Equilibrium and dynamical properties of rotating clouds of ultra cold atoms

J. D. Rodrigues[1]

[1] *IPFN, Instituto Superior Técnico, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal*

The multiple scattering of light in a gas of ultra cold atoms is responsible for many exciting features observed in magneto-optical traps including the collective behaviour forced by a Coulomb like potential. This field also induces plasma like phenomena in the cloud which allows the treatment of the system as a one component trapped plasma. With a fluid description and casting the thermodynamical behaviour in the form of a polytropic equation of state we investigate the equilibrium profiles of rotating clouds and its dependence on the experiment characteristics. Numerical solutions predict the formation of stable orbital modes both in rotating and non rotating clouds. We also investigate the normal modes on such rotating systems.

I. INTRODUCTION

The celebrated magneto-optical trap (MOT) [1] technology has allowed the study of many and exciting topics in atomic physics. Among them, much interest was devoted to the study of Bose-Einstein condensates [2, 3] which led to a decreasing investigation on the basic properties of MOT physics. Nevertheless, a new trend has recently begun that rekindled the interest on this subject. This is related to the increasing number of astrophysical phenomena that we can simulate and study using ultra cold atomic clouds. In particular, we can refer to a new mechanism associated with the laser cooling process, which can lead to the formation of static and oscillating photon bubbles inside the gas [4]. Photon bubbles have been considered in an astrophysical context [5–7] where huge photon densities are required to have any significant impact on high energy particles.

Moreover, Kaiser *et al* [8] were recently able to achieve random lasing in a cloud of ultra cold atoms under laboratory conditions. The effect was first seen decades ago in stellar clouds [9, 10] and in some planetary atmosphere [11, 12], when random lasing was first proposed to explain why certain specific emission lines in clouds of stellar gas are more intense than theoretically predicted [10, 13].

Finally we refer to the recent work of Terças *et al* [14], where it is investigated the hydrodynamic equilibrium and normal modes of cold atomic traps, combining the effects of multiple scattering and the thermal fluctuations inside the system, cast in the form of a polytropic equation of state. This analysis results in a generalized Lane-Emden equation to describe the equilibrium density profiles of the cloud, derived to study astrophysical fluids [15].

The process of multiple scattering of light, which typically became significant for a number of atoms above $N \simeq 10^5$, is responsible for the rich and complex behaviour of ultra cold atomic vapours. This mechanism has been described, since the early stages of MOTs, as the principal limitation for the compressibility of the cloud [16, 17]. In this regime, the atoms in the cloud are strongly correlated due to the presence of a Coulomb type long-range interaction [18] and the treatment and description of the system as a one component plasma becomes feasible and very fruitful [19–21].

In the present paper, we present an extension to the previously mentioned work of Terças *et al* [14] to include the interesting case of a rotating cloud of ultra cold atoms. We begin to derive the equilibrium density profiles of such systems. Numerical solutions show that stable orbital modes are possible for systems with a large number of trapped atoms and/or with a sufficiently high angular frequency. Satellite rings have been observed in rotating clouds since the early nineties [17], although their nature was not yet well understood. In section III we will investigate the dynamics of such systems by computing the normal oscillation modes that can be excited in the cloud.

II. POLYTROPIC EQUILIBRIUM

The starting point to compute the equilibrium profiles corresponds then to the setting of the fluid equations

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) = 0 \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{mn} + \frac{\boldsymbol{F}_T}{m}. \tag{2}$$

The fluid description of the system requires some relation between the pressure P and the atom density n and we assume the existence of a polytropic equation of state for the MOT

$$P(r) = C_\gamma n(r)^\gamma. \tag{3}$$

As we are dealing with rotating clouds, the total force acting an element of fluid is $\boldsymbol{F}_T = \boldsymbol{F}_{MOT} + \boldsymbol{F}_c + \boldsymbol{F}_r$, with $\boldsymbol{F}_{MOT} \simeq -\alpha\boldsymbol{v} - \kappa\boldsymbol{r}$, and $\boldsymbol{F}_c$ the collective force determined by

$$\boldsymbol{\nabla} \cdot \boldsymbol{F}_c = Qn. \tag{4}$$

Notice that in the expression for the collective force $\boldsymbol{F}_c$, $Q = (\sigma_R - \sigma_L)\sigma_L I_0/c > 0$ represents the square of the effective charge [18], with c the speed of light and I_0 the total intensity of the six cooling laser beams. The terms σ_R and σ_L represent the emission and absorption cross sections, respectively [17]. The term $\boldsymbol{F}_{MOT}$ corresponds to the contributions of the Doppler cooling force, with an equivalent damping coefficient α, and the trapping force, with an equivalent spring constant $\kappa = m\omega_0^2$. The difference with the non rotating case is the presence of the term $\boldsymbol{F}_r$. A rotation in the system can be easily achieved with a slight misalignment in four of the six laser beams and can be phenomenologically described by a force [17]

$$\boldsymbol{F}_r = \kappa' r \boldsymbol{e}_z \times \boldsymbol{e}_r = \kappa' r \boldsymbol{e}_\phi. \tag{5}$$

For this reason, we must, from now on, consider a cylindrical symmetry for the system, whose validity will be discussed later.

Assuming equilibrium conditions, $\partial/\partial t = 0$ and $\dot{r} = 0$, and adimensioning the system as

$$\theta(r) \equiv \left(\frac{n(r)}{n(0)}\right)^{\gamma-1} \tag{6}$$

and $\xi \equiv r/R$ with

$$R = \left(\frac{P(0)}{3m\omega_0^2 n(0)}\right)^{1/2} \tag{7}$$

we then get, in a dimensionless form

$$\frac{\gamma}{\gamma-1}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) + (1-\beta^2) - \Omega_p'^2\theta^{\frac{1}{\gamma-1}} = 0 \tag{8}$$

with

$$\Omega_p'^2 = \frac{Qn(0)}{3m\omega_0^2} \equiv \frac{\omega_p^2}{3\omega_0^2} \tag{9}$$

the effective plasma frequency and $\beta^2 = 2\left(\frac{\dot{\phi}}{\omega_0}\right)^2 \geq 0$ with $\dot{\phi} = \kappa'/\alpha$ the angular velocity of the fluid element. The term β^2 is associated with the ratio between the rotation's angular frequency and the angular frequency associated with the confinement, ω_0. The first is equivalent to an expansion force and the second one is related with a contraction one whereby this constant will be related with the stability of the cloud. In particular, as it will be clear when we derive the frequency of the allowed oscillation modes in section **III**, the system will become unstable for $\beta^2 > 1$. For non rotating clouds we have $\beta^2 = 0$. Eq. [8] depends then on the rotation state of the system. We can now realize that, by defining the parameter R as

$$R = \left(\frac{P(0)}{3m\omega_0^2 n(0)}\right)^{1/2} \to \left(\frac{P(0)}{3m\omega_0^2(1-\beta^2)n(0)}\right)^{1/2} \tag{10}$$

and introducing a redefined plasma frequency as

$$\Omega_p^2 = \frac{\Omega_p'^2}{1-\beta^2} \tag{11}$$

we get

$$\frac{\gamma}{\gamma-1}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) + 1 - \Omega_p^2\theta^{\frac{1}{\gamma-1}} = 0 \tag{12}$$

.

We have then an equation which is independent of the rotation state of the system, by incorporating the rotation angular frequency inside the definitions of R and Ω_p^2, which remarkably implies that rotating and non rotating systems share the some solutions for the equilibrium profiles, differing only by a scale factor $(1-\beta^2)^{1/2}$. The eq. [12] corresponds to a generalization of the Lane-Endem equation [15], derived to study astrophysical fluids. The density profiles only depend on the polytropic exponent γ and on the constant Ω_p^2 which accounts for the three forces in play. This term corresponds, then, to the "ratio" of the collective force, determined by ω_p^2, and the confining force (ω_0^2) "subtracted" with the centripetal force due to rotation (β^2). The confining force is then directly contour-posed by the centripetal one. This ratio is, then, sufficient to determine the density profiles. With this interpretation of Ω_p^2 in mind, we can easily realize that a stable cloud of atoms can only exist if $\Omega_p^2 < 1$ which implies an overall attractive force greater than the repulsive one (multiple scattering). Since we are interested in associating systems in different rotation states, we realize that degenerate density solutions exist if $\Omega_p^2 = \frac{\Omega_p'^2}{1-\beta^2} = \eta$, with $\eta < 1$ for a stable solution. Each of these sets of solutions (for each value of the constant η) differ only, then, by the scale factor $(1-\beta^2)^{1/2}$.

Let us now compute some analytical solutions for eq. [12] in some limiting cases. For small traps, typically $N < 10^5$ the effects of multiple scattering can be neglected, which corresponds to the limit $\Omega_p^2 \to 0$, reducing eq. [12] to

$$\frac{\gamma}{\gamma-1}\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) + 1 = 0 \tag{13}$$

which takes the solution $\theta(\xi) = \left[1 - \frac{\gamma-1}{4\gamma}\xi^2\right]$, or equivalently, in the the dimensional form

$$n(r) = n(0)\left(1 - \frac{\gamma-1}{4\gamma}\frac{r^2}{R^2}\right)^{\frac{1}{\gamma-1}}. \tag{14}$$

The isothermal case, which corresponds to $\gamma = 1$, simply corresponds to a Gaussian profile, in agreement with the Maxwell-Boltzmann equilibrium

$$n(r) = n(0)e^{-\frac{m\omega_0^2(1-\beta^2)r^2}{2k_BT}} = n(0)e^{-\frac{r^2}{4R^2}} \tag{15}$$

which also results from taking the formal limit $\gamma \to 1$ in eq. [14] [1]. This limit approximately describes the temperature limited traps, corresponding to a small number of particles, where the thermal effects determine the dynamics of the cloud, which does not correspond to the state-of-the

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[1] Remember the definition of the exponential function as

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

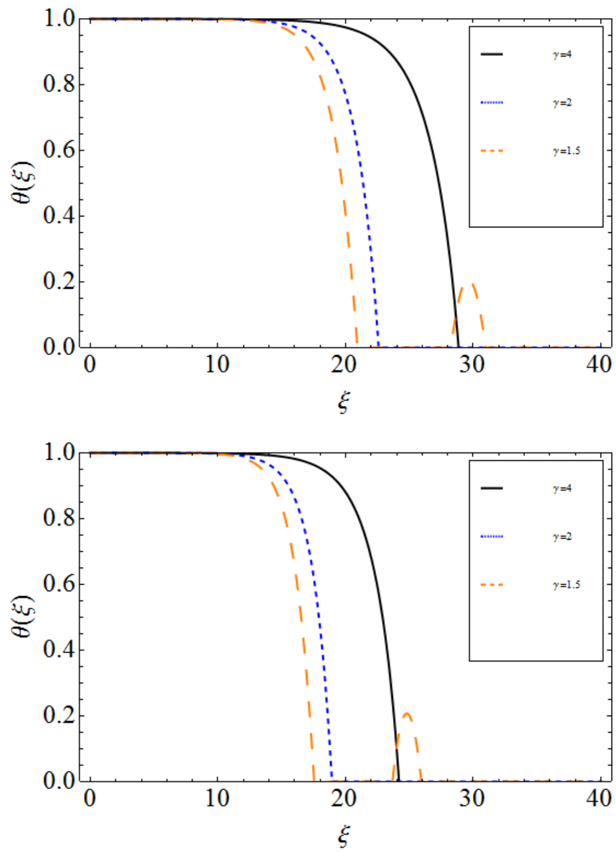

FIG. 1. Numerical solution for eq. [8] for high collective effects. For a sufficiently high Ω_p^2 the solutions start to exhibit orbital modes. From top to bottom $(\Omega_p'^2, \beta^2, \Omega_p^2) = (0.7, 0.29999, 0.999986)$ and $(\Omega_p'^2, \beta^2, \Omega_p^2) = (0.999986, 0, 0.999986)$. We only present the positive part of the function $\theta(\xi)$.

art traps, so the temperature limited regime is here considered only for completeness.

In the opposite case, for a large number of atoms trapped, the collective force dominates and temperature effects can be neglected, which corresponds to the limit $\gamma \to 0$ in eq. [12] giving rise to the solution $\theta(\xi) = 1/\Omega_p^2$ or, in the dimensional form,

$$n(r) = \frac{3m\omega_0^2(1-\beta^2)}{Q} H(a-r) \tag{16}$$

where a is the radius of the cloud.

One of the most interesting features observed in rotating clouds of atoms is the formation of stable orbital modes in the density profiles [17]. Up so far, it was widely believed that the existence of these orbital modes was intrinsically related with the rotation of the cloud. We are, now, in position to disregard this hypothesis since we now understand that rotating and non rotating systems must share the same solutions for the density profiles.

In figure [1] we present numerical solution to eq. [8]. These solutions represent two of the most important results of this work. First, we prove that the generalized Lane-Endem equation, derived with the simple assumption that there exists a polytropic equation of state for the gas, can account for solutions with orbital modes, for a sufficiently high Ω_p^2. Secondly, we have the up so mentioned degeneracy of the solutions in the $\Omega_p'^2$ and β^2 parameters, which is evidenced in figure [1], where we have two combinations of $(\Omega_p'^2, \beta^2)$ combining in the same effective plasma frequency $\Omega_p^2 = \Omega_p'^2/(1-\beta^2)$ to produce the same $\theta(\xi)$ solution, except for a scale factor. This feature can bring important advantages for experimentalists. In fact, we realise that to achieve an orbital mode we must have a very high Ω_p^2 which, for non rotating systems, implies a very large number of trapped atoms, conditions that cannot trivially be achieved experimentally. Introducing a rotation in the system, easily achieved experimentally, reduces the number of atoms and density needed to achieve an orbital mode. This is the reason why orbital modes have only been observed in rotating clouds, even with a small number of trapped particles, although its existence is now proven possible in non rotating systems.

Let us now discuss the validity of employing a cylindrical symmetry for the problem in hand. First of all, the confining magnetic field created by an anti-Helmholtz pair of coils is cylindrically symmetric, although this problem is not a major concern and a spherical symmetry still holds as a good approximation despite this issue. More important than the shape of the magnetic field is the relative intensity of the three pairs of laser beams. For instance, a cigar shaped cloud of atoms can be easily achieved by lowering the intensity of two collinear laser beams, reducing the confining force and allowing the cloud to expand in that direction. In this case, the cylindrical coordinate system is the appropriate one and the results presented here can be directly applied. On the other hand, in a system with three pairs of equal intensity laser beams, the cloud of atoms will present a spherical shape and we must be careful about how we apply the results presented here. Nevertheless, it is still possible to think about the cylindrical solutions as slices of $\theta = const$, where θ is the azimuthal angle of the spherical coordinates. In fact, in a spherical cloud, increasing the azimuthal angle θ corresponds to circular slices with lower central densities. In this interpretation of the cylindrical solutions, lower central densities are equivalent to a lower Ω_p^2 and then an orbital mode can exist in the equatorial plane, $\theta = 0$ and cease to exist for a given angle $\theta > 0$, reason why we get Saturn-like density profiles with orbital modes near the equatorial plane.

In figure [2] we present density plots for the atoms density profiles, along with the usual solution $\theta(\xi)$ for high values of Ω_p^2.

III. NORMAL MODES

We will now evaluate the nature of the localized oscillations, or normal modes of the system, with a perturbative analysis on the fluid mechanical description of the system. It will be useful to notice that the collective force can be derived from a potential, $\boldsymbol{F}_c = -\boldsymbol{\nabla}\phi_c \Leftrightarrow \nabla^2\phi_c = -Qn$. We now introduce small perturbations in the equilibrium quantities, labelled with the subscript 0,

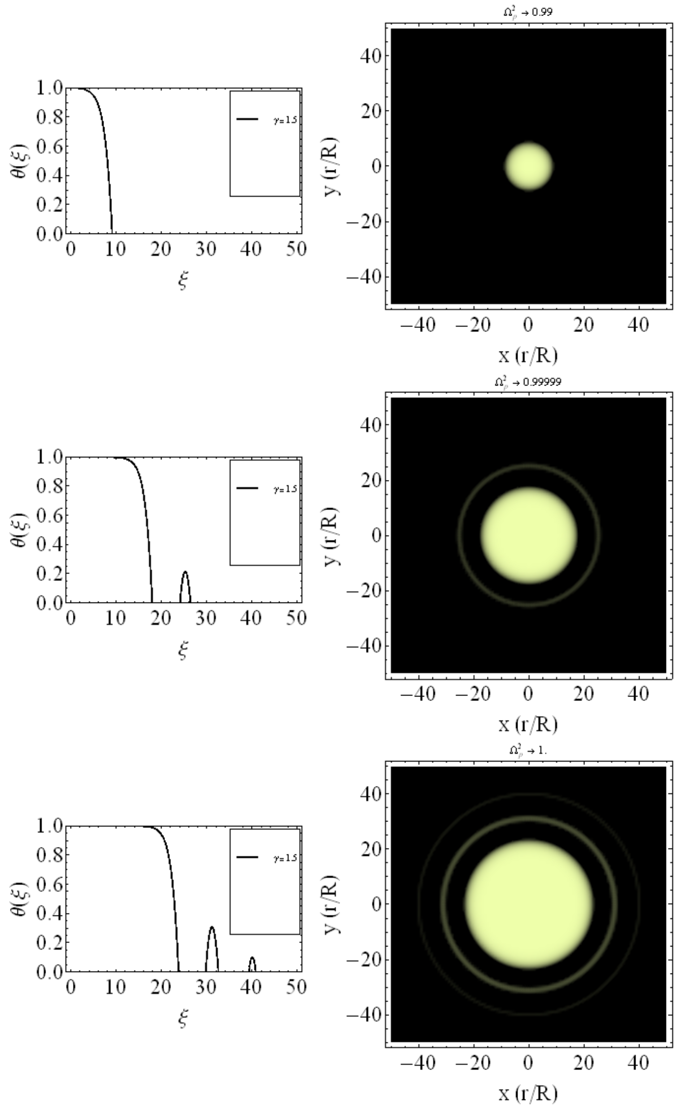

FIG. 2. Density plots for the atoms density profiles for high values of Ω_p^2. We observe that increasing even further the value of Ω_p^2 allows for a second orbital mode at an higher radius.

$$\begin{cases} n = n_0 + \delta n \\ P = P_0 + \delta P \\ \boldsymbol{v} = \boldsymbol{v}_0 + \delta \boldsymbol{v} \\ \phi^c = \phi^c_0 + \delta\phi^c \end{cases} \tag{17}$$

with the perturbations varying as $\delta a = \bar{\delta a} e^{i\omega t}$. This prescription allows us to linearise the set of fluid equations, neglecting quadratic terms of the form $\delta a \delta b \sim 0$, which yields

$$-\left(\omega^2 + i\frac{\alpha}{m}\omega\right)\delta n = \frac{\gamma C_\gamma}{m}\boldsymbol{\nabla}\cdot\left[n_0^{\gamma-1}(r)\boldsymbol{\nabla}\delta n\right] + \frac{1}{m}\boldsymbol{\nabla}\cdot\left[n_0(r)\boldsymbol{\nabla}\delta\phi_c\right] \tag{18}$$

$$\nabla^2 \delta\phi_c = -Qn \tag{19}$$

where we used $\boldsymbol{\nabla}\delta P \simeq \gamma C_\gamma n_0^{\gamma-1}\nabla\delta n$, from the polytropic equation of state. It is worth noticing that this final equation for the perturbation δn takes the same form of the one derived for non rotating clouds [14]. This result is, in fact, to be expected since in the derivation process we only considered radial perturbations, in the form $\delta\boldsymbol{v} = \bar{\delta v}(r)e^{i\omega t}\boldsymbol{e}_r$ and $\delta n = \bar{\delta n}(r)e^{i\omega t}$. The existence of the angular velocity is put in play by the equilibrium density profiles $n_0(r)$ from which the previous equation depends on. Recall that, due to the geometry of the problem, we must employ a cylindrical coordinate system to our equations. Now, in order to combine equations [18] and [19], we can introduce the auxiliary quantity η, defined as $\delta n = \frac{1}{2\pi r}\frac{d\eta}{dr}$, from which eq. [19] becomes $\nabla^2\delta\phi_c = -\frac{Q}{2\pi r}\frac{d\eta}{dr}$ and we have $\frac{d\delta\phi_c}{dr} = -\frac{Q}{2\pi r}\eta$. Making proper substitutions, the linearised equations can finally be put together resulting in a single expression

$$\left[\omega^2 + i\frac{\alpha}{m}\omega - \omega_p^2\frac{n_0(r)}{n_0(0)}\right]\eta(r) = -\frac{\gamma C_\gamma}{m}\left[r n_0^{\gamma-1}(r)\frac{d}{dr}\left(\frac{1}{r}\frac{d\eta}{dr}\right)\right] \tag{20}$$

with $\omega_p^2 = \frac{Qn(0)}{m}$. We then reduced the problem of finding the normal modes of oscillation to an eigenvalue problem. Despite the linear form of the differential equation this is a non-trivial problem and general solutions involve numerical simulations. For that reason we will examine some limiting cases. As a last remark, notice that the term $i\alpha/m$ in the previous equation corresponds to a damping of the modes, which is to be expected since it comes from the cooling Doppler force acting the atoms, which takes the form of a damping force, $\boldsymbol{F}_{Doppler} = -\alpha\boldsymbol{v}$. In fact, along with the damping of the modes, this term also causes a small variation in the frequency of the modes but, as it turns out, in typical experimental conditions, the frequency associated with this term is small in comparison with the trapping frequency ω_0 and the plasma frequency ω_p, for what we shall neglect this term from now on. For a more detailed analysis of this problem, refer to [19].

A. Temperature limited regime

For small clouds, typically with $N < 10^5$ atoms trapped, the effects of multiple scattering can be neglected and therefore we can set $\omega_p = 0$, or equivalently $\phi_c = 0$ in eq. [18], which simplifies to

$$\omega^2\delta n + \frac{\gamma C_\gamma}{m}\boldsymbol{\nabla}\cdot\left[n_0^{\gamma-1}(r)\boldsymbol{\nabla}\delta n\right] = 0 \tag{21}$$

Replacing the equilibrium density profile $n_0(r)$ given by eq. [14] into eq. [21] and writing the result in terms of the adimensional variable $\xi = r/R$, as defined before, we obtain

$$-\omega^2\delta n - \frac{3}{4}(\gamma-1)\omega_0^2\left(1-\beta^2\right)\frac{1}{\zeta}\frac{d}{d\zeta}\left[\left(1-\zeta^2\right)\zeta\frac{d\delta n}{d\zeta}\right] = 0 \tag{22}$$

where we performed another change of variables, namely $\xi = \sqrt{\frac{4\gamma}{\gamma-1}}\zeta$. For this differential equation we can try out solutions in the form of a power series

$$\delta n = \sum a_{nl} \zeta^{2n+l} \tag{23}$$

where we distinguish even (n) and odd (l) contributions because they correspond to slightly different types of oscillations. For the lowest radial modes, $n = 0$, the solutions correspond to surface excitations [23], and the even perturbations solutions, $l = 0$, correspond to breathing modes. Deriving the ansatz [23] and replacing in eq. [22] we obtain a recursive expression for the coefficients a_{nl} and the allowed modes of the system as

$$\omega = \omega_0 \sqrt{(1-\beta^2)\frac{3}{4}(\gamma-1)(2n+l+2)} \tag{24}$$

As mentioned before, setting $n = 0$ we get the surface modes

$$\omega = \omega_0 \sqrt{(1-\beta^2)\frac{3}{4}(\gamma-1)(l+2)} \tag{25}$$

Breathing modes, $l = 0$, are also possible even in small traps, with a spectrum given by

$$\omega = \omega_0 \sqrt{(1-\beta^2)\frac{3}{2}(\gamma-1)(n+1)} \tag{26}$$

Similar results are found in spherical systems [14]. These results are very interesting, because they allow a "measurement" of the equation of state for the ultra cold atoms which is simply determined by the polytropic exponent γ. In fact, experimental techniques usually make possible the measurement of frequencies in a very precise manner, and these results relate the polytropic exponent γ with the mode frequencies and the angular velocity of the cloud.

B. Multiple scattering regime

In contrast with the temperature limited regime for small traps, when we have a large number of particles trapped, the system is dominated by the collective effects and we can neglect the temperature effects by setting $\gamma = 0$ in eq. [20], yielding

$$\left(\omega^2 - \omega_p^2 \frac{n_0(r)}{n_0(0)}\right)\eta(r) = 0. \tag{27}$$

Introducing the water-bag solution for the equilibrium density profile given by eq. [16], found earlier, we obtain

$$\omega = \sqrt{3}\omega_0 \left(1-\beta^2\right)^{1/2} = \omega_p. \tag{28}$$

Remember that the multiple scattering regime corresponds to the limit $\Omega_p^2 \to 1$ with $\Omega_p^2 = \frac{Qn(0)}{3m\omega_0^2(1-\beta^2)} = \frac{\omega_p^2}{3\omega_0^2(1-\beta^2)}$ which implies $\sqrt{3}\omega_0\left(1-\beta^2\right)^{1/2} = \omega_p$. We then have a breathing mode in the system at the plasma frequency ω_p. The latter result is formally analogous to the well-known, in plasma physics, uncompressional monopole oscillation of the system at the classical plasma frequency. The same solution is found for systems with spherical symmetry [14]. We shall, however, notice that this solution is not unique. A simple manipulation of eq. [18] and [19] and introducing the equilibrium water-bag profile yields

$$\nabla^2\left[\varepsilon(\omega)\delta\phi_c\right] = 0 \tag{29}$$

with $\varepsilon(\omega) = 1 - 3\frac{\omega_0^2(1-\beta^2)}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$. This equation holds for the interior region of the cloud while for outside the cloud we shall have $\nabla^2\delta\phi_c = 0$, $\varepsilon(\omega) = 1$. The breathing mode obtained earlier can be derived from this formulation by setting the particular solution to eq. [29] as $\varepsilon(\omega) = 0$, which implies $\omega = \sqrt{3}\omega_0(1-\beta^2)^{1/2} = \omega_p$ as before. The general solution to eq. [29] in given by a simple power series in spherical coordinates. In cylindrical coordinates, the solution is more complicate but it can be given in terms of the Bessel functions. We must consider two regions for the solution, the inside region of the cloud, $r < a$, and the outside region, $r > a$, where a stands for the radius of the cloud. The solutions to the Laplace equation then read [24]

$$\varepsilon(\omega)\delta\phi_c^{in} = A_{lk}\frac{I_l(kr)}{I_l(ka)}e^{il\phi}e^{ikz} \tag{30}$$

and

$$\delta\phi_c^{out} = B_{lk}\frac{K_l(kr)}{K_l(ka)}e^{il\phi}e^{ikz} \tag{31}$$

where $I_l(kr)$ and $K_l(kr)$ are the modified Bessel functions and A_{lk} and B_{lk} are constants. We have also included the possibility of a perturbation propagating in the z direction with wavevector k. The solutions must satisfy regular continuity conditions at the surface of the cloud ($r = a$)

$$\varepsilon(\omega)\delta\phi_c^{in}(a) = \delta\phi_c^{out}(a) \tag{32}$$

$$\left.\frac{d}{dr}\varepsilon(\omega)\delta\phi_c^{in}(r)\right|_{r=a} = \left.\frac{d}{dr}\delta\phi_c^{out}(r)\right|_{r=a}. \tag{33}$$

The first continuity condition implies $A_{lk} = B_{lk}$ and the second condition yields the allowed modes as

$$\omega = \sqrt{3}\omega_0(1-\beta^2)^{1/2}\sqrt{\left(1 - \frac{I_l'(ka)K_l(ka)}{I_l(ka)K_l'(ka)}\right)^{-1}} \tag{34}$$

In figure [3] we present the dispersion relation given by eq. [34] for different values of l. Notice that a similar result can be found in the context of a cylindrical pore surface-plasmon modes [24].

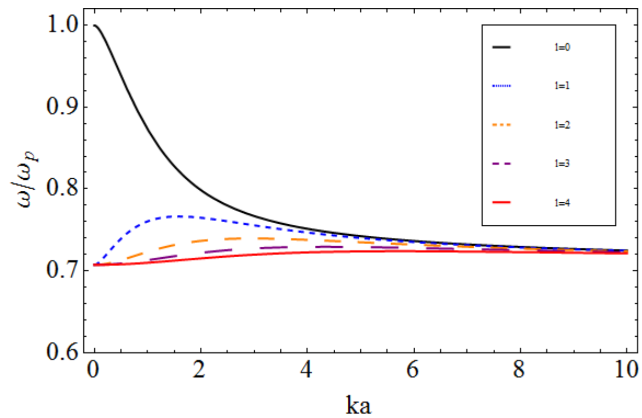

FIG. 3. Dispersion relation for oscillations in the multiple scattering regime, $\Omega_p^2 \to 1$.

We can easily realize that, for the low wavevector limit, $k \to 0$, we recover the breathing mode derived earlier at $\omega_B = \sqrt{3}\omega_0(1-\beta^2)^{1/2} = \omega_p$, for $l = 0$. In this same limit, all the remaining modes ($l \neq 0$) collapse in a single one at $\omega = \sqrt{3/2}\omega_0(1-\beta^2)^{1/2} = \omega_p/\sqrt{2}$, characteristic of a cylindrical system. It is worth mentioning that for a spherical system, the same problem also yields a breathing mode at $\omega_B = \omega_p$ and, solving the Laplace equation in spherical coordinates [14], we get $\omega = \omega_p\sqrt{\frac{l}{2l+1}} = \omega_0\sqrt{\frac{3l}{2l+1}}$ which collapses to our second breathing mode in the limit $l \to \infty$, when we get $\omega = \omega_p/\sqrt{2}$. As a last remark let's clarify the nature of these oscillations. These dispersive and propagating modes appear because we consider a system boundless in the z direction, reason why propagation is allowed. Such modes could be also be excited in bound extended systems, with dimensions in the z direction large compared with the typical wavelength. As mentioned before such a geometry is possible lowering the intensity of the trapping laser beams in the direction of the rotation axis, allowing the cloud to expand and form a cigar-like shape. However, these modes are determined by radial boundary conditions, in equations [32] and [33], whereby they differ from the plasma hybrid waves determined by [19]

$$\frac{\omega}{\omega_p} = \sqrt{1 + k^2u_s^2/\omega_p^2} \tag{35}$$

characteristic of infinite systems, where u_s can be identified with the sound speed. Nevertheless, in the small wavevector limit, the dispersion relation of eq. [34] and figure [3] takes the same form of these plasma waves $\omega/\omega_p \simeq \sqrt{1/2 + k^2u_l^2/\omega_p^2}$, clarifying the plasma nature of these oscillations. For higher wavevectors the radial boundary conditions alter the form of this dispersive relation.

IV. CONCLUSION

The investigation performed here relies on the simple premiss of the existence of a polytropic equation of state for the cold gas, which phenomenologically models a large class of MOT's, including the state of the art ones, that can trap a larger number of particles. The "measurement" of this equation of state, which only depends on the polytropic exponent γ, can then be performed through the density profiles of the gas or by determining the spectrum of the normal modes which, employing direct and absorption imaging techniques, can be done in a very precise manner.

On the other hand, we concluded that this simple polytropic model can account for the so long observed stable orbital modes, that up so far were though to be intrinsically related with the rotation of the cloud, which we can now disregard as orbital modes are also possible in non rotating systems, although requiring a larger number of trapped atoms.

We also discussed the validity of the results derived and, although there are systems with a well defined spherical or cylindrical symmetry, in the general case and specially for rotating systems, the appropriate set of coordinates lies somewhere in between. For this reason we are now trying to figure out how to employ an elliptical coordinate system that in the non rotating case would be reduced to the spherical geometry (zero eccentricity). This would be a more general and appropriate description of the cloud. Another important aspect ignored here is the toroidal oscillation modes that possibly could be excited in solutions with orbital modes, whose dynamics are not evaluated here. These investigations will be published in a future work.

Also related to these features, and following the annunciated new trend of studying astrophysical phenomena in the laboratory, the results present within this paper and in the previous work [14], allow for an investigation of the equilibrium and dynamical properties of the Lane-Endem equation, establishing another interesting connection between the community of ultra cold matter and astrophysics. In particular we can refer to the recent observation of a new class of variable stars [25]. These rapidly rotating main-sequence stars are not expected to pulsate or to have any other physical characteristic that would lead to periodic variations of their luminosity, according to current theories. One of the hypothesis considered is that fast rotation could alter the internal conditions of a star enough to sustain stellar pulsations. But there is currently no stellar model that can predict whether pulsation can be sustained in very fast rotating stars. The ability to achieve conditions in laboratory, namely with ultra cold atomic gases, with similar dynamics to stellar and astrophysical fluids, can prove being important in investigating the properties of these systems. In fact, it may be possible to tune the experimental parameters in order to achieve working conditions closer to those found in astrophysical scenarios.

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