A Not So Short Introduction To Grothendieck Topoi

João Frederico Pinto Basto de Carvalho

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Examination Committee

Chairperson: Prof. Miguel Tribolet Abreu
Advisor: Prof. Pedro Manuel Agostinho Resende
Co-Advisor: Prof. Wojciech Chachólski
Members of the Committee: Prof. Michael Joseph Paluch
Prof. Gustavo Rui Gonçalves Fernandes de Oliveira Granja

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To all those who have endured me over the years.

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Resumo

Nesta tese fazemos uma breve introdução à teoria de categorias e apresentamos algumas construções básicas nesta área. Introduzimos ainda a definição de feixe, quer num espaço topológico, quer num site de Grothendieck, chegando então à definição de topos de Grothendieck. Seguidamente focamo-nos em duas classes de topoi em particular, os topoi de feixes num espaço topológico e os topoi de feixes na categoria de espaços étale sobre um espaço topológico, e provamos que as categorias são equivalentes. Note-se que este resultado é falso no caso mais habitual das topologias de Zariski e étale em esquemas.

Uma vez que topoi de Grothendieck são uma generalização das categorias de feixes em espaços topológicos, faz sentido estudar os functores análogos aos que se obtêm a partir de funções contínuas. Estes chamam-se morfismos geométricos e dedicamos a estes o último capítulo. Na última secção fazemos a classificação, a menos de equivalência de categorias da categoria de morfismos geométricos chamados os pontos da categoria de feixes sobre um espaço topológico.

Palavras-chave: Categorias, Topoi, Grothendieck, Morfismos, Geométricos.

Abstract

In this thesis we introduce category theory and some of the basic constructions in this area. We also introduce the definition of sheaves, both on a topological space, and on a Grothendieck site, leading to the definition of a Grothendieck topos as a category equivalent to that of sheaves over a site. Next we focus on two particular classes of topoi, those of sheaves on a topological space and sheaves on the category of étale spaces over a topological space, and we prove that these are equivalent. This is known not to be the case in algebraic geometry where we use schemes rather than topological spaces. Since topoi are a generalisation of categories of sheaves on a topological space, it makes sense to study functors analogous to those which arise from continuous functions between spaces. These are called geometric morphisms and we dedicate to them the last chapter. In the last section we classify, up to equivalence, the category of geometric morphisms called points of the topos of sheaves on a topological space.

Keywords: Categories, Topoi, Grothendieck, Geometric, Morphisms.

Contents

Introduction

Category theory was introduced in the 1940s by Mac Lane and Eilenberg as the appropriate language for algebraic topology. This development did not go unnoticed by mathematics at large and thus started to leak to other areas, most notably so into algebraic geometry where, in the same decade, the notion of sheaf was developed. This notion was a very important feat as it allowed to give a precise definition of variety and other algebraic-geometric entities. Within this context Alexander Grothendieck was led to define the notions of scheme, and through the notion of site, he obtained a generalization of the concept of sheaves and established the first definition of a topos (now called a Grothendieck topos) as being a category of sheaves over some site.

In the present work we analyze two examples of Grothendieck sites associated to a topological space (defined, for example, in [Art62]), the small site (which we refer to as the Zariski site), and the étale site. But rather than doing so over a scheme we analyze them in the context of an arbitrary topological space (so when we say "étale", we really mean "local homeomorphism"), for this we try to keep our assumptions at a minimum and prove as much of the material as possible.

Along with the definition of a topos, as with most things in mathematics, comes a definition of a morphism of topoi, called geometric morphisms. In this context the topos Set has a special role: It acts as a "point" in the category of topoi. This is analogous to the way we can define a point of a topological space X as a continuous function from the one-point-set to X.

This was again first explored in the context of schemes, and we review it without making any such assumptions, in the context of the topoi of sheaves over the Zariski site and the étale site. We do not however restrict ourselves to analyze just the point functors, but proceed to infer upon the structure of the category of points (as a full subcategory of the appropriate functor category).

Organization

The present work intends to be as self-contained as possible. Because of this the proof of many well known theorems are included.

In the first chapter we give a brief introduction to category theory presenting both basic constructions such as comma-categories, limits and colimits, as well as more complex constructions such as adjunctions and Kan extensions. We also prove a few results that will be useful in the sequel such as the Yoneda lemma, criteria for completeness/cocompleteness of a category and the relation between Kan extension and adjoints.

In the second chapter we change subject, and focus on general topological concerns. Specifically we delve into properties of irreducible closed sets of a topological space and of étale maps. We define the category of étale spaces over a fixed topological space X, and having done that, we proceed with the definition of sheaf on a topological space and give some examples.

The third chapter provides the gateway to the generalization of the notion of sheaf, for this we present the concept of a Grothendieck topology J over a category $\mathcal{C}$ and we provide two examples of such topologies, the Zariski and the étale sites associated to a topological space X.

In the fourth chapter we generalize the notion of a sheaf over a site and we present the process of sheafification of a presheaf in both the contexts of sheaves over a topological space (by way of the equivalence between $\mathbf{\acute{E}t}(X)$ and Shv(X)) and in the more general case of a presheaf over a site (by way of the + construction). Finally we delve into the relationship between Zariski sheaves and étale sheaves and prove that the topos of sheaves on a topological space X is equivalent to the topos of sheaves on $\mathbf{\acute{E}t}(X)$.

In the fifth and last chapter we introduce the notion of a geometric morphism between topoi. We check that when specialized for topoi of sheaves over a topological space, every continuous function gives rise to a geometric morphism between the respective categories of sheaves, and that in some situations each geometric morphism between the categories of sheaves of topological spaces gives rise to a continuous function. We then specialize the notion of geometric morphism to that of a point, as a geometric morphism from $\boldsymbol{Set}$ and lastly, we give a description of the categories of points of a topos Shv(X), and prove that this category is equivalent to the poset of irreducible closed subsets of the space X.

Chapter 1

Categorical Introduction

In this chapter we present some of the main categorical ideas to be used throughout the thesis. We start with the basic definitions of category, functor and natural transformation, and proceed to define limits, colimits, adjoint functors and Kan extensions. In the mean time we present and prove important results such as the left exactness of filtered colimits and the ubiquitous Yoneda lemma which greatly simplifies the study of functor categories.

1.1 First Definitions

We start this section by introducing the definitions of category, functor, and of natural transformations of functors, presenting some illustrative examples along the way. We also provide the first properties of objects and arrows and give a first example of a rather naive method to obtain new categories from a predefined one which will motivate the definition of a presheaf, central to the remainder of this work.

Definition. A *category* $\mathcal{C}$ is a pair of classes $(\mathsf{ob}(\mathcal{C}), \mathsf{ar}(\mathcal{C}))$ called, respectively, the of class objects, and the class of arrows of $\mathcal{C}$, together with two (surjective) maps $\mathsf{d}_0, \mathsf{d}_1 : \mathsf{ar}(\mathcal{C}) \to \mathsf{ob}(\mathcal{C})$ (called respectively domain and codomain) a map, $1 : \mathsf{ob}(\mathcal{C}) \to \mathsf{ar}(\mathcal{C})$, (called the identity arrow map), and a partially defined associative binary operation on the class of arrows $\circ : \mathsf{ar}(\mathcal{C}) \times \mathsf{ar}(\mathcal{C}) \to \mathsf{ar}(\mathcal{C})$ called composition satisfying:

- For any pair $f, g \in \mathsf{ar}(\mathcal{C}), f \circ g$ is defined iff $\mathsf{d}_0(f) = \mathsf{d}_1(g)$.
- $\mathsf{d}_0(1_c) = \mathsf{d}_1(1_c) = c$ for any object $c \in \mathsf{ob}(\mathcal{C})$.
- For any arrow f, $f \circ 1_{\mathsf{d}_0(f)} = 1_{\mathsf{d}_1(f)} \circ f = f$.

Another way of introducing a category that may be more in touch with its algebraic roots, is in an arrows-only fashion, this approach has the advantage that it makes the notion of enriched categories, such as n-categories more intuitive.
As is common practice, we will confuse $\mathcal{C}$ with $\mathsf{ob}(\mathcal{C})$ (thus $c \in \mathcal{C}$ should be read as "c is an object of $\mathcal{C}$"). Also both *morphism*, and *homomorphism* will be used as aliases for the word "arrow". We will also write

$f : c \to c'$ to denote $c = \mathsf{d}_0(f)$ and $c' = \mathsf{d}_1(f)$ (this expression should be read as "f is a morphisms from c to c'). Finally define Mor(c, c') to be the class of arrows $f : c \to c'$. These will usually be sets and we call them *hom-sets*.

Example 1. Below we present some standard examples of categories.

- $\boldsymbol{Set}$: the category whose objects are sets and arrows are functions between sets.
- $\boldsymbol{Top}$: the category of topological spaces and continuous functions between them.
- $\boldsymbol{Grp}$: the category of groups and group homomorphisms.
- $\boldsymbol{Ab}$: the category of abelian groups and group homomorphisms.
- $\boldsymbol{fAb}$: the category of finite abelian groups and group homomorphisms.

Remark 2. The examples above are examples of what we call *large categories* that is, categories whose classes of objects and arrows are not sets.
Example 1 also illustrates how we usually define a category by simply stating what are the objects and what are the arrows, the verification that it is a category (existence of identity arrows and associativity of the composition operation) is normally a straightforward process.
By contraposition with "large category" we say a category is *small* if both the classes of objects and arrows are sets (as opposed to proper classes), and it is said to be *locally small* if for every pair of objects c, c' the class Mor(c, c') is a set.

Definition. A functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ between two categories is a pair of maps F_o : ob$(\boldsymbol{\mathcal{C}}) \to$ ob$(\boldsymbol{\mathcal{D}})$ and F_a : ar$(\boldsymbol{\mathcal{C}}) \to$ ar$(\boldsymbol{\mathcal{D}})$, which commute with the structure maps of the category, that is, they satisfy:

- For any arrow f, $F_o(\mathsf{d}_i(f)) = \mathsf{d}_i F_a(f)$, for $i = 0, 1$.
- For any pair of composable arrows f, g in $\boldsymbol{\mathcal{C}}$, $F_a(f \circ g) = F_a(f) \circ F_a(g)$.
- For any object $c \in \boldsymbol{\mathcal{C}}$, $F_a(1_c) = 1_{F_a(c)}$.

Remark 3. Again, just as we do not explicitly mention the classes of objects and arrows of a category, we do not distinguish graphically between F_o and F_a, we therefore just use F for both of them and let the context do the work of determining which one we mean.

Given a functor F between small categories, we say that a functor is *faithful* if it is injective on hom sets and that it is *full* if it is surjective onto hom sets, and naturally we say a functor is *fully faithful* if it is a bijection on hom sets.
Furthermore we say the functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ is *essentially surjective* if for every $d \in \boldsymbol{\mathcal{D}}$, there exists $c \in \boldsymbol{\mathcal{C}}$ so that $F(c) \cong d$. Analogously we say a functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ is *essentially injective* if $F(c) = F(c')$ implies $c \cong c'$.
Now just as we have a concept of subgroup of a group, subspace of a topological space, etc. . . We also have the concept of a *subcategory* of a category $\boldsymbol{\mathcal{C}}$. It is a category $\boldsymbol{\mathcal{C}}'$ whose objects and arrows lie inside the category $\boldsymbol{\mathcal{C}}$ and the structure maps are obtained by restriction. Every such category determines an inclusion functor $\boldsymbol{\mathcal{C}}' \subseteq \boldsymbol{\mathcal{C}}$.

A subcategory of $\mathcal{C}$ is said to be a full subcategory if $\mathrm{Mor}_{\mathcal{C}'}(c,c') = \mathrm{Mor}_{\mathcal{C}}(c,c')$ for every $c, c' \in \mathcal{C}'$ (i.e. the inclusion functor is full). It is easy to check that the image of a functor $F : \mathcal{C} \to \mathcal{D}$, $\mathrm{Im}(F)$ defined as the objects and arrows of $\mathcal{D}$ which are image of some object or arrow of $\mathcal{C}$ under F, form a subcategory of $\mathcal{D}$.

Example 4. The following are some examples of functors between the categories from Example 1.

1. $\mathscr{P} : \boldsymbol{Set} \to \boldsymbol{Set}$ defined on objects to take every set to their corresponding power set (the set of all subsets)[1], and every function to the induced set-valued function on the subsets.

2. $U : \mathbf{B} \to \boldsymbol{Set}$ the functor from any of the big categories ($\mathbf{B} = \boldsymbol{Top}, \boldsymbol{Grp}, \boldsymbol{Ab}, \boldsymbol{fAb}$ or any other of the usual examples) which takes any object $b \in \mathbf{B}$ and returns the underlying set, and takes any morphism in $\mathbf{B}$ to the underlying set function (U is usually called a forgetful functor).

3. The functor $\boldsymbol{Set} \to \boldsymbol{Top}$ that sends each set to the topological space S with topology $\{\emptyset, S\}$, and every function $S \to S'$ to itself is fully faithful.

4. The functor $F : \boldsymbol{Set} \to \boldsymbol{Ab}$ that associates to any set S the free abelian group generated by S (with elements formal sums $\sum_{s \in S} a_s s$) and to each map $f : S \to S'$ the homomorphism $f'(\sum_{s \in S} a_s s) = \sum_{s \in S} a_s f(s)$.

5. The inclusions: $\boldsymbol{fAb} \subseteq \boldsymbol{Ab}$ and $\boldsymbol{Ab} \subseteq \boldsymbol{Grp}$ (and by composition $\boldsymbol{fAb} \subseteq \boldsymbol{Grp}$).

6. The functor $F : \boldsymbol{Grp} \to \boldsymbol{Ab}$ that associates to any group G its abelianization $G/[G,G]$, and any group homomorphism to the homomorphism induced on the abelianizations.

7. $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ the identity functor on a category $\mathcal{C}$ which sends each object and arrow to itself.

8. Given an object d of a category $\mathcal{D}$, and a category $\mathcal{C}$ consider the functor $d : \mathcal{C} \to \mathcal{D}$ that acts on object by $d(c) = d$ and on arrows as $d(f) = 1_d$.

9. Given any locally small category $\mathcal{C}$, and an object $c \in \mathcal{C}$ we can define the representable functor $\mathrm{Mor}_{\mathcal{C}}(c, -) : \mathcal{C} \to \boldsymbol{Set}$ that associates to each object $d \in \mathcal{C}$ the set $\mathrm{Mor}(c,d)$ and to each arrow $f : d \to e$ the arrow $f_* : \mathrm{Mor}_{\mathcal{C}}(c,d) \to \mathrm{Mor}_{\mathcal{C}}(c,e)$ given by $f_* : g = f \circ g$.

Now given any category $\mathcal{C}$ we can associate to it another category $\mathcal{C}^{\mathrm{op}}$, called the opposite category of $\mathcal{C}$, with the same objects and arrows as $\mathcal{C}$ but with structure maps defined by the following identities (where the structure maps of $\mathcal{C}^{\mathrm{op}}$ are distinguished by a $*$):

$$\mathsf{d}_0^* = \mathsf{d}_1 \qquad \mathsf{d}_1^* = \mathsf{d}_0 \qquad f \circ^* g = g \circ f$$

Note that the map $I^{\mathrm{op}} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ taking objects and arrows to themselves in the opposite category, is not a functor because it doesn't commute with the structure maps.

Definition. Given a pair of categories $\mathcal{C}, \mathcal{D}$, a *contravariant* functor $F : \mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

[1] The ZFC axioms explicitly state that this is always a set.

A functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ is also called a *presheaf with values in* $\mathcal{D}$. When we let $\mathcal{D} = \boldsymbol{Set}$ we say that F is a *presheaf* on $\mathcal{C}$.

Example 5. Let $\mathcal{C}$ be a locally small category, then consider the class of contravariant functors for each $c \in \mathcal{C}$, $\text{Mor}_{\mathcal{C}}(-, c) : \mathcal{C} \to \boldsymbol{Set}$ sending c' to $\text{Mor}_{\mathcal{C}}(c', c)$. These functors are called *representable presheaves*.

Finally we establish some basic terminology concerning objects and arrows. Let $\mathcal{C}$ be a category then:

- An arrow $g : c \to c'$ is said to be a *monomorphism* if for every parallel pair of arrows $f, f' : c'' \rightrightarrows c$ we have $g \circ f = g \circ f'$ if and only if $f = f'$.
- An arrow $g : c' \to c$ is said to be an *epimorphism* if for every parallel pair of arrows $f, f' : c \rightrightarrows c''$ we have $f \circ g = f' \circ g$ if and only if $f = f'$.
- An arrow $g : c \to c'$ is said to be an *isomorphism* if there exists an arrow $g^{-1} : c' \to c$ so that $g \circ g^{-1} = 1_{c'}$ and $g \circ g^{-1} = 1_c$, the arrow g^{-1} is called the inverse arrow.
- An object i is called *initial* if for every other object c there exists a unique arrow $i \to c$.
- An object t is called *terminal* if for every other object c there exists a unique arrow $c \to t$.
- An object z is said to be a *zero* if it is both initial and terminal. Given two objects c, c' we call the unique arrow $c \to z \to c'$ the *zero arrow* between c and c'.

Finally, to end the section we introduce the notion of a morphism between functors:

Definition. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\mu : F \to G$ is a family of morphisms in $\mathcal{D}$, μ_c indexed by the objects of $\mathcal{C}$ so that the square below commutes for any $f : c \to c'$

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ \downarrow{\scriptstyle \mu_c} & & \downarrow{\scriptstyle \mu_{c'}} \\ G(c) & \xrightarrow{G(f)} & G(c') \end{array}$$

Remark 6. When we say a diagram commutes we mean that the result of composing arrows between nodes is path-independent, that is, it depends only of the initial and final position of the path, and not the path chosen between them. In the above diagram, commutativity comes down to saying

$$\mu_{c'} \circ F(f) = G(f) \circ \mu_c \quad \forall_{c, c' \in \mathcal{C}} \quad \forall_{f : c \to c'}$$

i.e. It represents a (possibly infinite) set of equations.

Note that if $(\mu_c)_{c \in \mathcal{C}} : F \to G$ and $(\nu_c)_{c \in \mathcal{C}} : G \to H$ are natural transformations then the family of morphisms $(\nu \circ \mu)_c = (\nu_c \circ \mu_c)$ is a natural transformation $F \to H$, and because the composition of arrows in $\mathcal{D}$ is associative, so is this composition of natural transformations. Also since any functor F has an identity natural transformation to itself, we obtain the following definition:

Definition. Given any two categories $\mathcal{C}, \mathcal{D}$ the category Func$(\mathcal{C}, \mathcal{D})$ is the category whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ and the arrows are natural transformations.

An important example is when $\mathcal{C} = \mathcal{I}^{\text{op}}$ and $\mathcal{D} = \boldsymbol{Set}$, in which case the category Func$(\mathcal{C}, \mathcal{D})$ is denoted PSh$(\mathcal{I})$ and called the category of presheaves on $\mathcal{I}$.

1.2 Yoneda Lemma

The goal of this section is to present a lemma in category theory known as the Yoneda Lemma. This is arguably one of the most important lemmas to keep in mind since many times we are interested in studying functors from a locally small category to $\boldsymbol{Set}$ and this result allows us to relate them to hom sets.

Lemma 7. *Given a locally small category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \boldsymbol{Set}$, there is a natural bijection:*

$$\gamma_{F,c} : \textit{Nat}(\textit{Mor}_{\mathcal{C}}(c, -), F) \cong F(c)$$

Proof. A natural transformation $\alpha : \text{Mor}_{\mathcal{C}}(c, -) \to F$ assigns, to each $d \in \mathcal{C}$, a function $\alpha_d : \text{Mor}_{\mathcal{C}}(c, d) \to F(d)$ in such a way that, given $f : d \to d'$, we have $\alpha_{d'} \circ f_* = F(f) \circ \alpha_d$[2]. In particular, if we take $d = c$ (and d' arbitrary) then by following the identity around the diagram below

$$\begin{array}{ccc} \text{Mor}_{\mathcal{C}}(c, c) & \xrightarrow{\alpha_c} & F(c) \\ \big\downarrow f_* & & \big\downarrow F(f) \\ \text{Mor}_{\mathcal{C}}(c, d) & \xrightarrow{\alpha_d} & F(d) \end{array}$$

We get that $\alpha_d(f) = F(f)(\alpha_c(1_c))$, so the natural transformations $\alpha : \text{Mor}_{\mathcal{C}}(c, -) \to F$ are in bijection with the possible choices of $\alpha_c(1_c)$, that is, $F(c)$.
Now it's plain to see that the maps $(F, c) \mapsto (F(c))$ and $(F, c) \mapsto$ Nat$(\text{Mor}_{\mathcal{C}}(c, -) \to F)$ define functors $E, N :$ Func$(\mathcal{C}, \boldsymbol{Set}) \times \mathcal{C} \to \boldsymbol{Set}$ (where we denote the arrows of Func$(\mathcal{C}, \boldsymbol{Set}) \times \mathcal{C}$ as pairs (α, f) with $\alpha : F \to F'$ a natural transformation and $f : c \to d$ an arrow in $\mathcal{C}$) where the arrow maps are determined by:

$$\begin{aligned} E(\alpha, 1_c) &= \alpha_c \qquad & E(1_F, f) &= F(f) \\ N(\alpha, 1_c) &= \alpha_* \qquad & N(1_F, f) &= \tilde{f}_* \end{aligned}$$

Where α_* is composition (on the left) with α and $\tilde{f}_*$ is defined by:

$$\tilde{f}_*(\alpha : \text{Mor}(c, -) \to K)_{c'}(g : d \to c') = \alpha(g \circ f)$$

With these definitions in place it's easy to see that the diagrams below commute:

[2]We denote by f_* the composition with f on the left.

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c,-),F) & \xrightarrow{\gamma_{F,c}} & F(c) \\ \Big\downarrow{\scriptstyle \alpha_*} & & \Big\downarrow{\scriptstyle \alpha_c} \\ \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c,-),F') & \xrightarrow{\gamma_{F',c}} & F'(c) \end{array} \qquad \begin{array}{ccc} \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c,-),F) & \xrightarrow{\gamma_{F,c}} & F(c) \\ \Big\downarrow{\scriptstyle \tilde{f}_*} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(d,-),F') & \xrightarrow{\gamma_{F,c'}} & F'(d) \end{array}$$

And so γ is natural in F and c. □

Remark 8. Note that a representable presheaf on $\mathcal{C}$, $\mathrm{Mor}_{\mathcal{C}}(-,c)$ is the same as a representable functor $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(c,-)$ and so we have the following version of the Yoneda lemma for contravariant functors:

Corollary 9. *Let $\mathcal{C}$ be a locally small category and $F : \mathcal{C}^{op} \to \boldsymbol{Set}$, then there is a natural bijection:*

$$\mathit{Nat}(\mathit{Mor}_{\mathcal{C}}(-,c),F) \cong F(c)$$

1.3 Comma Categories

We include this section mainly to establish terminology for Section 1.7 and for the definitions in Section 2.2 that will be used throughout the text (here we opt to use the notation in [Lan98] regarding slice categories).

Definition. Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}$ we define the comma categories of objects under and over c as:

- The category $c \downarrow \mathcal{C}$ (also denoted as $c/\mathcal{C}$) has objects, pairs (c', f) with $c' \in \mathcal{C}$; and $f : c \to c'$, and arrows $(c', f') \to (c'', f'')$ are arrows $g : c' \to c''$ so that $f'' = gf'$.
- The category $\mathcal{C} \downarrow c$ (or $\mathcal{C}/c$) has objects, pairs (c', f) with $c' \in \mathcal{C}$ and $f : c' \to c$; and arrows $(c', f') \to (c'', f'')$ are functions $g : c' \to c''$ so that $f''g = f'$.

A further generalization that is sometimes useful is when we replace the "base category" $\mathcal{C}$ with a functor $F : \mathcal{C} \to \mathcal{D}$. In this situation, given $d \in \mathcal{D}$, we can define the following categories:

- $d \downarrow F$ with objects (c, f) so that f is a morphism from d to Fc and arrows $g : (c, f) \to (c', f')$ those arrows $g : c \to c'$ in $\mathcal{C}$ so that $f' = F(g) \circ f$.
- $F \downarrow d$ with objects (c, f) so that f is a morphism from Fc to d and arrows $g : (c, f) \to (c', f')$ those arrows $g : c \to c'$ in $\mathcal{C}$ so that $f = f' \circ F(g)$.

Note that when $F = \mathrm{Id}_{\mathcal{C}}$, $\mathcal{C} \downarrow c = F \downarrow c$. The over and under categories of a functor $F : \mathcal{C} \to \mathcal{D}$ come equipped with projection functors, which in the case of the over category $P : F \downarrow d \to \mathcal{D}$ send objects (c, f) to $F(c)$ and each arrow $g : c \to c'$ to $F(g)$.

Remark 10. It's worth noting that given an over category $\mathcal{C} \downarrow c$, we have $(\mathcal{C}^{\mathrm{op}} \downarrow c) = (c \downarrow \mathcal{C})^{\mathrm{op}}$.

1.4 Limits and Colimits

Limits and colimits are generalizations of the notions of cartesian product, direct sum, direct products, quotients and other so-called universal constructions.

Recall that given two categories $\mathcal{C}, \mathcal{D}$, every object $d \in \mathcal{D}$ defines a constant functor $d : \mathcal{C} \to \mathcal{D}$ with $d(c) = d$ for every object $c \in \mathcal{C}$ and $d(f) = 1_d$ for every morphism f in $\mathcal{C}$.

Definition. Given a functor $F : \mathcal{C} \to \mathcal{D}$, a *cone* over F[3] is a pair (d, π) where $d \in \mathcal{D}$ is an object and π is a natural transformation $\pi : d \to F$. Thus for each arrow $f : c \to c'$ in $\mathcal{C}$, we have $\pi_{c'} = F(f) \circ \pi_c$.

Dually a *cocone*[4] over F, is a pair (d, ι) where $d \in \mathcal{D}$ and ι is a natural transformation $\iota : F \to d$ so that for each $f : c \to c'$ in $\mathcal{C}$, we have $\iota_c = \iota_{c'} \circ F(f)$.

With this said, a *limit* of a functor F (if there is one) is a cone over F which is universal among such cones. That is, a cone $(\text{Lim}\, F, \pi)$ over F with the property that: Given any other cone (d, μ) over F, there exists a unique morphism $m : d \to \text{Lim}\, F$ so that $\mu = \pi \circ m$ (or, put short, for every other such cone there is a unique morphism to the limiting object that factors the cone through the limit).

Dually, a *colimit* of a functor F (if it exists) is a cocone over F which is universal among cocones over F. Again in the sense that it is a cocone of the form $(\text{Colim}\, F, \iota)$ satisfying: For each cocone (d, ν) over F, there exists a unique arrow $n : \text{Colim}\, F \to d$, so that $\nu = n \circ \iota$ (this means that any other cocone over F factors uniquely through the colimiting object).

A category is said to be *complete* (resp. *cocomplete*) if it has all possible limits (resp. colimits), finitely complete (resp. finitely cocomplete) if it has all limits (resp. colimits) from finite categories, and small complete (resp. small cocomplete) if it has all limits (resp. colimits) from small categories

Note now that when the category $\mathcal{D}$ is complete (resp. cocomplete); if we compose (resp. precompose) the limiting cone (resp. colimiting cocone) over a functor $F : \mathcal{C} \to \mathcal{D}$ with a natural transformation $F \to G$ (resp. $G \to F$) we obtain a cone (resp. cocone) over G which must factor uniquely through the limiting cone (resp. colimiting cocone) of G, this assigns to a natural transformation $F \to G$ an arrow $\text{Lim}\, F \to \text{Lim}\, G$ (resp. $\text{Colim}\, F \to \text{Colim}\, G$). Since the identity natural transformation does not alter the cone it induces the identity at the level of the limits (resp. colimits). Uniqueness implies that this assignment preserves composition, so we have:

Lemma 11. *If $\mathcal{D}$ is a complete category then Lim and Colim define functors Func$(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$.*

Note however that, depending on the size of $\mathcal{C}$ and $\mathcal{D}$, and since the limit and colimit are determined only up to isomorphism, we have to assume a rather strong version of the axiom of choice.

Given a functor $F : \mathcal{C} \to \mathcal{D}$ between complete (resp. cocomplete) categories, we say that F is a *continuous* (resp. *cocontinuous*) functor if it commutes with limits (resp. colimits), that is, if for every functor $G : A \to \mathcal{C}$, we have $\text{Lim}\,(F \circ G) \cong F(\text{Lim}\, G)$ (resp. $\text{Colim}\,(F \circ G) \cong F(\text{Colim}\, G)$).

Given finitely complete (resp. finitely cocomplete) categories $\mathcal{C}, \mathcal{D}$ we say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *left exact* or finitely continuous (resp. *right exact* or finitely cocontinuous) if it preserves finite limits (resp. colimits). If a functor is both left and right exact we say that it is *exact*.

[3] Or "a cone to F" would be a more graphic nomenclature.

[4] As is usual we apply the prefix "co-" to mean that we take the dual notion, i.e. consider all arrows going the other way.

Remark 12. Above, in the definition of colimit we stated that a colimit is the dual construction to that of a limit. But as we have seen in the discussion of opposite categories, we have a very precise notion of "dual". In these terms, the limit of a diagram $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is the same as a colimit of the same diagram regarded as a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$.

Example 13. Now we give a brief description of three important types of limits and colimits that are particularly important when dealing with the general notion of limit and colimit.

- A set (or more generally, a class) I can be interpreted as a category whose only morphisms are the identities. A functor $F : I \to \mathcal{C}$ is the same as a choice of objects $\{c_i \mid i \in I\}$ of $\mathcal{C}$. So a limit for this functor is an object $c \in \mathcal{C}$ so that giving an arrow $d \to c$ is the same as choosing morphisms $d \to c_i$ for each $i \in I$.

  In $\boldsymbol{Set}, \boldsymbol{Top}, \boldsymbol{Grp}$ and other familiar categories this is the same as the cartesian product, the product of spaces and the direct product of groups. We call this kind of limit a *product* in $\mathcal{C}$ and denote it by $\prod_{i \in I} c_i$.

  The dual notion, the colimit of F, is an object c so that giving an arrow $c \to d$ is the same as choosing morphisms from $c_i \to d$ for each $i \in I$. In the categories $\boldsymbol{Set}, \boldsymbol{Top}, \boldsymbol{Ab}$ this becomes the disjoint union of sets, the disjoint union of spaces and the direct sum. Such a colimit is usually referred to as a *coproduct* and denoted by $\coprod_{i \in I} c_i$.

- Consider a category $\mathcal{C}$ with only two objects and two non-identity arrows parallel to each other (as in the diagram $a \rightrightarrows b$) then a functor $F : \mathcal{C} \to \mathcal{D}$ corresponds to a choice of two objects $a, b \in \mathcal{D}$ with two parallel arrows $f, g : a \rightrightarrows b$ between them. A cone on such a functor is a choice of an arrow $h : c \to a$ so that $f \circ h = g \circ h$. A limit if F is then a cone such that any other cone on F factors uniquely through it.

  In $\boldsymbol{Set}$, $\boldsymbol{Top}$ and $\boldsymbol{Ab}$, h corresponds to the inclusion of the subset of a where f and g coincide, although in $\boldsymbol{Ab}$ it can be thought of as the inclusion of the kernel of $f - g$ in a, so we commonly refer to the limit object as $\text{eq}(f, g)$ or the *equaliser* of f and g. If $\mathcal{D}$ has a zero arrow, 0 then we define the *kernel* of a morphism f as $\ker(f) = \text{eq}(f, 0)$.

  Dually we define the *coequaliser* as being a choice of an arrow $h : b \to c$ so that $h \circ f = g \circ f$ and any other such arrow factors through it. In $\boldsymbol{Set}$ and $\boldsymbol{Top}$ this represents the quotient of b by the equivalence relation generated by $f(x) = g(x)$ for all $x \in a$. We the coequaliser of f and g by $\text{coeq}(f, g)$.

- Let $\mathcal{C}$ be the category with only three objects and two non-identity arrows with distinct domains and common codomain (i.e. as in the diagram $x \to z \leftarrow y$). A functor $F : \mathcal{C} \to \mathcal{D}$ is determined by the choice of three objects x, y, z and two arrows $f : x \to z, g : y \to z$. A cone over this functor is a choice of two arrows $h_1 : d \to x, h_2 : d \to y$ so that $f \circ h_1 = g \circ h_2$. This cone is a limit of F if any other such cone factors uniquely through it. In both $\boldsymbol{Set}$ and $\boldsymbol{Top}$ a model for the limit is the subset/subspace of $x \times y$ given by $\{(a, b) : f(a) = g(b)\}$. The limit is denoted by $x \times_z y$, and called the *pullback* of f and g.

The dual notion, naturally involves the opposite category $\mathcal{C}^{\text{op}}$ which this time is not equivalent to $\mathcal{C}$ as the non-identity arrow now have a common domain. In both $\boldsymbol{Set}$ and $\boldsymbol{Top}$ a model for the colimit is given by the disjoint union of x and y quotiented by the equivalence relation generated by $f(a) = g(a)$ for all $a \in z$. If we take $z = \{*\}$, the colimit is called the wedge sum of x and y over the chosen points. This type of colimit is called a *pushout* of the two non-identity arrows, and we denote it by $x +_z y$ or $x \coprod_z y$.

At a first glance it might seem that pullbacks contain equalisers as a special case but they are in fact different: Let $f : A \to B$ be a map of sets, then consider the models for the pullback and equaliser

$$\text{eq}(f, f) = \{a \in A : f(a) = f(a)\} \qquad A \times_B A = \{(a, a') \in A \times A : f(a) = f(a')\}$$

In this case we always have $\text{eq}(f, f) = A$, (with the structure morphism given by the identity) while when f is not injective, $A \times_B A$ contains all pairs (a, a') so that $f(a) = f(a')$, and thus is not canonically isomorphic to A.

Lemma 14. *Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism in $\mathcal{C}$. Then f is a monomorphism if and only if the following diagram is a pullback square:*

$$\begin{array}{ccc} A & = & A \\ \| & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array} \tag{1.1}$$

Proof. The proof is a straight forward consequence of the properties of limits described above.

$(\Rightarrow)$ Let K be the pullback of f along f with projection arrows $k_1, k_2 : K \to A$. Then $f \circ k_1 = f \circ k_2$, which, since f is a monomorphism, implies that $k_1 = k_2$, thus setting k to be either of the arrows we see that the cone defined by k_1, k_2 factors uniquely through diagram 1.1.

$(\Leftarrow)$ Assuming the diagram in the lemma is a pullback then any pair of functions $g', g'' : C \to A$ satisfying $f \circ g' = f \circ g''$ factors through a unique arrow $g : \mathcal{C} \to A$, moreover $1_A \circ g = g'$ and $1_A \circ g = g''$ so $g' = g''$, whereupon we conclude f is a monomorphism. ☐

Naturally from this lemma we obtain also the dual statement: f is an epimorphism if and only if the diagram opposite to diagram (1.1) is a pushout.

1.5 Existence and Construction of Limits/Colimits

In this section we provide a convenient way of computing arbitrary limits and colimits in any category. While this may seem of little use when one can just give ad-hoc descriptions of limits and colimits, the results we prove are useful because they provide a minimal set of limits/colimits that must exist for a category to be complete/cocomplete. This simplifies the process of checking these properties for a category (as well as others that may depend on them).

Theorem 15. *Any category $\mathcal{D}$ with arbitrary products and equalisers of pairs of arrows is complete.*

Naturally (and this will be apparent from the proof, which follows [Lan98]) it is also true that a category with finite products and equalisers is finitely complete and, by duality, a category with all coproducts (resp. finite coproducts) and coequalisers of pairs of arrows is cocomplete (resp. finitely cocomplete)

Proof. Let $\mathcal{C}$ be a category with all products and equalisers of pairs of arrows and $F : \mathcal{C} \to \mathcal{D}$ be a functor, we are going to use these two constructs to obtain a limit for F.
Consider the diagram below, where the middle row is an equaliser and the two parallel arrows in the middle are defined so as to make the upper and lower squares commute.

$$
\begin{array}{ccccc}
 & F(d) & \xrightarrow{\;1_{F(d)}\;} & F(d) & \\
 & \uparrow{\scriptstyle \pi_d} & & \uparrow{\scriptstyle \pi_f} & \\
L \xrightarrow{\;e\;} & \prod_{c\in\mathcal{C}} F(c) & \overset{q}{\underset{p}{\rightrightarrows}} & \prod_{f:c\to d \text{ in } \mathcal{C}} F(d)_f & \\
 & \downarrow{\scriptstyle \pi_c} & & \downarrow{\scriptstyle \pi_f} & \\
 & F(c) & \xrightarrow{\;F(f)\;} & F(d) &
\end{array} \tag{1.2}
$$

We claim that $(L, \pi_c \circ e)$ is a limit for F:
Since every morphism to $e : L \to \prod_{c\in\mathcal{C}} F(c)$ is uniquely determined by its components $e_c := \pi_c \circ e$, and write $e = \prod_c e_c$. Since e is the equaliser of p and q, we have $p \circ e = q \circ e$, so for every $f : c \to d$ in $\mathcal{C}$ we have $\pi_f \circ p \circ e = \pi_f \circ q \circ e$, but by definition we have:

$$\pi_f \circ q \circ e = e_d \qquad \pi_f \circ p \circ e = F(f) \circ e_c$$

So $e_d = F(f) \circ e_c$ and indeed (L, e_c) is a cone over F. Now given any cone to F it's easy to see that it equalises p, q and thus it has to factor uniquely through the equaliser e, making L the limit of F as intended. □

It is now worthwhile noting that in the case of $\boldsymbol{Set}$ (which is both complete and cocomplete) limits have a simple representation as $\text{Lim}\, F \cong \text{Mor}_{\boldsymbol{Set}}(\{*\}, F)$. This is because choosing an element in the limit is the same as choosing elements in each set in the image of F which are compatible with the functions in the image of F.
As mentioned earlier, the dual to Theorem 15 is also true. The result then states that the colimit of F is canonically isomorphic to the coequaliser of the following pair of arows:

$$
\begin{array}{ccccc}
F(c) & \xrightarrow{\;1_{F(c)}\;} & F(c) & & \\
\downarrow{\scriptstyle \iota_f} & & \downarrow{\scriptstyle \iota_{F(c)}} & & \\
\coprod_{\substack{c,d\in\mathcal{C}\\ f:c\to d}} F(c)_f & \overset{p}{\underset{q}{\rightrightarrows}} & \coprod_{c\in\mathcal{C}} F(c) & \xrightarrow{\;j\;} & C \\
\uparrow{\scriptstyle \iota_f} & & \uparrow{\scriptstyle \iota_{F(d)}} & & \\
F(c) & \xrightarrow{\;F(f)\;} & F(d) & &
\end{array}
$$

The proof of Theorem 15 and its dual, yield the following Corollary:

Corollary 16. *A functor $F : \mathcal{C} \to \mathcal{D}$ is:*

- *Continuous if it commutes with terminal objects, products and equalisers.*
- *Left exact if it commutes with terminal objects, finite products, and equalisers.*
- *Cocontinuous if it commutes with initial objects, coproducts and coequalisers.*
- *Right exact if it commutes with initial objects, finite coproducts and coequalisers.*

1.6 Adjunctions

Given a pair of functors L, R with reversed domain and codomain as below:

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

we say L is a *left adjoint* to R if there exists a natural bijection[5]:

$$\psi_{a,b} : \text{Mor}_{\mathcal{D}}(La, b) \cong \text{Mor}_{\mathcal{C}}(a, Rb)$$

Equivalently an *adjunction* can be given by a pair of natural transformations $\eta : \text{Id}_{\mathcal{C}} \to RL$ and $\varepsilon : LR \to \text{Id}_{\mathcal{D}}$ so that the *triangular identities* below are satisfied

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & RLR \\ & \diagdown\!\!\diagdown & \big\downarrow R(\varepsilon) \\ & & R \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{L(\eta)} & LRL \\ & \diagdown\!\!\diagdown & \big\downarrow \varepsilon_L \\ & & L \end{array}$$

If instead of just natural transformations η, ε are natural isomorphisms (in which case we don't need to check the triangular identities) we say that the functors L, R define an *equivalence of categories* (in this case it actually happens that L is both left and right adjoint to R).
To prove the equivalence of the two formulations of the concept of adjunction, we will provide a way of going from one formulation to the other and back. Suppose we have a unit η and a counit ε, satisfying the triangular identities, then set:

$$\psi_{a,b}(f : La \to b) = R(f) \circ \eta_a : a \to Rb$$
$$\psi^{-1}_{a,b}(g : a \to Rb) = \varepsilon_b \circ L(g) : La \to b$$

Since naturality follows from the fact that we are only composing with natural transformations, it remains to prove that these are inverse to each other. Given morphisms $f \in \text{Mor}_{\mathcal{D}}(La, b)$ and $g \in \text{Mor}_{\mathcal{C}}(a, Rb)$ we have:

[5] Here naturality means as functors $\mathcal{C}^{\text{op}} \times \mathcal{D} \to \boldsymbol{Set}$.

$$\begin{aligned} \psi^{-1}_{a,b}(\psi_{a,b}(f)) &= \psi^{-1}_{a,b}(R(f)\circ\eta_a) \\ &= \varepsilon_b \circ L(R(f)\circ\eta_a) \\ &= \varepsilon_b \circ LR(f)\circ L(\eta_a) \\ &= f \end{aligned} \qquad \begin{aligned} \psi_{a,b}(\psi^{-1}_{a,b}(g)) &= \psi^{-1}_{a,b}(\varepsilon_b \circ L(g)) \\ &= R(\varepsilon_b \circ L(g))\circ\eta_a \\ &= R(\varepsilon_b)\circ RL(g)\circ\eta_a \\ &= g \end{aligned}$$

where the last step on each side is justified by the fact that the following diagrams (which are derived from the triangular identities) commute:

$$\begin{array}{ccccccc} La & \xrightarrow{L(\eta_a)} & LRLa & \xrightarrow{LR(f)} & LRb & \xrightarrow{\varepsilon_b} & b \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \varepsilon_{La}} & & \downarrow{\scriptstyle \varepsilon_b} & \diagup\!\!\diagup & \\ & & La & \xrightarrow{f} & b & & \end{array} \qquad \begin{array}{ccccccc} a & \xrightarrow{\eta_a} & RLa & \xrightarrow{RL(g)} & RLRb & \xrightarrow{R(\varepsilon_b)} & Rb \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \eta_a} & & \uparrow{\scriptstyle \eta_{Rb}} & \diagup\!\!\diagup & \\ & & a & \xrightarrow{g} & Rb & & \end{array}$$

Conversely let ψ be a natural isomorphism:

$$\psi : \mathsf{Mor}_{\mathcal{D}}(L-,-) \cong \mathsf{Mor}_{\mathcal{C}}(-,R-)$$

and define $\eta_a = \psi_{a,La}(1_{La})$ and $\varepsilon_b = \psi^{-1}_{Rb,b}(1_{Rb})$. We need to see that η and ε satisfy the triangular identities. To do this we only need to expand the naturality of ψ in two particular cases, where we let $f : La \to b$ and $g : a \to Rb$ be morphisms, and follow the identity around in the following commutative diagram:

$$\begin{array}{ccc} \mathsf{Mor}(La,La) & \xrightarrow{\psi_{a,La}} & \mathsf{Mor}(a,RLa) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle R(f)_*} \\ \mathsf{Mor}(La,b) & \xrightarrow{\psi_{a,b}} & \mathsf{Mor}(a,Rb) \end{array} \qquad \begin{array}{ccc} \mathsf{Mor}(Rb,Rb) & \xrightarrow{\psi^{-1}_{Rb,b}} & \mathsf{Mor}(LRb,b) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle L(g)^*} \\ \mathsf{Mor}(a,Rb) & \xrightarrow{\psi^{-1}_{a,b}} & \mathsf{Mor}(La,b) \end{array}$$

By doing this we get $R(f)\circ\psi_{a,La}(1_{La}) = \psi_{a,b}(f)$ and $\psi^{-1}_{Rb,b}(1_{Rb})\circ L(g) = \psi^{-1}_{a,b}(g)$ and thus the composites in the triangular identities become:

$$\varepsilon_{La}\circ L(\eta_a) = \psi^{-1}_{RLa,La}(1_{RLa})\circ L(\psi_{a,La}(1_{La})) = \psi^{-1}_{a,La}(\psi_{a,La}(1_{La})) = 1_{La}$$
$$R\varepsilon_b \circ \eta_{Rb} = R(\psi^{-1}_{Rb,b}(1_{Rb}))\circ\psi_{Rb,LRb}(1_{LRb}) = \psi_{Rb,b}(\psi^{-1}_{Rb,b}(1_{Rb})) = 1_{R_b}$$

We conclude that the formulations of the concept of adjunction via a natural isomorphism ψ or a unit and counit satisfying triangular identities are equivalent.

One of the useful aspects of adjunctions is that they give us a convenient way of expressing limits and colimits:

Given a pair of categories $\mathcal{C},\mathcal{D}$ consider the diagonal functor $\Delta : \mathcal{D} \to \mathsf{Func}(\mathcal{C},\mathcal{D})$ that sends each object $d \in \mathcal{D}$ to the functor $d : \mathcal{C} \to \mathcal{D}$. The universality of a limit says that $\mathsf{Nat}(d,F) \cong \mathsf{Mor}_{\mathcal{D}}(d,\mathsf{Lim}\,F)$ thus the diagonal functor is left adjoint to the limit functor. Similarly, we have $\mathsf{Mor}_{\mathcal{D}}(\mathsf{Colim}\,F,d) \cong \mathsf{Nat}(F,d)$ and so the colimit functor, if everywhere defined provides a left adjoint to the diagonal. This proves the following lemma:

Lemma 17. *Given categories $\mathcal{C}$ and $\mathcal{D}$ where, $\mathcal{D}$ is complete (resp. cocomplete) and $\mathcal{C}$ is small then Lim : Func($\mathcal{C}, \mathcal{D}$) $\to \mathcal{D}$ (resp. Colim : Func($\mathcal{C}, \mathcal{D}$) $\to \mathcal{D}$) is a right (resp. left) adjoint to the diagonal functor $\Delta : \mathcal{D} \to$ Func($\mathcal{C}, \mathcal{D}$).*

With this said we can further explore the relationship between limits, colimits and adjunctions by checking the continuity of the functors involved in the adjunction:

Lemma 18. *Every left (resp. right) adjoint between cocomplete (resp. complete) categories is cocontinuous (resp. continuous).*

Proof. Consider an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ with $\mathcal{C}$ and $\mathcal{D}$ complete, and let $H : J \to \mathcal{C}$ be a functor. Then for all $d \in \mathcal{D}$, $c \in \mathcal{C}$:

$$\begin{aligned} \mathrm{Mor}_{\mathcal{D}}(F\mathrm{Colim}\, H, d) &\cong \mathrm{Mor}_{\mathcal{C}}(\mathrm{Colim}\, H, Gd) \\ &\cong \mathrm{Nat}(H, Gd) \\ &\cong \mathrm{Nat}(FH, d) \\ &\cong \mathrm{Mor}_{\mathcal{D}}(\mathrm{Colim}\, FH, d) \end{aligned} \qquad \begin{aligned} \mathrm{Mor}_{\mathcal{C}}(c, G\mathrm{Lim}\, K) &\cong \mathrm{Mor}_{\mathcal{D}}(Fc, \mathrm{Lim}\, K) \\ &\cong \mathrm{Nat}(Fc, K) \\ &\cong \mathrm{Nat}(c, GK) \\ &\cong \mathrm{Mor}_{\mathcal{C}}(c, \mathrm{Lim}\, GK) \end{aligned}$$

Where we use completeness/cocompleteness of $\mathcal{D}$ in the last step. Note that when let $d = F\mathrm{Colim}\, H$ (resp. $c = G\mathrm{Lim}\, K$) and consider the sequence of isomorphisms on the left (resp. right) we get an isomorphism Colim $FH \to F$Colim H (resp. GLim $K \to$ Lim GK) as the image of $1_{F\mathrm{Colim}\, H}$ (resp. $1_{G\mathrm{Lim}\, K}$) and this assignment is natural in H (resp. K). □

1.7 Kan Extensions

Consider the inclusion functor $i : \mathcal{M} \subseteq \mathcal{C}$ of a subcategory. This defines, for any category $\mathcal{D}$ a functor $i^* :$ Func($\mathcal{C}, \mathcal{D}$) $\to$ Func($\mathcal{M}, \mathcal{D}$) which amounts to precomposition with the inclusion, or in other words, restriction to the subcategory $\mathcal{M}$.

Definition. Let $T : \mathcal{M} \to \mathcal{D}$ be a functor, where $\mathcal{D}$ is complete. The right Kan extension of T, is the functor $i_K(T) : \mathcal{C} \to \mathcal{D}$ given on objects by:

$$i_K(T)(c) = \underset{f \in (c \downarrow \mathcal{M})}{\mathrm{Lim}}\, T \circ Q(f)$$

where Q is the projection $(c \downarrow \mathcal{M}) \to \mathcal{M}$.

Given that limits (and colimits) are functorial in the indexing category of the diagrams in question this does in fact provide a functor from $\mathcal{C}$ to $\mathcal{D}$. Analogously we define the left Kan extension $i^K(T)$, of $T : \mathcal{M} \to \mathcal{D}$ along the inclusion $i : \mathcal{M} \to \mathcal{C}$.

Definition. If $\mathcal{D}$ is cocomplete and $T : \mathcal{M} \to \mathcal{D}$ is a functor defined on $\mathcal{M}$ a subcategory of $\mathcal{C}$, then the left Kan extension of T is given (on objects) by

$$i^K(T)(c) = \underset{f \in (\mathcal{M} \downarrow c)}{\mathrm{Colim}}\, T \circ Q(f)$$

where Q is the projection $(\mathcal{M} \downarrow c) \to \mathcal{M}$.

We now prove that the left and right Kan extensions i_K, i^K provide left and right adjoints to i^* (when they are defined for all functors). We will prove only the statement concerning let Kan extensions, as the other one follows from duality.

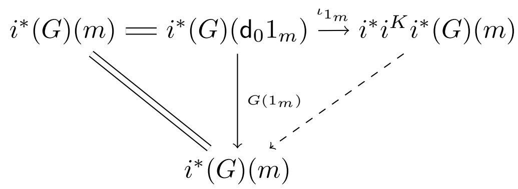
Proof. To prove the adjunction we provide a unit and counit:

$$\varepsilon : i^K i^* \to \mathsf{Id}_{\mathsf{Func}(\mathcal{C},\mathcal{D})} \qquad \eta : \mathsf{Id}_{\mathsf{Func}(\mathcal{M},\mathcal{D})} \to i^* i^K$$

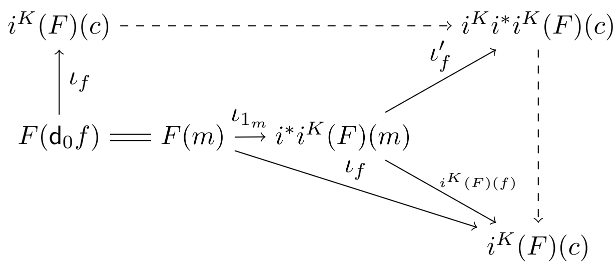
Note that components ε_G and η_F of ε and η are natural transformations, where for each $F \in \mathsf{Func}(\mathcal{M}, \mathcal{D})$, $G \in \mathsf{Func}(\mathcal{C}, \mathcal{D})$, $m \in \mathcal{M}$ and $c \in \mathcal{C}$ we have components:

$$\varepsilon_{G,c} : i^K i^*(G)(c) \to G(c) \qquad \eta_{F,m} : F(m) \to i^* i^K(F)(m)$$

So define $\varepsilon_{G,c}$ as the unique map $i^K i^*(G)(c) \to G(c)$ arising from the cone $G(\mathsf{d}_0 f) \overset{G(f)}{\to} G(c)$, and $\eta_{F,m} = \iota_{1_m}$ the inclusion of $F(\mathsf{d}_0 1_m)$ into the colimit $i^K(F)(m)$. The verification that these maps satisfy the triangular identities is indicated by the commutativity of the following diagrams.



And for every $f : m \to c \in (\mathcal{M} \downarrow c)$:



□

The existence of left and/or right Kan extensions is a very useful fact since it allows us to conclude that a functor of the form i^* is continuous and/or cocontinuous, simply by looking at the completeness/cocompleteness of the category $\mathcal{D}$.

1.8 Filtered Categories and Colimits

Filtered categories share properties with filtered posets. In this section we present the definition of filtered category and prove an important theorem which states that filtered colimits commute with finite limits to $\boldsymbol{Set}$.

Definition. A category I is said to be *filtered* if it satisfies:

1. For any objects $j, j' \in I$ there exists an object $k \in I$ so that $\text{Mor}(j,k), \text{Mor}(j',k) \neq \emptyset$
2. For any object $i \in I$ and any pair of parallel arrows $f, g : i \rightrightarrows j$ there exists an object k and an arrow $w : j \to k$ so that $w \circ f = w \circ g$

It is worth noting that every category with a terminal object satisfies axiom 1 and 2.
We call colimits from a filtered category, *filtered colimits*.

Theorem 19. *Filtered colimits commute with finite limits in $\boldsymbol{Set}$.*

Or in other words the functor $\underset{i\in I}{\text{Colim}} : \text{Func}(I, \boldsymbol{Set}) \to \boldsymbol{Set}$ given by $F \mapsto \underset{i\in I}{\text{Colim}}\, F(i)$ is left exact when I is filtered.

Proof. Let I be a filtered category and J a finite category, then consider a functor $F : I \times J \to \boldsymbol{Set}$, every object $j \in J$ defines a functor $F(-,j) : I \to \boldsymbol{Set}$, and every object $i \in I$ defines a functor $F(i,-) : J \to \boldsymbol{Set}$. We want to show that we have an isomorphism:

$$\underset{i\in I}{\text{Colim}}\, \underset{j\in J}{\text{Lim}}\, F(i,j) \cong \underset{j\in J}{\text{Lim}}\, \underset{i\in I}{\text{Colim}}\, F(i,j)$$

For this purpose we start by defining:

$$F_J(i) := \underset{j\in J}{\text{Lim}}\, F(i,j) \qquad F^I(j) := \underset{i\in I}{\text{Colim}}\, F(i,j)$$

First note that there is a canonical map $\underset{i\in I}{\text{Colim}}\, F_J(i) \to \underset{j\in I}{\text{Lim}}\, F^I(j)$ determined by the diagram below where the solid arrows are canonical projections and inclusions associated to the appropriate limit or colimit objects, and the dashed arrows are determined by the universal properties.

$$\begin{array}{ccccc} \underset{i\in I}{\text{Colim}}\, F_J(i) & & \dashrightarrow & & \underset{j\in J}{\text{Lim}}\, F^I(j) \\ \uparrow & & \nearrow & & \downarrow \\ F_J(i) & \longrightarrow & F(i,j) & \longrightarrow & F^I(j) \end{array} \tag{1.3}$$

All we need to do now is to show that this canonical map is an isomorphism when J is finite and I is filtered, to do this we have to further investigate the structure of the colimit:
Specialising the construction of colimits in Section 1.5 to the case of $\boldsymbol{Set}$ we have that:

$$\underset{i\in I}{\text{Colim}}\, F(i,j) = \frac{\coprod_i F(i,j)}{E}$$

Where E is the equivalence relation described as follows:
Represent the elements of $\coprod_i F(i,j)$ (with j fixed) as pairs (x,i) with $x \in F(i,j)$, then E is the equivalence relation generated by $(x,i)E(x',i')$ if there exists some object $i'' \in I$ and arrows $f : i \to i''$, $f' : i' \to i''$ so that $F(f, 1_j)(x) = F(f', 1_j)(x')$. And so two elements $(x,i), (x',i')$ are equivalent if and only if there exists a connection in I:

$$i = i_0 \xleftarrow{f_1} i_1 \xrightarrow{g_2} i_2 \xleftarrow{f_3} \dots \xleftarrow{f_{n-1}} i_{n-1} \xrightarrow{g_n} i_n = i' \tag{1.4}$$

And elements $x_l \in i_l$ for each $l = 0, \dots, n$ such that $x_0 = x$ and $x_n = x'$ and $F(f_l, 1_j)(x_l) = x_{l-1}$, and $F(g_l, 1_j)(x_{l-1}) = x_l$. Now we want to prove that if (x, i) and (x', i') are equivalent then there exists an object i'' and morphisms $f : i \to i''$ and $f' : i' \to i''$ so that $F(f, 1_j)(x) = F(f', 1_j)(x')$:

If (x, i) and (x', i') are equivalent, then there exists a connection as in (1.4), and note that these connections have even length (if we have a connection with an odd length we can extend it to an even length connection by an identity map). So if $(x, i), (x', i')$ have a length 2 connection then by the first axiom of filtered categories they have we have arrows $g : i \to i''$ and $g' : i' \to i''$, and by applying the second axiom of a filtration there exists some morphism $w : i'' \to i'''$ with $f = w \circ g$ and $f' = w \circ g'$ being the intended arrows. On the other hand if $(x, i), (x', i')$ have a length $2n$ connection then, again by the same argument, there exist morphisms $h_0 : i_0 \to i''$ and $h_2 : i_2 \to i''$ such that $F(h_0, 1_j)(x) = F(h_2, 1_j)(x_2)$, applying this procedure n times and composing the resulting morphisms yields the intended result.

Now an element of $\operatorname*{Lim}_{j \in J} F^I(j)$ is determined by a family of elements $(x_j, i_j) \in F(i_j, j)$ required to satisfy:

$$F^I(f)(x_j, i_j) E(x_{j'}, i_{j'})$$

for every morphism $f : j \to j'$ in J. So consider the the set:

$$\{(x_j, i_j) \mid j \in J\} \cup \{(F(1_{i_j}, f)(x_j), i_j) \mid f : j \to j' \text{ in } J\}$$

Since J is finite, at each $j \in J$ this is a finite list of elements $(x_1, i_{1,j}), \dots, (x_{n_j}, i_{n_j,j})$ which are equivalent among themselves. The first axiom of a filtered catgory states that there exists some object $i \in I$ with morphisms $f_{l,j} : i_{l,j} \to i$ for each $j \in J$ and $l = 1, \dots, n_j$, and hence we can consider the i–component to be the same across all $j \in J$. Now note that each morphism $w : i \to i'$ in I determines a natural transformation $F(i, -) \to F(i', -)$, so ordering the elements of J as $j_1, \dots, j_m$, the observations on the previous paragraph guarantee that there exists $i_1 \in I$ and a morphism $w_1 : i \to i_1$ such that:

$$F(w_1, 1_{j_1})(x_l) = F(w_1, 1_{j_1})(x_k)$$

for all $l, k = 1, \dots, n_{j_1}$. By the same observations there exists $i_2 \in I$ and a morphism $w_2 : i_1 \to i_2$ such that:

$$F(w_2 \circ w_1, 1_{j_2})(x_l) = F(w_2 \circ w_1, 1_{j_2})(x_k)$$

for all $l, k = 1, \dots, n_{j_2}$. Proceeding this way, at the m^{th} step we obtain a list of m elements

$$x_1 \in F(i_m, j_1), \dots, x_m \in F(i_m, j_m)$$

such that for every $f : j_l \to j_k$ in J, $F(1_{i_m}, f)(x_l) = x_k$, which corresponds to an element of $\operatorname*{Lim}_{j \in J} F(i_m, j)$ and finally, the canonical inclusion into the colimit provides an element of $\operatorname*{Colim}_{i \in I} \operatorname*{Lim}_{j \in J} F(i_m, j)$. This map is well defined and provides a bilateral inverse to the canonical map from diagram (1.3). $\square$

Chapter 2

Topological Introduction

In the sequel we concern ourselves with some constructions from topos theory, but instead of using general elementary topoi we will focus simply on Grothendieck topoi of sheaves over sites which are constructed from topological spaces. As such, this section intends to be a concise, yet thorough exposition of the topological constructs we will come across in the text, as well as to give an introduction to the concept of sheaves on a topological space (we will later see how these generalise to the concept of sheaves on a site). We therefore employ as little category theory as possible making only from time to time parallels between the topological intuition and the corresponding categorical construct.

2.1 Irreducibility

Definition. A topological space, T is said to be *irreducible* if it cannot be written as the union of two (and hence, any finite number of) proper closed subsets.
We say that T is an *irreducible closed* subset of X if it is a closed subset of X, and is an irreducible space when endowed with the subspace topology.

In terms of open sets we say that an open set U is an *irreducible open* subset if its complement U^{c} is an irreducible closed set. Alternatively, U is an *irreducible open subset* of X if given any pair of open sets $U_1, U_2 \subseteq X$ such that $U = U_1 \cap U_2$, then either $U = U_1$ or $U = U_2$ (that is, U cannot be written as the intersection of two open sets that properly contain it).

Lemma 20. *Let X be a topological space. The following assertions are equivalent:*

1. *X is irreducible.*
2. *Given any pair of non-empty open sets $U, V \subseteq X$, $U \cap V \neq \emptyset$.*
3. *Any non-empty open set $U \subset X$ is dense.*

Proof.
$(1 \Rightarrow 2)$ Assume $U, V \subseteq X$ and $U \cap V = \emptyset$ then $X = (U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$, and since both U^{c} and V^{c} are closed and X is irreducible, either U^{c} or V^{c} is equal to X, hence either $U = \emptyset$ or $V = \emptyset$.

$(2 \Rightarrow 3)$ Given a non empty open set U and any point $x \in X$, for any open neighborhood V of x, we have $U \cap V \neq \emptyset$ and thus $x \in \overline{U}$, since x is arbitrary we get $\overline{U} = X$.

$(3 \Rightarrow 1)$ Let F, G be two closed proper subsets of X, and let $U = F^{\mathsf{c}}$ and $V = G^{\mathsf{c}}$ then $F \cup G = (U \cap V)^{\mathsf{c}}$, since U and V are non-empty open sets, and thus, dense, $U \cap V \neq \emptyset$ and thus $F \cup G \neq X$. □

The irreducible closed subsets $T \subseteq X$ of a topological space form a partially ordered set (or poset) under inclusion, this then becomes a category to which we call Irr(X).

Example 21.

1. If X is T_1 then all points are irreducible closed sets.
2. If X is Hausdorff, the only irreducible closed sets are points.
3. Let $X = \mathbb{C}^n$ with the Zariski topology, then irreducible closed sets correspond to irreducible varieties, i.e. the zero loci of prime ideals of $\mathbb{C}[X_1, \ldots, X_n]$.

2.2 Étale Maps and Étale Spaces

Étale spaces are another of the constructs in which we will focus in the subsequent chapters. These are locally, open embeddings, and since we can identify every open embedding with the inclusion of its image in X, these provide to some extent, an extension of the topology of X (the meaning of this will become apparent in the next chapter).

Definition. Given two topological spaces Y and X we say that a continuous map $p : Y \to X$ is *étale*, or a *local homeomorphism*, if for every $y \in Y$ there exist open neighborhoods $V \subseteq Y$ of y and $U \subseteq X$ of $p(y)$ so that $p|_V$ is a homeomorphism between V and U.

Given a map $p : Y \to X$ we call the set $p^{-1}(x)$ for $x \in X$, the *fiber over* x, and we call a space Y with an étale map $p : Y \to X$, an *étale space* over X.
From the definition of étale map we immediately obtain that the fiber of an étale map over any point $x \in X$ must be a discrete set: Since $p^{-1}(x)$ is a subspace of Y with the induced topology, and for any point $y \in p^{-1}(x)$ there exists an open neighborhood U_y so that $p|_{U_y}$ is a homeomorphism onto the image, then $U_y \cap p^{-1}(x) = \{y\}$ is open in $p^{-1}(x)$, and so $p^{-1}(x)$ has the discrete topology.
Note also that for every open set $U \subseteq Y$, we can take for each point $y \in U$ a neighborhood $U_y \ni y$ so that $p|_{U_y}$ is an open embedding and so $p(U) = p(\bigcup_{y \in U} U_y \cap U) = \bigcup_{y \in U} p(U_y \cap U)$. Finally since $U_y \cap U \subseteq U_y$, is open and $p|_{U_y}$ is an open embedding, $p(U_y \cap U)$ is also open whereupon we conclude $p(U)$ is open, proving the following lemma:

Lemma 22. *Every étale map is an open map.*

Now given two étale maps $f : Y \to X$ and $g : Z \to Y$, their composite fg is clearly étale, as we can for any point $z \in Z$ consider a neighborhood U'_z of z so that g is a homeomorphism on U'_z to an open set W'_z, now consider a neighborhood $W_z \subseteq W'_z$ of $g(z)$ where f is a homeomorphism onto some open set V_z. Then, setting $U_z = U'_z \cap g^{-1}(W_z)$, allows us to conclude U_z is a neighborhood of z for which fg is a homeomorphism onto V_z, therefore fg is étale and we arrive at the following theorem:

Theorem 23. *The class of topological spaces with étale maps between them, form a subcategory of* $\boldsymbol{Top}$.

We call the category of topological spaces and étale morphisms $\mathbf{\acute{E}t}$. From this point on, when we are dealing with an étale map $p : Y \to X$, we will refer to neighborhoods U of $y \in Y$ so that the $p|_U$ is a homeomorphism onto its image, as *étale neighborhoods* of y.

Example 24.

1. The identity map $1_X : X \to X$ is trivially an étale map, making X into an étale space over itself.
2. The inclusion of an open set $U \subseteq X$ is an open embedding and therefore an étale map; More generally every open embedding is an étale map.
3. Consider the line with two origins given by $(\mathbb{R} \coprod \mathbb{R})/\sim$ where we identify every non-zero element on the first copy of $\mathbb{R}$ with its counterpart in the second copy. It has an obvious projection onto $\mathbb{R}$ which makes it into a non-Hausdorff étale space over $\mathbb{R}$.

Now we proceed with some considerations about the category of étale spaces over a fixed space X by taking the category $\mathbf{\acute{E}t}(X) := (\mathbf{\acute{E}t} \downarrow X)$.
Giving a continuous map between two étale spaces $p : Y \to X$ and $q : Z \to X$ is (by definition of comma category) the same as giving a continuous map $f : Y \to Z$ so that $p = q \circ f$. Since p is étale, for every point $y \in Y$ we can consider étale neighborhoods U of y and V of $f(y)$ so that $p|_U$ and $q|_V$ are both open embeddings; thus they are both open embeddings when restricted to $U' = U \cap p^{-1}(p(U) \cap q(V))$ and $V' = V \cap q^{-1}(p(U) \cap q(V))$, whereupon we conclude f restricts to a homeomorphism from U' to V'. Hence every map between étale spaces is also étale, from which we conclude:

Proposition 25. *Given any topological space* X, *the category* $\mathbf{\acute{E}t}(X)$ *is a full subcategory of* $\boldsymbol{Top}/X$.

2.3 Presheaves and Sheaves on a Topological Space

On this section we introduce one of the most important pieces of machinery to be used in the rest of the thesis, presheaves and sheaves on a topological space. This is a geometric construction that was developed in the 1940's in the context of algebraic geometry to study certain objects with a local character on spaces. A further generalization due to Grothendieck will be considered in Chapter 4.
We start by giving the definition of a presheaf and proceed to define sheaf and stalks of presheaves.

Definition. Given a topological space X, we define a presheaf on X to be a functor $F : O(X)^{\text{op}} \to \boldsymbol{Set}$, where $O(X)$ is the poset category of open sets.

This definition, in non-categorical terms, says that for each open set $U \subset X$ we have a set $F(U)$ and for any two open sets $V \subseteq U \subseteq X$ we have a map, called a *restriction map* $\rho_{U,V} : F(U) \to F(V)$, satisfying, $\rho_{U,U} = 1_{F(U)}$, and for each triple $W \subseteq V \subseteq U$ we have $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$.
We call the elements of $F(U)$ the *sections* of F on U, furthermore we call an element of $F(X)$ a *global section* of F. We will indulge in the habit of denoting the image of a section $s \in F(U)$ through a restriction map $\rho_{U,V}$ by $s|_V$. And also if we have open sets U_i indexed in a set I we will denote their intersections by $U_{ij} := U_i \cap U_j$.
Along with the concept of presheaf we have a more restrictive family of functors which we call "sheaves", these are presheaves where sections can be obtained by "gluing together smaller sections".
The usual examples are that of sheaves of functions with some nice properties (usually continuous, C^∞ or polynomial functions) on some topological space.
This intuition, nonetheless needs to be formalized and so we provide two equivalent definitions, one entirely topological, and the other resorting to abstract nonsense, the verification of the equivalence is straight forward.

Definition. A presheaf F over X is said to be a *sheaf* if given any open covering $\{U_i\}_{i\in I}$ of U and any family of sections $s_i \in F(U_i)$ so that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all $i, j \in I$, there exists a unique section $s \in F(U)$ so that $s|_{U_i} = s_i$.

Definition. A presheaf F over X is said to be a *sheaf*, if for any covering $\{U_i\}_{i\in I}$ the following diagram is exact.

$$F(U) \xrightarrow{e} \prod_{i\in I} F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j\in I} F(U_{ij}) \tag{2.1}$$

Where each of the maps is determined by their projections to the the components of the product as:

- e projects onto the component i through the restriction ρ_{U,U_i}.
- p projects onto the component i, j through the map $\rho_{U_i,U_{ij}} \circ \pi_i$.
- q projects onto the component i, j through the map $\rho_{U_j,U_{ij}} \circ \pi_j$.

Exactness simply says that e is universal among maps making the diagram commute, i.e. e is an equalizer.
The categorical definition, although seemingly more cryptic than the topological one, is useful in the sense that it allows for further generalization of the notion of sheaf from a topological space to other objects such as categories, once we translate the concepts of open cover and intersection of open sets into our desired context.
To understand how one goes about to use the diagram (2.1) to check if a particular presheaf happens to be a sheaf, we present the skyscraper sheaves as standard examples, which we will call upon later.

Let S be any set, and $x \in X$ be any point. The presheaf S^x (called the "skyscraper" of S at x) by:

$$S^x(U) = \begin{cases} S & \text{if } x \in U \\ \{*\} & \text{otherwise} \end{cases}$$

Where the restriction maps $S^x(U) \to S^x(V)$ are the identity if $x \in V$ or $x \notin U$, and the unique map $S \to \{*\}$ if $x \in U \setminus V$.

Proposition 26. *The skyscraper presheaf S^x is a sheaf.*

Proof. To prove this we employ the categorical definition by proving the diagram (2.1) is an equalizer for every cover.

Let U be an open set and $\{U_i\}_{i \in I}$ be an open covering of U. If $x \notin U$, $F(U) = F(U_i) = F(U_{ij}) = \{*\}$ for all $i, j \in I$, since $\prod_i \{*\} \cong \{*\}$ and since $\{*\}$ is terminal in $\boldsymbol{Set}$ there exists a unique function $\{*\} \to \{*\}$ and so $p = q$ and so the equalizer must be an isomorphism, which in this case, e must be.

If $x \in U$ then take $s, s' \in S^x(U)$ so that $e(s) = e(s')$ then for all $i \in I$, $e(s)_i = e(s')_i$, now since x is in U and the U_i cover U, $x \in U_i$ for some i and, and for that index $S^x(U_i) = S$ and the restriction map is the identity, which implies $s = e(s)_i = e(s')_i = s'$ and therefore e is injective.

The fact that S^x is a presheaf already guarantees that e maps into the equalizer, so all we need to do now is to check that it maps surjectively onto the equalizer. For this let $(s_i) \in \prod_i S^x(U_i)$ be so that $p(s_i) = q(s_i)$, by definition this means that for all $l, k \in I$, $p(s_i)_{lk} = q(s_i)_{lk}$. Now $p(s_i)_{lk} = s_l|_{U_{lk}}$ and $q(s_i)_{lk} = s_k|_{U_{lk}}$. Now if $x \notin U_l$ then $S^x(U_l) = \{*\}$, thus we have an isomorphism $\prod_{i \in I} S^x(U_i) \cong \prod_{i : x \in U_i} S^x(U_i) \cong \prod_{i : x \in U_i} S$, now since, when we restrict ourselves to the subset $\{i : x \in U_i\}$ all the restrictions are the identity, it is plain to see that $p(s) = q(s)$ if and only if $s_i = s_j$ for all i, j with $x \in U_i \cap U_j$ and therefore we can just take the section $s_i \in S^x(U)$. □

The key to the previous proof is that whenever the sheaf is non-trivial on both U and V, that means that $x \in U$ and $x \in V$ and so $x \in U \cap V$, hence the sheaf is not trivial in $U \cap V$. We might want to generalize the condition for sets $D \subset X$ substituting the condition $x \in U$ by $D \cap U \neq \emptyset$. This only gives rise to a sheaf in the case where D is an irreducible closed set because otherwise it may happen that $U \cap D, V \cap D \neq \emptyset$ but $U \cap V \cap D = \emptyset$, and in this situation diagram (2.1) becomes:

$$S \times S \rightrightarrows S \times \{*\} \times \{*\} \times S$$

But in this case $p = q$ and so the equalizer is isomorphic to $S \times S$, which is not canonically isomorphic to S for any $S \neq \{*\}, \emptyset$.

Lemma 27. *Given a set $D \subseteq X$ and any set $S \neq \emptyset, \{*\}$, the skyscraper presheaf S^D defined as:*

$$S^D(U) = \begin{cases} S & \text{if } U \cap D \neq \emptyset \\ * & \text{otherwise} \end{cases}$$

is a sheaf if and only if D is irreducible closed.

As a corollary we observe that the presheaf on X constantly equal to S is a sheaf if and only if $S \cong \{*\}$, $S = \emptyset$, or X is itself irreducible.

Now denote by h_c the representable presheaf on any category Mor$(-, c)$. In our present setting this category is $O(X)$ which is a partially ordered set (seen as a category).

For any $V \in O(X)$, either $V \subseteq U$, in which case $h_U(V) = \{*\}$ or $h_U(V) = \emptyset$. In the first case, for any covering $\{V_i\}_{i \in I}$ we have $V_i \subseteq U$ and $V_i \cap V_j \subseteq U$, hence the diagram (2.1) becomes trivial with all components isomorphic to $\{*\}$. Otherwise, if $V \not\subseteq U$ we have $V_i \not\subseteq U$ for some $i \in I$, this necessarily yields that both $h_U(V)$ and $\prod h_U(V_i)$ are isomorphic to $\emptyset$ and hence $p = q$ and e is an isomorphism. This then proves:

Proposition 28. *For any topological space X, and any open subset $U \subseteq X$ the representable presheaf h_U is a sheaf on X.*

Furthermore, since any open set U has a unique inclusion into X we have $h_X(U)$ is constant equal to $\{*\}$. Also given any set S there is a unique arrow $S \to \{*\}$, and hence for any sheaf (or presheaf) F, there is a unique map $F(U) \to h_X(U)$ which is trivially natural; this leads us to conclude that h_X is a terminal sheaf over X.

Now, another important construction in this case comes from considering, for any irreducible closed set $D \subseteq X$ the left adjoint to the functor $(-)^D : \boldsymbol{Set} \to \text{Shv}(X)$. Note that, to give a map from a sheaf F to a skyscraper sheaf is the same as giving, for every open set U intersecting D, a map $F(U) \to S$ which is invariant under restriction (to other open sets intersecting D). This is the same as giving a map to S out of the colimit:

$$\operatorname*{Colim}_{U \cap D \neq \emptyset} F(U) \qquad D \subseteq X \text{ irreducible closed set}$$

This leads to define the stalk of F at an irreducible closed set D to be $F_D = \operatorname*{Colim}_{U \cap D \neq \emptyset} F(U)$. Since colimits are functorial and $\boldsymbol{Set}$ is cocomplete, the map $F \mapsto F_D$, is a functor called the *stalk* at D, and we have:

Theorem 29. *For any irreducible closed set $D \subseteq X$ the functor of stalks at $\mathcal{D}$, $F \mapsto F_D$ is left adjoint to the skyscraper sheaf functor $S \mapsto S^D$.*

Another way of getting the stalk of a sheaf in a topological, rather than categorical way is to consider the following:

Given a sheaf F over X and an irreducible closed set, or a point D we say that sections $s \in F(U)$ and $s' \in F(U')$ define the same germ at D if there exists some non-empty open set $V \subseteq U \cap U'$ so that $V \cap D \neq \emptyset$ and $s|_V = s'|_V$. It is trivial to check that the relation $s \sim s'$ iff s, s' define the same germ, is an equivalence relation and we denote the equivalence class of s by $[s]_D$ (where $[s]_D$ should be read "the germ of s at D"). Finally the stalk F_D is given by the set

$$F_D := \bigcup_{\substack{s \in F(U) \\ U \cap D \neq \emptyset}} [s]_D$$

Intuitively, what the stalk does is to "catalog all possible behaviors of sections" at neighborhoods of the irreducible closed set or point D. The verification that these two definitions of stalk are equivalent is a straight forward exercise using the definition of the colimit in Set as a quotient of a disjoint union.

Chapter 3

Grothendieck Topologies

In this chapter we focus on Grothendieck topologies, these are generalisations of the usual concept of a topology where instead of dealing just with a poset of open sets, we deal with a category where we instil the notion of a covering family of morphisms. A covering family plays the role of an open cover in point set topology. This is done by first defining the notion of a sieve over an object, and then the notion of a covering sieve. Finally we describe the two Grothendieck topologies we will focus on and prove that they satisfy the axioms for a Grothendieck topology.

Definition. Given a small category $\mathcal{C}$, we define a *sieve*, S over some element $X \in \mathcal{C}$ as a set of morphisms with codomain X, chosen to satisfy the condition that, if $f \in S$ then $f \circ g \in S$ for any g in $\mathcal{C}$ so that the composition is defined.
We say that a family of morphisms in $\mathcal{C}$, $\{f_i : X_i \to X | i \in I\}$ *generates* S if

$$S = \bigcup_{i \in I} \{f_i \circ g \mid \mathsf{d}_1 g = X_i\}$$

Note that there a bijection between sieves S over X and subfunctors of the representable functor $h_X = \text{Mor}_{\mathcal{C}}(-, X)$.

Definition. Given a set of morphisms S with a common codomain X and a morphism $g : Y \to X$ we define the *pullback* of S along g as:

$$g^*(S) := \{f : Z \to Y \mid f \circ g \in S, Z \in \mathcal{C}\}$$

Note that in the case where S is a sieve on X, $g^*(S)$ is also a sieve on Y. To justify the nomenclature, we can obtain $g^*(S)$ as being the pullback $S \times_{h_X} h_Y$ when we regard S as a subfunctor of h_X and we map h_Y to h_X by composition on the left with g.
Finally we can define a Grothendieck topology on a category $\mathcal{C}$, which intuitively tells us how one can "cover" an object in the category with other objects, in much the same way as the topology on a topological space tells us how one can cover an open subset of the space with open subsets.

Definition. A *Grothendieck topology* on a category $\mathcal{C}$ is a map J that assigns to each object $X \in \mathcal{C}$ a collection of *covering sieves* $J(X)$ which are sieves on X satisfying:

1. For all $X \in \mathcal{C}$, $h_X \in J(X)$.
2. If S is a sieve in $J(X)$ and $h : Y \to X$ is an arrow in $\mathcal{C}$, then $h^*(S) \in J(Y)$.
3. If $S \in J(X)$ and R is a sieve on X, so that for every arrow $h : Y \to X \in S$, $h^*(R) \in J(Y)$, then $R \in J(X)$.

Alternatively we can define a pretopology or a base for a Grothendieck topology:

Definition. A *Grothendieck pretopology* on a category $\mathcal{C}$, or a *Grothendieck coverage*[1] is a map J' where for each $X \in \mathcal{C}$, $J'(X)$ is a collection of *covering families* of X, which are families of morphisms $\{f_i : X_i \to X \mid i \in I\}$ satisfying:

1. If $\phi : X' \to X$ is an isomorphism then $\{\phi\} \in J'(X)$.
2. If $\{f_i : X_i \to X\} \in J'(X)$ then for any $g : Y \to X$ in $\mathcal{C}$ there exists $\{h_l : Y_l \to Y\} \in J'(Y)$ such that each of the $g \circ h_j$ factor through some f_i.
3. If $\{f_i : X_i \to X \mid i \in I\} \in J'(X)$ and $\{f_{i,j} : X_{i,j} \to X_i \mid j \in I_i\} \in J'(X_i)$ for each $i \in I$ then $\{f_i \circ f_{i,j} \mid i, \in I, j \in I_i\} \in J'(X)$.

Proposition 30. *Given a Grothendieck pretopology J', the map J that, to each $X \in \mathcal{C}$, assigns the collection of those sieves on X which contain some covering family in $J'(X)$ is a Grothendieck topology on $\mathcal{C}$. J is the* topology generated by the Grothendieck pretopology J'.

Proof. Let $\mathcal{C}$ be a category and J' a pretopology on $\mathcal{C}$, we want to check that the family of sieves $J(X)$ defined as those sieves containing some covering family in $J'(X)$ forms a topology on $\mathcal{C}$.

1. Any sieve S on X containing and isomorphism $\phi : X' \to X$, also contains the identity $1_X = \phi\phi^{-1}$, and thus is the maximal sieve.
2. If S is a sieve on X containing a covering family $\{f_i : X_i \to X\}$, then let $g : Y \to X$ be an arrow in $\mathcal{C}$ and $\{h_l : Y_l \to Y\} \in J'(Y)$ be a covering family such that each of the $g \circ h_l$ factor through some f_i, then $g^*(S)$ contains $\{h_l : Y_l \to Y\}$, and so is a covering sieve.
3. If S contains $\{f_i : X_i \to X\} \in J'(X)$ and R is a sieve such that for every i, $f_i^*(R)$ contains a covering family $\{f_{i,j} : X_{i,j} \to X_i\} \in J'(X_i)$ then R contains $\{f_i \circ f_{i,j}\}$, which is a covering family, hence R is a covering sieve.

Thus J is a Grothendieck topology on $\mathcal{C}$. □

We call a pair $(\mathcal{C}, J)$ of a category with a topology on it, a *site*, or a *Grothendieck site*, and we often omit the topology and simply refer to the category $\mathcal{C}$ as the site $\mathcal{C}$ (although it should be kept in mind that in such situations there is an implied topology).

[1] The more general notion of coverage admits only the second axiom.

Now note that given a site $(\mathcal{C}, J)$ and any object $X \in \mathcal{C}$, $J(X)$ has a natural structure of category: It is the subcategory of subfunctors of h_X with morphisms the inclusion morphisms.

Lemma 31. *For any object $X \in \mathcal{C}$ and J a topology on $\mathcal{C}$, both $J(X)$ and $J(X)^{op}$ are filtered categories.*

Proof. Since h_X is a terminal object in $J(X)$, $J(X)$ is necessarily filtered.
Now since given two sieves S, S' in $J(X)^{\mathsf{op}}$ there exists at most one morphism between them, the second axiom of the filtration is trivial, furthermore since for any two covering sieves S, S', $S \cap S'$ is a covering sieve and we have arrows $S \to S \cap S' \leftarrow S'$ in $J(X)^{\mathsf{op}}$, $J(X)^{\mathsf{op}}$ satisfies the first axiom of a filtration. □

3.1 The Zariski and Étale Sites

In this section we aim to describe the topologies we will be focusing on. These are topologies, or sites, one can associate to a topological space, one is usually referred as the small site of a topological space, and the other one is known as the étale site.

Given a set X and a family of functions $\{X_i \xrightarrow{p_i} X \mid i \in I\}$ we say that this family is a *surjective family of functions* if $X = \bigcup_{i \in I} p_i(X_i)$.
More generally for an arbitrary category with coproducts we say that the family of morphisms $\{f_i : X_i \to X \mid i \in I\}$ is a *jointly epimorphic family of morphisms* if the induced morphism $\coprod_i f_i : \coprod_i X_i \to X$ is an epimorphism.

Definition. Given a topological space X, the *Zariski site* $X_{\mathbf{Zar}}$ on X is the site where:

- The underlying category is $O(X)$, the poset of open subsets of X (ordered by inclusion).
- The Grothendieck topology is generated by surjective families of inclusions $\{U_i \subseteq U \mid i \in I\}$.

Note that to give a surjective family of inclusions of open sets into an open set U, is the same as to give an open cover of U.
Recall that a continuous map $f : Y \to X$ is étale if for every point $y \in Y$ there exist open neighborhoods U of y and V of $f(y)$, so that $f|_U$ is a homeomorphism onto V.

Definition. Given a topological space X, we define the *étale site* $X_{\mathbf{\acute{E}t}}$ on X where:

- The underlying category, $\mathbf{\acute{E}t}(X)$ is the subcategory of $\boldsymbol{Top}/X$ whose objects are étale maps to X.
- The topology is generated by surjective families of étale morphisms.

Now for the sake of completeness we include the proof that both the Zariski and étale sites are indeed sites:

Proof. (That $X_{\mathbf{Zar}}$ is a site) We prove that the open covers form a Grothendieck pretopology:

1. For any open set U the inclusion of U into itself is an open cover.

2. Let $\{U_i \subseteq U \mid i \in I\}$ be an open cover of U. If $V \subseteq U$ is an open inclusion, then since the $U_i \subseteq U$ cover U, the $U_i \cap V$ cover V and for each i the inclusion $V \cap U_i \subseteq U$ is the same as $U_i \cap V \subseteq U_i \subseteq U$, hence factors through $U_i \subseteq U$.

3. Let $\{U_i \subseteq U \mid i \in I\}$ be an open cover of U and for each i let $\{U_{i,j} \subseteq U_i \mid j \in I_i\}$ be an open cover of U_i, then $\{U_{i,j} \subseteq U \mid i \in I, j \in I_i\}$ is a family of open inclusions satisfying $\bigcup_{i\in I} \bigcup_{j \in I_i} U_{i,j} = \bigcup_{i \in I} U_i = U$, and thus is an open cover.

□

Proof. (That $X_{\mathbf{\acute{E}t}}$ is a site) We again prove that these covers satisfy the axioms for a Grothendieck pretopology:

1. For any Y étale space over X, an isomorphism $\phi : Y' \to Y$ is, in particular, a surjective étale map, and so $\{\phi\}$ is a cover.

2. Let $\{f_i : Y_i \to Y \mid i \in I\}$ be an étale cover of Y, and $g : Z \to Y$ an étale map, then for some point $z \in Z$, let U_z be an open neighborhood of z so that $g|_{U_z}$ is an open embedding. Because the $f_i : Y_i \to Y$ cover Y, $g(z) \in f_i(Y_i)$ for some i, so choose $y \in Y_i$ so that $f_i(y) = g(z)$. Since f_i is étale, there exists an open neighborhood V_y of y so that $f_i|_{V_y}$ is an open embedding, and so consider the inclusion $h : f_i(V_y) \cap g(U_z) \hookrightarrow Z$ into a neighborhood of z, $g \circ h$ factors through f_i, and since we can do this for every $z \in Z$ this forms a cover of Z by étale maps.

3. Let $\{f_i : Y_i \to Y \mid i \in I\}$ and $\{f_{i,j} : Y_{i,j} \to Y_i \mid j \in I_i\}_{i \in I}$ be étale covers, since the composition of étale maps is étale, the maps $f_i \circ f_{i,j}$ are étale, and because the f_i and the $f_{i,j}$ are jointly surjective families we have $\bigcup_{i\in I} \bigcup_{j\in I_i} f_i \circ f_{i,j}(Y_{i,j}) = \bigcup_{i\in I} f_i(\bigcup_{j\in I_i} f_{i,j}(Y_{i,j})) = \bigcup_{i\in I} f_i(Y_i) = Y$.

□

Chapter 4

Introduction to The Theory of Sheaves

In this chapter we generalise the notions developed in Section 2.3 to include sheaves over a site, as defined in Chapter 3. We then proceed to describe how in both cases we can associate a sheaf to any presheaf, and we end the chapter by analysing the relationship between the categories of sheaves over the sites defined in Section 3.1 whereupon we conclude these are equivalent notions.

Recall that a functor $F : \mathcal{C}^{\mathrm{op}} \to \boldsymbol{Set}$ (i.e. a contravariant functor from $\mathcal{C}$ to $\boldsymbol{Set}$) is called a *presheaf* (of sets) on $\mathcal{C}$, and since we are considering presheaves with values in $\boldsymbol{Set}$, then for every object x of $\mathcal{C}$ we have a notion of element $s \in F(x)$, which we call, as in Section 2.3 *sections* of F over x. Furthermore any arrow $f : x \to y$ in $\mathcal{C}$ defines a *restriction* map along f, $F(f) : F(y) \to F(x)$.
When instead of just a category we take a Grothendieck site, we are led to the definition of a sheaf of sets on $\mathcal{C}$ as being a presheaf F where all sections over an object, arise from "gluing together compatible sections" over the coverings of this object. This idea is made explicit in the following definition.

Definition. Let $\mathcal{C}$ be a category and J a topology on $\mathcal{C}$, we say that a presheaf F on $\mathcal{C}$ is a sheaf, if for every $x \in \mathcal{C}$ and every sieve $S \in J(x)$ we have:

$$F(x) \cong \operatorname*{Lim}_{f \in S} F(\mathsf{d}_0 f) \tag{4.1}$$

If we assume that $\mathcal{C}$ has pullbacks then we can reformulate the sheaf condition in terms of covering families instead of sieves:

Definition. Let $\mathcal{C}$ be a category, F a presheaf on $\mathcal{C}$, and S a sieve on some object x we say that the family of sections $(s_f)_{f \in S} \in \prod_{f \in S} F(\mathsf{d}_0 f)$ is *S-compatible* if for every composable pair f, g with $f \in S$, we have $s_{f \circ g} = F(g)(s_f)$
If $\mathcal{C}$ is a category with pullbacks and $\{f_i : x_i \to x\}$ is a covering family then we say that (s_i) is a *compatible family of sections* if for every pair i, j, $F(\pi^1_{i,j})(s_i) = F(\pi^2_{i,j})(s_j)$ where the $\pi^1_{i,j}, \pi^2_{i,j}$ are the projections from the pullback $x_i \times_x x_j$ to x_i and x_j, respectively.

Proposition 32. *Let $\mathcal{C}$ be a category with pullbacks and J' a pretopology on $\mathcal{C}$, then a presheaf F over*

$\mathcal{C}$ is a sheaf, if and only if for any object $x \in \mathcal{C}$ and every covering family $S \in J'(x) = \{x_i \xrightarrow{\alpha_i} x \mid i \in I\}$, the following diagram is an equalizer:

$$F(x) \xrightarrow{e} \prod_{i\in I} F(x_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j\in I} F(x_i \times_x x_j) \tag{4.2}$$

Where we let $\beta_{ij} : x_i \times_x x_j \to x_i$ and $\beta'_{ij} : x_i \times_x x_j \to x_j$ and the maps are defined by:

- $\pi_i \circ e = F(\alpha_i) : F(x) \to F(x_i)$.
- $\pi_{ij} \circ p = F(\beta_{ij}) \circ \pi_i : \prod_{i\in I} F(x_i) \to F(x_i \times_x x_j)$.
- $\pi_{ij} \circ p = F(\beta'_{ij}) \circ \pi_j : \prod_{i\in I} F(x_i) \to F(x_i \times_x x_j)$.

Remark 33. Recall that an open cover of an open set U in a topological space X is the same as a covering family in $X_{\mathbf{Zar}}$ for the object U (on the Zariski site), and that the pull-back of open sets in the underlying category of $X_{\mathbf{Zar}}$ coincides with the intersection, thus when we consider a sheaf over the site $X_{\mathbf{Zar}}$ we recover the definition of a sheaf given in Section 2.3.

Proof. Let $\mathcal{C}$ be a category, J' a Grothendieck pretopology on $\mathcal{C}$ and J the topology generated by J', and let S be the sieve on x generated by the covering family $\{f_i : x_i \to x \mid i \in I\}$. We want to show that $\mathrm{eq}\left(\prod_{i\in I} F(x_i) \rightrightarrows \prod_{i,j\in I} F(x_i \times_x x_j)\right)$ and $\underset{f\in S}{\mathrm{Lim}}\, F(\mathsf{d}_0 f)$ are canonically isomorphic.
Recall from the proof of Theorem 15, that the limit (4.1) can be described as the equalizer of:

$$\prod_{f\in S} F(\mathsf{d}_0 f) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{f, f\circ g\in S} F(\mathsf{d}_0 f \circ g) \tag{4.3}$$

Where, denoting the projections onto the component $(f, f\circ g)$ by $\pi_{f,g}$, we set $\pi_{f,g} \circ p = \pi_{f\circ g}$ and $\pi_{f,g} \circ q = F(g) \circ \pi_f$. So in particular consider the pullbacks of the generators f_i:

$$\begin{array}{ccc} x_i \times_x x_j & \xrightarrow{\pi^2_{i,j}} & x_j \\ \downarrow{\scriptstyle \pi^1_{i,j}} & & \downarrow{\scriptstyle f_j} \\ x_i & \xrightarrow{f_i} & x \end{array}$$

Thus let $(s_f)_{f\in S} \in \prod_{f\in S} F(\mathsf{d}_0 f)$ be an element of the equaliser of (4.3), then because $f_j \circ \pi^2_{i,j} = f_i \circ \pi^1_{i,j}$ we have:

$$F(\pi^1_{i,j})(s_{f_i}) = s_{f_i\circ\pi^1_{i,j}} = s_{f_j\circ\pi^2_{i,j}} = F(\pi^2_{i,j})(s_{f_j})$$

Which implies (s_{f_i}) is in the equaliser of diagram (4.2). So we only need to prove that each $(s_i)_{i\in I}$ in the equaliser of (4.2) corresponds to a unique element $(s'_f)_{f\in S} \in \prod_{f\in S} F(\mathsf{d}_0 f)$ that equalises the diagram (4.3) and satisfies $s'_{f_i} = s_i$ for all $i \in I$. Then let $(s_i)_{i\in I}$ be an element of the equaliser, and define for each $g : y \to x_i$, $s'_{f_i\circ g} = F(g)(s_{f_i})$, this map is well defined because, if for some pair i, j we have morphisms g_i, g_j so that $f_i \circ g_i = f_j \circ g_j$ then there exists a unique map $g : \mathsf{d}_0 g_i \to x_i \times_x x_j$ such that $\pi^1_{i,j} \circ g = g_i$ and $\pi^2_{i,j} \circ g = g_j$ and thus:

$$\begin{aligned} s_{f_i \circ g_i} &= F(g_i)(s_{f_i}) \\ &= F(\pi^1_{i,j} \circ g)(s_{f_i}) \\ &= F(g)(F(\pi^1_{i,j})(s_{f_i})) \\ &= F(g)(F(\pi^2_{i,j})(s_{f_j})) \\ &= F(g_j)(s_{f,j}) \\ &= s_{f_j \circ g_j} \end{aligned}$$

Now $(s_f)_{f \in S}$ satisfies $s_{f\circ\ g} = F(g)(s_f)$ for every $f, f \circ g \in S$, thus it is in the equaliser of (4.3). Furthermore since S is generated by $\{f_i : x_i \to x\}$ every morphism of S is of the form $f_i \circ\ g$ for some $i \in I$ and some g composable with f_i, so any two elements $(s_f)_{f\in S}, (s'_f)_{f\in S}$ that project onto the same $(s_i)_{i\in I}$ satisfy $s_{f_i \circ g} = F(g)(s_i) = F(g)(s'_i) = s'_{f_i \circ g}$. □

We denote PSh($\mathcal{C}$) the category of contravariant functors from $\mathcal{C}$ to $\boldsymbol{Set}$ (i.e. presheaves on $\mathcal{C}$) and Shv($\mathcal{C}, J$) the (full) subcategory of PSh($\mathcal{C}$) consisting of sheaves on the site $(\mathcal{C}, J)$ and natural transformations between them[1]. Furthermore, given a topological space X, we write, just as in Section 2.3, PSh(X) and Shv(X) to mean presheaves or sheaves on the Zariski site associated to X, $X_{\mathbf{Zar}}$ discussed in Section 3.1.

Recall that if $\mathcal{C}$ is locally small, we call a contravariant functor representable if it is isomorphic to some functor of the form Mor$_{\mathcal{C}}(-, x)$ with $x \in \mathcal{C}$. A site is said to be *subcanonical* if all representable presheaves are sheaves.

Proposition 34. *The Zariski and étale sites associated to a topological space X are subcanonical.*

Proof. The fact that the Zariski site is subcanonical was proved in Proposition 28 so it only remains to show the same result for the étale site $X_{\mathbf{\acute{E}t}}$.

Let $q : Z \to X$ be an étale map, we want to check that h_q is a sheaf. So given an étale map $p : Y \to X \in \mathbf{\acute{E}t}(X)$, and $\{f_i : Y_i \to Y \mid i \in I\}$ a cover of Y and write $Y_{ij} := Y_i \times_Y Y_j$ (where the associated étale maps are the obvious ones induced by p and denoted p_i and p_{ij}). We start by showing that the map $e : h_q(p) \to \prod_{i\in\ I} h_q(p_i)$ is injective.

Let $g, g' \in h_q(p) =$ Mor$_{\mathbf{\acute{E}t}(X)}(Y, Z)$ be such that $e(g) = e(g')$ then for every $i \in I$, $g \circ f_i = g' \circ f_i$, but since the f_i cover Y, for every $y \in Y$ $g(y) = g'(y) \Rightarrow g = g'$.

Now let $\pi^1_{i,j} : Y_{ij} \to Y_i$ and $\pi^2_{i,j} : Y_{ij} \to Y_j$ be the canonical projections of the pullback onto its components, and let $(g_i)_{i\in I} \in \prod_{i\in I} h_q(p_i)$ satisfy $g_i \circ \pi^1_{ij} = g_j \circ \pi^2_{ij}$. For any $y \in Y$ such that $y \in f_i(Y_i) \cap f_j(Y_j)$ there exists a point $y' \in Y_{ij}$ such that $f_i \circ \pi^1_{ij}(y') = f_j \circ \pi^2_{ij}(y') = y$ and hence $g_i(f_i^{-1}(y)) = g_j(f_j^{-1}(y))$, so the function $g : Y \to Z$ defined for each $y \in Y$ by $g_i(f_i^{-1}(y))$ if $y \in f_i(Y_i)$ is well defined and satisfies $F(f_i)(g) = g_i$ for all $i \in I$. □

[1] Note that the image of a sheaf under a natural transformation need not to be a sheaf.

4.1 Sheafification of Presheaves

In general we have a canonical way of constructing a sheaf associated to any given presheaf F. This can be done in a particularly simple way by using étale spaces, and we will briefly describe this approach following [Moe92]. However this does not readily generalise to presheaves over a general Grothendieck site, thus we need a more general definition of sheafification which we will describe in Section 4.2.

Definition. Given a presheaf F on a topological the étale space $E(F)$ over X associated to F is:

$$E(F) = \coprod_{p \in X} F_p$$

With basis for the topology given by open sets $V(s, U)$ with $U \subseteq X$ open and $s \in F(U)$ defined as:

$$V(s, U) = \{ \ [s]_p \in F_p \mid p \in U \ \}$$

It's trivial to see that this is an étale space, since every point of $E(F)$ is a germ of a section s at some point p, thus there exists some open set $U \subseteq X$ containing p, on which s is defined, whence the (basic) open set $V(s, U)$ is homeomorphic to U.

Example 35. Consider the skyscraper sheaf over the reals $\mathbf{2}^{\{0\}}$ it has two sections over every open set containing the origin and one section over every other. In particular it has two global sections which have the same germ at every point $p \neq 0$ but are different when restricted to neighborhoods of 0. The étale space is given by two real lines glued at every point except the origin, with the quotient topology, this is the line with two origins discussed in Example 24.

This construction provides a functor $E : \mathsf{PSh}(X) \to \mathbf{\acute{E}t}(X)$. There is a functor in the opposite direction that associates to an étale space $Y \xrightarrow{p} X$ the presheaf of continuous sections:

$$\mathscr{F}(Y)(U) = \{ f : U \hookrightarrow Y \text{ continuous} \mid p \circ f = 1_U \}$$

Clearly this presheaf is, in fact, a sheaf since giving a continuous function from an open set U is the same as giving a continuous function from each of the open sets U_i of any open cover which agree on the intersections.

Now when we restrict our attention to the category of sheaves we have (natural) isomorphisms:

$$\eta : \mathsf{Id}_{\mathsf{Shv}(X)} \cong \mathscr{F} \circ E \qquad \varepsilon : E \circ \mathscr{F} \cong \mathsf{Id}_{\mathbf{\acute{E}t}(X)}$$

Given by $(\eta_F)_U(s) = (x \mapsto [s]_x)$ and $\varepsilon_Y([s]_x) = s(x)$, which proves the following theorem:

Theorem 36. *Given a topological space X, the categories $\mathbf{\acute{E}t}(X)$ and Shv(X) are equivalent.*

Definition. Given a presheaf $F \in \mathsf{PSh}(X)$ then define $F^{\#}$ the *sheaf associated* to the presheaf F, or the *sheafification* of F as:

$$F^{\#} = \mathscr{F} \circ E(F)$$

The map $F \mapsto F^{\#}$ is a functor since it is the composition of two functors.

Theorem 37. *The functor $F \mapsto F^{\#}$ is a left adjoint to the inclusion Shv$(X) \subseteq$ PSh(X).*

Proof. To prove the adjunction we have to prove the bijection

$$\mathsf{Mor}_{\mathsf{Shv}(X)}(F^{\#}, G) \cong \mathsf{Mor}_{\mathsf{PSh}(X)}(F, G)$$

To this effect, notice that we have a canonical morphism from $\iota : F \to F^{\#}$ given by $\iota(s)(x) = [s]_x$ for each $s \in F(U)$ and $x \in U$. It suffices to show that any morphism $\alpha : F \to G$ factors uniquely as $\alpha = \alpha' \circ \iota$.
Recall the obstructions for a presheaf F to be a sheaf:

- There exists an open cover $\{U_i \subseteq U \mid i \in I\}$ and a pair of sections $s \neq s' \in F(U)$ so that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$.
- There exists compatible family of sections $s_i \in F(U_i)$ so that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j and yet there exists no $s \in F(U)$ so that $s|_{U_i} = s_i$ for all i.

Now for any pair of sections $s, s' \in F(U)$ satisfying $s|_{U_i} = s'|_{U_i}$ we have:

$$\alpha(s)|_{U_i} = \alpha(s|_{U_i}) = \alpha(s'|_{U_i}) = \alpha(s')|_{U_i}$$

And so, since G is a sheaf $\alpha(s) = \alpha(s')$. Also, for any compatible family of section $s_i \in F(U_i)$, the family $\alpha(s_i)$ is a compatible family of sections and thus there exists a unique section $\hat{s} \in G(U)$ so that $\hat{s}|_{U_i} = \alpha(s_i)$.
Finally let $s' \in F^{\#}(U)$, and recall that s' is a continuous function $U \to E(F)$, then either there exists some $s \in F(U)$ satisfying $[s]_x = s'(x)$ for all $x \in U$, in which case $s' = \iota(s)$ and we define $\alpha'(s') = \alpha(s)$; or there exists an open cover $\{U_i \subseteq U \mid i \in I\}$ and a compatible family of sections $s_i \in F(U_i)$ such that $\iota(s_i) = s'|_{U_i}$, in which case we define $\alpha'(s') = \hat{s}$ where $\hat{s}|_{U_i} = \alpha(s_i)$. The uniqueness of the choice of $\hat{s}$ guarantees that α' is well defined, and so we get that any $\alpha : F \to G$ factors through $F^{\#}$.
Now if $\alpha \neq \beta$ are two natural transformations $F \to G$ then for some open set U, and some section $s \in F(U)$, $\alpha_U(s) \neq \beta_U(s)$ hence if we let s' denote the section $x \mapsto [s]_x$, then by definition $\alpha'_U(s') \neq \beta'_U(s')$, thus $\alpha' \neq \beta'$, establishing the required bijection. □

4.2 Sheafification of Presheaves (cont.) The + Construction

The + construction is a way to construct a sheaf out of a presheaf in the general sense of presheaves over a site. This construction is summarily described in [Jar87], [Moe92] and in [Joh02b] (pp. 551–552).

To understand this construction, we need to establish yet more terminology.

Definition. A presheaf $F \in \mathsf{PSh}(\mathcal{C})$ is said to be a *separated presheaf* if for every covering sieve $R \in J(X)$ the canonical map:

$$F(X) \to \operatorname*{Lim}_{f \in R} F(\mathsf{d}_0 f)$$

Is a monomorphism.

Now, given a presheaf $F \in \mathsf{PSh}(\mathcal{C})$ and a sieve R over $X \in \mathcal{C}$ define:

$$F(X)_R := \operatorname*{Lim}_{U \to X \in R} F(U)$$

Where R is seen as a subcategory of $\mathcal{C} \downarrow X$, note that an element of $F(X)_R$ corresponds to an R-compatible family of sections of F. With this said we define F^+ as:

$$F^+(X) := \operatorname*{Colim}_{R \in J(X)^{\mathsf{op}}} F(X)_R$$

Which is a presheaf since for any morphism $g : Y \to X$ and any sieve $R \in J(X)$ we have an obvious map $F(X)_R \to F(Y)_{g^*(R)}$.

Lemma 38. *For any presheaf F on the site $(\mathcal{C}, J)$ the presheaf F^+ defined above, is a separated presheaf.*

Proof. Note that F^+ is separated if and only if for any covering sieve S on X and any S-compatible family of sections $(s_f)_{f \in S}$ there exists at most one section $s \in F^+(X)$ so that $F^+(f)(s) = s_f$ for all $f \in S$. So let $\hat{s}, \hat{s}'$ be sections of $F^+(X)$ that agree on every restriction along some covering sieve S. This means that if we let $(s_f)_{f \in R} \in F(X)_R, (s'_f)_{f \in R'} \in F(X)_{R'}$ be representatives of $\hat{s}$ and $\hat{s}'$ respectively, then for each $g \in S$ there exists a covering sieve on $S_g \subseteq g^*(R) \cap g^*(R')$ on $\mathsf{d}_0 g$ so that for all $h \in S_g$, $s_{g \circ h} = s'_{g \circ h}$. Now since S_g covers $\mathsf{d}_0 g$ for all $g \in S$ then $S' = \bigcup_{g \in S} g_*(S_g)$ is a covering sieve on X contained in $R \cap R'$. Thus if we denote $s'' = (s_f)_{f \in S'} \in F(X)_{S'}$, $\hat{s}''$ the equivalence class of s'' in $F^+(X)$ and $\pi_{R,S'} : F(X)_R \to F(X)_{S'}$ the morphism induced by the inclusion morphism of $S'' \subseteq R$ then:

$$\pi_{R,S'}((s_f)_{f \in R}) = \pi_{R',S'}((s_f)_{f \in R}) = (s_f)_{f \in S'}$$

And so $\hat{s} = \hat{s}' = \hat{s}''$, hence F^+ is separated. ☐

Lemma 39. *If F is any separated presheaf then F^+ is a sheaf.*

Proof. Let $S = \{f : X_f \to X\}$ be a covering sieve on X, and let $(s_f)_{f \in S}$ and $(s'_f)_{f \in S'}$ be representatives of the same element in F^+, then since for any $f \in S \cap S'$ $s_{f \circ g} = s'_{f \circ g}$ for all g in some covering sieve of $\mathsf{d}_0 f$ and F is separated, $s_f = s'_f$ for all $f \in S \cap S'$. Thus each element $\hat{s} \in F^+(X)$ has a canonical representative $(\tilde{s}_f)_{f \in S_s}$ where S_s is the union of all covering sieves S such that $\hat{s}$ has a representative in $F(X)_S$.

Now let R be a covering sieve on X and $(s_f)_{f \in R}$ be an R-compatible family of sections of F^+, and for each f let r^f be the canonical representative of s_f then by the same reasoning as before $\tilde{r}^f_g = \tilde{r}^{f'}_{g'}$

whenever $f \circ g = f' \circ g'$, and thus let $S' = \bigcup_{f \in R} f_*(S_{s_f})$, $(\tilde{r}^f_g)_{f \in R, g \in S_{s_f}}$ is an S'-compatible family of morphisms of F and thus an element of $F(X)_{S'}$ whose equivalence class restricts to $(s_f)_{f \in R}$. □

Theorem 40. *Given any presheaf F the presheaf $F^\# := F^{++}$ is a sheaf, and furthermore $\#$ is a left exact left adjoint to the inclusion $i_{\mathit{Shv}(\mathcal{C})} : \mathit{Shv}(\mathcal{C}) \to \mathit{PSh}(\mathcal{C})$*

Proof. The functor $F \mapsto F^+$ is left exact since it is a filtered colimit (by Lemma 31), so necessarily $F \mapsto F^\# = F^{++}$ is left exact.

Now it's also evident from the preceding results that $F \mapsto F^\#$ is a functor $\mathrm{PSh}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$. So it remains to show the adjunction. There exists a canonical map $\iota : F \to F^\#$, and we only need to check that the map:

$$\iota^* : \mathrm{Mor}_{\mathrm{Shv}(\mathcal{C})}(F^\#, G) \to \mathrm{Mor}_{\mathrm{PSh}(\mathcal{C})}(F, G)$$

Defined as $\iota^*(\mu) = \mu \circ \iota$ is a bijection.

Let $F \in \mathrm{PSh}(\mathcal{C})$, $G \in \mathrm{Shv}(\mathcal{C})$, and $\alpha : F \to G$ be a natural transformation; then, given any pair of sections $s, s' \in F(X)$ and a covering sieve $S = \{f : U_f \to X\}$ so that $s|_f = s'|_f$ for all $f \in S$:

$$\alpha_X(s)|_f = \alpha_{U_f}(s|_f) = \alpha_{U_f}(s'|_f) = \alpha_X(s')|_f$$

Whence, since G is a sheaf, and S is a covering sieve, $\alpha_X(s) = \alpha_X(s')$. Also, because this is true for any sieve, we conclude that for every $X \in \mathcal{C}$ and every sieve $R \in J(X)$, α_X factors uniquely through $F(X)_R$. And since F^+ is the colimit of the $F(X)_R$, α factors uniquely through F^+, yielding the intended bijection. □

4.3 Zariski and Étale Sheaves

Given a topological space we have several "natural" sites we may consider which may give different notions of sheaf. In the present work we will be concerned with just two of them, the Zariski and Étale sites introduced in Section 3.1, and the sheaves over them.

Giving a Zariski sheaf is, by definition, the same as giving a sheaf over a topological space, whereas giving an étale sheaf requires, in all likelihood, more information. One might ask oneself if there is a way to relate these two concepts of sheaf, in the present section we show that there is and explicitly give an equivalence between the two sheaf categories.

As we noted before every inclusion map $U \subseteq X$ is an étale map, and so every open subset of X can be viewed as an étale space over X. This provides us with an inclusion functor $O(X) \to \mathbf{\acute{E}t}(X)$ where we send an open subset $U \subseteq X$ to the inclusion map $U \subseteq X$.

Remark 41*.* The same line of thought that proves $\mathbf{\acute{E}t}(X)$ is a full subcategory of $\boldsymbol{Top}/X$ proves that the inclusion functor $i : O(X) \to \mathbf{\acute{E}t}(X)$ is a fully faithful functor.

Let $F : \mathbf{\acute{E}t}(X)^{\mathsf{op}} \to \boldsymbol{Set}$ be an étale sheaf then, noting that every open cover is also an étale cover, F restricts to a sheaf on the subcategory $i(O(X)) \subseteq \mathbf{\acute{E}t}(X)$. This implies that the inclusion functor i defines a functor:

$$i^* : \mathsf{Shv}(X_{\mathbf{\acute{E}t}}) \to \mathsf{Shv}(X_{\mathbf{Zar}}) \qquad \text{given by} \qquad F \mapsto F \circ i$$

Now in Section 1.7 we saw how to push a functor from a category $\mathcal{M}$ forward along a functor $i : \mathcal{M} \to \mathcal{C}$, and we saw that we had two canonical ways of doing this: The right and left Kan extensions.
As we have also seen in Section 1.7, a left Kan extension of a functor $F : O(X)^{\mathsf{op}} \to \boldsymbol{Set}$ along $i^{\mathsf{op}} : O(X)^{\mathsf{op}} \to \mathbf{\acute{E}t}(X)^{\mathsf{op}}$ can be computed on an étale space $p : Y \to X$ by calculating the colimit over the comma category $(O(X)^{\mathsf{op}} \downarrow Y)$. The elements of this category are of course maps $f : U \to Y$ in $\mathbf{\acute{E}t}(X)^{\mathsf{op}}$ so that the following diagram commutes in $\mathbf{\acute{E}t}(X)^{\mathsf{op}}$:

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & Y \\ & \searrow \quad \swarrow_{p} & \\ & X & \end{array} \tag{4.4}$$

But since this is the opposite category to $\mathbf{\acute{E}t}(X)$, f is an étale map $\hat{f} : Y \to U$. Substituting f by $\hat{f}$ in diagram (4.4) (and inverting the direction of this arrow) we get that $f(Y) \subseteq U$ and $f(Y) = p(Y)$, whence we conclude $p(Y) \subseteq U$. Thus the formula for the left Kan is:

$$i^K(F)(Y) = \underset{U \in (O(X)^{\mathsf{op}} \downarrow Y)}{\mathsf{Colim}} F(U) = \underset{U \supseteq p(Y)}{\mathsf{Colim}} F(U) \cong F(p(Y))$$

Where the last isomorphism is derived from the fact that the inclusion $p(Y) \subseteq X$ is the terminal object of the subcategory of $\mathbf{\acute{E}t}(X)^{\mathsf{op}}$ comprised of open sets containing $p(Y)$.
The procedure above, however, only gives us a presheaf which we can readily observe is not (in general) a sheaf:

Example 42. For any space X, let X_1 and X_2 be copies of X, and F a Zariski sheaf with 2 or more global sections, then consider the étale space $X_1 \coprod X_2 \to X$ with the obvious projection map. Now consider the cover of $X_1 \coprod X_2$ given by the inclusions $f_i : X_i \hookrightarrow X_1 \coprod X_2$, with $i = 1, 2$. The pullback of f_1 and f_2 is empty, and so the pair of arrows from the sheaf condition (4.2) for this cover of X becomes:

$$i^K(F)(X_1) \times i^K(F)(X_2) \rightrightarrows i^K(F)(X_1) \times i^K(F)(\emptyset) \times i^K(F)(\emptyset) \times i^K(F)(X_2)$$

However $i^K(F)(X_1 \coprod X_2) = F(X)$ while the equalizer of the diagram above is $F(X) \times F(X)$.

The general reason why the sheaf condition does not hold for $i^K(F)$ is the following: Let $f_i : Y_i \to Y$ be a cover, where the Y_i are equipped with projection maps p_i, Y has a projection p and $Y_i \times_Y Y_j$ have projections $p_{i,j}$, and consider the sheaf condition for $i^K(F)$ and this cover:

$$\prod i^K(F)(Y_i) \rightrightarrows \prod i^K(F)(Y_i \times_Y Y_j)$$

Depending on F the equalizer of this diagram will in general, be different from $i^K(F)(Y) \cong F(p(Y))$ because usually $q_{i,j}(Z_i \times_Y Z_j) \ncong q_i(Z_i) \cap q_j(Z_j)$ and so $i^K(F)(Z_i \times_Y Z_j) \cong F(q_i(Z_i) \cap q_j(Z_j))$.
In order to obtain the left adjoint of the functor i^* we resort to sheafification, but since sheafification is itself left adjoint to the inclusion Shv($X_{\textbf{Ét}}$) $\hookrightarrow$ PSh($X_{\textbf{Ét}}$), then the functor:

$$i_* := \# \circ i^K : \text{Shv}(X_{\mathbf{Zar}}) \to \text{Shv}(X_{\textbf{Ét}})$$

is a left adjoint to i^*. Since i^K is left exact (because it can be described point-wise as a filtered colimit) we have proven the theorem:

Theorem 43. *The inclusion map $i : O(X) \to \textbf{Ét}(X)$ induces an adjunction:*

$$i_* \dashv i^* : \textit{Shv}(X_{\mathbf{Zar}}) \rightleftarrows \textit{Shv}(X_{\textbf{Ét}})$$

Where i_* is left exact.

Theorem 44. *Any sheaf $F \in$ Shv(X) is a colimit of representable sheaves.*

Proof. Recall the equivalence of categories discussed in Theorem 36:

$$\textbf{Ét}(X) \underset{E}{\overset{\mathscr{F}}{\rightleftarrows}} \text{Shv}(X)$$

Note that every étale space $Y \xrightarrow{p} X$ is isomorphic to the colimit of $\coprod_{i,j} U_{ij} \rightrightarrows \coprod_i U_i$ where $\{U_i \hookrightarrow Y\}$ is an open cover of Y by open sets so that $p|_{U_i}$ is an open embedding and $U_{ij} = U_i \cap U_j$.
Since $\mathscr{F}$ is cocontinuous and any sheaf F is isomorphic to $\mathscr{F}(Y)$ for some Y (as it is an equivalence), then F is isomorphic to the coequalizer of $\coprod_{i,j} \mathscr{F}(U_{ij}) \rightrightarrows \coprod_i \mathscr{F}(U_i)$. This is the coequalizer of $\coprod_{i,j} h_{U_{ij}} \rightrightarrows \coprod_i h_{U_i}$[2], so it is a colimit of representable sheaves. □

Now we provide a useful result when thinking about étale sheaves and the morphisms between them.

Proposition 45. *Let $F, G \in$ Shv$(X_{\textbf{Ét}})$ and $\varphi : F \to G$ be a morphism so that for all $U \subseteq X$, $\varphi_U : F(U) \to G(U)$ is an isomorphism, then φ is an isomorphism.*

Proof. Let $Y \xrightarrow{p} X$ be an étale morphism, and let $\{U_i \subseteq Y\}$ be an open cover of Y so that $p|_{U_i}$ is an open embedding for all i, then since φ is a natural transformation, the diagram below commutes.

$$\begin{array}{ccc} F(p(U_i)) & \xrightarrow{\varphi_{U_i}} & G(p(U_i)) \\ {\scriptstyle F(p)}\downarrow & & \downarrow{\scriptstyle G(p)} \\ F(U_i) & \xrightarrow[\sim]{\varphi_{p(U_i)}} & G(U_i) \end{array}$$

Since both vertical arrows are isomorphisms (because $p|_{U_i}$ is a homeomorphism) so is φ_{U_i}, similarly for

[2]Note that here the colimits are sheaf colimits, which correspond to the sheafification of the corresponding presheaf colimits.

$\varphi_{U_{ij}}$. Now consider the sheaf condition for both F and G:

$$\begin{array}{ccccc} F(Y) & \longrightarrow & \prod_i F(U_i) & \rightrightarrows & \prod_{i,j} F(U_{ij}) \\ \downarrow \varphi_Y & & \downarrow \prod_i \varphi_{U_i} & & \downarrow \prod_{ij} \varphi_{U_{ij}} \\ G(Y) & \longrightarrow & \prod_i G(U_i) & \rightrightarrows & \prod_{i,j} G(U_{ij}) \end{array} \tag{4.5}$$

Where commutativity on the right square means that the top arrow of the top row commutes with the top arrow of the bottom row (and the same for the bottom arrows). Since the φ_{U_i} and $\varphi_{U_{ij}}$ are isomorphisms, so are $\prod_i \varphi_{U_i}$ and $\prod_{ij} \varphi_{U_{ij}}$, hence φ_Y is an isomorphism. $\square$

Now recall the construction of the right Kan extension of a functor from Section 1.7; which we repeat here specialised for the case where i is the inclusion of $O(X)^{\mathsf{op}}$ into $\mathbf{\acute{E}t}(X)^{\mathsf{op}}$.

$$i_K(F)(Y) := \underset{f \in (Y \downarrow O(X)^{\mathsf{op}})}{\mathsf{Lim}} F(\mathsf{d}_1 f) = \underset{\hat{f} \in (O(X) \downarrow Y)}{\mathsf{Lim}} F(\mathsf{d}_0 \hat{f}) \tag{4.6}$$

Proposition 46. *If F a sheaf over the Zariski site, then its right Kan extension to the étale site is a sheaf $i_K(F) \in$ Shv($X_{\mathbf{\acute{E}t}}$).*

Proof. First let $Y = U$ be an open subset of X, the canonical map $F(U) \to i_K(F)(U)$ is an isomorphism because $i_K(F)(U)$ is by definition the set of compatible families of F on the covering sieve $\mathsf{Mor}_{O(X)}(-, U)$, and F is a sheaf on $O(X)$.

Now for the more general case, let $Y \xrightarrow{p} X$ be an étale space and $U \subseteq X$ an open set, we call an arrow $f : U \to Y$ where $p|_{f(U)}$ is an open embedding, a lift of U to Y. So consider the cover of Y by all possible lifts open sets $\{f_i : U_i \hookrightarrow Y \mid i \in I\}$ and let $U_{ij} = U_i \times_Y U_j$ with inclusion $f_{ij} : U_{ij} \to Y$. By definition, a section of $i_K(F)(Y)$ is a family of sections $(s_i)_{i \in I} \in \prod_{i \in I} F(f_i(U_i))$ such that $F(f_{ij})(s_i) = s_{ij}$, then necessarily $i_K(F)(Y)$ is the equalizer of $\prod_{i \in I} i_K(F)(U_i) \rightrightarrows \prod_{i,j \in I} i_K(F)(U_{ij})$, but because F is a sheaf on $O(X)$ such a compatible family of sections is uniquely determined by a compatible family of sections on any open cover of Y by lifts of open sets, and so $i_K(F)$ satisfies the sheaf condition for covers by lifts of open subsets of X.

Now consider any open cover $\{U_i \subseteq Y \mid i \in I\}$ then each of the U_i can be covered by lifts of open subsets of X, $\{V_k^i \mid k \in I_i\}$ and it's plain to see that:

$$\mathsf{eq}\left(\prod_i i_K(F)(U_i) \rightrightarrows \prod_{i,j} i_K(F)(U_i \cap U_j)\right) = \mathsf{eq}\left(\prod_{i,k} i_K(F)(V_k^i) \rightrightarrows \prod_{i,j,k,l} i_K(F)(V_k^i \cap V_l^j)\right)$$

And since we've already seen that the equalizer on the right is $i_K(F)(Y)$ we get that Y satisfies the sheaf condition for arbitrary open covers. Finally since any étale map $f : Z \to Y$ factors uniquely through the inclusion $f(Z) \hookrightarrow Y$, given any étale cover $R = \{f : Z_f \to Y\}$ we get:

$$\mathsf{eq}\left(\prod_i i_K(F)(Z_f) \rightrightarrows \prod_{i,j} i_K(F)(Z_f \times_Y Z_g)\right) = \mathsf{eq}\left(\prod_{f \in R} i_K(F)(f(Z_f)) \rightrightarrows \prod_{f,g \in R} i_K(F)(f(Z_f) \cap g(Z_g))\right)$$

And since the equalizer on the right is, once again $i_K(F)(Y)$, we conclude $i_K(F)$ is an étale sheaf. $\square$

Proposition 46 immediately implies the following corollary:

Corollary 47. *The adjunction $i^* \dashv i_K$: PSh($X_{\mathbf{\acute{E}t}}$) $\to$ PSh($X_{\mathbf{Zar}}$) (resulting from the Kan extension along $i : O(X) \hookrightarrow \mathbf{\acute{E}t}(X)$) restricts to an adjunction between the sheaf categories:*

$$i^* \dashv i_K : \mathit{Shv}(X_{\mathbf{\acute{E}t}}) \to \mathit{Shv}(X_{\mathbf{Zar}})$$

Note from the proof of Proposition 46 that the right Kan extension satisfies $i_K(F)(U) \cong F(U)$ for any open set $U \subseteq X$, thus applying Proposition 45 we obtain the corollary:

Corollary 48. *If $F \in$ Shv($X_{\mathbf{\acute{E}t}}$) then the unit of the adjunction $i^* \dashv i_K$ from Corollary 47 is an isomorphism $F \cong i_K i^*(F)$.*

The observation that $i_K(F)(U) \cong F(U)$ for any sheaf $F \in$ Shv($X_{\mathbf{Zar}}$) coupled with the fact that i^* is only the restriction, implies that the counit $i^* i_K \to \mathrm{Id}_{\mathrm{Shv}(X_{\mathbf{Zar}})}$ is also an isomorphism and thus proves the following theorem:

Theorem 49. *The restriction functor i^* : Shv($X_{\mathbf{\acute{E}t}}$) $\to$ Shv($X_{\mathbf{Zar}}$) is an equivalence of categories.*

Remark 50. This is an instance of the “comparison lemma” (see [Joh02b] Theorem 2.2.3 in section C2.2).
Note that this is not true for the case where we take the Zariski and étale topoi of schemes, as can be noted by the fact that the large Zariski topos is the classifying topos of local rings (see for example [Moe92]) whereas the large étale topos is the classifying topos of strict henselian rings (first proved in [Hak72] although an easier reference may be [Wra79]).

Another thing to note is that this equivalence makes i_K into both a right and left adjoint to i^* and hence it is naturally isomorphic to $i_* = \# \circ i^K$.

Chapter 5

Describing the Category of Point Functors

In this chapter (especially in Section 5.1) we concern ourselves with the question of determining what is the natural definition of a morphism between Grothendieck topoi, and we base our definition in the case where the site of definition is $O(X)$ for some topological space X. In the final section we focus entirely on a particular class of morphisms, called the points of a topos, and study the structure of the category of points of in the setting of both the site Shv(X) for some topological space X.

5.1 Geometric Morphisms

As we have seen Grothendieck topoi are generalizations of the categories of sheaves on topological spaces. Therefore when considering the definition of morphisms of topoi we should keep in mind that these should relate somehow to continuous functions. This leads to the following question:
What does a continuous function $f : X \to Y$ induce at the level of the topoi of sheaves Shv(X) and Shv(Y)?

Proposition 51. *Given two topological spaces X, Y and a continuous function $f : X \to Y$, the function f naturally induces a functor called the* direct image functor*:*

$$f_* : \textit{Shv}(X) \to \textit{Shv}(Y) \qquad f_*(F)(U) := F(f^{-1}(U))$$

Proof. Given a sheaf $F \in$ Shv(X) the presheaf f_*F on Y given by $(f_*F)(U) := F(f^{-1}(U))$ for each $U \subseteq Y$ open. This is well defined because, since f is continuous $f^{-1}(U)$ is open. Note that f^{-1} commutes with arbitrary unions and finite intersections.
Now given any open cover $\{U_i \subseteq U\}$ of an open set $U \subseteq Y$ the preimages of the U_i naturally form a cover $\{f^{-1}(U_i) \subseteq f^{-1}(U)\}$, and since F satisfies the sheaf condition for any open cover, it does in particular for preimages of open covers in Y, it follows that f_*F is a sheaf, and thus the map $F \mapsto (f_*F)$ defines a functor f_* : Shv$(X) \to$ Shv(Y). □

Just as we were able to give a functor $f_* : \mathsf{Shv}(X) \to \mathsf{Shv}(Y)$ associated to any continuous function $f : X \to Y$ we can show there exists a functor back. For this effect recall the functors $E, \mathscr{F}$ from Theorem 36 that comprise an equivalence between the categories $\mathsf{Shv}(X)$ and $\mathbf{\acute{E}t}(X)$:

Definition. Let $f : X \to Y$ be a continuous function. The *inverse image functor* $f^* : \mathsf{Shv}(Y) \to \mathsf{Shv}(X)$ is defined by:

$$f^*(F) := \mathscr{F}(X \times_Y E_F)$$

Since the pullback of étale maps is étale f^* is well defined, and since $\mathscr{F}$, E and pullbacks are all functorial, f^* is indeed a functor.

Now note that for any two sheaves G, H over Y, choosing a section of $E_{G\times H}$ over the open set $U \subseteq X$ is the same as giving a pair of sections (s,t) where s is a section of E_G and t is a section of E_H, both over the open set U, therefore we have:

$$\begin{aligned}
E_{f^*(G\times H)} &\cong \Big\{(x, [(s,t)]_{f(x)}) : (s,t) \in G(U) \times H(U), f(x) \in U\Big\}_{x\in X} \\
&\cong \Big\{(x, [s]_{f(x)}, [t]_{f(x)}) : s \in G(V), t \in H(V'), f(x) \in V \cap V'\Big\}_{x\in X} \\
&\cong \Big\{(x, [s]_{f(x)}) : s \in G(V), f(x) \in V\Big\}_{x\in X} \times_X \Big\{(x, [t]_{f(x)}) : t \in H(V'), f(x) \in V'\Big\}_{x\in X} \\
&\cong E_{f^*G} \times_X E_{f^*H}
\end{aligned}$$

Where the pullback is taken along the projections onto the first component, X. Note also that the categorical product in $\mathbf{\acute{E}t}(X)$ corresponds to the pullback over X, so applying $\mathscr{F}$ on both sides followed by the counit of the adjunction $\mathscr{F} \dashv E$ (which is an isomorphism) gives us:

$$f^*(F \times G) \cong f^*(F) \times f^*(G) \qquad \text{for all } F, G \in \mathsf{Shv}(X)$$

If we now take G to be the terminal sheaf h_X then $F \times h_X \cong F$, hence we have $f^*(F) \cong f^*(F \times h_X) \cong f^*(F) \times f^*(h_X)$ and we conclude $f^*(h_X) \cong \{*\}$, implying that f^* commutes with terminal objects.

Note also that we can apply the same principle to equalizers:

Given $G, H \in \mathsf{Shv}(X)$ and two parallel morphisms between them, consider the morphisms induced on the level of the étale spaces, $k, k' : E_G \rightrightarrows E_H$ and their equalizer $e : E_F \to E_G$. Finally consider the diagram induced by the inverse image functor, E_F has an explicit model given by:

$$E_F \cong \{(y, [s]_y) : V \ni y, s \in G(V), k[s]_y = k'[s]_y\}$$

So naturally $E_{(f^*F)}$ satisfies:

$$E_{(f^*F)} \cong \{(x, [s]_{f(x)}) : V \ni f(x), s \in G(V), k[s]_{f(x)} = k'[s]_{f(x)}\}$$

Note also that $E_{(f^*G)} \cong \{(x, [s]_{f(x)}) : f(x) \in V\}$ and that the map $l : E_{(f^*G)} \to E_{f^*(H)}$ induced by k

sends $(x, [s]_{f(x)})$ to $(x, k[s]_{f(x)})$ (and similarly for l'). Thus the equalizer of l, l' becomes:

$$\{(x, [s]_{f(x)}) : f(x) \in V, s \in G(V), k[s]_{f(x)} = k'[s]_{f(x)}\} \cong E_{(f^*F)}$$

And so, once again, applying the equivalence between $\mathbf{\acute{E}t}(X)$ and Shv(X) we get $f^*(\mathrm{eq}(k, k')) \cong \mathrm{eq}(f^*k, f^*k')$. Finally applying Corollary 16 (for finite products) we obtain:

Proposition 52. *The inverse image functor f^* : Shv$(Y) \to$ Shv(X) is left exact.*

Now as expected, these functors are related by an adjunction:

Proposition 53. *The inverse image functor f^* : Shv$(Y) \to$ Shv(X) is a left adjoint to the direct image functor f_* : Shv$(X) \to$ Shv(Y).*

Proof. We adapt the proof from [Moe92]. Recall that f^*G for some sheaf $G \in$ Shv(Y) is defined by the pullback square:

$$\begin{array}{ccc} E(f^*G) & \xrightarrow{k} & E(G) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

Where once again $E(f^*G) \cong \{(x, [s]_{f(x)}) \mid s \in G(U), f(x) \in U\}$, k is the projection onto the second component and p is the projection onto the first component.

Note also that if r is a section of p over some open set U, then $q \circ k \circ r = f \circ p \circ r = f|_U$ and so giving a section over U is equivalent to giving a map $t : U \to E(G)$ lifting $f|_U$. In particular, given any section s over an open set $V \subseteq Y$, let $s' = s \circ f$ (defined in $f^{-1}(V)$) then by what we just observed, s' defines a section $t_s : f^{-1}(V) \to E(f^*G)$ of p over $f^{-1}(V)$. Furthermore $t_s(f^{-1}(V))$ is an open set in $E(f^*G)$ and these sets form an open cover $E(f^*G)$, which then implies that for any topological space L, a function $l : E(f^*G) \to L$ is continuous if and only if $l \circ t_s$ is continuous for all $V \subseteq Y$ and $s \in G(V)$. Now let L be an étale space over X, then define:

$$K(E(f^*G), L) := \{k : E(f^*G) \to L \mid k \circ t_s \text{ is an étale morphism.}\}$$

With this definition we break down the proof of the proposition into proving the following sequence of isomorphisms:

$$\begin{aligned} \mathsf{Mor}_{\mathsf{Shv}(X)}(f^*G, F) &\cong \mathsf{Mor}_{\mathbf{\acute{E}t}(X)}(E(f^*G), E(F)) \\ &\cong K(E(f^*G), E(F)) \\ &\cong \mathsf{Mor}_{\mathsf{Shv}(Y)}(G, f_*\mathscr{F}E(F)) \\ &\cong \mathsf{Mor}_{\mathsf{Shv}(Y)}(G, f_*F) \end{aligned}$$

From what we have already seen, all we need to prove is $K(E(f^*G), E(F)) \cong \mathsf{Mor}_{\mathsf{Shv}(Y)}(G, f_*\mathscr{F}E(F))$ and for this we provide an explicit formula:

To each $k \in K(E(f^*G), E(F))$ we can associate a natural transformation τ_k defined for each $V \subseteq Y$ and $s \in G(V)$ as $(\tau_k)_V(s) = k \circ t_s : f^{-1}(V) \to E(f^*G)|_{f^{-1}(V)} \to E(F)|_{f^{-1}(V)}$ which is then a section of $\mathscr{F}E(F)(f^{-1}(V)) = f_*\mathscr{F}E(F)(V)$.

Now recall once again that the étale spaces $E(F)$, and $E(f^*G)$ have canonical representations with elements $(x, [s]_x)$ and $(x, [s]_{f(x)})$ respectively. Then given a natural transformation $\tau : G \to f_*\mathscr{F}E(F)$ we define $k_\tau \in K(E(f^*G), E(F))$ pointwise for each $(x, [s]_{f(x)}) \in E(f^*G)$ as $k_\tau(x, [s]_{f(x)}) = \tau_V(s)(x) = (x, [\tau_V(s)]_x)$ where $s \in G(V)$ (note that if $[s]_{f(x)} = [s']_{f(x)}$ then $s|_U = s'|_U$ for some open $U \subseteq Y$ containing $f(x)$; and so $\tau_U(s|_U) = \tau_U(s'|_U)$, which implies k_τ is well defined).

Finally we check that $\tau \mapsto k_\tau$ and $k \mapsto \tau_k$ are mutually inverse:

$$k_{\tau_k}(x, [s]_{f(x)}) = (\tau_k)_V(s)(x) = k \circ t_s(x) = k(x, [s]_{f(x)})$$
$$(\tau_{k_\tau})_V(s)(x) = k_\tau \circ t_s(x) = k_\tau(x, [s]_{f(x)}) = \tau_V(s)(x)$$

And so $k_{\tau_k} = k$ and $\tau_{k_\tau} = \tau$ giving the intended isomorphism. □

Lastly we want to check that an adjoint pair as in the statement of Proposition 53 gives rise to a continuous function. We do this in the setting where one of the spaces is assumed to be Hausdorff (a more general version of this result is presented in [Moe92] yet we do it this way so it becomes more intuitive).

Theorem 54. *If Y is Hausdorff then there is a bijection, up to natural isomorphism between adjoint pairs $\varphi \dashv \psi$: Shv$(Y) \to$ Shv(X) where φ is left exact, and continuous functions $f : X \to Y$.*

Proof. As we have seen each continuous function $f : X \to Y$ gives rise to an adjunction $f^* \dashv f_*$: Shv$(Y) \to$ Shv(X) where f^* is left exact.

On the other hand, given an adjunction $\varphi \dashv \psi$: Shv$(Y) \to$ Shv(X) with φ left exact, we note that the map $O(Y) \hookrightarrow$ Shv(Y) defined by $U \mapsto h_U$ commutes with finite products (which correspond to intersections), equalizers (which are all trivial) and terminal objects, hence we get a left exact functor $\hat{\varphi} : O(Y) \to$ Shv(X) given by $\hat{\varphi}(U) = \varphi(h_U)$.

Since $\hat{\varphi}$ is left exact, we have $\hat{\varphi}(Y) \cong h_X$ and, by Lemma 14, $\hat{\varphi}$ preserves monomorphisms, so it takes all inclusions $U \subseteq Y$ to monomorphisms $\hat{\varphi}(U) \hookrightarrow h_X$.

Note now that a subsheaf F of h_X, is such that for any open set $V \subseteq X$, $F(V)$ is either the empty set or $\{*\}$, and consider the set $\{V_i \subseteq X : F(V_i) \cong \{*\}\}$. Let $V = \bigcup_i V_i$. For any open set $U \subseteq V$, $\{V_i \cap U : V_i \cap U \neq \emptyset\}$ is an open cover, and since we have a map $\{*\} \cong F(V_i) \to F(V_i \cap U)$, $F(V_i \cap U) \cong \{*\}$ for all i. Since F is a sheaf, it follows that $F(U) \cong \{*\}$, and so $F \cong h_V$.

Hence we conclude $\hat{\varphi}(U) \cong h_V$ for some unique $V \subseteq X$ and so we obtain a functor $g^{-1} : O(Y) \to O(X)$ defined as $h_{g^{-1}(V)} \cong \varphi(h_v)$. This functor preserves finite intersections:

$$h_{g^{-1}(V \cap U)} \cong \varphi(h_{V \cap U}) \cong \varphi(h_V \times h_U) \cong \varphi(h_V) \times \varphi(h_U) \cong h_{g^{-1}(V)} \times h_{g^{-1}(U)} \cong h_{g^{-1}(V) \cap g^{-1}(U)}$$

The functor g^{-1} also preserves unions because the union $\bigcup_i U_i$ can be thought of as the colimit in sheaves of $\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i$, so let $U = \bigcup_i U_i$, then $h_{g^{-1}(U)}$ is a coequalizer of representables:

$$\begin{aligned}\varphi(h_U) &= \varphi\left(\text{coeq}\left(\coprod_{i,j} h_{U_i \cap U_j} \rightrightarrows \coprod_i h_{U_i}\right)\right) \\ &= \text{coeq}\left(\coprod_{i,j} h_{g^{-1}(U_i \cap U_j)} \rightrightarrows \coprod_i h_{g^{-1}U_i}\right) \\ &= h_{\bigcup g^{-1}(U_i)}\end{aligned}$$

Now, for any point $y \in Y$, define $\hat{g}^{-1}(y) := \bigcap_{y \in U} g^{-1}(U)$. We want to show that $\{\hat{g}^{-1}(y) \mid y \in Y\}$ is a partition of X:

Since Y is Hausdorff, for every two points $y \neq y'$ there exist disjoint neighborhoods $U_y, U_{y'}$ of y, y'. Then $g^{-1}(U_y) \cap g^{-1}(U_{y'}) = g^{-1}(U_y \cap U_{y'}) = \emptyset$, and therefore $\hat{g}^{-1}(y) \cap \hat{g}^{-1}(y') = \emptyset$. Suppose now $x \notin \hat{g}^{-1}(y)$ for all $y \in Y$. Then every point $y \in Y$ has a neighborhood U_y such that $x \notin g^{-1}(U_y)$ and therefore $x \notin g^{-1}(\bigcup_{y \in Y} U_y) = g^{-1}(Y) = X$. So the partition $\{\hat{g}^{-1}(y) \mid y \in Y\}$ defines a function $g : X \to Y$ with $g(x)$ defined to be the unique y such that $x \in \hat{g}^{-1}(y)$.

The function g induces the same inverse image function g^{-1} on the poset of open subsets of Y:

Let $U \subseteq Y$ be an open set then $\bigcup_{y \in U} \hat{g}^{-1}(y) \subseteq g^{-1}(U)$ (because U is an open neighborhood of any $y \in U$), and now let $x \notin \bigcup_{y \in U} \hat{g}^{-1}(y)$, then each $y \in U$ has an open neighborhood $U_y \subseteq U$ so that $x \notin g^{-1}(U_y)$, and so $x \notin \bigcup_{y \in U} g^{-1}(U_y) = g^{-1}(U)$, hence $g^{-1}(U) = \bigcup_{y \in U} \hat{g}^{-1}(y)$, and g is continuous.

Since by Theorem 44 every sheaf is a colimit of representable sheaves, left adjoints are determined by their values on representables and hence the inverse image functor g^* associated to g is naturally isomorphic to φ. □

This short survey thus provides our motivation for the definition of geometric morphism, by extrapolating to general topoi.

Definition. Given two topoi $\mathcal{E}, \mathcal{F}$, a *geometric morphism* from $\mathcal{E}$ to $\mathcal{F}$ is an adjoint pair $f^* \dashv f_* : \mathcal{F} \to \mathcal{E}$ (where $f^* : \mathcal{F} \to \mathcal{E}$) so that f^* is left exact.

Definition. For a topos $\mathcal{E}$ a functor $x^* : \mathcal{E} \to \boldsymbol{Set}$ is said to be a *point functor* of $\mathcal{E}$ if x^* is a left exact left adjoint.

Here we disregard the right adjoint since this introduces no further structure and is determined up to isomorphism, so equivalently we can say a point of $\mathcal{E}$ is a geometric morphism $x : \boldsymbol{Set} \to \mathcal{E}$.

Example 55.

- Note that Theorem 43 establishes the pair $i_* \dashv i^*$ as a geometric morphism, this is a particular case of the more general fact that an equivalence of categories between two topoi is always a geometric morphism.

- Given a small category $\mathcal{C}$ we can define a pretopology J' on $\mathcal{C}$ whose only covering families are isomorphisms, for this pretopology every presheaf is a sheaf and so PSh($\mathcal{C}$) is a Grothendieck topos.

- For any site $(\mathcal{C}, J)$ the sheafification discussed in Section 4.2 defines (half of) a geometric morphism Shv$(\mathcal{C}) \to$ PSh$(\mathcal{C})$.

- Theorem 54 implies that in the case where X is Hausdorff the isomorphism classes of point functors of Shv(X) correspond bijectively to points of the space X.

5.2 Point Functors of the Zariski Site

Throughout this section x will denote a point of the topos Shv(X) for X some topological space, that is, a geometric morphism $\boldsymbol{Set} \to$ Shv(X) with underlying adjunction $x^* \dashv x_*$.

Given any sheaf $F \in$ Shv(X), and any open set $U \subseteq X$, the Yoneda lemma states that.

$$\text{Mor}_{\text{Shv}(X)}(h_U, F) = \text{Nat}(\text{Mor}_{O(X)}(-, U), F) \cong F(U)$$

Note also that since point functors are left adjoints, they are cocontinuous and since every sheaf $F \in$ Shv(X) is, by Theorem 44 a colimit of representables, the value of x^* on representable sheaves completely determines x^*.
For any $V \subseteq X$ we have a unique arrow $h_V \to h_X$ which is a monomorphism, and because x^* is left exact, by Lemma 14, this provides a monomorphism $x^*(h_V) \to x^*(h_X)$. Again, because x^* is left exact and h_X is the terminal object in Shv(X), $x^*(h_X)$ is terminal in $\boldsymbol{Set}$ and so is isomorphic to $\{*\}$.
Now denote by $\mathbf{2} = \{0, 1\}$ a two-point set. Any $V \subseteq X$ induces a surjection[1]:

$$\mathbf{2} \cong \text{Mor}_{\boldsymbol{Set}}(\{*\}, \mathbf{2}) \cong \text{Mor}_{\boldsymbol{Set}}(x^*(h_X), \mathbf{2}) \to \text{Mor}_{\boldsymbol{Set}}(x^*(h_V), \mathbf{2}) \tag{5.1}$$

Now consider the right adjoint $x_* : \boldsymbol{Set} \to$ Shv(X) to x^*, on the right-hand side of equation (5.1) we can apply the isomorphism from the adjunction and the Yoneda lemma to obtain:

$$\text{Mor}_{\boldsymbol{Set}}(x^*(h_V), \mathbf{2}) \cong \text{Mor}_{\text{Shv}(X)}(h_V, x_*(\mathbf{2})) \cong x_*(\mathbf{2})(V)$$

And hence we have a surjection $\mathbf{2} \to x_*(\mathbf{2})(V)$. Thus, for any open set $V \subseteq X$ and any point functor x^*:

$$x_*(\mathbf{2})(V) = \begin{cases} \text{either} & \mathbf{2} \\ \text{or} & \{*\} \end{cases}$$

Now consider the family of open sets $\{W_\alpha \mid x_*(\mathbf{2})(W_\alpha) \cong \{*\}\}$ and let $W = \bigcup W_\alpha$. Since $x_*(\mathbf{2})$ satisfies the sheaf condition (2.1) and $\{W_\alpha \subseteq W\}$ is an open covering of V we conclude $x_*(\mathbf{2})(W) \cong \{*\}$.
Taking now into consideration that any open subset $U \subseteq W$ induces an injection $h_U \to h_W$ and therefore,

[1] Note that a monomorphism $f : a \to b$ induces a natural epimorphism $f^* : \text{Mor}(b, -) \to \text{Mor}(a, -)$.

we obtain a surjection:

$$x_*(\mathbf{2})(W) \cong \text{Mor}_{\boldsymbol{Set}}(x^*(h_W), \mathbf{2}) \to \text{Mor}_{\boldsymbol{Set}}(x^*(h_U), \mathbf{2}) \cong x_*(\mathbf{2})(U)$$

From whence we conclude that $x_*(\mathbf{2})(U) \cong \{*\}$. This implies that setting $D := X \setminus W$, we can fully determine $x_*(\mathbf{2})$ as:

$$x_*(\mathbf{2})(U) = \begin{cases} \mathbf{2} & \text{if } D \cap U \neq \emptyset \\ \{*\} & \text{otherwise} \end{cases}$$

And this sets $x_*(\mathbf{2})$ as isomorphic to the skyscraper presheaf $\mathbf{2}^D$. By the fact that $x_*(\mathbf{2})$ is a sheaf and Lemma 27, D must be an irreducible closed subset of X.

Now recall that we denote the power set of S by $\mathscr{P}(S)$, (where $\mathscr{P}$ is the power set functor). Notice also that $S \cong S' = \{\{s\} : s \in S\} \subseteq \mathscr{P}(S)$, and taking $\chi_{S'} : \mathscr{P}(S) \to \mathbf{2}$ to be the characteristic function of S' on $\mathscr{P}(S)$ ($\chi_{S'}(x) = 0$ if $x \subseteq S, |x| = 1$ and $\chi_{S'}(x) = 1$ otherwise) and 0 to be the constant function $\mathscr{P}(S) \to \mathbf{2}$ equal to 0 we obtain:

$$S \cong \text{eq}\left(\mathscr{P}(S) \underset{0}{\overset{\chi_{S'}}{\rightrightarrows}} \mathbf{2}\right)$$

Noting also that $\mathscr{P}(S) \cong \text{Mor}_{\boldsymbol{Set}}(S, \mathbf{2}) \cong \prod_{s \in S} \mathbf{2}$ and that x_* commutes with arbitrary limits:

$$x_*(S) \cong x_*\left(\text{eq}\left(\prod_{s \in S} \mathbf{2} \rightrightarrows \mathbf{2}\right)\right) \cong \text{eq}\left(\prod_{s \in S} x_*(\mathbf{2}) \rightrightarrows x_*(\mathbf{2})\right) \cong \text{eq}\left(\prod_{s \in S} \mathbf{2}^D \rightrightarrows \mathbf{2}^D\right) \tag{5.2}$$

Since the product of skyscrapers over some (fixed) irreducible closed set D forms a skyscraper, also over D, and when $U \cap D = \emptyset$ the diagram in (5.2) becomes trivial, we get:

$$x_*(S)(U) \cong \begin{cases} \text{eq}\left(\prod_{s \in S} \mathbf{2} \rightrightarrows \mathbf{2}\right) & \text{if } D \cap U \neq \emptyset \\ \{*\} & \text{otherwise} \end{cases}$$

Note that since the parallel maps in the equalizer are induced by $\chi_{S'}$ and 0 defined on $\mathscr{P}(S)$ the equalizer eq $\left(\prod_{s \in S} \mathbf{2} \rightrightarrows \mathbf{2}\right)$ is isomorphic to S. So we conclude:

$$x_*(S) \cong S^D$$

By uniqueness of the adjoint (up to isomorphism), the associated point functor x^* must be (naturally isomorphic to) the functor of stalks $F \mapsto F_D$ which, as we have seen in theorem 29 is the left adjoint to $S \mapsto S^D$.This brings us finally to the theorem:

Theorem 56. *Any point functor $x^* : \mathit{Shv}(X_{\mathbf{Zar}}) \to \boldsymbol{Set}$ is isomorphic to a functor of stalks over some irreducible closed set $D \subseteq X$.*

Now let Pt$(X_{\mathbf{Zar}})$ be the full subcategory of Func(Shv$(X_{\mathbf{Zar}}), \boldsymbol{Set})$ whose objects are point functors and arrows are natural transformations. Also, let Irr(X) be the poset category of irreducible closed subsets of X.

Recall that the stalk over an irreducible closed set D is defined as the colimit:

$$F_D = \operatorname*{Colim}_{U \cap D \neq \emptyset} F(U)$$

If $D \subseteq D'$ and $U \cap D \neq \emptyset$ then $U \cap D' \neq \emptyset$ so we have a map $F_D \to F_{D'}$ which is natural in F. Let St : Irr$(X) \to$ Pt$(X_{\mathbf{Zar}})$ be the functor defined as:

$$\text{St}(D)(F) := F_D$$

Proposition 57. *The functor St* : *Irr*$(X) \to$ *Pt*$(X_{\mathbf{Zar}})$ *is an equivalence of categories.*

Proof. Theorem 56 guarantees that St is essentially surjective. Since the domain of St is a poset category it is also faithful. It remains to prove that it is injective on objects and full.

Let D, D' be two distinct irreducible closed sets of X so that $D \not\subseteq D'$, and let $V = X \setminus D'$. Then:

$$(h_V)_D \cong \{*\}$$
$$(h_V)_{D'} = \emptyset$$

Hence we conclude that St$(D) \neq$ St(D') and hence that St is injective on objects. Moreover under the above hypothesis that there can be no natural transformation St$(D) \to$ St(D'), i.e there can only be a natural transformation St$(D) \to$ St(D') if $D \subseteq D'$.

Finally since any sheaf $F \in$ Shv$(X_{\mathbf{Zar}})$ is a colimit of representable sheaves $F = \operatorname*{Colim}_i F_i$ and stalk functors are cocontinuous, a map $F_D \to F_{D'}$ is determined by a natural transformation $(F_i)_D \to (F_i)_{D'}$. These natural transformations are unique since $F_i \cong h_{U_i}$ and the maps $(h_{U_i})_D \to (h_{U_i})_{D'}$ are uniquely determined by U_i, D and D'. $\square$

Remark 58. Theorem 49 implies that the category of points for the topos of sheaves on the étale site is also equivalent to the poset of irreducible closed sets of X.

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