} \text{sgn}(\beta_z) \frac{\partial^2 u}{\partial \tau^2} = 0 \]  \hspace{1cm} (4.51)

where \( u \) is the pulse, and \( \zeta \) and \( \tau \) represents the normalization of variables according to:

\[
\begin{align*}
\tau &= \frac{t - \beta_z z}{\tau_0} \\
\zeta &= \frac{z}{L_0} = z \left| \frac{\beta_z}{\tau_0} \right|
\end{align*}
\]

Working in the anomalous zone, i.e, when: \( \text{sgn}(\beta_z) = -1 \) the equation (4.51) results on:

\[ \frac{\partial u}{\partial \zeta} = -i \frac{\partial^2 u}{\partial \tau^2} \]  \hspace{1cm} (4.52)

Being this equation the foundation for the following simulations

4.4.1. “Sech” Pulse
The “sech” pulse to be studied in this section is represented by:

\[ u(0, \tau) = \text{sech}(\tau) \]  \hspace{1cm} (4.53)

where \( \tau \) represents the normalized time. The chosen range for the normalized time for this simulation was \([-20 ; 20]\) ps. Simulating the “sech” pulse under these considerations the following results were obtained:

![Graph showing the absolute value of the "sech" pulse and the relation between input and output.](image)

**Figure 14** - Absolute value of the "sech" pulse. Relation between the input and output
In order to show the pulse variation along the communication channel the following figures were obtained.

![Figure 15 - "Sech" pulse broadening along the fiber](image)

Observing the figures from the simulation it is possible to conclude that although there is no visible attenuation it is clear that the absolute value of the pulse decreases throughout the fiber having this effect a fairly simple reason which is the GVD.

A closer look at the figures also reveals a very interesting fact: the energy of the pulse stays constant along the fiber despite of the broadening which means that the medium is considered as perfect since the energy is maintained.

### 4.4.2. Gaussian pulse

For a Gaussian pulse:

\[
A(0, t) = \exp \left( -\frac{t^2}{2\tau_0^2} \right) \tag{4.54}
\]

Where \( \tau_0 \) is the initial pulse width.

Applying equation (4.54) on equation (4.44) and integrating taking in consideration the following equality:
\[ \int_{-\infty}^{\infty} \exp(-ax^2 + bx) \, dx = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right) \]  
\[(4.55)\]

We have that \( A(z,t) \) is given by:

\[ A(z,t) = \frac{\tau_0}{(\tau_0^2 - i\beta_2 z)^{1/2}} \exp\left[-\frac{t^2}{2(\tau_0^2 - i\beta_2 z)}\right] \]  
\[(4.56)\]

And taking in consideration the dispersion length equation, \( L_\beta = \frac{\tau_0^2}{|\beta_2|} \) we can see that a Gaussian pulse maintains its form although it suffers an enlargement, as verified in the following equation:

\[ \tau(z) = \tau_0 \sqrt{1 + \left(\frac{z}{L_\beta}\right)^2} \]  
\[(4.57)\]

The equation (4.57) shows that the GVD enlarges the pulse.

In order to demonstrate the evolution of the Gaussian pulse the following figures were obtained as result of the simulation:

\[ \text{Figure 16} - \text{Absolute value of the Gaussian pulse for C=0. Relation between the input and output} \]
In the figure above is seen that the pulse is broadened and its amplitude decreases, due to the energy conservation, being the pulse width variation of a factor of $\sqrt{2}$. The maximum amplitude of the pulse occurs near the entrance of the fiber and as it moves along the fiber the pulse loses amplitude and broadens. This phenomenon is observed because of the GVD which affects the characteristics parameters of the pulse like the amplitude and the width. The different frequency components will travel at different velocities inciting, this way, the ISI. If we simulate the evolution of the pulse's spectrum for $C = 0$ the following figure is obtained:
It is verified that the pulse spectrum is the same along the fiber since the attenuation was neglected in the fiber and this way none of the spectral components is attenuated. The GVD changes the phase of each spectral component of the pulse, which depends on the frequency and also the distance, and these phase variations don’t affect the pulse spectrum but only the pulse width.

4.4.3. Gaussian pulse with *chirp*

\[
A(0,t) = \exp \left( -\frac{(1 + iC) t^2}{2 \tau_0^2} \right)
\]  

(4.58)

Where \( C \) is the chirp parameter already introduced briefly in previous sections.

With equation (4.59), it is verified that the instantaneous frequency increases linearly from one end to another (*up-chirp*) for \( C > 0 \), meanwhile the opposite happens for \( C < 0 \) (*down-chirp*). It is usual to refer *chirp* as positive or negative, depending upon its value.

The numerical value of \( C \) can be estimated using the spectral width of the Gaussian pulse

Therefore, given:

\[
\tilde{\Delta}(0,\omega) = \frac{2 \pi \tau_0^2}{\sqrt{1 + iC}} \exp \left[ -\frac{\omega^2 \tau_0^2}{2(1 + iC)} \right]
\]  

(4.59)

The half spectral width is:

\[
\Delta \omega = \frac{\sqrt{1 + C^2}}{\tau_0}
\]  

(4.60)

In the absence of *chirp* frequency (\( C = 0 \)), the spectral width is constant and satisfies the relation: \( \Delta \omega \tau_0 = 1 \).

It is safe to imply that the spectral width is reinforced by the factor: \( \sqrt{1 + C^2} \) in the presence of linear *chirp*.

It is possible to estimate \( |C| \) using measurements of \( \Delta \omega \) and \( \tau_0 \), leading to the following equation for the transmission field:
Therefore, even a Gaussian pulse with chirp tends to maintain its Gaussian shape through the propagation. The width \( \tau_1 \) after a certain distance \( z \) of propagation is directly related with the width \( \tau_0 \) as follows:

\[
\frac{\tau_1}{\tau_0} = \left[ 1 + \left( \frac{C \beta_z z}{\tau_0^2} \right)^2 + \left( \frac{\beta_z z}{\tau_0^2} \right)^2 \right]^{1/2}
\]  

(4.62)

It is common to define the normalized distance \( \zeta \) as \( \zeta = \frac{z}{L_D} \), where \( L_D = \frac{\tau_0^2}{|\beta_z|} \) is the dispersion length introduced earlier.

Following the previous sections and simulating this pulse to obtain the amplitude at entrance and exit of the fiber and also the pulse evolution along the fiber for \( C = -2 \) we get

**Figure 7 - Absolute value of the Gaussian pulse for \( C=-2 \). Relation between the input and output**
Due to energy conservation it is observable in the figures above that there is a broadening of the pulse along the fiber resulting in a decrease of the amplitude at the exit when comparing to the entrance. The maximum amplitude occurs near the entrance of the fiber and throughout the fiber the pulse loses amplitude and broadens, being this broaden variation more sudden comparing with the pulse without chirp, since the effects of the chirp parameter sum up with the GVD effects. Considering this it is highly recommend to avoid its application.

The evolution of the pulse spectrum for $C = -2$ is given next:

It is possible to conclude that even with the introduction of the chirp parameter the pulse spectrum maintains the same along the fiber once again because the attenuation was
neglected. Hence, the GVD changes and broadens the pulse width and amplitude but doesn’t have any impact on its spectrum.

4.4.4. Super Gaussian pulse

The Super Gaussian pulse is a special case of the following general expression:

\[ A(0,t) = \exp \left( -\frac{1 + i C}{2} \left( \frac{t}{\tau_0} \right)^{2m} \right) \]  

(4.63)

Considering \( m = 3 \) and for the case in which the chirp parameter is zero the first Super Gaussian pulse is obtained for our study purpose.

![Figure 10 - Absolute value of the SuperGaussian pulse (m=3) for C=0. Relation between the input and output](image)

**Figure 10** - Absolute value of the SuperGaussian pulse (m=3) for C=0. Relation between the input and output
It is clear that the pulse will spread due to GVD. It is also estimated that that for non-values of Chirp the pulse will act with GVD which will cause a higher spreading of the pulse.

In order to compensate the dispersion that occurs in the initial branch it is often used another branch with opposite GVD characteristics.

4.5. Dispersion Compensating Fiber

One of the techniques to compensate dispersion in linear regimes in order to minimize its effect is using dispersion compensating fibers.

This technique allows to combine fibers with different characteristics so that the average GVD of the link stays as low as possible.

In practical terms it is used a periodic dispersion map, with a period equal to the spacing between the two amplifiers. In each pair there are two types of fibers: one with a higher length $L_1$ operating in the anomalous zone and another with a shorter length $L_2$ operating in the normal zone, however with GVD coefficients values quite different.

This technique has an advantage of linear kind since we consider an optical pulse which propagates in two sections of the fiber and which propagation equation is given by [1]:

![Figure 19 - Super Gaussian pulse with C=0 broadening along the fiber](attachment:image.png)
\[
A(L, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{A}(0, \psi) \exp\left[-\frac{i}{2} \psi^2 (\beta_{21} L_1 + \beta_{22} L_2) - i\psi \tau\right] d\psi
\]  

(4.64)

Where \( L = L_1 + L_2 \), \( \beta_{21} \) and \( \beta_{22} \) represent the dispersion coefficients of section 1 and 2 respectively.

The length of the compensating fiber can be obtained as follows:

\[
\beta_{21} L_1 + \beta_{22} L_2 = 0 \Rightarrow L_2 = -\frac{\beta_{21}}{\beta_{22}} L_1
\]  

(4.65)

In this approach we force that \( |\beta_{22}| >> |\beta_{21}| \), so that \( L_1 << L_2 \).

When equation(4.65) is verified, \( A(L, \tau) = A(0, \tau) \) the pulse recovers its initial shape after two consecutive sections even if the pulse width varies significantly in each section.

Figure 20 – Gaussian pulse at entrance and exit of the fiber for C=0
Figure 21 - Gaussian pulse at entrance and exit of the fiber for C=0

Figure 22 - Pulse evolution along the Fiber and DCF for the GVD compensation
Chapter 5

Optical Fiber: Non Linear regime

The analysis in previous chapters was made considering optical fibers operating in linear mediums, however, it is verified that this consideration can’t be done for all applications. This approximation for high powers of the input signal of for higher connection lengths loses its validity. It is possible to confirm that every material behaves in a nonlinear way for high electromagnetic filed intensities by investigating the increase of the refractive index with the also increasing intensity [1, 2]. This effect, which establishes the refractive index dependence on the field intensity, is called nonlinear Kerr effect and pulse propagation in nonlinear regime cannot be mentioned without referencing this effect since it is thanks to this effect that pulse propagation in nonlinear regime is governed by the self-phase modulation (SPM) and the group-velocity dispersion (GVD) simultaneously apart from other high order effects. There is a permanent balance under certain circumstances, when fundamental solitons propagate, i.e, when the effect of losses is neglected, the pulses do not change their form during propagation.
5.1. Nonlinear Kerr effect in an optical fiber

It becomes important to study the consequences of the Kerr effect in pulse propagation in nonlinear regime. As already mentioned, this effect modifies the refractive index of the optical fiber, induced by very intense optical fields. Being $\beta$ the linear propagation constant and $n$ the correspondent modal refractive index, we have [4]:

$$\beta = \bar{n}k_o$$

(5.1)

in which $k_o$ is the propagation constant in vacuum.

In a transversal plane it is possible to relate the dielectric relative constant, $\varepsilon$, with the refractive index of the fiber, $n$, as follows:

$$\varepsilon(x, y) = n^2(x, y)$$

(5.2)

In linear regime, the Helmholtz equation allows to express:

$$\nabla_i^2 F + [n^2(x, y)k_0^2 - \beta^2]F(x, y) = 0$$

(5.3)

which in rectangular coordinates is given by:

$$\nabla_i^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

(5.4)

In the LP modes approximation for low dielectric contrast fibers equation (4.14) can be considered for the electric field. Supposing there is a perturbation in the relative electric constant:

$$\varepsilon' = \varepsilon(x, y) + \Delta\varepsilon$$

(5.5)

This way, the new longitudinal propagation constant is given by:

$$\beta' = \beta + \Delta\beta$$

(5.6)

With

$$\Delta\beta = \frac{k_0^2}{2\beta} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta\varepsilon |F(x, y)|^2 \, dx \, dy \right]$$

(5.7)

Having in account equation (5.2) it comes that:
\[ \Delta \varepsilon = 2n(x,y)\Delta n \] (5.8)

Admitting the approximation \( n(x,y) \approx \bar{n} \) and substituting equation (5.1) in (5.7) we get:

\[
\Delta \beta = k_0^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta n |F(x,y)|^2 \, dx \, dy \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 \, dx \, dy
\] (5.9)

In a silica optical fiber, the nonlinear Kerr effect determines that:

\[ n' = n(x,y) + n_2' |E_0|^2 \] (5.10)

in which \( E_0 \) is a fictitious field and \( n_2' = 3 \times 10^{-20} m^2 / W \). Considering that:

\[ |E_0|^2 = y_0 |E|^2 = I \] (5.11)

Where \( I \) stands for the optical intensity and \( y_0 \) an appropriate admittance. Hence,

\[ \Delta n = n_2' = |E_0|^2 \] (5.12)

According to the electric field definition and the longitudinal field variation we have:

\[ |E_0|^2 (x,y,z,t) = y_0 F(x,y)^2 |A(z,t)|^2 \] (5.13)

Substituting equations (5.12) and (5.13) in equation (5.9) we have:

\[
\Delta \beta = y_0 n_2' k_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta n |F(x,y)|^2 \, dx \, dy \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 \, dx \, dy |A(z,t)|^2
\] (5.14)

and introducing a new amplitude

\[ Q(z,t) = A(z,t) \sqrt{y_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 \, dx \, dy} \] (5.15)

Equation (5.14) can be rewritten as follows:

\[ \Delta \beta = \gamma |Q(z,t)|^2 \] (5.16)

Where the nonlinear coefficient \( \gamma \) is given by:
\[
\gamma = \frac{n_2' k_0}{A_{\text{eff}}} = \frac{2\pi n_2'}{\lambda A_{\text{eff}}}
\]  
\hspace{1cm} (5.17)

where \( A_{\text{eff}} \) is the effective area defined by:

\[
A_{\text{eff}} = \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 \, dx \, dy \right)^{\frac{1}{2}}
\[
\hspace{1cm} \frac{1}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x,y)|^2 \, dx \, dy}
\]  
\hspace{1cm} (5.18)

Representing \( |Q(z,t)|^2 \) for the transported power \( P(z,t) \), equation (5.16) modifies for:

\[
\Delta \beta = \gamma P(z,t)
\]  
\hspace{1cm} (5.19)

Being:

\[
P(z,t) = P_{\text{in}}(t) \exp(-\alpha z)
\]  
\hspace{1cm} (5.20)

In which \( \alpha \) is the attenuation constant and \( P_{\text{in}} \) is the peak power of the pulse at the entrance of the fiber.

The nonlinear phase generated by the Kerr effect is given by:

\[
\phi_{NL}(t) = \int_0^L (\beta' - \beta) \, dz = \int_0^L \Delta \beta \, dz = \gamma \int_0^L P(z,t) \, dz
\]  
\hspace{1cm} (5.21)

Thus

\[
\phi_{NL} = \gamma P_{\text{in}}(t) L_{\text{eff}}
\]  
\hspace{1cm} (5.22)

Where \( L_{\text{eff}} \) is the effective length so that:

\[
L_{\text{eff}} = \frac{1}{\alpha} \left(1 - \exp(-\alpha L)\right)
\]  
\hspace{1cm} (5.23)

It is verified that there is a shift in nonlinear phase which is called self-phase modulation (SPM) which gives origin to an instantaneous frequency deviation along the pulse propagation [4]. If there are used amplification sections, the nonlinear phase at the exit of the total group of amplification sections is given by:

\[
\phi_{NL_{\text{tot}}} = N \phi_{NL}
\]  
\hspace{1cm} (5.24)
Where $N_A$ represents the number of amplification sections.

Taking in consideration the nonlinear effect due to SPM, it is possible to obtain the frequency deviation:

$$
\delta \omega(t) = -\frac{\partial \phi_{NL,\text{total}}}{\partial t} = \gamma L_{\text{eff}} N_A \frac{\partial P_m}{\partial t}
$$

(5.25)

Hence at the pulse front we have:

$$
\frac{\partial P_m}{\partial t} > 0 \Rightarrow \delta \omega(t) < 0
$$

(5.26)

This means a negative frequency deviation (red zone).

In an analogous way applying the same process for the tail of the pulse:

$$
\frac{\partial P_m}{\partial t} < 0 \Rightarrow \delta \omega(t) > 0
$$

(5.27)

In which the tail of the pulse will suffer a positive frequency deviation (blue zone).

Considering the case when the product $\beta_2 C < 0$ and also the contraction of the pulse it is possible to obtain:

$$
\beta_2 = -\frac{1}{v_g(\omega_0)} \frac{\partial v_g}{\partial \omega}
$$

(5.28)

And having in mind the the anomalous region, where $\beta_2 < 0$

$$
\frac{\partial v_g}{\partial \omega} > 0
$$

(5.29)

Accordingly the higher frequencies will travel quicker than lower frequencies. As already mentioned SPM do exactly the opposite of GVD being this the reason why bright solitons appear. If the normal dispersion region is taken in account ($\beta_2 > 0$) it is possible to obtain other type of solitons, commonly named as dark solitons.
5.2. Nonlinear regime pulse propagation equation

In the last section there were discussed the basic conditions for propagation of solitons and in this section the corresponding propagation equation will be derived.

Considering for this purpose the term \( P_{NL} \) that was neglected in previous chapters we have a new term \( i\gamma |A|^2 A \) which considers the nonlinear effects. Adding the new term to equation (4.43) we get:

\[
\frac{\partial A}{\partial z} + \beta_i \frac{\partial A}{\partial t} + i \frac{1}{2} \beta_2 \frac{\partial^2 A}{\partial t^2} - \frac{1}{6} \beta_3 \frac{\partial^3 A}{\partial t^3} = \frac{-\alpha}{2} A + i\gamma |A|^2 A
\]  
\tag{5.30}

This is a differential equation for the pulse propagation in nonlinear regime, designed as nonlinear Schrodinger equation and represents the mathematical description of the behavior of the solitons with respect to GVD and SPM.

Notice that when the parameter \( \beta_2 < 0 \) we are operating the anomalous dispersion zone and it is possible for solitons to propagate.

To describe a soliton mathematically it is necessary that NLS equation is satisfied by the pulse \( A(z,t) \) in the presence of both GVD and SPM being also necessary to consider a fiber without losses and neglecting once again the high order dispersion effects [14].

Making some variable changes of \((z,t) \rightarrow (\zeta, \tau)\) so that:

\[
\zeta = \frac{z}{L_d} \quad \quad \tau = \frac{t - \beta_i}{\tau_0}
\]

And considering: \( A(z,t) = U(z,t) \sqrt{P_0} \exp(-\alpha z / 2) \) where the exponential factor accounts the loss effects.

We get:

\[
\frac{\partial A}{\partial z} = \frac{\partial A}{\partial \zeta} \frac{\partial \zeta}{\partial z} + \frac{\partial A}{\partial \tau} \frac{\partial \tau}{\partial z} = \frac{1}{L_d} \frac{\partial A}{\partial \zeta} - \frac{\beta_i}{\tau_0} \frac{\partial A}{\partial \tau}
\]

\[
\frac{\partial A}{\partial t} = \frac{\partial A}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{1}{\tau_0} \frac{\partial A}{\partial \tau}
\]
\[
\frac{\partial^2 A}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial \tau} \frac{\partial \tau}{\partial t} \right) = \frac{\partial \tau}{\partial \tau} \frac{\partial}{\partial \tau} \left( \frac{\partial A}{\partial \tau} \right) = \left( \frac{\partial \tau}{\partial \tau} \right)^2 \frac{\partial^2 A}{\partial \tau^2} = \frac{1}{\tau_0^2} \frac{\partial^2 A}{\partial \tau^2}
\]

By substituting on equation (5.30) it comes:

\[
\frac{\partial A}{\partial \zeta} = -i \frac{\beta_z L_d}{\tau_0} \frac{\partial^2 A}{\partial \tau^2} + i \gamma |A(z,t)|^2 A(z,t) \tag{5.31}
\]

and since \( L_d = \tau_0^2 / |\beta_z| \) and \( \text{sgn}(\beta_z) = \beta_z / |\beta_z| \) it results in

\[
\frac{\partial A}{\partial \zeta} = -i \text{sgn}(\beta_z) \left( \frac{\partial^2 A}{\partial \tau^2} \right) + i \gamma |A(z,t)|^2 A(z,t) \tag{5.32}
\]

And simplifying equation (5.32) we can obtain:

\[
\frac{\partial A}{\partial \zeta} = -i \text{sgn}(\beta_z) \left( \frac{\partial^2 U}{\partial \tau^2} \right) + i N^2 \exp(-\alpha z / 2) |U(z,t)|^2 U(z,t) \tag{5.33}
\]

in which \( N^2 = \gamma L_d P_0 = L_d / L_{NL} \) is the square of the relation between the dispersion length and the nonlinear length being the last given by: \( L_{NL} = 1 / \gamma P_0 \).

This equation can be written in the following form [15]:

\[
i \frac{\partial U}{\partial \zeta} = \frac{\text{sgn}(\beta_z) \frac{\partial^2 U}{\partial \tau^2}}{2L_d} - \frac{\exp(\alpha z / 2)}{L_{NL}} |U|^2 U \tag{5.34}
\]

Presenting the equation in function of \( L_d \) and \( L_{NL} \) it can be performed a simple evaluation of when dispersive and nonlinear effects are relevant on pulse propagation. Depending on the values of the fiber length, \( L \), \( L_d \) and \( L_{NL} \) the propagation behavior can be classified in four different categories.

- When \( L \) is such that \( L \ll L_{NL} \) and \( L \ll L_d \) both dispersive and nonlinear effects can be neglected resulting in a pulse that maintains its form during propagation \( U(z,t) = U(0,t) \) which means that the fiber only carries optic pulses. This regime is called non-dispersive linear regime (NDLR).
- When $L$ is such that $L \ll L_{NL}$ and $L \approx L_{D}$, the pulses are mainly affected by GVD and the nonlinear effects can be neglected. The principal effect of GVD is the pulse broadening being this regime called dispersive linear regime (DLR).

- When $L$ is such that $L \approx L_{NL}$ and $L \ll L_{D}$, the dispersive effects can be neglected compared to the nonlinear effects. The pulse is mainly affected by the SPM which causes alterations in the pulse spectrum due to the phase variations that it implies. This regime is designated non-dispersive linear regime (NLDLR).

- Finally when $L$ is such that $L \approx L_{NL}$ and $L \approx L_{D}$, the pulse is influenced by both dispersive and nonlinear effects. The simultaneous interaction between GVD and SPM leads to different behaviors when compared to when the effect is manifested singly. In the anomalous zone, when both effects cancel mutually we assist the soliton propagation and in the normal zone the propagation of dark solitons [9]. This regime is called nonlinear dispersive regime (NLDR).

The last situation NLDR is the most usual in an optical fiber and the NDLR corresponds to the utopic situation, rarely observable.

For the case in which $\text{sgn}(\beta_2) = -1$ (anomalous zone when $\beta_2 < 0$) we have:

$$i \frac{\partial U}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 U}{\partial \tau^2} + N^2 |U|^2 U = 0$$  \hspace{1cm} (5.35)

### 5.3. Solitons

Equation (5.35) is a nonlinear partial differential equation which only has a solution in specified cases in which it is possible to apply the inverse scattering method (IST) [10, 20]. From the application of this method it can be conclude that for a pulse given by:

$$U(0,t) = N \sec h(t)$$  \hspace{1cm} (5.36)

Propagating in an optical fiber operating in the anomalous region, it maintains its form when $N = 1$ presenting a periodic pattern of $\pi/2$ for integer values of $N > 1$. The special case when $N = 1$ it is called fundamental soliton and equation (5.35) can be written as:

$$i \frac{\partial U}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 U}{\partial \tau^2} + |U|^2 U = 0$$  \hspace{1cm} (5.37)

It is observable that this pulse has a phase shift of $\zeta/2$ but maintains its amplitude along the propagation, which means the effects of GVD and SPM are compensated mutually. Although
This is an important result it has to been clarified that the situation described only exists in a nonlinear regime when the losses are neglected.

The analytical expressions of high order solitons are really complex being almost impossible to present them in a closed form. The only case we can still describe with an analytical expression is the second order soliton which is given by:

\[
U(ζ, τ) = \frac{4 \left[ \cosh(3τ) + \exp(4iζ) \cosh(τ) \right]}{\cosh(4τ) + 4\cosh(2τ) + 3\cosh(4ζ)} \exp\left( i \frac{ζ}{2} \right)
\]  

(5.38)

**5.3.1. Dark Solitons**

As previously mentioned, there are innumerable solutions for equation (5.35). These different solutions take in account the different cases of dispersion and nonlinear properties. To study and obtain dark solitons it is necessary to operate in the normal region \( \text{sgn}(β_2) = 1 \).

The discovery of these solitons dates back in 1973 and the demand on this field has been increasing since. The name itself is related to the intensity profile which represents an “hole” in a uniform background.

The equation that sums up the behavior of the dark solitons is:

\[
i \frac{∂u}{∂τ} - \frac{1}{2} \frac{∂^2 U}{∂τ^2} + |u|^2 u = 0
\]  

(5.39)

This equation derives directly from equation (5.35) using \( \text{sgn}(β_2) = 1 \) and \( u = NU \). It can be solved using the IST imposing a boundary condition: \( [u(ζ, τ)] \), for high values of \( |τ| \) must tend to a constant different than zero.

Another possible solution is to assume \( u(ζ, τ) = V(τ) \exp[iϕ(ζ, τ)] \) and then solve the differential equations in order to \( V \) and \( ϕ \). The final solution can be written as:

\[
u(ζ, τ) = η \left| B \tanh(ζ) - i\sqrt{1 - B^2} \right| \exp(iη^2ζ)
\]  

(5.40)

Where \( ζ = ηB(τ - τ_s - ηB\sqrt{1 - B^2}) \) [12]. The parameters η and \( τ_s \) represent the amplitude and the minimum, respectively. The parameter \( B \) regulates the depth of the “hole”. When this parameter is equal to 1 then a dark soliton appears.
5.3.2. Bright Solitons

As previously introduced, in order to obtain bright solitons it is necessary to operate in the anomalous zone \( \text{sgn}(\beta) = -1 \). With this consideration and using the normalized amplitude, which is \( u = NU \), equation (5.35) becomes:

\[
i\frac{\partial u}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + \left| u \right|^2 u = 0
\]  
(5.41)

For the linear regime. This can be solved using the IST. Although there aren’t present any further details about this method in this dissertation the general solution for equation (5.41) using this technique is given by:

\[
u(\xi, \tau) = \eta \text{sech} \left( \eta (\tau - \tau_s + \delta \xi) \right) \exp \left[ i(\eta^2 - \delta^2) \xi / 2 - i\delta \tau + i\phi \right]
\]  
(5.42)

The full demonstration on how to obtain expression (5.42) is fully developed in [12] but for our interpretation and study it is important to understand that the soliton characterization requires four parameters:

- \( \eta \) - amplitude
- \( \delta \) - frequency
- \( \tau_s \) - representative of position
- \( \phi \) - representative of phase

In order to simplify some considerations must be made. The factor \( \phi \) is a absolute phase constant which has no physical relevance. The factor \( \tau_s \) refers to the peak position of the soliton, which can be 0 considering the peak at its origin.

Looking carefully at equation (5.42) it becomes clear that parameter \( \delta \) represents the frequency deviation of the soliton in the carrier frequency \( \omega_0 \).

Having in account \( \exp(-i\omega_0 t) \) the new frequency needs to be written as \( \omega_0' = \omega_0 + \delta / \omega_0 \).

With this the soliton suffers a change on its speed. Using \( \tau = (t - \beta z) / T \) and making the proper substitution in equation (5.42) we have:

\[
u(\xi, \tau) = \eta \text{sech} \left[ \eta (t - \beta z) / T \right]
\]  
(5.43)

Where \( \beta = \beta + \delta |\beta| / T \).
As discussed previously the group velocity variation is a direct consequence of dispersion.

If the carrier frequency is well chosen, \( \delta \) can be let out of the equation, which means equation (5.42) can be rewritten in its final form as:

\[
u(\xi, \tau) = \eta \sec h(\eta \tau) \exp(i\eta^2 \xi / 2)\]

(5.44)

Based on this equation an important conclusion can be taken: the parameter \( \eta \) not only alters the soliton amplitude but also its width. This fact is demonstrated by what happens with the relation \( T_0 / \eta \). The soliton amplitude doesn’t need to be equal to 1.

### 5.4.2.1. Fundamental soliton

Having at the fiber entrance a pulse of the type:

\[
u_0(\tau) = \sec h(\tau)\]

(5.45)

The following figures are obtained:
This pulse corresponds to the fundamental soliton. As expected, from the theoretical analysis, the pulse of this type maintains its form during propagation without having any change in its amplitude or width.

5.4.2.2. Second order soliton

Having at the fiber entrance a pulse of the type:

\[ u_0(\tau) = 2 \sec h(\tau) \]  \hspace{1cm} (5.46)

The following figures are obtained:
This pulse corresponds to the second order soliton given by expression (5.38). In this situation the soliton presents variations in its shape and amplitude throughout the propagation along the fiber. The fact that SPM and GVD effects vary across the length justify this phenomenon. It is also possible to observe the energy conservation principle by verifying the appearance of a peak when the pulse straightens. In the final of the analyzed distance the soliton recovers its initial conditions.
5.4.2.3. Third order soliton

Finally having at the fiber entrance a pulse of the type:

\[ u_0(\tau) = 3 \text{sech}(\tau) \]  \hspace{1cm} (5.47)

The following figures are obtained:
The third order soliton presents different characteristics of the solitons analyzed so far. The energy conservation principle is observable once again when the pulse contraction happens looking at the peaks.

In this case the effects of SPM and GVD don’t compensate each other and therefore there is an initial broadening of the spectrum with a deviation to blue in the tail and to red in the front of the pulse.

The existence of SPM originates high frequency in the tail and on the other hand the GVD makes those frequencies shift more rapidly to the front of the pulse appearing this way the first peak. Due to the simultaneous existence of SPM and GVD there are two identical and parallel parts in the center of the pulse. Then, there is a new peak and in the end of the propagation distance the pulse regains its initial shape.

5.4.2.4. Gaussian Pulse

The Gaussian pulses are often used to describe dispersion management solitons in dispersion maps. These maps are intended to lower the average GVD of the connection and at the same time maintain the GVD of each section high enough so that the third order dispersion can be ignored.
The characteristics of the dispersion management solitons depend on several factors. If the dispersion map is a fraction of the non-linear length the nonlinear effects can be ignored and the pulse evolves linearly in that section of the map. For higher lengths it is possible the existence of solitons, but its peak power, form and length oscillates periodically. In this situation the pulse equation is given by:

\[ i \frac{\partial U}{\partial z} + \frac{\beta_2(z)}{2} \frac{\partial^2 U}{\partial z^2} + \gamma(z) |U|^2 U = 0 \]  

Equation (5.48) doesn't have a trivial solution, because \( \beta_2 \) and \( \gamma \) vary with \( z \). The pulse width of a dispersion management soliton can be generalized for a Gaussian pulse of the following type:

\[ U(z,t) = a \exp\left[ \frac{i\phi - i\Omega(t-T) - (1+iC)(t-T)^2}{2\tau^2} \right] \]  

A Gaussian pulse was then set to propagate through the fiber:

\[ u_0 = \exp\left( \frac{\tau^2}{2} \right) \]  

![Figure 26 - Gaussian pulse at entrance and exit](image-url)
It is observable that the amplitude reduces and consequently there is a temporal broadening when $\zeta \to \infty$, again due to the energy conservation principle.

This simulation intended to confirm the robustness of the solution since it’s confirmable that the Gaussian pulse is converted into a pulse of the following type:

$$u(\xi, \tau) = \eta \text{sec}h(\eta \tau) \exp(i\eta^2 \xi/2)$$  \hspace{1cm} (5.51)

We can verify that the value of $\eta$ when the pulse stabilizes is 0.7541.

By obtaining the area of the final pulse it is possible to confirm that the Gaussian pulse is converted in to a fundamental soliton since its area is well defined and known. There is a critical value for the Chirp upon which it is impossible to obtain a fundamental soliton. This value differs from author to author and can be obtained by using inverse scattering methods $C_{\text{crit}} = 0.68$ \cite{2, 15}. Unfortunately using the SSFM technique in this dissertation it was impossible to confirm that a Gaussian pulse degenerates into a fundamental soliton for any kind of situation.

This result is important because fundamental soliton represents the ideal scene to implement in the fiber-optic system, in a lossless situation, since, as seen, it maintains its initial characteristics during the propagation along the fiber.

When a laser emits a pulse it is often chirped, therefore it is important to consider the initial frequency chirp on soliton formation. The chirp can be detrimental simply because it imposes on
the SPM induced chirp and disturbs the balance between the GVD and SPM effects necessary to the solitons.

Its effect on solution formation can be studied by solving equation (5.48) numerically with an input amplitude, which for the Gaussian pulse was obtained previously. For the super-Gaussian initial shape:

\[ u(0, \tau) = N \exp\left(-\frac{\tau^2}{2}\right) \exp\left(-\frac{iC\tau^2}{2}\right) \]  \hspace{1cm} (5.52)

For \( C > 0 \) the instantaneous frequency increases linearly from the leading to the trailing edge of the pulse (up-chirp) while for \( C < 0 \) occurs the opposite (down-chirp). The chirp parameter \( C \) reaches the critical value \( C_{\text{crit}} \) if the stationary peak amplitude \( |u|_{\text{max,stationary}} \) of the fundamental soliton is equal to zero.

### 5.4.3 – Soliton Interaction

In an ideal world communications aren’t executed using only one pulse at a time. Instead, a “train” of pulses is constantly propagation through the fiber. In order to achieve this soliton “train” it is necessary that they are separated. Typically, this separation is equivalent to four times the bandwidth. This means that each soliton uses only a fraction of its bit slot. However, the interaction between them is inevitable.

In order to study the interaction between solitons the following equation is used:

\[ u_0(\tau) = \sec h(\tau - q_0) + r \sec h[r(\tau + q_0)] \exp(i\theta) \]  \hspace{1cm} (5.53)

Where \( r \) is the relative amplitude between two solitons, \( \theta \) the phase and \( \tau \) the initial normalized separation. The following figures demonstrate two solitons interacting in the same channel:
Two solitons on the same channel don’t interact with each other, thus there will be no interferences and the information will arrive with no issues to the receptor. The parameter $q_0$ influences the separation between both solitons.

Next the influence of this parameter is shown with more detail:
The figure above demonstrates the interaction and in this situation it is seen that the lower the parameter \( q_0 \) is the higher the interaction between the solitons will be, and from a certain distance they will collide.

The parameter \( r \) also has some importance since for a value higher than for this parameter solitons oscillate but will not collide as seen next:

![Figure 30 - Comparing interaction between two solitons for \( Q_0 = 3 \) and a) \( R=1.1 \) b) \( R=1.7 \)](image)

It becomes obvious that the collapse effect between solitons is undesirable. The most effective and direct way to avoid such effect is to increase the initial separation. As suggest in [3] the suggested value for parameter \( q_0 = 8 \) since it is big enough to avoid the collisions. Although the parameter of relative amplitude \( r \) is relevant and influences the interaction if a proper value for \( q_0 \) is chosen it becomes less relevant.

### 5.5 – Dispersion-managed solitons

As seen in previous sections, dispersion management is most performed for modern wavelength-division multiplexed (WDM) systems. However, soliton based systems benefit if the GVD parameter varies along the length. The following section will be focusing on dispersion-managed solitons by considering first the dispersion-decreasing fibers and finally focusing on dispersion maps that consist of two or more sections of constant-dispersion fibers.

#### 5.5.1 – Dispersion decreasing fiber
A few decades ago an interesting method was proposed as a way to reduce the restriction \( L_A \ll L_{Q_0} \) imposed in loss-managed solitons. This scheme suggested a decrease of the GVD along the fiber length.

The existence of solitons depends on the balance of GVD and SPM phenomena in an optical fiber. This balance is destructed by the existence of losses that attenuate the pulse and decrease the effect of SPM. To solve this problem, it can be implemented a solution with a decreasing dispersion fiber (DDF) in which the decreasing GVD counteracts the reduced SPM experienced but the weakened solitons from fiber losses.

It’s known that the relation between \( L_D \) and \( L_{NL} \) is given by:

\[
N^2 = \frac{L_D}{L_{NL}}
\]  

(5.54)

For the fundamental soliton case we have that \( N = 1 \), and also knowing that \( L_D = \tau_0^2 / \beta_2 \) and \( L_{NL} = 1/\gamma P \), solving in order to \( \beta_2 \) comes that:

\[
\beta_2 = \tau_0^2 \gamma P
\]  

(5.55)

Assuming that the power profile is given by \( P = P_0 \exp(-\alpha z) \). We can conclude that:

\[
\beta_2 = |\beta_2^0| \exp(-\alpha z)
\]  

(5.56)

With \( |\beta_2^0| = \tau_0^2 \gamma P_0 \) and \( \alpha \) representing the losses along the fiber.

This \( \beta_2 \) profile represents the ideal solution to solve these problems. However, in practical terms its implementation isn’t easy at all making it necessary to adopt a solution that approximates the curve expression (5.56) by a step function.
Figure 31 - Staircase approximation of the ideal DDF (with 4 stairs)

Figure 32 - Staircase approximation of the ideal DDF (with 8 stairs)
Figure 33 – Fundamental soliton pulse at entrance and exit using the staircase approximation

Figure 34 - Evolution of the fundamental soliton pulse along the DDF
It is noticeable the dispersion compensation for the sections defined in the staircase approximation of the $\beta_2$ profile. This compensation is more efficient as we increase the number of stairs used to approximate the $\beta_2$ profile. However, the increase of stairs will eventually lead to a higher processing capacity.

### 5.5.2 Periodic dispersion maps

The DDF solution studied in the previous section the disadvantage of having an average dispersion along the link considerably large. And since operating solitons in the region of low average GVD improves the system performance it becomes essential to come up with a solution in order to overcome the frequency shifts.

It is common to use dispersion management techniques applying fiber sections with alternately different dispersion coefficients, so that during propagation the dispersion mean value is maintained inside the anomalous zone in order to permit the pulse propagation whose form contracts and broadens during the propagation along the fiber [12]. The use of dispersion management is important since it forces each soliton to propagate in the normal-dispersion regime of a fiber during each period.

In order to minimize the *jitter*, the dispersion coefficient values must be chosen so that the average dispersion $\bar{D}$ given by:

$$D = \frac{D_1 L_1 + D_2 L_2}{L_1 + L_2}$$  \hspace{1cm} (5.57)

is practically null. This allows not only to increase the peak power, which leads to an increase of the signal-noise value relation of the system, and decrease of the *jitter* but also to increase the maximum transmission distance of the soliton based systems. Considering the dispersion variation, the equation that rules the soliton propagation, neglecting the high order terms, is defined as follows:

$$i \frac{\partial u}{\partial \zeta} + \sigma(\zeta) \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = -i \frac{\Gamma}{2} u$$  \hspace{1cm} (5.58)

Being: $\sigma(\zeta) = \frac{\partial \zeta}{\partial D}$
Where $\sigma(\zeta)$ is the normalized dispersion coefficient and $D(\zeta)$ the dispersion parameter at certain local in the connection.

Parameter $D(\zeta)$ is given by:

$$D(\zeta) = -\frac{2\pi c}{\lambda^2} \beta_2(\zeta)$$

(5.59)

One of the most used dispersion maps has $D_1 > 0$ and $D_2 < 0$. Using the map described, it is possible to illustrate the evolution of the dispersion management throughout a dispersion period. By neglecting the losses the main conclusions taken are that the pulse width broadens initially since $D_1 > \overline{D}$ and in the next section since $D_2 < \overline{D}$ the pulse contracts recovering its initial shape after the total period: $L_1 + L_2$

Figure 35 - Evolution of the fundamental soliton for a system with dispersion management
Chapter 6

Conclusions & Future work

During this dissertation there were analyzed many aspects relative to the pulse propagation in optical fibers.

In chapter 2, we started by studying the semiconductor lasers as emitting devices. These devices have the capacity to produce light by converting the signal of the electric current. In a first approach, it was possible to conclude that when the laser is emitting, the population of electrons and photons has an oscillatory character that lasts during the pulse duration. When it ends, the values tend to stabilize. The second simulation, in which the laser is emitting in a short time span, there is decrease in the electrons population as well as a delay in the response of the device compared to the case when it is emitting freely.

Next, and after a brief approach about the optical fibers, in chapter 3 there is an early approach to the modal theory as a way to describe the propagation of rays in an optical fiber basing on the electromagnetic theory. Some important conclusions were taken in this chapter in particular the fact fibers can be classified as single mode and multimode depending on how many modes are propagating being this number related to the core radius. According to the analysis it was verified that a fiber is in single mode regime when its normalized frequency is lower than 2.4048. For these fibers the second order dispersion coefficient (group velocity dispersion - GVD) is responsible for the pulse broadening during propagation. This temporal broadening will
cause inter symbolic interference limiting the bit rate and connection length. It is practicable to
dimension optical fibers in order to obtain zero dispersion by manipulating the optical fiber
characteristics such as the core radius and the dielectric contrast.

The fourth chapter is essentially dedicated to study the pulse propagation in linear regime.
Starting with the development of the linear regime propagation equation some interesting
results and understand that for higher values of group-velocity dispersion and high order effects
the pulse will experience temporal dispersion which causes inter symbolic interference. The
chirp parameter was better discussed in this chapter and with this was possible to infer that its
presence can cause pulse spreading or pulse contraction when the product of group velocity
dispersion parameter with chirp is positive or negative, respectively. Also in this chapter many
simulations recurring Fast Fourier Transform technique were presented. For the pulses studied
it was clear to visualize time dispersion as well as a reduction of the pulse’s amplitude. In the
particular case of the Gaussian pulse the effect of the chirp parameter was evident which
provokes an increase of the spreading. All the conclusions taken by the simulations were in
agreement with the analytical approach. Having to deal with the dispersion effect it was
somehow mandatory to come up with a numerical solution which resulted in the application of
the dispersion compensating fiber. This technique is pretty efficient and worked accurately to
compensate the group velocity dispersion by matching the fiber with another fiber of opposite
sign dispersion so that the effect is canceled.

In the last chapter the nonlinear regime and its model based on the nonlinear Schrodinger
equation was handled. This regime has a major difference due to the nonlinear Kerr effect and
because new frequencies are generated. This new effect is known as Self Phase Modulation
(SPM). Analyzing both SPM and GVD they have contradictory properties which cancel mutually
creating the necessary conditions for the appearance of solitons which are pulses that don’t
suffer any changes in their form during the propagation along the fiber making them ideal for
optical communications. Using the Split-Step Fourier Method several simulations were made:
fundamental soliton, second order soliton, and third order soliton. As expected, the fundamental
soliton maintained its width and amplitude during the propagation. The second and third order
soliton revealed a periodicity recovering its initial shape after each cycle and although they
achieve a periodic propagation, the conservation of energy is present since a perfect medium
(without perturbations) was simulated. The Gaussian pulse was also tested and as expected
theoretically it was observed how the initial Gaussian tends to origin a fundamental soliton.
Although this isn’t very clear in the simulations and it wasn’t possible to find a critical value for
the chirp in order to degenerate into a fundamental soliton this is one of the most interesting
facts in the nonlinear regime. The interaction between solitons was also addressed in this
chapter and an important conclusion was taken from the simulations and theoretical analysis: it
is possible to avoid the undesirable interaction if the initial separation between both solitons is
higher than a certain limit ($q_0 = 5.1$). Obviously for lower values of $q_0$ the interaction will occur. The biggest problem of the interaction is when solitons collide because this situation could lead to a change of the information carried by the initial message. It is important to clarify that although the existence of solitons depends on the balance between the SPM and GVD this balance can be destructed by the existence of losses. In the last sections of this chapter there was presented a way to overcome this effect by recurring to the decreasing dispersion fibers (DDF). It was verified that the profile of this dispersion is a negative exponential being its application in practical levels extremely difficult. This solution effectiveness depends on the number of ‘stairs’ used to approximate the dispersion profile curve.

**Future work:**

- Dispersion management: other techniques (linear and nonlinear regimes)
- Higher-order effects: higher-order dispersion; Raman effect; self-steepening
- Numerical simulations taking into account losses (namely the effect of the chirp parameter)
- WDM techniques
- Nonlinear photonic switching
- Further semi-analytical techniques (using variational methods)
Appendix A

Geometric interpretation of guided propagation in optical fibers

Equations (3.8), (3.9) and (3.10) introduced in chapter 3 have a fairly simple geometrical interpretation in terms of radiuses. Inside the core \( r \leq a \) the propagation constant is \( k_1 = n_1 k_0 \). Analogously, in the cladding \( r > a \) we have \( k_2 = n_2 k_0 \). Being the longitudinal propagation constant \( \beta \) the same in both mediums (in order to verify the boundary conditions core-cladding) it becomes possible to define two transversal propagation constants (throughout the \( r \) coordinate): \( \theta \) (core constant) and \( \phi \) (cladding constant).

For a better analysis the clear visualization of the core-cladding interface the following figure is presented:
According to the figure:

\[ \beta = n_1 k_0 \sin \theta_1 = n_2 k_0 \sin \theta_2 \]  \hspace{1cm} (A.1)

Most commonly known as Snell law in the literature this equation leads analogously to:

\[ h = n_1 k_0 \cos \theta_1 \]  \hspace{1cm} (A.2)

\[ q = n_2 k_0 \cos \theta_2 \]  \hspace{1cm} (A.3)

Hence from equations (A.1), (A.2) and (A.3):

\[ h^2 + \beta^2 = n_1^2 k_0^2 \]  \hspace{1cm} (A.4)

\[ q^2 + \beta^2 = n_2^2 k_0^2 \]  \hspace{1cm} (A.5)

Which from a geometrical perspective are no more than applications of the Pitagoras theorem. From these equations we end up with equations (3.8), (3.9) and (3.10) used in chapter 3. When, \( \theta_1 = \pi/2 \), the total reflection phenomenon in the core-cladding interface occurs. And according to (A.1) and (A.5) we have that \( \beta = n_2 k_0 \) and \( q = 0 \). There will be a guided mode in the core as
long as the total reflection in the interface $r = a$ is verified. This means that, in order to have a superficial guided wave, the following condition has to be verified ($\alpha > 0$):

$$q = i\alpha$$  \hspace{1cm} (A.6)

According to equation (A.3) implies for equation (A.6) that:

$$\theta_2 = \frac{\pi}{2} - i\psi$$  \hspace{1cm} (A.7)

And using the following trigonometry relations:

- $\sin(ix) = i\sinh x$
- $\cos(ix) = \cosh x$
- $\cos \left( \frac{\pi}{2} - x \right) = \sin x$
- $\sin \left( \frac{\pi}{2} - x \right) = \cos x$

We end up with:

$$\beta = n_2k_0\cosh\psi$$  \hspace{1cm} (A.8)

$$q = in_2k_0\sinh\psi$$  \hspace{1cm} (A.9)

In particular from equations (A.6) and (A.9):

$$\alpha = n_2k_0\sinh\psi$$  \hspace{1cm} (A.10)

Knowing the relation: $\cosh^2\psi - \sinh^2\psi = 1$ from (A.8) and (A.10) it is possible to confirm equation (3.10): $\alpha^2 = \beta^2 - n_2k_0^2$. The guided wave propagation corresponds to the total reflection. Hence:

$$\theta_1 \geq \theta_{crit} = \sin^{-1} \left( \frac{n_2}{n_1} \right)$$  \hspace{1cm} (A.11)

When $\theta_1 < \theta_{crit}$ there is refraction and the ray passes the interface $r = a$ and the energy is radiated to the cladding, having in this situation a radiation mode.
Appendix B

Gaussian pulse broadening with higher-order dispersion effects

Having neglected the effect of the source’s spectral width it is applied the general pulse broadening equation for the case of a Gaussian pulse with chirp propagating in a single mode linear regime. Having in mind that parameter $C$ represents a variation of the instantaneous frequency of the carrier along the fiber which causes a pulse scattering that can be harmful for long distance communications.
With:

\[ A(z,t) = A_0 \exp \left[ -\frac{1+iC}{4} \left( \frac{t}{\tau_0} \right)^2 \right] \]  \hspace{1cm} (B.1)

And being its Fourier transform defined as follows:

\[ \tilde{A}(0,\Omega) = \int_{-\infty}^{\infty} A_0 \exp \left[ -\frac{1+iC}{4} \left( \frac{t}{\sigma_0} \right)^2 + i\Omega t \right] dt \]  \hspace{1cm} (B.2)

By knowing the definition of integral:

\[ \int_{-\infty}^{\infty} \exp\left[-(at^2 + bt)\right] dt = \frac{\pi}{\sqrt{a}} \exp \left[ \frac{b^2}{4a} \right] \]  \hspace{1cm} (B.3)

Establishing that \( a = 1 + iC/4\sigma_0^2 \) and \( b = -i\Omega \) in equation (B.2) it is possible to solve the integral in equation (B.2) applying the relation above:

\[ \tilde{A}(0,\Omega) = A_0 \sqrt{\frac{4\pi\sigma_0}{1+C^2}} (1-iC) \exp \left[ -\frac{\Omega^2\sigma_0^2}{1+iC} (1-iC) \right] \]  \hspace{1cm} (B.4)

Knowing that:

\[ 1-iC = \sqrt{1+C^2} \exp \left[ i \tan^{-1}(-C) \right] \]  \hspace{1cm} (B.5)

The result obtained is:

\[ \tilde{A}(0,\Omega) = A_0 \sqrt{\frac{4\pi\sigma_0}{1+C^2}} \exp \left[ -\frac{\Omega^2\sigma_0^2}{1+iC} \right] \exp \left[ i \left( -\frac{1}{2} \tan^{-1}(C) + C \frac{\Omega^2\sigma_0^2}{1+iC} \right) \right] \]  \hspace{1cm} (B.6)

To calculate \( A_0 \) we have to use the following relation:

\[ \int_{-\infty}^{\infty} |A(z,t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |A(z,\Omega)|^2 d\Omega = 1 \]  \hspace{1cm} (B.7)

And with that:
\[
\int_{-\infty}^{\infty} A_0 \exp \left[ -\frac{1+iC}{4} \left( \frac{t}{\sigma^2} \right)^2 \right] dt = 1 \iff \int_{-\infty}^{\infty} A_0^2 \exp \left( -\frac{t^2}{2\sigma^2} \right) dt = 1 \quad (B.8)
\]

Solving this integral using the relation in equation (B.3):

\[
A_0 = \frac{1}{\sqrt{2\pi\sigma_0^2}} \quad (B.9)
\]

Applying this result in equation (B.6):

\[
\tilde{A}(0,\Omega) = \sqrt{\frac{8\pi\sigma_0^2}{1+C^2}} \exp \left[ -\frac{\Omega^2\sigma_0^2}{1+C^2} \right] \exp \left[ i \left( -\frac{1}{2}\tan^{-1}(C) + C \frac{\Omega^2\sigma_0^2}{1+C^2} \right) \right] \quad (B.10)
\]

And when comparing with: \( \tilde{A}(0,\Omega) = S(\Omega)\exp[i\theta(\Omega)] \) we can conclude that:

\[
S(\Omega) = \sqrt{\frac{8\pi\sigma_0^2}{1+C^2}} \exp \left[ -\frac{\Omega^2\sigma_0^2}{1+C^2} \right] \quad (B.11)
\]

\[
\theta_\Omega = -\frac{1}{2}\tan^{-1}(C) + C \frac{\Omega^2\sigma_0^2}{1+C^2} \quad (B.12)
\]

For the sake of the demonstration it becomes import to define the following results:

\[
\left| S \right|^2 = \sqrt{\frac{8\pi\sigma_0^2}{1+C^2}} \exp \left[ -\frac{2\Omega^2\sigma_0^2}{1+C^2} \right] \quad (B.13)
\]

\[
\theta_\Omega = \frac{\partial \theta}{\partial \Omega} = 2 \frac{C\Omega^2\sigma_0^2}{1+C} \quad (B.14)
\]

Applying the Taylor series which is independent of \( z \) till the third term for the \( \beta \) approximation:

\[
\tau_s(\Omega) = \int_0^L \frac{\partial \beta(z,\omega)}{\partial \omega} dz = \left( \beta_i + \beta_2\Omega + \frac{1}{2}\beta_3\Omega^2 \right) L \quad (B.15)
\]

The equation for \( \langle \tau_s \rangle \) is given by:
\[ \langle \tau_s \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tau_s| \, d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \beta_1 + \beta_2 \Omega + \frac{1}{2} \beta_3 \Omega^2 \right) L |S|^2 \, d\Omega \quad (B.16) \]

The individual integrals become:

\[ \int_{-\infty}^{\infty} \Omega^0 |S|^2 \, d\Omega = 0 \quad (B.18) \]

\[ \int_{-\infty}^{\infty} \Omega^2 |S|^2 \, d\Omega = 2\pi \left( \frac{1 + C^2}{4\sigma_0^2} \right) \quad (B.19) \]

Thus

\[ \langle \tau_s \rangle = \beta_1 L + \frac{1}{2} \beta_3 L \left( \frac{1 + C^2}{4\sigma_0^2} \right) \quad (B.22) \]

Which implies:

\[ \langle \tau_s \rangle^2 = (\beta_1 L)^2 + \beta_1 \beta_3 L^2 \left( \frac{1 + C^2}{4\sigma_0^2} \right)^2 \quad (B.23) \]

Applying the same approach to obtain \( \langle \tau_s \rangle^2 \)

\[ \langle \tau_s \rangle^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \beta_1 + \beta_2 \Omega + \frac{1}{2} \beta_3 \Omega^2 \right)^2 L^2 |S|^2 \, d\Omega \quad (B.24) \]

And using the integrals from equations (B.17) to (B.21), \( \langle \tau_s \rangle^2 \) can be rewritten:
\[ \langle \tau_x \rangle^2 = (\beta L)^2 + (\beta_2^2 + \beta_3^2) L^2 \left( \frac{1 + C^2}{4 \sigma_0^2} \right) + \frac{3 \beta_3^2 L^2}{4} \left[ \frac{1 + C^2}{4 \sigma_0^2} \right] \]  
(\text{B.25})

The next step is to calculate the term \( \langle \tau_x \theta_{\Omega} \rangle \) from the general equation of propagation by adjusting the terms obtained.

\[ \langle \tau_x \theta_{\Omega} \rangle = \frac{1}{2 \pi} \int_{-\infty}^{\infty} \left( \beta_1 + \beta_2 \Omega + \frac{1}{2} \beta_3 \Omega^2 \right) L \left( \frac{2 C \sigma_0^2}{1 + C^2} \right) \Omega |S|^2 d\Omega \]  
(\text{B.26})

Again from equations equation (B.17) to (B.21) it is possible to infer that:

\[ \langle \tau_x \theta_{\Omega} \rangle = \frac{1}{2} \beta_2 LC \]  
(\text{B.27})

As for the term \( \langle \tau_x \rangle \langle \theta_{\Omega} \rangle \) attending the definition of \( \theta_{\Omega} \) expressed in equation (B.14):

\[ \langle \theta_{\Omega} \rangle = \frac{1}{2 \pi} \int_{-\infty}^{\infty} 2 C \sigma_0^2 \frac{|S|^2}{1 + C} d\Omega \]  
(\text{B.28})

Being the integral in equation (B.18) zero we have \( \langle \theta_{\Omega} \rangle = 0 \) and for this reason

\[ \langle \tau_x \rangle \langle \theta_{\Omega} \rangle = 0 \]  
(\text{B.29})

Consequently the equation for the effective temporal width \( \sigma \) for this case becomes:

\[ \sigma = \sqrt{\sigma_0^2 + LC \beta_2 + \frac{L^2 \beta_2^2}{4 \sigma_0^2} + \left( \frac{L \beta_3}{2 \sigma_0^2} \right)^2 + (1 + C^2)^2 \left( \frac{L \beta_3}{4 \sqrt{2 \sigma_0^2}} \right)^2} \]  
(\text{B.30})

Finally, it is possible to obtain the expression for Gaussian pulse broadening, \( \sigma / \sigma_0 \) in linear regime considering the high order dispersion:

\[ \frac{\sigma}{\sigma_0} = \sqrt{\left( 1 + \frac{LC \beta_2}{2 \sigma_0^2} \right)^2 + \left( \frac{L \beta_3}{2 \sigma_0^2} \right)^2 + (1 + C^2)^2 \left( \frac{L \beta_3}{4 \sqrt{2 \sigma_0^2}} \right)^2} \]  
(\text{B.31})
Appendix C

Numerical solution of the nonlinear Schrödinger equation: split-step Fourier method

The nonlinear regime propagation equation does not offer a numerical solution and hence it is necessary to make some approximations. The main techniques in order to solve the propagation equation are the FFT and the IFFT.

In a first approach it is possible to write the following equation:

\[
\frac{\partial A}{\partial z} = (\hat{D} + \hat{N}) A
\]  \hspace{1cm} (C.1)

Where, \( \hat{D} \) is the differential operator which considers dispersion and losses within the linear medium and \( \hat{N} \) is a nonlinear operator that accounts the fiber nonlinearities on pulse propagation.

Both \( \hat{D} \) and \( \hat{N} \) can be written as follows:

\[
\hat{D} = -\frac{i\beta_2}{2} \frac{\partial^2}{\partial T^2} + \frac{\beta_4}{6} \frac{\partial^3}{\partial T^3} + \frac{\alpha}{2}
\]  \hspace{1cm} (C.2)

\[
\hat{N} = i\gamma |A|^2
\]  \hspace{1cm} (C.3)
This method considers that dispersion and nonlinearity phenomenon act independently although in reality they both act together along the fiber assuming also that the step $h$ is very small.

The approach of this method is to divide the optical fiber into two sub-lengths, were, in the first part, only the nonlinearities manifest and, then, in the second sub-length, only the dispersion effect is taken in account. Mathematically speaking this can be written as follows:

$$A(z+h,T) = \exp(h\hat{D})\exp(h\hat{N})A(z,T) \quad (C.4)$$

Being:

$$\exp(h\hat{D})B(z,T) = F_t^{-1}\exp[h\hat{D}(-i\omega)]F_tB(z,T) \quad (C.5)$$

where $F_t$ represents the Fourier transform operation, $\hat{D}(-i\omega)$ is taken from equation (C.2) and of course $\omega$ is the frequency in the Fourier domain. By using the FFT method the numerical evaluation of equation (C.5) becomes fast [8].

By estimating the accuracy of this method it is possible to write the exact solution of equation (C.4), becoming:

$$A(z+h,t) = \exp[h(\hat{D} + \hat{N})]A(z,T) \quad (C.6)$$

Having considered that $\hat{N}$ is independent of $z$.

By applying the Baker-Hausdorff formula to both non-commuting operators $\hat{a}$ and $\hat{b}$ then we end up with:

$$\exp(\hat{a})\exp(\hat{b}) = \exp\left(\hat{a} + \hat{b} + \frac{1}{2}[\hat{a},\hat{b}]_{...}\right) \quad (C.7)$$

Where $[\hat{a},\hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}$. Carefully analyzing equations (C.5) and (C.7) it is possible to conclude that this method ignores the non-commutating nature order of both operators. The resultant error of this approximation can be found when considering $\hat{a} = h\hat{D}$ and $\hat{b} = h\hat{N}$ which is equal to $\frac{1}{2}h^2[\hat{D},\hat{N}]$. The method used in our simulations is called asymmetric and a representation is shown in the following figure:
Figure 37 - SSFM along the fiber

It is important to have in mind that this method is an approximation given the fact that the non-commuting order of the operators isn't considered and also the way integrals are dealt using the FFT.

In order to better understand the numerical algorithm the following scheme is made to clearly demonstrate the procedure to solve computationally equation (C.1).

![Algorithm Scheme](image)

Figure 38 - SSFM algorithm

The algorithm above was used in all the simulations that required this method and although this is sufficiently good to handle all types of tested pulses (due to the low value of $h$) there is also another method used today, more efficient which is symmetrical. The difference for the one used in this dissertation is that it considers two dispersion zones and the nonlinear effect between them. Although this variant of the method is more efficient it has an error proportional to $h^3$. 
References


