Abstract—This paper aggregates some contributions for the development of a localization method for autonomous vehicles based on pairwise distance measures. The challenges have been proposed under the European project MORPH, short for Marine System of Self Organizing Logically Linked Physical Nodes, although the results are of general applicability. The contributions of this thesis can be summarized in four major topics.

a) Refinement of the experimental results of the MORPH'12 trials.
b) Suggestion of an alternative interrogation scheme that allows a reduction on the communications burden during the ranging process.
c) Performance evaluation of three convex formulations under noisy and incomplete measurements, namely the classical EDM with squared ranges (EDM-SR), the EDM with plain ranges (EDM-R) [1] and the novel Lower bound EDM with plain ranges (EDM-RLB).
d) Development of an optimization algorithm prior to the semidefinite relaxation where the goal is to find the set of missing distances that minimizes the embedding dimension of the problem. Results indicate that EDM-RLB achieves a lower positioning error although the classical EDM-SR remains as the most robust method. Plus, numerical simulations show that the proposed completion algorithm can achieve a significant reduction on the positioning error. Unfortunately, this is gained at the expense of a larger computational effort.

I. INTRODUCTION

Localization in formations of autonomous vehicles is an essential task for precise navigation and to properly georeference the gathered data. While Global Positioning System (GPS) is one of the most popular and widely accessible positioning technologies, it is not a universal solution; the GPS signals might not be available (ex. indoor, underwater), the cost of implementing receivers in large formations of vehicles might not be acceptable or the required accuracy is beyond GPS capabilities. A viable alternative is the positioning using pairwise distance measures. Given the adequate number of distances between pairs, it is possible to reconstruct the original constellation up to an isometric transformation mismatch. To remove the remaining ambiguities (translation, rotations and reflection) a subset of elements in the network must have known coordinates. Since the dynamics of each vehicle is ignored, the vehicles can be seen as generic nodes in a network which reduces the problem to the well-known Sensor Network Localization (SNL).

The MORPH Project

The kick off of the MORPH Project, an European Project in which IST/ISR plays an important role, brought an interesting opportunity for the developing and testing of range-based localization systems. MORPH advances the novel concept of an underwater robotic system composed of a number of spatially separated mobile robot-modules, carrying distinct and yet complementary resources (see Figure 1). Instead of being physically coupled, the modules are connected via virtual links that rely on the flow of information among them. Localization underwater is challenging, as autonomous systems cannot rely on radio communications and global positioning. Such signals propagate only short distances and therefore communications must depend on acoustic transmissions. The MORPH networks are composed of surface and underwater vehicles, with the reception of Global Navigation Satellite System (GNSS) signals limited to the surface modules. Positioning of the remaining elements relies on measured pairwise distances to the surface vehicle(s).

The scope of the work presented in this paper embraces underwater, terrestrial or aerial autonomous vehicles as long as a communication modem is available. However, the challenges addressed tried to answer to some of the needs found in the MORPH localization system designing process.

Problem Statement - Assumptions and Constraints

The challenge consists in localizing a formation of vehicles given a set of observed pairwise distances and a subset of nodes with known locations usually called anchors. The assumptions and constraints considered throughout the work will be listed next.

Dimensional Space: The networks can assume tridimensional or planar geometries. The proposed techniques can be
applied to both cases although simulations have considered only the two dimensional scenario.

**Transmission Range:** It is considered that all the nodes are within the communication range of each other, which means that the maximum transmission range can be considered infinite. However, this does not imply that all the messages arrive at all the listeners. Due to interferences or signal blockage, some of the packets may be lost or corrupted, in which case the messages will be discarded.

**Noise:** It is assumed that range measurements between nodes are affected by white noise. The ratio between the noise covariance of each measurement and the respective distance is considered constant, following the procedure in [3].

**Clock Synchronization:** All the techniques proposed in this work do not require clock synchronization since the time differences of arrival consider timestamps recorded on the same node. This implies that although clock offsets may assume any value, clock skews must be negligible.

**Line-of-sight:** The communications are assumed to travel in straight line and therefore the pairwise distances are only a linear function of time intervals and the sound speed, considered to be constant. In practice, refraction due to variations in the properties of the medium and ambiguities between primary and reflected signals might introduce measurement errors (outliers) that are not accurately modeled by the white noise.

**Small Scale Networks:** The convex optimization algorithms extensively used throughout the work are computationally demanding and therefore not suitable for networks with more than a few dozen nodes. This is not a serious limitation for the scenarios envisaged in the MORPH project, and in most foreseeable deployments of underwater vehicles in the near future.

**Simulation Environment:** The numerical simulations conducted during this work were performed using a 32-bit version of Matlab 7.12(R2011a) on a laptop running Windows 8, with a 2.40 GHz Intel Core i3-3110M processor and with 4 GB of RAM. The convex optimization problems (detailed later) were solved using CVX (see [4]).

This paper is structured as follows. Section II introduces the Time-of-arrival (TOA) and Time-difference-of-arrival (TDOA) ranging schemes and proposes an alternative interrogation procedure to the one implemented for the MORPH’12 trials. Section III details the framework reconstruction procedure given an Euclidean distance matrix (EDM) and a subset of anchor nodes. The results for the MORPH’12 trials are further refined through an alternative framework reconstruction approach that outperforms the one used in [5]. Section IV introduces some convex relaxations methods for the Euclidean distance matrix completion problem (EDMCP) and evaluates the performance of three different Semidefinite programming (SDP) formulations on a set of numerical simulations assuming that range measurements may be noisy and incomplete. Section V presents a novel optimization algorithm prior to the SDP instance that searches for a low-rank completion of the pre-distance matrix. Simulation results show a significant reduction on the positioning error. Finally, Section VI summarizes the main conclusions and discusses directions for future research. This work has also been reported in a short paper submitted to the Underwater Communications Conference and Workshop (UCOMMS’14) [6] with a planned extension as a submission to the Journal of Oceanic Engineering [7].

II. RANGING

Ranging can be based on signal strength, time-of-flight, or alternatively, angle-of-arrival measurements (see [8], [9], [10]). Time-of-flight based methods have received much attention lately since do not require additional equipment other than a communications modem (which an autonomous robot is expected to have). These methods record the time-of-arrival (TOA) or the time-difference-of-arrival (TDOA). The propagation time can be directly translated into distance, based on the (approximately) known signal propagation speed. TOA methods are based on Round-trip times (RTT) and may not be the most efficient solution for ranging measurements. Traditional TDOA methods are based on the difference between times-of-arrival of the same signal at different receivers, in which case clock synchronization is vital. An alternative approach, adopted in this work, considers the differences between times-of-arrival of signals from different sources at the same receiver. Figures 2a and 2b depict both range measurements schemes.

The interrogation scheme must also take into account the goals and constraints of the network itself such as latency, noise, vulnerability to interferences, among others. In high latency and very limited bandwidth networks, such as underwater acoustic networks, the communications burden plays a critical role and must be minimized, in particular, for large networks where the localization process requires a huge amount of data. The packet exchange scheme implemented for the MORPH’12 trials, from now on denominated as Master/Slaves, will be briefly described next as well as an alternative proposal who aims to reduce the amount of data transmitted, the Symmetric Interrogation Scheme.

A. Master/Slaves Interrogation Scheme

The Master/Slaves scheme is described in [5] and was evaluated in a real scenario during the MORPH’12 trials. In each cycle, each one of the nodes has a position in the hierarchy, with rotating roles between cycles. The hierarchy (Master, Slave#1,...,Slave#N-1) defines the sequence of responses, with

![Fig. 2. Illustration of TOA (a) and non-classical TDOA (b) ranging methods.](image-url)
increasing amount of data transmitted by each node. Figure 3 illustrates the interrogation process for a 5 node network. The Master sends a Query at $t_0$, Slave#1 replies after receiving the Query from the Master (the processing times are ignored) which arrives back at the Master at $t_1$. After receiving the reply from Slave#1, Slave#2 replies including in the packet the time difference between the arrivals of the reply from Slave#1 and the Query from the Master. The process goes on until all the nodes have answered back. In the next cycle, the positions in the hierarchy shift until all the nodes have the chance to be the Master since, in each cycle, only the Master is able to determine the location of its peers. This way the number of queries needed is $N$ and the number of time differences transmitted is $N(N-1) = N^2 - N$. Therefore

where each one of the nodes sends the $N-1$ time differences between arrivals. Figure 4 illustrates the process for a 3 node network. As in the Master/Slaves scheme there is no need of clock synchronization.

Let $\Delta$ be the vector grouping the time intervals and $x$ the vector of unknown variables, containing not only the pairwise distances divided by the signal speed but also the instants at which each one of the nodes have sent its Query ($t_{ii}$). Vectors $\Delta$ and $x$ are related through a matrix $A$ such that $Ax = \Delta$. $A$ has dimensions $\left( N^2 - N \right) \times \left( N^2 + N \right)$ and therefore the system is overdetermined when all the $\Delta$ entries are known for networks with more than 3 nodes. In that case, $x$ results from $x = A^+ \Delta$. Note that one of the variables $t_{ii}$ must be assumed as the origin of the time axis, causing $x$ to lose a variable and $A$ to lose a column. Typically, each node will consider the moment at which have sent the Query ($t_{ii}$) as the origin of the time axis. After this step, $A$ becomes a full rank matrix when the sufficient $\Delta$ entries are known.

In the case where some of the entries of $\Delta$ are unknown, the rows of $A$ corresponding to the unknown $\Delta$ entries are eliminated. If due to the row eliminations, $A$ becomes rank deficient, then for some of the variables in $x$ the solution will not be unique as described before.

**C. Communications Burden Analysis**

The minimum amount of data exchanged so that all the nodes are able to determine the complete set of pairwise distances depends on the interrogation scheme in use.

**Master/Slaves Interrogation Scheme:** Only one node per cycle (Master) is able to determine the location of its peers which implies that the amount of data transmitted for one cycle needs to be multiplied by $N$. There is only one query and $\sum_{n=2}^{N} (n-2)$ time differences transmitted per cycle,

$$N \sum_{n=2}^{N} (n-2) = N \frac{(N-1)}{2} (N-2) = \frac{N^3 - 3N^2 + 2N}{2}$$

Therefore the amount of data transmitted depends on $O \left( N^3 \right)$.

**Symmetric Interrogation Scheme:** In this scheme all the nodes determine the network configuration in just one cycle. The number of queries needed is $N$ and the number of time differences transmitted is $N(N-1) = N^2 - N$. Therefore
the amount of data transmitted depends on $O(N^2)$. This is a significant reduction, suggesting that the Symmetric scheme is a much more suitable alternative.

III. FRAMEWORK RECONSTRUCTION

The framework reconstruction procedure used in this work is based on the properties of the EDM. Consider a matrix $E$, where each entry $E_{ij}$ is the squared distance between the nodes $i$ and $j$ (with $i, j = 1, \ldots, N$ where $N$ is the number of nodes). It only represents a feasible constellation if and only if $E$ is an EDM, in which case it obeys to the following set of rules,

$$E \in \text{EDM} : E_{ii} = 0, \quad E_{ij} \geq 0, \quad \mathbf{J}E\mathbf{J} \preceq 0 \quad i, j = 1, \ldots, n$$

(2)

where $E \in \text{EDM}$ means that matrix $E$ lies in the EDM cone and $\mathbf{J}$ is the orthogonal projection on the orthogonal complement of 1 given by $\mathbf{J} = \mathbf{I} - \frac{1}{N} \mathbf{1} \mathbf{1}^T$, with $\mathbf{I}$ as the identity matrix as usual. Let $\tilde{A}$ be the matrix whose columns hold all node coordinates, translated so that the centroid of the constellation lies in the origin. Then the Gram matrix $\mathbf{G}$, given by $\tilde{A}^T \tilde{A}$, can be obtained from $\mathbf{G} = -(1/2) \mathbf{J}^T E \mathbf{J}$ (see [11]). Since $\mathbf{G}$ is semidefinite by construction,

$$\tilde{G} = V \Sigma \Sigma^T$$

$$= V \sqrt{\Sigma} \sqrt{\Sigma}^T$$

$$= \tilde{A}^T \tilde{A}$$

(3)

and therefore coordinates $\tilde{A}$ can be extracted by Singular Value Decomposition (SVD). The rank of the coordinate matrix returned by the SVD may be higher than the embedding dimension of the problem. In that case, a usual procedure is to truncate the SVD to the appropriate rank. To achieve global positioning, the remaining ambiguities must be removed with the help of anchor nodes. Briefly, the rotation and reflection ambiguities are eliminated by solving an Orthogonal Procrustes problem. The solution is the rotation matrix $\mathbf{Q}$ which closely maps the subset of $\tilde{A}$ corresponding to the anchor nodes with the anchor true positions. Obviously, both coordinate matrices must be translated so that their centroids lie in the origin. Finally, the constellation can be translated to the correct location using the centroid of the true anchor coordinates.

MORPH’12 - Framework Reconstruction

The MORPH’12 trials implemented and tested a range-based localization method using time-of-flight measurements. The estimated positions were then compared against GPS data. Details about the experimental setup can be found in [5]. Here, we will only focus on solving the translation, rotation and flip ambiguities, ignoring the methods employed to reach that point. The network was composed of three underwater nodes at approximately the same depth. Nodes 1 and 2 were attached to the same boat with its axis perpendicular to the vessel’s longitudinal axis. Node 3 was in a second boat. GPS data from node 2 was considered to be unreliable. In [5], the authors forced the $1-2$ axis to be aligned with the horizontal axis and the third node to lie in the positive semi-plane. The vessel’s attitude was computed based on tangent vectors to the trajectory of node 1. The estimated constellations were then rotated to match the vessel’s attitude. Finally, the translation ambiguity was resolved taken into account the positioning of node 1 in each instant. A different approach is proposed in this paper. The rotation ambiguity was solved through a Procrustes problem using only nodes 1 and 3 ”true” positions (GPS). The reflection ambiguity was tackled assuming that the positioning error from node’s 2 GPS was sufficiently low so that it could be used to decide whether the constellation should be flipped or not over the $1-3$ axis. Results are presented in Figure 5 where it is clear a better performance from the second approach, even if a quantitative evaluation cannot be performed due to the inaccuracy of the GPS data from node 2.

IV. EDM COMPLETION VIA SEMI-DEFINITE RELAXATIONS

When the ranging process is non-ideal, the set of pairwise distances may be noisy and incomplete. In that case, before the framework reconstruction can be applied, one must find the closest EDM to the given set of measurements, a problem usually known as the nearest EDMCP or in particular, low-dimensional EDMCP if the rank of the solution is constrained. Let $D$ be the pre-distance matrix with zero diagonal entries and with some nonnegative elements equal to the square of the available observed ranges. Formally,

$$\begin{align*}
\text{minimize} & \quad \|W \odot (E - D)\|_F \\
\text{subject to} & \quad E \in \text{EDM} \\
& \quad \text{rank}(JEJ) = r
\end{align*}$$

(4)

where $W$ is the $0-1$ mask matrix with zeros in the entries corresponding to the free elements of $D$ and ones elsewhere. As usual, $(\odot)$ denotes the Hadamard product. The rank constraint ensures that the related constellation lies in $\mathbb{R}^{r \times N}$. The low-dimensional EDMCP was shown to be NP-hard (see [12] for the proof in the noiseless case) and therefore alternative ways to tackle the problem must be found. One popular strategy is to turn to convex relaxations which can be solved efficiently, but may not solve the original problem. The three convex formulations introduced next have dropped the rank constraint leading to a SDP problem that can be handled by a standard convex optimization software.

**EDM with Squared Ranges**

The classical EDM-SR (see [13]) applies the Frobenius norm in (4) leading to,

$$\begin{align*}
\text{minimize} & \quad \|W \odot (E - D)\|_F^2 \\
\text{subject to} & \quad E \in \text{EDM}
\end{align*}$$

(5)

By squaring the observed ranges in the cost function in (5) the impact of noise will increase (see [5]). To avoid this, one may modify the cost function to accept plain distances leading to the EDM-R formulation.
The EDM-R formulation was proposed in [1], and can be written as,

$$\min_{E,T} \sum_{i,j} (E_{ij} - 2T_{ij} \sqrt{D_{ij}})$$

subject to $T_{ij}^2 \leq E_{ij}$

where $T$ is an epigraph variable that ideally should be equal to $\sqrt{E_{ij}}$. The EDM-R formulation doubles the number of variables relative to EDM-SR. Even so, the authors in [5] found it to have better numerical properties enabling the use of a faster solver.

**Lower bound EDM with plain ranges**

Here, a new SDP relaxation is proposed that closely resembles the EDM-R. The difference is in fixing a lower bound for $T$, restricting the search space. Instead of imposing an absolute value upper bound such as in EDM-R, where $T_{ij}^2 \leq E_{ij}$, variable $T$ is now restricted to assume positive values, upper bounded once again by $\sqrt{E_{ij}}$ and lower bounded by the line between 0 and the estimated maximum value for $\sqrt{E_{ij}}$ given by $\sqrt{E_{max}}$ for $E_{ij} = E_{max}$. Figure 6 illustrates this reasoning. The slope of the linear lower bound is given by $\sqrt{E_{max}}$ and therefore the constraints in $T$ result in $E_{ij} \sqrt{E_{max}} \leq T_i \leq \sqrt{E_{ij}}$. The EDM-RLB formulation is given by,

$$\min_{E,T} \sum_{i,j} (E_{ij} - 2T_{ij} \sqrt{D_{ij}})$$

subject to $T_{ij} \leq \sqrt{E_{ij}}$

$T_{ij} \geq \frac{E_{ij}}{E_{max}}$

$E \in \text{EDM}$

A. Simulation Results

The performance of the three formulations presented before was evaluated under different simulation scenarios against other simpler methods such as the Least-Squares Method (LS) (see [14] for details) and a raw data approach. By raw data approach, it is meant a simple framework reconstruction procedure over a complete pre-distance matrix $D$ that might not have the properties of an EDM.

The convex optimization problems were solved using CVX with the standard SDPT3 solver. For the Root Mean Square Error (RMSE) only the solved runs were accounted for, this is, results from the CVX other than solved were ignored (more on the CVX numerical results in [4]). All the simulations considered a planar constellation with 10 nodes randomly placed on a unit square area with randomly chosen anchors. These scenarios only contemplate the presence of Gaussian noise affecting the real distances. The level of noise in each
run is controlled by the average noise power defined as
\[ \sigma^2 = 1/P \sum_{i=1}^{P} \sigma_i^2 \] (see [3]), where \( \sigma_i^2 \) is the noise covariance for the distance \( i \) and \( P \) is the number of pairs in the network \( P = C_2^N = N(N-1)/2 \). It is assumed that \( \sigma_i^2 \) is related to the distances according to the path loss law, so that \( \sigma_i^2/d_i^2 \) is a constant ratio.

The performance criterion is the RMSE of the estimated positions over a maximum of 100 Monte Carlo runs which can be expressed as

\[ RMSE = \sqrt{\frac{1}{MN} \sum_{j=1}^{M} \| \hat{X} - X \|_F^2} \] (8)

where \( M \) is the number of successful (solved) Monte Carlo runs, \( \| \cdot \|_F \) denotes the Frobenius norm as usual, \( \hat{X} \) the estimated constellation and \( X \) the true constellation.

The first experiment considered a scenario where the set of observed distances is complete but noisy, with average noise powers ranging from 80 to 0 decibels. The goal is to evaluate the ability of each formulation to deal with noisy measurements. The number of anchor nodes is fixed, 3 out of 10 nodes, and the constellation changes randomly between runs. The results over 100 Monte Carlo runs are shown in Figure 7. The accuracy clearly degrades with the increase in noise. Since the experiments consider a square of unit area, a RMSE near 1 means the framework reconstruction has no practical value. As expected, the LS and the Raw data approaches exhibit lower accuracy than the convex optimization methods. And although the performance gap between the Raw method and the alternative convex approaches is small, it is worthwhile to recall that a direct reconstruction from the set of observed ranges can only be performed when all the pairwise ranges are available, thus making it an infeasible option for the general case. EDM-SR and EDM-R exhibit similar results. In fact, for almost every noise level the points of the two methods are coincident. For EDM-RLB however, the estimates are slightly better, which suggests the use of this formulation over the others, at least, in the absence of missing ranges.

An interesting phenomenon is related to the dependency with the scale verified in the formulations based on plain ranges, namely EDM-R and EDM-RLB. In fact, once the distances between nodes start growing, the results achieved by the convex optimization solver may no longer be reliable even in the absence of noise. This implies that, in practical situations, a preprocessing step must be carried out before the EDM-MCP to ensure proper scaling of measured distances for the solver.

The effect of the number of anchors in the accuracy of the results is presented in Figure 8. As expected, expanding the set of anchor nodes is advantageous only when the number of existing anchors is significantly small, after which it has no significant effect. In this particular case, for LS, when the number of anchors changes from 3 to 5 the RMSE decreases more than one order of magnitude. For the other methods, however, only the first change, from 3 to 4 anchors, brings some significant reduction in RMSE. The higher dependence of LS with the number of anchors can be explained by the role that anchor nodes play in each method. LS is based on a triangulation procedure between anchors and ordinary nodes, whereas the remaining techniques only make use of the anchors after the framework reconstruction step to remove the translation and rotation/reflection ambiguities.

The convex optimization techniques were also tested assuming that some of the distances were unknown, this is, when for some reason the set of observed ranges is incomplete. As in the previous case, the simulations encompass a set of 100 Monte Carlo runs with randomly generated constellations and an average noise level ranging from 80 to 40 decibels. Figures 9a, 9b and 9c consider the cases where 5, 10 and 20 randomly chosen distances are missing. The results for 5 distances missing (Figure 9a) resemble those for a complete set in Figure 7, showing a slight performance advantage of plain ranges techniques (EDM-R and EDM-RLB) over EDM-SR. In subsequent simulations, with 10 and 20 distances missing, the performance of the three methods can be considered to be approximately equivalent. It is worthwhile to point out that for 20 missing distances (out of 45) the RMSE is constant and relatively high throughout all the noise levels, indicating that
these methods are no longer reliable. A quick remark should also be made on the ratio of solved instances presented in Tables I, II and III. EDM-SR stands out as the most robust formulation, while the plain ranges methods (EDM-R and EDM-RLB) were unable to solve a significant share of the problems when the number of missing distances is high (10 and 20).

V. PRE-DISTANCE MATRIX COMPLETION ALGORITHM

The performance of the convex optimization techniques analyzed in Section IV was found to be quite satisfactory. However, since the rank constraint was dropped in the nearest EDMCP, the solver is free to search for optimal solutions in higher dimensions, which for some network geometries results in poor position estimates. The algorithm proposed here tries to find the completion for the pre-distance matrix \( D \) that minimizes the embedding dimension of the problem prior to the SDP instance.

Consider the Symmetric interrogation scheme introduced in II-B, recall that the pairwise distances (vector \( d \)) and the time intervals (vector \( \Delta \)) share a linear relation through a matrix \( A \) such that \( Ad = \Delta \). When some of the entries in the pre-distance matrix are missing so that matrix \( A \) has a nontrivial null-space, the solution of the non-homogeneous system of equations \( Ad = \Delta \) is given by

\[
d = d_0 + c_1 v_1 + \ldots + c_s v_s
\]

where \( \{c_1, \ldots, c_s\} \in \mathbb{R}^s \) and \( \{v_1, \ldots, v_s\} \) is the basis for the null space of \( A \) with dimension \( s \). The idea is now to find the optimal set \( \{c_1, \ldots, c_s\} \) such that \( JDJ \) has minimum rank. To achieve this, it is proposed to minimize the \((r + 1)\)th largest singular value and neglect the remaining ones by expecting that close to the optimal point their magnitudes are also negligible. For simplicity, let \( c \) be the vector containing \( \{c_1, \ldots, c_s\} \), \( v \) the basis for the null space of \( A \), and \( f(c) = \sigma_1(\mathbf{J}D\mathbf{J}) \) where \( \sigma_{r+1}(\cdot) \) is the \((r + 1)\)th largest singular value. The problem can be formulated as,

\[
\begin{align*}
\text{minimize} & \quad f(c) \\
\text{subject to} & \quad d = d_0 + vc \\
& \quad D_{ii} = 0 \\
& \quad D_{ij} = d_{ij}^2
\end{align*}
\]

Figure 10 exemplifies the shape of \( f(c) \) for a 5 node planar network \((r = 2)\). In this case, specifically chosen \( \Delta \) entries are missing (3 pairwise distances affected) so that the null space of \( A \) has two dimensions \((c = [c_1, c_2])\). Besides the presence of a global minimum at \( c = [-1.07, -0.82] \) it is also evident a local minimum in the vicinity. The existence of local minimums will be the main drawback in this approach.
since it will be highly sensitive to the starting point for the numerical search. The main strategy is briefly summarized in algorithm 1 and will be denominated as Pre-distance matrix rank minimization (PMRM). The procedure starts with a SDP instance using the available observed ranges. From the optimal EDM we can extract the respective vector $c$ from a simple least squares procedure,

$$\begin{align*}
\text{minimize } \quad & ||d - d_{EDM}||_2 \\
\text{subject to } \quad & d = d_0 + vc
\end{align*}$$

(11)

where $d_{EDM}$ is the vector of pairwise distances from the optimal EDM at step 2. The algorithm follows with a numerical optimization algorithm that aims to minimize $f(c)$ using the solution of (11) as the initialization point. The optimal $c$ given by the numerical search is then used to construct a new pre-distance matrix $D$ (step 16). Finally, a second SDP instance returns the closest EDM to matrix $D$ (step 17). The procedure ends with a framework reconstruction routine as described in Section III. It should be noted that steps 2 and 17 do not specify which SDP formulation should be used. Any of the three methods described in IV remains as a valid choice.

The core of the PMRM is the numerical minimization of $f(c)$. Since $f(c)$ is clearly non-convex we can no longer rely on the SDP relaxations introduced before. Instead, two alternative optimization algorithms were implemented; a zero-order procedure, the Powell’s Method and a Quasi-Newton method, the BFGS. For the sake of brevity, these techniques will not be discussed in detail. Some topics worth of remark will be discussed next. For more information, including the advantages, drawbacks and a description of each one of the algorithms see [15] and [16].

Dealing with Local Minima: The presence of local minima (see Figure 10) implies that both methods have to be initialized sufficiently near the solution in order not to get stuck in neighbor valleys. Therefore the first initialization attempt (step 3) is based on the closest EDM to the available set of distances. In most cases, this procedure returns a vector $c$ sufficiently close to the solution. However, if the residue for function $f(c)$ is still too high after the numerical search (above threshold $\epsilon$), a new search is conducted using a randomized initialization point (step 11) until threshold $\epsilon$ is achieved or the maximum number of cycles is exceeded.

Line Search Method: Powell’s and BFGS methods require the choice of a line search technique. While the Powell’s method does not impose any conditions, the BFGS demands the fulfillment of the Wolfe conditions (see [17]) in order to guarantee its self-correcting properties. In spite of that, both procedures were implemented using a simple Golden Search technique whenever a line search was needed, since it was verified that the respect for the Wolfe conditions in the BFGS would frequently lead to infinite loops.

Gradient of the Objective Function: Powell’s method as a zero-order procedure does not require the derivation of the objective function. On the other hand, the BFGS as a Quasi-Newton technique, requests the gradient of $f(c)$, $\nabla f$.

Let for simplicity $Z = JDJ$ and $\phi(Z) = \lambda_r + 1(Z)$, such that $f(c) = |\phi(Z)|$. The real absolute value function $\varphi(x) = |x|$ has a derivative for every $x \neq 0$, but is not differentiable at $x = 0$. Its derivative for $x \neq 0$ is given by the step function,

$$d|x| = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(12)

By the chain rule,

$$df(c) = \begin{cases} d\phi(Z) & \text{if } \phi(Z) > 0 \\ -d\phi(Z) & \text{if } \phi(Z) < 0 \end{cases}$$

(13)

The expressions for the differential of eigenvalues can be found in [18]. The proofs will be omitted here for the sake of brevity. Being $Z$ a real symmetric matrix,

$$d\phi = d\lambda_r + 1(Z) = u_r^T(dZ)u_r$$

$$= \left(u_r^T \otimes u_r^T\right) \text{vec}(dZ)$$

$$= (u_r^T \otimes u_r^T) vec(dZ)$$

(14)

where $(\otimes)$ denotes the Kronecker product and $u_{r+1}$ is the normalized eigenvector associated with $\lambda_{r+1}$ such that

$$Zu_{r+1} = \lambda_{r+1}u_{r+1} \quad u_{r+1}^T u_{r+1} = 1$$

(15)

Let $D'$ be the pointwise square root of $D$ such that $D = D' \odot D'$ and recall that the vector of pairwise distances can be written as $d = d_0 + c_1v_1 + \ldots + c_s v_s$. The generalization to the matrix form yields $D' = D_0 + \sum_{i=1}^s c_i D_i$. From [18, pp. 168], the differential of the Hadamard product between two matrix functions $U$ and $V$ can be written as $d(U \odot V) = (dU) \odot V + U \odot dV$. Hence,

$$dZ = 2J(D' \odot dD')J$$

$$= 2J \left(D' \odot \sum_{i=1}^s D_i dc_i\right)J$$

$$= \sum_{i=1}^s (2J(D' \odot D'_i)J) dc_i$$

(16)

Let $V_i = (2J(D' \odot D'_i)J)$, then,

$$\text{vec}(dZ) = \sum_{i=1}^s \text{vec}(V_i) dc_i = V dc$$

(17)

Therefore $d\phi = (u_i \otimes u_i)^T V dc$. From the first identification theorem in [18, pp. 199], the derivative can be obtained from the differential such that,

$$D\phi = (u_i \otimes u_i)^T V$$

(18)
where in this case \( D \) stands for the derivative operator and should not be mistaken with the pre-distance matrix. Finally, the gradient results from the derivative [18, pp. 99],

\[
\nabla \phi = D \phi^T
\]

and the gradient of \( f(c) \) can be written as,

\[
\nabla f(c) = \begin{cases} 
(u_i \otimes u_i)^T V^T & \text{if } \phi(Z) > 0 \\
-(u_i \otimes u_i)^T V^T & \text{if } \phi(Z) < 0
\end{cases}
\]  

(20)

Algorithm 1 PMRM algorithm.

1: procedure PMRM
2:     Solve a SDP problem with the available ranges.
3:     From the optimal EDM extract \( c \).
4:     \( c_{\text{original}} \leftarrow c \)
5:     \( c_0 \leftarrow c_{\text{original}} \)
6:     \( k \leftarrow 1 \)
7:     while \( k < N_c \) do
8:         Find the vector \( c \) that minimizes \( f(c) \) with \( c_0 \) as the initialization point.
9:         if \( f(c) < \epsilon \) then break
10:     else
11:         \( c_0 \leftarrow c_{\text{rand}} \) where \( c_{\text{rand}} \) is a vector given by random values in the range \([0, 2c_{\text{original}}]\)
12:     end if
13:     \( k \leftarrow k + 1 \)
14: end while
15: \( c_{\text{sol}} \leftarrow c \)
16: Complete the pre-distance matrix using \( c_{\text{sol}} \).
17: Solve a SDP problem with the optimal pre-distance matrix from step 16.
18: Reconstruct the framework based on the optimal EDM from step 17.
19: end procedure

Fig. 10. Example of a plot of \( f(c) \) for a 5 node planar network with missing distances between nodes 3 - 4, 2 - 5, 3 - 5 and 4 - 5. Null-space of \( A \) is two-dimensional. Average noise level of 80 decibels.

Fig. 11. Example of a plot of \( f(c) \) for a 5 node planar network with missing distances between nodes 3 - 4, 3 - 5 and 4 - 5. Null-space of \( A \) is one-dimensional. Average noise level of 80 decibels.

A. Performance Evaluation of the Pre-distance Matrix Rank Minimization

The performance of the PMRM was evaluated against a simple EDM-SR technique on 5 node networks disposed in a unit square area. For the sake of brevity, the parameters ruling each one of the search procedures will be omitted. The results for each scenario have considered 100 Monte Carlo runs with randomly created constellations for each run. The evaluations comprises the RMSE, the ratio of solved instances, the Mean Processing Time (MPT) and the ratio between the RMSE of both methods. Follows a description and the results for each scenario.

Scenario 1: One Dimensional Null Space: This scenario considers the elimination of 4 time intervals (affecting distances between nodes 3 - 4, 3 - 5 and 4 - 5) so that the null space of \( A \) is one dimensional. Figure 11 exemplifies the shape of function \( f(c) \) for the one dimensional case. The results are presented in Table IV for a 60 decibels noise power. It is clear that the accuracy improves when PMRM is applied, both with Powell’s and BFGS search. The difference between RMSE values for the two procedures is not significant and therefore does not allow to draw a definite conclusion regarding which search method is best suited for this application.

<table>
<thead>
<tr>
<th>Scenario 1: One Dimensional Null Space. Noise Level: 60 Decibels</th>
</tr>
</thead>
<tbody>
<tr>
<td>EDM-SR</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>PMRM-Powell</td>
</tr>
<tr>
<td>PMRM-BFGS</td>
</tr>
</tbody>
</table>

Scenario 2: Two Dimensional Null Space: This scenario considers the elimination of 5 time intervals (affecting distances 3 - 4, 2 - 5, 3 - 5 and 4 - 5) so that the null space of \( A \) is two dimensional. The results are presented in Table V for a 60 decibels noise power. Again, the completion of the pre-distances matrix before the convex optimization problem allowed a reduction on the positioning error with both
search methods at the expense of a considerable increase in the MPT. In this case, the BFGS seems to perform slightly better, although the difference is not significant enough to justify any conclusions. More test cases would be needed to definitely prefer one method over the other. Finally, one remark should be made about the error metric. The use of RMSE is very common and it makes an excellent general purpose error metric for numerical predictions. But we must keep in mind that when compared to the similar Mean Absolute Error, the RMSE amplifies and punishes large errors. In fact, a closer look at the results shows that the PMRM is extremely accurate for a great number of instances. The value of the RMSE is almost exclusively due to a small number of runs where the search procedure could not find the global minimum of $f(c)$.

### TABLE V

<table>
<thead>
<tr>
<th>Scenario 2: Two Dimensional Null Space, Noise Level: 60 Decibels</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>EDM-SR</td>
</tr>
<tr>
<td>PMRM-Powell</td>
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<tr>
<td>PMRM-BFGS</td>
</tr>
</tbody>
</table>

### VI. Conclusion

The presented work puts forward some contributions for the localization of vehicles based on distance measures covering both the ranging and the location estimation steps. The solutions proposed here tried to answer to some of the challenges found in the development of the MORPH project although remain valid for other kind of vehicle formations.

The ranging methods were addressed in Section II were a novel TDOA interrogation scheme was introduced allowing a reduction on the communications burden when compared to the ranging scheme used during the MORPH’12 trials [5]. Furthermore, an alternative framework reconstruction procedure is suggested in Section III using the data from the same trials, achieving a better agreement between the estimated positions and the GPS data than the approach suggested in [5].

When the set of measurements is noisy and incomplete, the problem translates into a low-dimensional EDMCP which was shown to be NP-hard [12]. A popular strategy to overcome this challenge is to perform SDP relaxations. Three SDP formulations are evaluated in Section IV under different noise power and packet loss conditions. The proposed EDMRLB has shown better accuracy than the remaining ones, the classical EDM-SR and the EDM-R, although the EDM-SR has proven to be the most robust. Finally, it was developed in Section V a pre-distance matrix completion algorithm, the PMRM, that aims to find the completion of the pre-distance matrix that minimizes the embedding dimension of the problem. Numerical simulations have shown that the proposed completion algorithm can achieve a significant reduction on the positioning error (up to 84%). Unfortunately, this is gained at the expense of a larger computational effort.

The natural continuation of this work involves the further refinement of the proposed PMRM through a fine tuning of the search parameters and the study of alternative numerical optimization algorithms and initialization routines. Plus, a great effort should be made to develop an effective convex formulation capable of attaining low-rank solutions in an elegant mathematical framework. The development of heuristics based on nuclear norm minimization and the fully inclusion of the time-difference-of-arrivals directly into the convex formulation appear as the the two most appealing research directions.

### References