Quadratic invariants for discrete non-resonant clusters of interacting waves in fully nonlinear regimes

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Dedicated to my sister

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Resumo

Nesta tese estuda-se ondas dispersas não-lineares, tais como ondas atmosféricas Rossby e ondas de deriva em plasmas, descritas pela equação de Charney-Hasegawa-Mima (CHM). Esta equação tem soluções na forma de três ondas em interacção mais conhecidas por um tríade, consequentemente quando estes tríades partilham ondas comuns formam-se os chamados clusters. No limite de pequena não-linearidade apenas clusters em exacta ressonância interagem, enquanto que no limite de não-linearidade finita, clusters não-ressonantes também podem interagir.

Num primeira parte estuda-se em detalhe estes clusters não-resonantes, e desenvolve-se um formalismo para se encontrar um grupo de invariantes quadráticos funcionalmente independentes em termos de um problema de álgebra linear associado ao espaço nulo de uma matriz, mais conhecida pela matriz não-ressonante. De seguida mostra-se uma nova forma de contar os graus de liberdade da equação de evolução de um cluster arbitrário em termos da matriz não-ressonante e da matriz ressonante, esta última tendo sido introduzida numa recente publicação por Bustamante e colaboradores. Numa segunda parte aplica-se a teoria desenvolvida a exemplos conhecidos de clusters não-ressonantes resultantes da equação de CHM. Assume-se que clusters com poucos modos de interacção podem ser integráveis se o seu número de graus de liberdade for reduzido. Tal redução surge quando os vectores de onda de um cluster obedecem a certas relações de modo a que o grau da matriz não-ressonante varie. Os resultados desta tese constituem o ponto de partida na direcção da determinação de cascatas turbulentas de energia e enstrofia nos regimes discreto e mesoscópico de ondas turbulentas.

Palavras-chave: Dinâmica de Fluidos, Sistemas não lineares, Ondas turbulentas, Equação de CHM, Invariantes Quadráticos.

Abstract

In this thesis we study nonlinear dispersive waves such as atmospheric Rossby waves, and drift waves on plasmas modeled by the nonlinear Charney-Hasegawa-Mima (CHM) equation. This equation admits solutions of interacting triplets of waves called triads, and when sharing common modes, these triads couple together to form large clusters. In the limit of small nonlinearity, clusters only interact via exact-resonances, however for finite nonlinearity clusters can interact non-resonantly.

We study in detail the construction of the evolution equation governing these non-resonant clusters, and develop a formalism to find a set of functionally independent quadratic invariants for these clusters, in terms of a basic linear algebra problem consisting of finding the null space of the so called non-resonant matrix. We show how to count the number of effective degrees of freedom of the evolution equations using the non-resonant cluster matrix and the so-called resonant cluster matrix. The latter matrix was introduced in a recent publication by Bustamante and co-workers, but the non-resonant case had not been understood until this thesis. We apply this theory to discuss selected examples of non-resonant clusters arising in numerical simulations of the CHM equation. Clusters involving a few interaction modes can be integrable if the number of degrees of freedom is reduced. Such reduction is attained when wave vectors in the cluster satisfy certain relations so that the cluster matrices change their rank. The results of this thesis are staring point towards the modeling of turbulent cascades of energy and enstrophy in discrete and mesoscopic wave turbulence.

Keywords: Fluid Dynamics, Nonlinear Systems, Wave Turbulence, CHM equation, Quadratic Invariants.

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Chapter 1

Introduction

Wave

In physics a wave is a recognizable signal feature of a disturbance such as an abrupt change in some quantity provided that it can be clearly recognized at any time and any location [1]. It travels through space and matter from one part of the medium to another, accompanied by a transfer of energy. Wave motion transfers energy from one point to another, often with no permanent displacement of the particles of the medium, that is, with little or no associated mass transport. They consist of oscillations or vibrations around almost fixed locations. Waves are described by a wave equation which sets out how the signal proceeds over time, where the mathematical form of this equation varies depending on the type of wave.

Nonlinear Dispersive Wave

A mathematical differential equation describing a system of nonlinear waves is called dispersive if its linear part has wave solutions $\psi(\mathbf{x}, t)$, that depend on space and time as follows:

$$\psi(\mathbf{x}, t) = A_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t + \varphi). \tag{1.1}$$

Here, $A_{\mathbf{k}}$ is a constant wave amplitude, φ the constant wave phase, $\mathbf{k}$ the constant wave vector associated with the wave's wavelength λ by $k = |\mathbf{k}| = \frac{2\pi}{\lambda}$, and $\omega_{\mathbf{k}} \equiv \omega(\mathbf{k})$ the angular wave frequency which is a quantity determined by the particular wave system, commonly known as the wave dispersion relation. The general solution of the differential equation will consist of the superposition of several modes of the form (1.1) for different wave vectors $\mathbf{k}$. Waves are said to be dispersive if their phase velocity $\frac{\omega}{k}$, depends on k. If the phase velocity is not the same for all wave vectors, the modes Eq. (1.1) with different wave vectors will propagate at different velocities, in consequence they will disperse or spread.

1.1 Wave Turbulence

Wave turbulence is defined as out of equilibrium statistical mechanics of random nonlinear waves . Wave turbulence is at the same time a physical phenomenon and a mathematical formalism of systems commonly observed in nature that have not been explained rigorously yet. These include systems of weakly nonlinear dispersive waves, such as planetary Rossby waves which describe weather and climate evolutions [2–4]; drift waves in magnetically confined plasmas [5]; Alfvén waves in astrophysical magnetohydrodynamic turbulence [6, 7]; waves in Bose-Einstein condensates and in nonlinear optics [8, 9] among many other physical wave systems. The name wave turbulence may appear paradoxical since turbulence is associated mostly with vortices, and the waves are only secondary. However, a system of weakly nonlinear waves behaves very similar to the classical turbulence system, even in absence of vortices. The first approach to wave turbulence traces back to a paper published by Rudolph Peierls in 1929 [10], where in order to describe the kinetics of phonons in anharmonic crystals, the wave kinetic equation was derived for describing evolution of the wave spectrum. However, the solutions obtained described thermodynamic equilibria and small deviations from thermodynamic states, which are not the most interesting and relevant to the wave turbulence systems. Later in 1965, Vladimir Zakharov [11] discovered a new type of solutions of the kinetic equation corresponding to a constant energy flux through scales. These solutions are called Kolmogorov-Zakharov spectra because they are analogous to the Kolmogorov spectrum of hydrodynamic turbulence which describes the energy distribution in turbulent systems. These solutions put wave turbulence systems into the domain of general turbulence, i.e. strongly nonequilibrium statistical systems with many degrees of freedom whose state is determined by a flux through phase space rather than by temperature and thermodynamic potentials as in the case of equilibrium or weakly non-equilibrium systems [12].

Wave turbulence described by the kinetic equation in nonlinear wave systems assumes weak nonlinearity and infinite box limit. Numerical and laboratory experiments confirm the Kolmogorov-Zakharov spectrum predicted by the wave turbulence theory for capillary waves on a fluid surface (water, ethanol, liquid hydrogen or liquid helium) [13]. However, for Rossby and drift waves, systems become sensitive to the finite box effects. In laboratory experiments, the finite size effects and presence of breaking coherent waves obscure the Kolmogorov-Zakharov state, and the latter is not observed in its pure form. Moreover, similar behavior often occurs in nature when waves are bounded, e.g. for planetary Rossby waves bounded by the finite planet radius. When systems are bounded, the set of interacting wave vectors $\mathbf{k}$ are no longer continuous variables, but instead become discrete variables. For waves in a $d-$periodic box with all side lengths $L = 2\pi$, the normal modes are given by (1.1) with a discrete set of wavenumbers $\mathbf{k} = \frac{2\pi\mathbf{m}}{L}$ where $\mathbf{m} \in \mathbb{Z}^d$. Systems that are in a state where $\mathbf{k}$ are discrete variables, and intermediate states where both discrete and continuous $\mathbf{k}$ exist are called Discrete and Mesoscopic Wave Turbulence. Therefore, three different regimes are singled out in Wave Turbulence: $Kinetic\ Wave\ Turbulence$, $Discrete\ Wave\ Turbulence$, and $Mesoscopic\ Wave\ Turbulence$. In this thesis we focus on aspects of discrete and mesoscopic wave turbulence [14–18].

1.1.1 Three-Wave Resonances

In this thesis we deal with a system of weak Rossby and drift waves described by the so called Charney-Hasegawa-Mima (CHM) equation (see Section 1.2). This equation is nonlinear, so modes with different vectors couple together and exchange energy. If the nonlinearity is weak, one finds that this energy exchange is generally quite slow and occurs most efficiently between groups of modes which are in resonance. This equation has quadratic nonlinearity therefore such resonances involve three modes [15]. On the other hand if the nonlinearity was cubic, resonances of four interacting modes would be predominate. Three wave vectors $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ satisfying the resonance conditions,

$$\begin{aligned} &\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = 0, \\ &\omega_1 + \omega_2 - \omega_3 = 0, \end{aligned} \tag{1.2}$$

are referred to as a resonant triad. Although energy exchange is maximized when triads are in exact resonance, it is often necessary to rely on approximate resonances to account for energy transfer. Approximate resonance is possible due to the phenomenon known as nonlinear resonance broadening. This is an effect whereby the frequency of a wave acquires a correction to its linear value which depends on the amplitude. Triads which are not exactly in resonance can then interact at finite amplitude if the frequency mismatch is less than this correction. Such triads are known as quasi-resonant triads or non-resonant triads and satisfy the broadened resonance conditions,

$$\begin{aligned} &\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = 0, \\ &|\omega_1 + \omega_2 - \omega_3| < \delta, \end{aligned} \tag{1.3}$$

where δ is a characteristic value for the resonance broadening taken to be positive.

A triad (resonant or non-resonant) can be grouped together with other modes from other triads to form resonant clusters of various sizes such as butterflies, kites, stars, chains, etc. Isolated triads are integrable systems where its motion is regular and periodic, but for large clusters the dynamics can be irregular and chaotic, where the systems are not integrable.

Integrable Systems

In this section we show a theorem on integrability presented in [19, 20]. Considering the Einstein notation $f_{,i} \equiv \partial f / \partial x^i$, a n-dimensional system of autonomous evolution equations of the form,

$$\frac{dx^i}{dt}(t) = \Delta^i(x^j(t)), \qquad i = 1, ..., n, \tag{1.4}$$

where any scalar function $f(x^i, t)$ that satisfies $\frac{d}{dt}(f(x^i(t), t)) = \frac{\partial}{\partial t} f + \Delta^i f_{,i} = 0$ is called a conservation law. Therefore there are two types of conservation laws: an explicit time dependent one called a dynamical invariant of the system, and an explicit time independent one called simply a conservation law. The system Eq. (1.4) is integrable if there are n functionally independent dynamical invariants. If it possesses $n-1$ functionally independent conservation laws, then it is constrained to move along a $1-$dimensional manifold, and the way it moves is dictated by one dynamical invariant. This dynamical invariant can be obtained from the knowledge of the $(n-1)$ conservation laws and the explicit form of the system, i.e. Eq.(1.4) is integrable then. It follows from the Theorem below that in many cases the knowledge of only $(n-2)$ conservation laws is enough for the integrability.

Theorem 1. *Let us assume that the system* (1.4) *possesses a standard Liouville volume density*

$$\rho(x^i) : (\rho\Delta^i)_{,i} = 0 \tag{1.5}$$

and (n-2) functionally independent conservation laws, $H^1, ..., H^{n-2}$. Then a new conservation law can be constructed, which is functionally independent of the original ones, and therefore the system is integrable.

1.1.2 Wave Turbulence Regimes

The finite size effects in Wave Turbulence can be characterized by considering the nonlinear frequency broadening Γ which is the inverse correlation time of wave packets. Such a correlation time is roughly equal to the characteristic time of nonlinear evolution. Below it is seen the three different regimes due to the finite box size, which are described by different relationships between Γ and the frequency spacing in the finite box $\Delta\omega$ (see e.g. [16]),

$$\Delta\omega = \left|\frac{\partial\omega_{\mathbf{k}}}{\partial\mathbf{k}}\right|\frac{2\pi}{L} \sim \frac{\omega_{\mathbf{k}}}{kL}. \tag{1.6}$$

When wave amplitudes are very small, the nonlinear frequency broadening is much less that the frequency spacing,

$$\Gamma \ll \Delta\omega, \tag{1.7}$$

and this is the regime of discrete Wave Turbulence and only waves that are in exact resonance can interact and exchange energy. Very large clusters are rare and there are usually a large number of small clusters, the simplest being an isolated triad. If the energy of the system is initially concentrated in these small clusters, then an energy cascade cannot take place. An extreme version of such a situation is when there are no resonant triads at all, like in the case of the capillary surface waves, in which case turbulence is "frozen". For larger amplitudes, the nonlinear frequency broadening gets bigger and originally isolated clusters may become connected via quasi-resonances. This will allow energy to be transferred between waves which are not exactly resonant. If the wave system is forced weakly but continuously, the amplitudes will eventually become sufficiently large and the resonance broadening will become approximately the same size as the frequency spacing:

$$\Gamma \sim \Delta\omega, \tag{1.8}$$

this regime corresponds to the mesoscopic wave turbulence: both types of wave turbulence kinetic and discrete exist with the system oscillating between them. For much larger levels of forcing the resonance broadening Γ will always greatly exceed $\Delta\omega$,

$$\Gamma \gg \Delta\omega, \tag{1.9}$$

in which case the wave system will be in the kinetic regime and an energy cascade between the forcing and dissipation scales gets triggered.

1.1.3 Hamiltonian Evolution Equation

Wave turbulence formalism deals with dispersive and weak nonlinearity wave systems. A general class of non-dissipative nonlinear waves is usually described within the framework of the classical Hamiltonian approach. After a proper change of variables, the Hamiltonian equation can be presented in the universal form with canonical variables that characterize the wave amplitudes. The Hamiltonian equations for the space-homogeneous systems are most conveniently written in Fourier space lattice, because it is a natural space for describing the wave solutions (see e.g. [21]). Introducing the Fourier transform of the canonical variables the Hamiltonian equation can be written as follows,

$$i\dot{a}_{\mathbf{k}} = \frac{\partial \mathcal{H}}{\partial a^*_{\mathbf{k}}} \tag{1.10}$$

where $\dot{a}_{\mathbf{k}}$ is notation for time derivative, $a_{\mathbf{k}}$ is the amplitude of the Fourier mode corresponding to the wave vector $\mathbf{k}$ and $a^*_{\mathbf{k}}$ denotes the complex conjugate of $a_{\mathbf{k}}$. For the waves of small amplitudes the Hamiltonian can be expanded in powers of $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$:

$$\begin{aligned} &\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + ..., \\ &\mathcal{H}_2 = \sum_{n=1}^{\infty} \omega_{\mathbf{k}} |a_{\mathbf{k}}|^2, \\ &\mathcal{H}_3 = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} V^3_{12} a_1 a_2 a^*_3 \delta^3_{12} + c.c., \end{aligned} \tag{1.11}$$

where the following notation is introduced: $\omega_{\mathbf{k}} \equiv \omega(\mathbf{k})$, $a_j \equiv a_{\mathbf{k}_j}$, $\delta^3_{12} \equiv \delta(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)$ and $V^3_{12} \equiv (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is the nonlinear interaction coefficient characteristic of a three-wave interaction. The Hamiltonian part $\mathcal{H}_0$ is independent of $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$ therefore it is omitted here because it does not contribute to the equation of motion. Only waves excited about steady state equilibrium are considered, thus the linear Hamiltonian $\mathcal{H}_1 = 0$. The first term of the expansion, $\mathcal{H}_2$ produces a linear equation of motion,

$$i\dot{a}_{\mathbf{k}} = \omega_{\mathbf{k}} a_{\mathbf{k}}, \tag{1.12}$$

thus describes noninteracting waves with the dispersion relation $\omega_{\mathbf{k}}$. The second term $\mathcal{H}_3$ has quadratic nonlinearity and describes the process of decaying of a single wave into two waves or confluence of two waves into a single one. The full Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$ with Eq. (1.10) yields the system of evolution equations,

$$i\dot{a}_{\mathbf{k}} = \omega_{\mathbf{k}} a_{\mathbf{k}} + \sum_{\mathbf{k}_1,\mathbf{k}_2} (V^{\mathbf{k}}_{12} a_1 a_2 \delta^{\mathbf{k}}_{12} + 2V^{1*}_{\mathbf{k}2} a_2 a^*_1 \delta^1_{\mathbf{k}2}), \tag{1.13}$$

Three-wave interactions take place in wave systems where the leading order of nonlinearity is quadratic, e.g. for Rossby and drift waves, etc. In the CHM equation the main contribution to the evolution equation in the limit of weak wave amplitudes comes from modes satisfying the three-wave resonance conditions

Eq. (1.2). When restricting the allowed modes to these resonant modes, the Hamiltonian equations of the form Eq. (1.13) are satisfied. However, when non-resonant modes are allowed to interact, the evolution equation governing the system is still quadratic but it cannot be written in the form of Eq. (1.13).

1.2 CHM Equation

The barotropic vorticity equation [22] also known as the Charney-Hasegawa-Mima equation [23, 24] describes geophysical Rossby atmospheric waves [25] and drift waves in magnetized plasmas [26]. This is the simplest two-dimensional model of the large scale dynamics of a shallow layer of fluid on the surface of a strongly rotating sphere. The surface of the sphere is approximated locally by a plane, $\mathbf{x} = (x, y)$, with x varying in the longitudinal (meridional) direction and the y varying in the latitudinal (zonal) direction. The equation gives:

$$\frac{\partial}{\partial t}(\Delta\psi - F\psi) + \beta\frac{\partial \psi}{\partial x} + J[\psi, \Delta\psi] = 0. \tag{1.14}$$

For Rossby waves the wave field $\psi(\mathbf{x}, t)$ is the geopotential height; β is the Coriolis parameter ($f = 2|\Omega|\sin\varphi$ with Ω the normal component of the earth's rotation rate and φ the earth latitude) measuring the variation of the Coriolis force with latitude $\beta = \frac{\partial f}{\partial y} = 2|\Omega|\cos\varphi_0$; $F = \frac{1}{\rho^2}$ where ρ is the Rossby deformation radius. For drift waves the wave field $\psi(\mathbf{x}, t)$ is the electrostatic potential; β is the gradient of the logarithm of the plasma density $\beta = \frac{\partial}{\partial y}\ln n_0$; and $F = \frac{1}{\rho^2}$ where ρ is the Larmour radius. $\Delta \equiv \nabla^2$ is the Laplacian operator. We also introduced the notation for the Jacobian operator where,

$$J[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{1.15}$$

This equation has an anisotropic dispersion relation,

$$\omega_{\mathbf{k}} = -\frac{k_x\beta}{k^2 + F} \tag{1.16}$$

the nonlinear term shows the mixing of modes, and is the starting point in the construction of so-called resonant triad solutions, which are approximate solutions, valid in the asymptotic limit when the oscillation amplitudes are small.

1.2.1 CHM Equation in Fourier representation

The evolution equation for systems of Rossby and drift waves obtained from the CHM equation, similarly to the Hamiltonian evolution equation. According to the WT formalism, the first step is to write the CHM equation in a Fourier **k**-space such that the PDE is reduced into a simple ODE. Using the Fourier transform of $\psi(x, y)$,

$$\hat{\psi_{\mathbf{k}}} = \frac{1}{L^2}\int_{Box}\psi(x, y)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}, \tag{1.17}$$

in physical space is equivalent to,

$$\dot{\hat{\psi}}_{\mathbf{k}} + i\omega_{\mathbf{k}}\hat{\psi}_{\mathbf{k}} = \frac{1}{2}\sum_{1,2}\delta^{\mathbf{k}}_{12}\hat{\psi}_1\hat{\psi}_2\frac{(k_2^2 - k_1^2)}{k^2 + F}(k_{1x}k_{2y} - k_{2x}k_{1y}), \tag{1.18}$$

where the sum is over all modes that satisfy $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and the term $\frac{1}{2}(k_2^2 - k_1^2)$ is a result of the symmetrization of the nonlinear term of the PDE. In order to separate time scales, we introduce variables which do not change in the linear approximation (i.e. when the nonlinear term is zero and no interactions take place), these are called the interaction representation variables $\hat{b}_{\mathbf{k}} = \hat{\psi}_{\mathbf{k}}e^{i\omega_{\mathbf{k}}t}$ and Eq. (1.18) is written as

$$\dot{b}_{\mathbf{k}} = \frac{1}{2}\sum_{1,2}U^{\mathbf{k}}_{12}\delta^{\mathbf{k}}_{12}b_1b_2e^{-i\omega^{\mathbf{k}}_{12}t}, \tag{1.19}$$

where $\omega^{\mathbf{k}}_{12} = \omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}$ is the frequency detuning parameter or the linear frequency mismatch of the triad. The coefficient $U^{\mathbf{k}}_{12}$ is called the nonlinear interaction coefficient of the mode $\mathbf{k}$ which gives,

$$U^{\mathbf{k}}_{12} = \frac{k_2^2 - k_1^2}{k^2 + F}(k_{1x}k_{2y} - k_{2x}k_{1y}), \tag{1.20}$$

this coefficient can be decomposed into two factors, where the first one (the left factor) is only dependent on the wave vectors length squares, and the second factor involves the third component of the cross product of two wave vectors. The interaction coefficient is symmetric on its lower indices and is anti-symmetric under reflection of any of its lower wave vectors:

$$U^{\mathbf{k}}_{12} = U^{\mathbf{k}}_{21} = -U^{\mathbf{k}}_{-12}. \tag{1.21}$$

For previously studied Hamiltonian systems [27], the limit of small enough amplitude oscillation is considered, such that efficiently-interacting triads could only be in exact resonance. However most systems described by Eq. (1.14) are non-resonant, where $\omega^{\mathbf{k}}_{12} \neq 0$. Such systems are much more sensible and observed in nature, in a way that the most efficient energy transfer mechanism involves the coupling of two non-resonant triads sharing two modes, where the non-linear frequency oscillation of the wave amplitudes $b_{\mathbf{k}}$ is comparable with the linear frequency mismatch $\omega^{\mathbf{k}}_{12}$ [28]. In other words, the most efficient mechanism of energy transfer occurs at intermediate amplitudes such that non-linear time scales are comparable with linear time scales.

1.2.2 Energy and Enstrophy

The CHM equation conserves both energy E and enstrophy Ω, for any system of Rossby and drift waves. Rearranging the terms of Eq. (1.14), multiplying by ψ, and integrating over a square periodic domain of side L, while using the fact that boundary terms vanish upon integration by parts, gives

$$E \equiv \frac{1}{2}\int\big((\nabla\psi)^2 + F\psi^2\big)d\mathbf{x} = const., \tag{1.22}$$

this equation represents the conservation of energy of the system. In Fourier **k**-space this reads

$$E = \frac{1}{2}\sum_{\mathbf{k}}(|\mathbf{k}|^2 + F)|\hat{\psi}_{\mathbf{k}}|^2, \tag{1.23}$$

with dimensions of the square velocity of the flow. Similarly, multiplying Eq. (1.14) by $\Delta\psi$, gives the conservation of enstrophy, defined as

$$\Omega \equiv \int \big((\Delta\psi)^2 + F(\nabla\psi)^2\big)d\mathbf{x} = const., \tag{1.24}$$

with dimensions of a quartic velocity. In Fourier space conservation of enstrophy is equivalent to,

$$\Omega = \sum_{\mathbf{k}} |\mathbf{k}|^2\big(|\mathbf{k}|^2 + F\big)|\hat{\psi}_{\mathbf{k}}|^2. \tag{1.25}$$

These two conservation properties are characteristic of most wave systems and for different types of regimes of Wave Turbulence, and it will be shown that they are conserved for any generic cluster of connected triads satisfying Eq. (1.19).

Chapter 2

Clusters

As a consequence of the discreteness of the wave vectors **k**, any mode can be part of few triads. This case of groups of triads sharing common modes is called a cluster. For waves with very small amplitudes, interaction between waves occur only when modes are in exact resonance, therefore only resonant clusters will exist. Big clusters are rare since in them each mode belongs to one or more exact resonant triads. The dynamics of resonant clusters has been studied in [27]. In the case of finite amplitudes non-resonant clusters are formed because the modes nonlinear frequencies depend on the waves' amplitudes. In this case any mode may belong to a bigger group of non-resonant triads (quasi-resonant triads) than in the resonant case, therefore bigger clusters will be formed and energy will be more successfully transferred nonlinearly through different paths along the non-resonant clusters [28].

2.1 Conserved quantities in clusters general setting

Consider a conserved quantity I with quadratic dependence on the wavemodes amplitudes,

$$I = \sum_{n=1}^{N} \varphi_n |b_n(t)|^2, \tag{2.1}$$

written as a sum over all modes in the cluster, and $\varphi_n \equiv \varphi_{k_n}$ is a real function of the wavenumbers of the modes (as in the case of energy and enstrophy). This quantity will be conserved under the evolution of the system if and only if

$$\dot{I} = \sum_{n=1}^{N} \varphi_n (b_n^* \dot{b}_n + b_n \dot{b}_n^*) = 0. \tag{2.2}$$

Substituting the evolution equations Eq. (1.19) for the modes amplitudes $\dot{b}_n$ in Eq. (2.2) we get that

$$\dot{I} = \sum_{123} (U_{12}^3 \varphi_3 - U_{23}^1 \varphi_1 - U_{13}^2 \varphi_2)(b_1 b_2 b_3^* e^{i\omega_{12}^3 t} + c.c.) = 0, \tag{2.3}$$

where the sum is over all triads in the cluster that satisfy $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. Consider each term in the sum in Eq. (2.3), (i.e. consider each triad in the sum). The factor $b_1 b_2 b_3^* e^{i\omega_{12}^3 t} + c.c.$ is a function of time, typically independent of the corresponding factors for the other triads. Consequently, the conservation of I is possible only if the factor $(U^3_{12}\varphi_3 - U^1_{23}\varphi_1 - U^2_{13}\varphi_2)$ is equal to zero for each triad in the cluster. This is equivalent to a problem of a linear system of equations.

2.1.1 Energy and Enstrophy conservation

We show that energy and enstrophy are both conserved quantities. Substituting Eq. (1.23) in Eq. (2.3) for the energy, we get

$$\dot{E} = \sum_{123}(k_{1x}k_{2y} - k_{2x}k_{1y})\left(k_2^2 - k_1^2 + k_3^2 - k_2^2 - k_3^2 + k_1^2\right)(b_1 b_2 b_3^* e^{i\omega^3_{12}t} + c.c.) = 0. \tag{2.4}$$

Substituting Eq. (1.25) in Eq. (2.3) for the enstrophy, we get

$$\dot{\Omega} = \sum_{123}(k_{1x}k_{2y} - k_{2x}k_{1y})\left(k_2^2k_3^2 - k_1^2k_3^2 + k_3^2k_1^2 - k_2^2k_1^2 - k_3^2k_2^2 + k_1^2k_2^2\right)(b_1 b_2 b_3^* e^{i\omega^3_{12}t} + c.c.) = 0. \tag{2.5}$$

Therefore energy and enstrophy are shown to be conserved for the generic non-resonant system of connected triads.

Linear independence of Energy and Enstrophy

Considering a cluster with N modes and M triads, energy Eq. (1.23) and enstrophy Eq. (1.25) are linearly dependent if and only if

$$\Omega = \xi E, \tag{2.6}$$

where ξ is a constant, this implies that

$$\mathbf{k}_n = const., \qquad \text{with} \quad n = 1, ..., M, \tag{2.7}$$

where $\mathbf{k}_n$ is equal to a different constant, other than ξ, in other words all wave vectors $\mathbf{k}_n$ of the cluster must have the same size. Therefore in generic clusters with at least one wave vector of different size, there are at least two linearly independent invariants E and Ω. Notice that in a general cluster there might be more independent invariants. This is studied in detail in Section 2.2.

2.1.2 Connection to exact resonant clusters

Let us consider in this subsection, only resonant clusters (i.e. $\omega^3_{12} = 0$). Then it is possible to obtain a relation to the work presented in [27] on Hamiltonian resonant systems. The linear systems of equations shown in Eq. (2.3) can be manipulated to be equivalent to the linear systems of equations in [27]. In order to show this we do the transformation on the variable $\varphi_{\mathbf{k}}$ (related to the transformation to normal variables presented in [14]),

$$\varphi_{\mathbf{k}} = \gamma_{\mathbf{k}} \frac{(k^2 + F)^2}{\beta k_x}, \tag{2.8}$$

where zonal modes $k_x = 0$ are excluded from the analysis. Eq. (2.1) now reads

$$I = \sum_{n=1}^{N} \gamma_n \frac{(k^2+F)^2}{\beta k_x} |b_n|^2, \tag{2.9}$$

and taking its time derivative, we get

$$\dot{I} = \sum_{123} (\mathbf{k}_1 \times \mathbf{k}_2)_z \left[\gamma_3 (k_2^2 - k_1^2) \frac{(k_3^2+F)}{\beta k_{3x}} + \gamma_1 (k_3^2 - k_2^2) \frac{(k_1^2+F)}{\beta k_{1x}} + \gamma_2 (k_1^2 - k_3^2) \frac{(k_2^2+F)}{\beta k_{2x}} \right] (b_1 b_2 b_3^* + c.c.), \tag{2.10}$$

where the three cross terms of the interaction coefficients are written in terms of the same cross product $(\mathbf{k}_1 \times \mathbf{k}_2)_z$ using the relation $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$ for the x and y components of the wave vectors. The resonance condition of the frequency gives

$$\frac{k_{1x}}{k_1^2+F} + \frac{k_{2x}}{k_2^2+F} = \frac{k_{3x}}{k_3^2+F}, \tag{2.11}$$

and using the resonance condition $k_{3x} = k_{1x} + k_{2x}$ terms can be manipulated and rearranged such that Eq. (2.11) becomes equivalent to

$$k_{1x}(k_1^2 - k_3^2)(k_2^2 + F) = k_{2x}(k_3^2 - k_2^2)(k_1^2 + F), \tag{2.12}$$

or

$$k_{1x}(k_2^2 - k_1^2)(k_3^2 + F) = k_{3x}(k_3^2 - k_2^2)(k_1^2 + F). \tag{2.13}$$

Substituting these last two relations in (2.10), we get

$$\dot{I} = \sum_{123} (k_1 \times \mathbf{k}_2)_z \frac{k_3^2 - k_2^2}{\beta k_{1x}} (k_1^2 + F) \Big(\gamma_1 + \gamma_2 - \gamma_3 \Big) (b_1 b_2 b_3^* + c.c.), \tag{2.14}$$

and this gives the linear system of equations $\gamma_1 + \gamma_2 - \gamma_3 = 0$ obtained in [27] for Hamiltonian resonant clusters. The proof that the linear system $\gamma_1 + \gamma_2 - \gamma_3 = 0$ is equivalent to the linear system $(U_{12}^3 \varphi_3 - U_{23}^1 \varphi_1 - U_{13}^2 \varphi_2) = 0$ for all triads in the case of exact resonances, can be achieved by using the fact that it is not possible to find non-zero resonant triads with $\mathbf{k}_1 \parallel \mathbf{k}_2$ or $|\mathbf{k}_1|^2 = |\mathbf{k}_2|^2 = |\mathbf{k}_3|^2$ if zonal modes are excluded. It is important to stress that the mapping from the system $(U_{12}^3 \varphi_3 - U_{23}^1 \varphi_1 - U_{13}^2 \varphi_2) = 0$ to the general system $\gamma_1 + \gamma_2 - \gamma_3 = 0$ is possible only in the case when all triads are resonant.

Momentum conservation

Resonant clusters also conserve both components of momentum k_x and k_y. It is evident from Eq. (2.14) when $\gamma_n = k_{nx}$ and $\gamma_n = k_{ny}$ that $\dot{I} = 0$ because the resonance conditions, but the case $\gamma_n = k_{nx}$ is actually derived from conservation of E and Ω because Eq. (2.8) gives a conserved quantity which is a linear combination of E and Ω. As this conservation of E and Ω is present also in the non-resonant case, we can say that k_x momentum is generally conserved in non-resonant clusters.

The above points raise the question: Is k_y momentum conserved for non-resonant clusters? In the non-resonant case, putting $\gamma_n = k_{ny}$ in Eq. (2.8) gives $\varphi_n = \frac{k_{ny}}{\beta k_{nx}}(k_n^2 + F)^2$. This gives rise to a quadratic function

$$\dot{I} = 2(k_1 \times \mathbf{k}_2)_z \Big(k_{1x}(k_1^2 - k_3^2)(k_2^2 + F) - k_{2x}(k_3^2 - k_2^2)(k_1^2 + F) \Big) Re\Big[b_1 b_2 b_3^* e^{i\omega_{12}^3 t} \Big], \tag{2.15}$$

but this is not conserved. In fact, the frequency resonance condition now gives,

$$k_{2x}(k_3^2 - k_2^2)(k_1^2 + F) = k_{1x}(k_1^2 - k_3^2)(k_2^2 + F) + \omega_{12}^3 \sum_i^3 (k_i^2 + F), \tag{2.16}$$

and substituting in Eq. (2.3), we get

$$\dot{I} = 2(k_1 \times \mathbf{k}_2)_z \sum_i^3 (k_i^2 + F)\omega_{12}^3 Re\Big[b_1 b_2 b_3^* e^{i\omega_{12}^3 t} \Big], \tag{2.17}$$

so

$$\begin{aligned} \dot{I} &= 2 \sum_{triads} (k_1 \times \mathbf{k}_2)_z \sum_i^3 (k_i^2 + F)\omega_{12}^3 Re\Big[b_1 b_2 b_3^* e^{i\omega_{12}^3 t} \Big] = \\ &= \frac{\partial}{\partial t} \sum_{triads} (k_1 \times \mathbf{k}_2)_z \sum_i^3 (k_i^2 + F) Im\Big[e^{i\omega_{12}^3 t} b_1 b_2 b_3^* \Big], \end{aligned} \tag{2.18}$$

where $\frac{\partial}{\partial t}$ means derivative with the explicit time dependence.

2.2 Cluster Matrices: Resonant and Non-Resonant

2.2.1 Resonant Cluster Matrix

Considering resonant clusters, the problem of finding a complete set of independent invariants[1] I satisfying $\dot{I} = 0$ in Eq. (2.14) i.e. invariants

$$I = \sum_{n=1}^{N} \gamma_n |b_n|^2, \tag{2.19}$$

with $(\gamma_1 + \gamma_2 - \gamma_3) = 0$ for each triad is equivalent to the problem of finding the maximal set of linearly independent vectors in the null space of a matrix, called the resonant cluster matrix or simply resonant matrix A [27]. The resonant matrix A is a $N \times M$ matrix formed by N columns and M rows, where N is equal to the number of modes, and M the number of triads of the resonant cluster. Each row of A contains exactly two elements with value 1, one element with value -1 and the remaining elements are equal to zero. In other words the m-th row of A corresponds to the coefficients in the resonant conditions for the m-th triad, i.e.

$$A_{mn_1}\mathbf{k}_{n_1} + A_{mn_2}\mathbf{k}_{n_2} + A_{mn_3}\mathbf{k}_{n_3} = 0, \tag{2.20}$$

where m is fixed (m-th triad in the cluster), $m = 1, ..., M$, and (n_1, n_2, n_3) correspond to the three modes in the cluster which belong to the m-th triad, with $1 \leq n_1, n_2, n_3 \leq N$. In this way the resonance

[1]For all types of clusters, resonant and non-resonant, a set of quadratic invariants of the form Eq. (2.19) or Eq. (2.1) is linearly independent if and only if it is functionally independent (in terms of the complex amplitudes $b_{\mathbf{n}}$).

conditions of the resonant cluster can be obtained from the resonant cluster matrix in the form:

$$\sum_{n=1}^{N} A_{mn}\mathbf{k}_n = 0, \qquad \sum_{n=1}^{N} A_{mn}\omega_{\mathbf{k}_n} = 0, \qquad m = 1, ..., M. \tag{2.21}$$

Finding the invariants Eq. (2.19) is then equivalent to solving the linear problem

$$\sum_{n=1}^{N} A_{mn}\gamma_n = 0, \qquad m = 1, ..., M, \tag{2.22}$$

for the M triads in the cluster. In general this is a set of M equations for N unknowns, and the maximal set of linearly independent solutions generate the null space of A, so the number of independent solutions is equal to the number of independent quadratic conserved quantities of the resonant cluster [27]. As an example the linear system of equations of an isolated triad are $\varphi_1 + \varphi_2 - \varphi_3 = 0$, therefore the resonant matrix reads

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \tag{2.23}$$

which gives a maximal set of two linearly independent vectors in the null space of A, in other words, a triad has two quadratic conserved quantities.

2.2.2 Non-Resonant Cluster Matrix

For non-resonant clusters, the problem of finding a complete set of independent quadratic invariants[1] corresponds to the problem of finding solutions of the linear system of equations in Eq. (2.3), $(U_{12}^3\varphi_3 - U_{23}^1\varphi_1 - U_{13}^2\varphi_2) = 0$ for all triads. This is analogous to the case of resonant clusters, in the sense that it is about finding the maximal set of linearly independent vectors in the null space of a new matrix, called the non-resonant cluster matrix or simply non-resonant matrix B. In other words, solving

$$\sum_{n=1}^{N} B_{mn}\varphi_n = 0, \qquad m = 1, ..., M, \tag{2.24}$$

where similarly to the resonant matrix, φ_n $\gamma = (\gamma_1, \gamma_2, ..., \gamma_N)^T$ is the maximal set of linearly independent vectors in the null space of the non-resonant matrix B. This matrix has the same dimension properties as the resonant matrix, such that for a non-resonant cluster with M triads and N modes, the non-resonant matrix B is a $N \times M$ matrix of N columns and M rows. However, a given m-th row of B $m = 1, .., M$ (i.e. m-triad of the cluster) has slightly different elements than the resonant matrix. Where each element of the n-mode $n = 1, ..., N$ of the m-th triad in the resonant matrix is given by the respective interaction coefficient of the triads mode. In other words, the element of the non-resonant matrix of the mode n of triad m is the interaction coefficient of mode n of triad m U_{ij}^n, where b_i and b_j are the other two modes of the triad m. Another difference, is that the signs of the elements in B are the opposite as for the elements in A. Such that, considering a similar row for both matrices, where there is a positive element 1 in A, the respective element in B will be an interaction coefficient with a minus sign, and vice-versa for a negative element in A. Also similarly to A any remaining elements of the rows are equal to zero.

An example of the non-resonant matrix can be given for the isolated triad, where the linear system of equations is given by $(U_{12}^3\varphi_3 - U_{23}^1\varphi_1 - U_{13}^2\varphi_2) = 0$ and the non-resonant cluster matrix B reads,

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 \end{bmatrix}. \tag{2.25}$$

2.3 Counting of degrees of freedom

Integrability of resonant and non-resonant clusters is directly related to the number of degrees of freedom of the respective cluster systems. According to the theorem of integrability in section 1.1.1 an n-dimensional system is integrable if it possesses $(n-2)$ functionally independent conservation laws. Integrable dynamical systems can be reduced to one or two dimensional manifolds by using these conservation laws. The dimension of the reduced manifold is called the effective number of degrees of freedom of the reduced system of equations. In general, a system can have an effective number of degrees of freedom that is greater than two, which leads to potentially chaotic behavior. We show a new way of counting the degrees of freedom for any type of non-resonant clusters of the CHM equation.

Consider a non-resonant cluster of M connected triads and N modes, with dynamical system given by Eq. (1.19) for the N modes. The amplitudes of the non-resonant interacting modes $b_{\mathbf{k}}$ are complex variables, therefore the dynamical system of the cluster corresponds to a system of $2N$ equations (i.e. $b_{\mathbf{k}}$ and their complex conjugates). It is useful to use the amplitude-phase representation for the modes' complex amplitudes

$$b_n = c_n e^{i\theta_n}, \qquad n = 1, ..., N, \tag{2.26}$$

where $c_n = |b_n|$ are real amplitudes and θ_n individual phases of the interacting modes. The effective number of degrees of freedom can be computed as follows. First, notice that the only combination of phases that affects the dynamics of a given triad in a cluster, is called the dynamical phase of the triad, and is defined as

$$\varphi_m = \theta_1 + \theta_2 - \theta_3, \qquad m = 1, ..., M, \tag{2.27}$$

where the individual phases $(\theta_1, \theta_2, \theta_3)$ of the triad are called "slave variables" because they can be obtained by quadratures [20, 29]. The system is then partially reduced to a system of $N + M$ equations of the real amplitudes and dynamical phases of the cluster. However if a cluster has a linearly dependent triad, i.e. a triad where its resonance condition $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ can be obtained as a linear combination of the other resonance conditions of the other triads in the cluster, then its dynamical phase can be obtained as a linear combination of the dynamical phases of the other triads. Therefore the number of effective dynamical phases that contribute to the cluster dynamics are equal to the number of linearly independent resonance conditions in the cluster, which according to the properties of the resonant cluster matrix of a cluster in Subsection 2.2.1, is equal to the number of linearly independent rows of the resonant matrix A. The number of degrees of freedom of the cluster is then reduced to $N + M_A^*$ where M_A^* is the number of linearly independent rows of A.

In Section 2.2.2 it was shown that the number of linearly independent quadratic invariants of a non-resonant cluster is equal to the maximal set of linearly independent vectors in the null space of the non-resonant matrix B. Still considering the example of a non-resonant cluster of M triads and N modes with a $N \times M$ non-resonant matrix B, of N columns and M rows, the maximal set of linearly independent vectors in the null space (i.e the number of linearly independent quadratic invariants) is equal to

$$J = N - M_B^*, \tag{2.28}$$

where M_B^* is the number of linearly independent rows of the non-resonant matrix B. This set of linearly independent quadratic invariants of the cluster reduce even further the degrees of freedom of the cluster as it consists of conservation laws of the system. In that sense the effective number of degrees of freedom (d.o.f.) of the initially considered non-resonant cluster is equal to

$$\begin{aligned} \#d.o.f. &= N + M_A^* - (N - M_B^*) \\ &= M_A^* + M_B^*. \end{aligned} \tag{2.29}$$

This established that the effective number of $d.o.f.$ of non-resonant clusters is related to linearly independence of the rows of both resonant and non-resonant cluster matrices.

Note that a similar count of the degrees of freedom for resonant clusters can be deduced from Eq. (2.29), where in this case #d.o.f. reads

$$\#d.o.f. = 2M_A^*, \tag{2.30}$$

this is possible because of the mapping from A to B presented in Section 2.1.2 in terms of the respective linear system of equations. In other words the non-resonant matrix B is equal to A when the "non-resonance" clusters described by Eq. (1.19) are in exact resonance, (i.e. $\omega_{12}^{\mathbf{k}} = 0$) consequently $M_B^* = M_A^*$. So this shows that for a given exact-resonant cluster the number of $\#d.o.f$ is only dependent on M_A^*.

Chapter 3

Low Dimensional Clusters

In this Section we give a brief description on the dynamics of a few examples of low-dimensional non-resonant clusters governed by Eq. (1.19), with special emphasis on the cluster properties presented in Sections 2.2 and 2.3. Namely the search for quadratic invariants using the non-resonant cluster matrix and counting of degrees of freedom, of generic and special cases of non-resonant clusters obtained from representative initial conditions.

3.1 Triad

An isolated triad is the primary cluster in the case of 3-wave resonances Eq. (1.3), described by a truncated system of three modes.

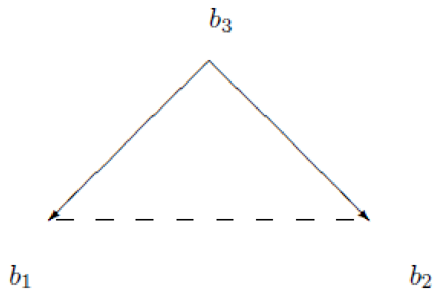

Figure 3.1: Isolated Triad - Arrows mean b_3 is active and b_1, b_2 are passive (see text for explanation).

The dynamical system can be obtained from Eq. (1.19) for the 3-wave resonance condition Eq. (1.3):

$$\begin{aligned} \dot{b_1} &= -b_2^* b_3 U_{23}^1 e^{-i\omega_{12}^3 t} \\ \dot{b_2} &= -b_1^* b_3 U_{13}^2 e^{-i\omega_{12}^3 t} \\ \dot{b_3} &= b_1 b_2 U_{12}^3 e^{i\omega_{12}^3 t}, \end{aligned} \tag{3.1}$$

where each vertex consists of an interacting mode of the triad, and the arrows correspond to the dominant energy transfer direction. The mode b_3 is special in that if the energy is initially mainly localized in mode b_3 (with a little energy in b_1 and b_2) then energy will be efficiently transfered to modes b_1 and b_2. Therefore mode b_3 is said to be the active mode A of the triad, whereas modes b_1 and b_2 are called passive modes P. The notions of A- and P-modes accounts for a very useful description of energy percolation within generic resonance clusters, for a better description see [20, 29–31].

The resonant matrix A of the non-resonant isolated triad reads

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \tag{3.2}$$

whereas the non-resonant matrix B gives

$$B = \begin{bmatrix} -U^1_{23} & -U^2_{13} & U^3_{12} \end{bmatrix}. \tag{3.3}$$

The difference between both matrices was explained in Section 2.2 as an example, where it was shown the explicit dependence of the non-resonant matrix on the representative initial conditions in the form of the interaction coefficients.

The null space of B is encoded in

$$\Phi_B = \begin{bmatrix} -\frac{U^2_{13}}{U^1_{23}} & \frac{U^3_{12}}{U^1_{23}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.4}$$

where Φ_B is a matrix whose columns correspond to the complete set of linearly independent vectors of the null space of B. According to the definition of (2.24), a set of two linearly independent quadratic invariants of the non-resonant triad is obtained from the pair of linearly independent vectors of the null space of B, where the invariants read,

$$\begin{aligned} I_{13} &= U^1_{23}|b_3|^2 + U^3_{12}|b_1|^2, \\ I_{12} &= U^1_{23}|b_2|^2 - U^2_{13}|b_1|^2, \end{aligned} \tag{3.5}$$

called the two Manley-Rowe invariants [32, 33]. This means that the set of linearly independent quadratic invariants obtained from Eq. (2.24) for a non-resonant cluster depend on the modes' wave vectors, particularly the interaction coefficients.

An isolated triad, in both cases resonant and non-resonant, is an integrable system with periodic exchange of energy between modes (see e.g. [17]). When considering the complex wave amplitudes Eq. (3.1) of the interacting modes of the triad in the standard amplitude-phase representation Eq. (2.26), and discarding 'slave' phases, we reduce the dynamical system to a total of four equations:

$$\begin{aligned} \dot{c}_1 &= -c_2 c_3 U^1_{23} \cos\varphi \\ \dot{c}_2 &= -c_1 c_3 U^2_{13} \cos\varphi \\ \dot{c}_3 &= c_1 c_2 U^3_{12} \cos\varphi \\ \dot{\varphi} &= c_1 c_2 c_3 \sin(\varphi)\left(\frac{U^1_{23}}{c_1^2} + \frac{U^2_{13}}{c_2^2} - \frac{U^3_{12}}{c_3^2}\right), \end{aligned} \tag{3.6}$$

with three equations for the real amplitudes c_1, c_2, c_3 and one equation for the dynamical phase $\varphi = \omega^3_{12}t + \theta_1 + \theta_2 - \theta_3$. Considering the two conserved quantities, the Manley-Rowe quadratic invariants (3.5), the number of degrees of freedom is reduced to two: the system is reduced to a 2-dimensional system. In addition, an isolated non-resonant triad has a Hamiltonian, which may be considered as an extra functionally independent conserved quantity which reduces the dynamical system to a 1-dimensional manifold, in other words $\#d.o.f. = 1$. This means that an isolated triad is an integrable system [17, 19, 20, 34]. Therefore, the dynamical system of the isolated non-resonant triad has regular and periodic motion, and is shown for the wave amplitudes b_1, b_2, b_3 in figure 3.2

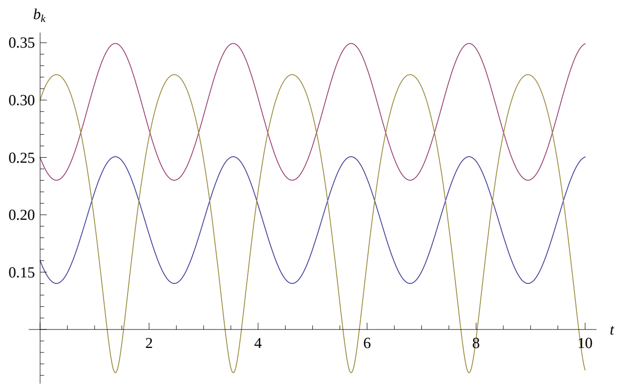

Figure 3.2: Complex wave amplitudes b_1, b_2, b_3 evolution of the non-resonant isolated triad for $0 \leq t \leq 10s$, with the initial conditions for the wave amplitudes, $b_1(0) = .16, b_2(0) = .25, b_3(0) = .3e^{i\pi/2}$, and the interaction coefficients $U^1_{23} = \frac{25}{9}, U^2_{13} = \frac{40}{9}, U^3_{12} = \frac{45}{7}$, $\beta = 10$ and $F = 1$.

3.2 Kite

The second lowest dimensional clusters observed consist of two connected triads. There are two types of double-triad clusters: a butterfly and a kite (for more on double clusters see [19, 27, 28]). A butterfly has both triads connected via one common mode whereas a kite is connected via two common modes. The kite is shown in figure 3.3.

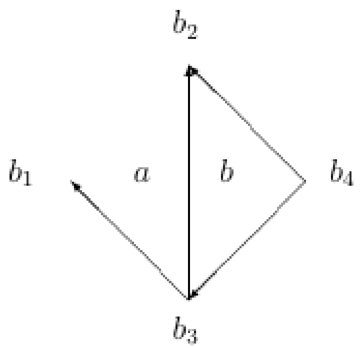

Figure 3.3: Kite: the only combination of modes for a kite is AP-PP because of the resonance conditions and the fact that for systems of Rossby and drift waves the wave function ψ is real.

A kite has only one type of connection possible AP-PP, this notation means that mode b_3 behaves as an active mode in triad a and as a passive mode in triad b, while b_2 is passive on both triads a and b, where b_2 and b_3 are the two connection modes of the cluster.

Similarly to the isolated triad, the dynamical system of a kite is obtained from Eq. (1.19) but for two systems of 3-wave resonances with two common modes, and gives

$$\begin{aligned}
\dot{b_1} &= -b_2^* b_3 U_{23}^1 e^{-i\omega_{12}^3 t} \\
\dot{b_2} &= -b_1^* b_3 U_{13}^2 e^{-i\omega_{12}^3 t} - b_3^* b_4 U_{34}^2 e^{-i\omega_{23}^4 t} \\
\dot{b_3} &= b_1 b_2 U_{12}^3 e^{i\omega_{12}^3 t} - b_1^* b_3 U_{24}^3 e^{-i\omega_{23}^4 t} \\
\dot{b_4} &= b_2 b_3 U_{23}^4 e^{i\omega_{23}^4 t}.
\end{aligned} \tag{3.7}$$

The non-resonant kite is the first example of a cluster where the number of linearly independent quadratic invariants of the system depend on the initial conditions of the CHM equation, i.e. location of the cluster in wave vector space. More precisely, it depends on whether a few selected interaction coefficients Eq. (1.20) are equal to zero or not. Therefore we consider two cases: generic case where all interaction coefficients $U_{23}^{\mathbf{k}} \neq 0$; special case where some of the interaction coefficients $U_{23}^{\mathbf{k}}$ depending on the wave vectors, are equal zero. The non-resonant matrix A is the same in both cases, as it is only dependent on the resonance conditions Eq. (1.2), and reads

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \tag{3.8}$$

although the resonant matrix A accommodates the description of resonant clusters (Section 2.2.1), it is also important for non-resonant clusters because of M_A^* dependence in Eq. (2.29): matrix A determines the set of independent dynamical phases φ_m defined in Eq. (2.27).

Generic case

In this case the non-resonant matrix B reads

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & U_{23}^4 \end{bmatrix}, \tag{3.9}$$

and the null space Φ_B gives,

$$\Phi_B = \begin{bmatrix} U_{13}^2 U_{24}^3 + U_{12}^3 U_{34}^2 & -U_{13}^2 U_{23}^4 \\ -U_{23}^1 U_{24}^3 & U_{23}^1 U_{23}^4 \\ U_{23}^1 U_{34}^2 & 0 \\ 0 & U_{23}^1 U_{34}^2 \end{bmatrix}, \tag{3.10}$$

the two rows of (3.9) are easily noticeable to be linearly independent, so according to Eq. (2.28) a generic non-resonant kite has $J = N - M_B^* = 3$ linearly independent quadratic invariants,

$$\begin{aligned} I_{123} &= U_{23}^1 U_{34}^2 |b_3|^2 - U_{23}^1 U_{24}^3 |b_2|^2 + (U_{13}^2 U_{24}^3 + U_{12}^3 U_{34}^2)|b_1|^2, \\ I_{124} &= U_{34}^2 |b_4|^2 + U_{23}^1 U_{23}^4 |b_2|^2 - U_{13}^2 U_{23}^4 |b_1|^2, \end{aligned} \tag{3.11}$$

which is the same number of invariants as for resonant kites.

The number of degrees of freedom of the non-resonant kite can be obtained by representing the complex amplitudes of the wave modes Eq. (3.7) in the amplitude-phase representation Eq. (2.26), such that the dynamical system is reduced to a system of six linearly independent differential equations:

$$\begin{aligned} \dot{c}_1 &= -c_2 c_3 U_{23}^1 \cos\varphi_1 \\ \dot{c}_2 &= -c_1 c_3 U_{13}^2 \cos\varphi_1 - c_3 c_4 U_{34}^2 \cos\varphi_2 \\ \dot{c}_3 &= c_1 c_2 U_{12}^3 \cos\varphi_1 - c_2 c_4 U_{24}^3 \cos\varphi_2 \\ \dot{c}_4 &= c_2 c_3 U_{23}^4 \cos\varphi_2 \\ \dot{\varphi}_1 &= c_1 c_2 c_3 \sin(\varphi_1)\left(\frac{U_{23}^1}{c_1^2} + \frac{U_{13}^2}{c_2^2} - \frac{U_{12}^3}{c_3^2}\right) + c_2 c_3 c_4 \sin(\varphi_2)\left(\frac{U_{34}^2}{c_2^2} - \frac{U_{24}^3}{c_3^2}\right) \\ \dot{\varphi}_2 &= c_1 c_2 c_3 \sin(\varphi_1)\left(\frac{U_{13}^2}{c_2^2} + \frac{U_{12}^3}{c_3^2}\right) + c_2 c_3 c_4 \sin(\varphi_2)\left(\frac{U_{34}^2}{c_2^2} + \frac{U_{24}^3}{c_3^2} - \frac{U_{23}^4}{c_4^2}\right). \end{aligned} \tag{3.12}$$

Four equations for the real amplitudes and three for the dynamical phases (i.e. one for each triad in the cluster), where $\varphi_1 = \omega_{12}^3 t + \theta_1 + \theta_2 - \theta_3$ and $\varphi_2 = \omega_{23}^4 t + \theta_2 + \theta_3 - \theta_4$. The number of linearly independent quadratic invariants Eq. (3.11) of the non-resonant kite for the generic case is $J = N - M_B^* = 2$, therefore the effective number of degrees of freedom is

$$\#d.o.f. = M_A^* + M_B^* = 4. \tag{3.13}$$

Particular case: $k_2^2 = k_3^2$

In order to understand which set of possible wave vectors of the non-resonant modes b_n with $n = 1, ..., 4$, satisfy the condition $k_2^2 = k_3^2$, we use the complex representation of 2-dimensional wave vectors as a tool. The resonance conditions of the kite (which can be deduced from the resonant matrix A) are written as

$$\begin{aligned} Re^{i\alpha_3} &= Re^{i\alpha_2} + |\mathbf{k}_1| e^{i\alpha_1} \\ |\mathbf{k}_4| e^{i\alpha_4} &= Re^{i\alpha_3} + Re^{i\alpha_2}, \end{aligned} \tag{3.14}$$

where $R = |\mathbf{k}_2| = |\mathbf{k}_3|$ is a natural number. We can reduce the number of the unknown variables with the following change of variables:

$$
\begin{aligned}
p &= \frac{|\mathbf{k}_1|}{R} \\
q &= \frac{|\mathbf{k}_4|}{R} \\
\zeta &= \alpha_1 - \alpha_2 \\
\sigma &= \alpha_1 - \alpha_2 \\
\rho &= \alpha_4 - \alpha_2,
\end{aligned}
\tag{3.15}
$$

where $p, q \in \mathbb{Q}$ and Eq. (3.14) can be written as

$$
\begin{aligned}
pe^{i\sigma} &= e^{i\zeta} - 1 \\
qe^{i\rho} &= e^{i\zeta} + 1,
\end{aligned}
\tag{3.16}
$$

we take the square complex modulus of each of these equations and we obtain the following condition:

$$
p^2 + q^2 = 4, \tag{3.17}
$$

so the variables p and q are mapped into an ellipse. Replacing back $p = \frac{|\mathbf{k}_1|}{R}$ and $q = \frac{|\mathbf{k}_4|}{R}$ we get

$$
|\mathbf{k}_1|^2 + |\mathbf{k}_4|^2 = 4R^2, \tag{3.18}
$$

this is an equation for 4 integer variables (two components of $\mathbf{k}_1$ and two components of $\mathbf{k}_4$) and can be solved by using Lagrange's four square theorem [35].

The non-resonant cluster matrix under the assumption $k_2^2 = k_3^2$ reads,

$$
B = \begin{bmatrix} 0 & -U_{13}^2 & U_{12}^3 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & 0 \end{bmatrix}, \tag{3.19}
$$

where $U_{23}^1 = 0, U_{23}^4 = 0$, $U_{13}^2 = U_{12}^3$ and $U_{34}^2 = -U_{24}^3$. These conditions on the interaction coefficients imply that there is only one linearly independent row in B, so $M_B^* = 1$.

Explicitly, the null space reads,

$$
\phi_B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{3.20}
$$

which gives three linearly independent quadratic invariants $J = N - M_B^* = 3$, and they read

$$\begin{aligned} I_1 &= |b_1|^2, \\ I_{23} &= |b_3|^2 + |b_2|^2, \\ I_4 &= |b_4|^2, \end{aligned} \tag{3.21}$$

this is shows that the amplitudes $|b_2|$ and $|b_3|$ exchange energy periodically as $|b_3|^2 + |b_2|^2$=const., and that the amplitudes $|b_1|$ and $|b_4|$ are constants of motion. This means that $|b_1|$ and $|b_4|$ are no longer interacting amplitudes, and consequent decouple in the cluster of modes b_1 and b_4 from modes b_2 and b_3 occurs. The decouple is evident, and it is shown in the equations of the dynamical system for this particular case,

$$\begin{aligned} \dot{b_1} &= 0 \\ \dot{b_2} &= -b_1^* b_3 U_{13}^2 e^{-i\omega_{12}^3 t} - b_3^* b_4 W_2 e^{-i\omega_{23}^4 t} \\ \dot{b_3} &= b_1 b_2 U_{12}^3 e^{i\omega_{12}^3 t} + b_1^* b_3 W_2 e^{-i\omega_{23}^4 t} \\ \dot{b_4} &= 0, \end{aligned} \tag{3.22}$$

where both parts ($b_\mathbf{k}$ and complex conjugate) of the complex amplitudes of b_1 and b_4 are constants.

The amplitude-phase representation Eq. (2.26) of the complex amplitudes (3.22) in this particular case of $k_2^2 = k_3^2$, gives

$$\begin{aligned} \dot{c}_1 &= 0 \\ \dot{c}_2 &= -c_1 c_3 U_{13}^2 \cos\varphi_1 - c_3 c_4 U_{34}^2 \cos\varphi_2 \\ \dot{c}_3 &= c_1 c_2 U_{13}^2 \cos\varphi_1 - c_2 c_4 U_{24}^3 \cos\varphi_2 \\ \dot{c}_4 &= 0 \\ \dot{\varphi}_1 &= c_2 c_3 c_4 \sin(\varphi_2) U_{34}^2 \left(\frac{1}{c_2^2} + \frac{1}{c_3^2} \right) \\ \dot{\varphi}_2 &= c_1 c_2 c_3 \sin(\varphi_1) U_{34}^2 \left(\frac{1}{c_2^2} + \frac{1}{c_3^2} \right), \end{aligned} \tag{3.23}$$

where number of degrees of freedom after reduction of the "slave" phases is equal to $N + M_A^* = 4 + 2 = 6$. However the number of linearly independent quadratic invariants is $J = N - M_B^* = 3$, therefore the effective number of degrees of freedom is equal to

$$\#d.o.f. = M_A^* + M_B^* = 2 + 1 = 3. \tag{3.24}$$

In contrast, notice that for the generic case we have

$$\#d.o.f. = M_A^* + M_B^* = 2 + 2 = 4. \tag{3.25}$$

So we might ask the question: is the dynamical system (3.22) integrable? As mentioned before not only the amplitudes $|b_1|$ and $|b_4|$ are constant but also the full complex functions b_1 and b_4 are constant. Therefore, equations for b_2 and b_3 become a linear system of ODEs with essentially time-dependent periodic coefficients, where

$$\begin{bmatrix} \dot{b}_2 \\ \dot{b}_2^* \\ \dot{b}_3 \\ \dot{b}_3^* \end{bmatrix} = M(t) \begin{bmatrix} b_2 \\ b_2^* \\ b_3 \\ b_3^* \end{bmatrix} . \tag{3.26}$$

The equations can be interpreted as a generalization of the swing equation [36]. The solutions are not trivial and can show the so-called parametric instability [37], however Floquet theory [38] ensures the construction of solutions. Moreover since $|b_2|^2 + |b_3|^2$ is bounded, there should be no instability.

Chapter 4

High Dimensional Clusters

In this Section we study three different non-resonant clusters of larger sizes than the previous, with each having specific properties. For simplicity we will omit the equations of the dynamical systems of the clusters since they can be obtained from the resonance conditions of the cluster. As well as the matrices of the null space of the non-resonant clusters, since they can easily be obtained computationally through the non-resonant matrix Eq. (2.24), and the respective equations for the quadratic invariants of each cluster since we will be more focused specifically on the number of linearly independent quadratic invariants of the clusters. Despite not shown in all cases below, numerical studies were performed with generic initial conditions for the clusters, and the results were validated for the quadratic invariants.

4.1 Five-Triad Cluster

First we consider a fully connected non-resonant cluster of $N = 7$ modes and $M = 5$ triads.

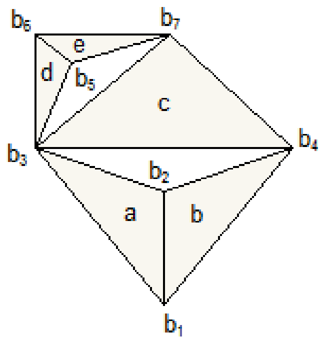

Figure 4.1: Five-Triad Cluster.

The resonant matrix of this cluster reads

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \tag{4.1}$$

there are no linearly dependent triads in this cluster, which means that the number of linearly independent rows of this matrix is equal to the number of rows, $M_A^* = M = 5$. Similarly to what was done in the analysis of the kite, we will consider two different cases for each cluster: generic case where all interaction coefficients $U_{23}^{\mathbf{k}} \neq 0$; special case where some of the interaction coefficients $U_{23}^{\mathbf{k}}$ depending on the wave vectors, are equal zero.

Generic case

In the generic case the non-resonant cluster matrix reads,

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 & 0 \\ -U_{24}^1 & U_{14}^2 & 0 & -U_{12}^4 & 0 & 0 & 0 \\ 0 & 0 & -U_{47}^3 & -U_{37}^4 & 0 & 0 & U_{34}^7 \\ 0 & 0 & -U_{56}^3 & 0 & -U_{36}^5 & U_{35}^6 & 0 \\ 0 & 0 & 0 & 0 & -U_{67}^5 & -U_{57}^6 & U_{56}^7 \end{bmatrix}, \tag{4.2}$$

this matrix has $M_B^* = M = 5$ linearly independent rows, which means that from Eq. (2.28) this cluster in the generic case has $J = N - M_B^* = 2$ linearly independent quadratic invariants that correspond to the energy Eq. (1.23) and enstrophy Eq. (1.25) of the cluster, showed in Section 2.1. Since $M_A^* = M_B^* = M$ the number of degrees of freedom Eq. (4.2) of the cluster in the generic case is given by

$$\#d.o.f. = M_A^* + M_B^* = 10, \tag{4.3}$$

where the large number of $d.o.f.$ is usually explained as a characteristic of large clusters of many modes and triads. Note however, that Eq. (2.29) is solely dependent of M^*, this means that the large number of degrees of freedom is a cause of the large number of triads in a cluster. Large clusters of many triads consequently have many modes, nevertheless it is remarkable that the number of degrees of freedom of a cluster only depends on the number of triads.

It is possible to form, in (4.2), a smaller square 2×2 matrix B' located in the top left corner of B with only zeros on the remaining entries of the respective columns, where B' is

$$B' = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 \\ -U_{24}^1 & U_{14}^2 \end{bmatrix}. \tag{4.4}$$

As it will be shown, the matrix B' will determine whether there will be any extra linearly independent quadratic invariants in the cluster, or not, (similarly to what was done for resonant matrices in [27]). If we compute the determinant of B', we obtain that

$$DetB' = (k_1^2 - k_2^2)(k_3^2 - k_4^2), \tag{4.5}$$

where two options must be considered:

- $k_1^2 = k_2^2$,
- $k_3^2 = k_4^2$.

In both cases, the determinant of B' is equal to zero.

Particular case: $k_1^2 = k_2^2$

In this case, the non-resonant matrix B reads,

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & 0 & 0 & 0 & 0 & 0 \\ -U_{24}^1 & U_{14}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -U_{47}^3 & -U_{37}^4 & 0 & 0 & U_{34}^7 \\ 0 & 0 & -U_{56}^3 & 0 & -U_{36}^5 & U_{35}^6 & 0 \\ 0 & 0 & 0 & 0 & -U_{67}^5 & -U_{57}^6 & U_{56}^7 \end{bmatrix}, \tag{4.6}$$

where the condition $k_1^2 = k_2^2$ affects the interaction coefficients, such that: $U_{12}^3 = 0$, $U_{12}^4 = 0$, $U_{23}^1 = -U_{13}^2$ and $U_{24}^1 = U_{14}^2$. Therefore the non-resonant matrix B decomposes as a direct sum of B' with the bottom right 3×5 submatrix in (4.6). This implies that modes b_1 and b_2 no longer interact with the other modes of the cluster[2]. In other words the two modes b_1 and b_2 get decoupled from the rest of the cluster, and the dynamics of the original cluster is determined by the dynamics of two separate smaller clusters. One containing modes b_1 and b_2 and a second with the remaining modes of the cluster b_3, b_4, b_5, b_6 and b_7. The smaller cluster of modes b_1 and b_2 contribute with one linearly independent quadratic invariant to the main cluster. Where because of the identities $U_{23}^1 = -U_{13}^2$ and $U_{24}^1 = U_{14}^2$, both rows of B' become linearly dependent, therefore modes b_1 and b_2 contribute with an invariant given by,

$$I = |b_1|^2 + |b_2|^2. \tag{4.7}$$

On the other hand, the remaining part of B on the other hand has all its rows linearly independent. One way to detect the linearly dependence, can be done by eliminating linearly independent rows/columns until we are left with the two or more rows/columns linearly dependent. The third column of B has only one element $-U_{37}^4$, consequently the third row of B is necessarily linearly independent (because rows/columns with only one element non-zero necessarily are linearly independent). While the last two rows of B will be linearly dependent unless the cluster, in addition to satisfying the first condition $k_1^2 = k_2^2$,

[2]the evolution equations Eq. (1.19) of modes b_1 and b_2 are only dependent on the complex amplitudes b_1 and b_2 and the equations for modes b_3 to b_7 are only dependent on the amplitudes b_3 to b_7

also satisfies a second condition $k_5^2 = k_6^2$. However, it is not possible to find such cluster that satisfies both conditions and the clusters resonance conditions[3]. This means that the remaining part of the cluster has always $J = 2$ quadratic invariants for the condition $k_1^2 = k_2^2$. Thus the total number of invariants of the original cluster in this case is $J = 3$, one more than in the generic case $J = 2$. As a consequence, the number of $d.o.f.$ will also be less than the generic case, $\#d.o.f. = M_A^* + M_B^* = 9$.

Particular case: $k_3^2 = k_4^2$

The non-resonant cluster matrix B in this case read

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 & 0 \\ -U_{24}^1 & U_{14}^2 & 0 & -U_{12}^4 & 0 & 0 & 0 \\ 0 & 0 & -U_{47}^3 & -U_{37}^4 & 0 & 0 & 0 \\ 0 & 0 & -U_{56}^3 & 0 & -U_{36}^5 & U_{35}^6 & 0 \\ 0 & 0 & 0 & 0 & -U_{67}^5 & -U_{57}^6 & U_{56}^7 \end{bmatrix}. \tag{4.8}$$

Under this condition, the only coefficient equal to zero is U_{34}^7. The cluster in this particular case does not separate as in the previous case, leading to an easy observation of a linearly dependent row in B which gives an additional invariant. Let us count the number of independent rows in B, in the same way as we did for the previous case (eliminating linearly independent rows/columns). The last row of (4.8) can be eliminated since the interaction coefficient U_{56}^7 is the only element of the last column and is non-zero. It follows that the fourth row is also linearly independent because U_{36}^5 and U_{35}^6 are non-zero. We are then left with the 3×4 matrix in the top left corner of (4.8). Note that for the case $k_3^2 = k_4^2$, the determinant of the 2×2 submatrix in the top left corner of B is equal to zero, which means that the first column can be deleted (because the number of independent rows of any rectangular matrix equal the number of independent columns), leaving the 3×3 matrix, B'':

$$B'' = \begin{bmatrix} -U_{13}^2 & U_{12}^3 & 0 \\ U_{14}^2 & 0 & -U_{12}^4 \\ 0 & -U_{47}^3 & -U_{37}^4 \end{bmatrix}. \tag{4.9}$$

Notice that because of the condition $k_3^2 = k_4^2$, some interaction coefficients become equal: $U_{13}^2 = U_{14}^2$, $U_{12}^3 = U_{12}^4$ and $U_{47}^3 = -U_{37}^4$. This implies that in (4.9) $row1 + row2$ is proportional to $row3$. Therefore the non-resonant matrix (4.8) has one linearly dependent row, so the cluster in this particular case also has an extra invariant, i.e. $J = 3$. Thus, the cluster in the case of $k_3^2 = k_4^2$ has the same number of d.o.f. as in the previous case $\#d.o.f. = 9$.

It is possible to obtain more clusters of that satisfy the resonance conditions in (4.1) and additional cases (e.g. $k_5^2 = k_6^2$), however for such cases the number of linearly independent quadratic invariants will remain remain between two and three.

[3]It can be showed from the resonance conditions given in (4.1), that the condition $k_1^2 = k_2^2$ implies that $k_6^2 = k_5^2 + k_3^2$.

4.2 Five-Triad Triangle

In this section we consider a cluster of $N = 6$ modes and $M = 5$ triads presented in [28] for non-resonant clusters.

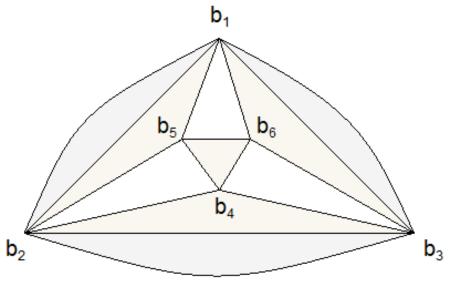

Figure 4.2: Five-Triad Triangle.

The resonant cluster matrix reads

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}. \tag{4.10}$$

The cluster has one linearly dependent resonant condition, that can be proved by showing that

$$row5 = row4 + row3 - row2, \tag{4.11}$$

so that, for example, row5 is linearly dependent $M_A^* = M - 1$.

Generic case

The non-resonant matrix of this cluster reads

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & U_{23}^4 & 0 & 0 \\ U_{25}^1 & -U_{15}^2 & 0 & 0 & -U_{12}^5 & 0 \\ -U_{36}^1 & 0 & -U_{16}^3 & 0 & 0 & U_{13}^6 \\ 0 & 0 & 0 & -U_{56}^4 & -U_{46}^5 & U_{45}^6 \end{bmatrix}. \tag{4.12}$$

The relation (4.11) is not observed in the non-resonant matrix (4.12), nevertheless row 5 of the non-resonant matrix B is linearly dependent $M_B^* = M - 1$. Explicitly, we have

$$\alpha\, row1 - \frac{U_{56}^4}{U_{23}^4} row2 + \frac{U_{46}^5}{U_{12}^5} row3 + \frac{U_{45}^6}{U_{13}^6} row4 = row5, \tag{4.13}$$

where α is a constant:

$$\alpha = 3\frac{(k_6^2 - k_3^2)(k_5^2 - k_4^2)(k_2^2 - k_1^2) - (k_5^2 - k_2^2)(k_6^2 - k_4^2)(k_3^2 - k_1^2)}{(k_3^2 - k_1^2)(k_3^2 - k_2^2)(k_2^2 - k_1^2)}, \tag{4.14}$$

dependent on the wave vectors of the cluster. This means that despite the cluster being in the generic case (i.e. all interaction coefficients non-zero), the minimum number of quadratic invariants is $J = N - M_B^* = 2$, where $M_B^* = M_A^* = M - 1 = 4$. This can can be explained (and generalized for all clusters) because of the linear independence of the two quadratic invariants: energy (1.23) and enstrophy (1.25). The number of $d.o.f.$ in this cluster in the generic form is equal to

$$\#d.o.f. = M_A^* + M_B^* = 8. \tag{4.15}$$

Particular case: $k_2^2 = k_3^2$

It is interesting to consider if it is possible to obtain extra quadratic invariants when some interaction coefficients are zero so that the non-resonant matrix (4.12) has $M_B^* < M_A^* < M$. For this purpose we consider a particular case of this cluster when $k_2^2 = k_3^2$.

The non-resonant matrix in this case reads,

$$B = \begin{bmatrix} 0 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & 0 & 0 & 0 \\ U_{25}^1 & -U_{15}^2 & 0 & 0 & -U_{12}^5 & 0 \\ -U_{36}^1 & 0 & -U_{16}^3 & 0 & 0 & U_{13}^6 \\ 0 & 0 & 0 & 0 & -U_{46}^5 & U_{45}^6 \end{bmatrix}, \tag{4.16}$$

with $U_{23}^1 = 0$, $U_{23}^4 = 0$, $U_{56}^4 = 0$, $U_{46}^5 = U_{45}^6$, $U_{13}^2 = U_{12}^3$ and $U_{34}^2 = -U_{24}^3$. It can be deduced from the resonances conditions in (4.1), that the condition $k_2^2 = k_3^2$ implies that $k_6^2 = k_5^2$, thus the change in the interaction coefficients of the last row. Consequently the fourth column of (4.16) is equal to zero[4] which means that the amplitude $|b_4|$ will be a constant of motion.

Due to the equalities $U_{13}^2 = U_{12}^3$ and $U_{34}^2 = -U_{24}^3$ the first two rows of (4.16) are linearly dependent, therefore the sum of first two terms in Eq. (4.14) is equal to zero. Thus Eq. (4.14) gives

$$\frac{U_{46}^5}{U_{12}^5} row3 + \frac{U_{45}^6}{U_{13}^6} row4 = row5, \tag{4.17}$$

which means that the last row of (4.16) is also linearly dependent. Consequently, the matrix (4.16) has $M_B^* = M - 2 = 3$ independent rows, that gives $J = 3$ number of linearly independent quadratic invariants for this cluster in the particular case of $k_2^2 = k_3^2$, therefore one more than the general case. Therefore the number of degrees of freedom is also reduced by one $\#d.o.f. = 7$, in comparison to the generic case where $\#d.o.f. = 8$.

[4]This requires that mode b_4 is disconnected from the cluster, and treated separately, similarly to the particular case of the kite in Section 3.2 where $k_2^2 = k_3^2$, modes b_1 and b_4 become non-interacting modes of the cluster

4.3 Four-Triad Triangle

In this section we consider a cluster of $N = 6$ modes and $M = 4$ triads presented in [27]. This cluster is very similar to the previous cluster in the Section 4.2 with the exception that has one less triad, and is shown in figure 4.3

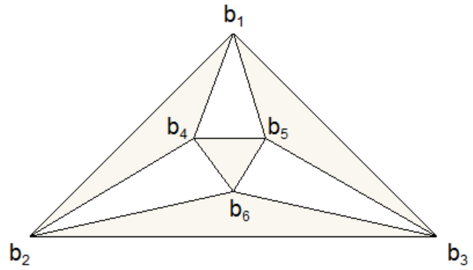

Figure 4.3: Four-Triad Triangle.

The resonant matrix A reads

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}. \tag{4.18}$$

Where the resonant matrix has a linearly independent row (i.e. the cluster has one resonance condition linearly dependent) that can be proved by showing that $row4 = row1 - row2 - row3$, so that A has $M_A^* = M - 1 = 3$. This means that in the resonant case, this cluster would have $J = 3$ independent quadratic invariants.

Considering the cluster only in the generic case, the non-resonant matrix B reads

$$B = \begin{bmatrix} -U_{24}^1 & -U_{14}^2 & 0 & U_{12}^4 & 0 & 0 \\ -U_{35}^1 & 0 & -U_{15}^3 & 0 & U_{13}^5 & 0 \\ 0 & U_{36}^2 & -U_{26}^3 & 0 & 0 & -U_{23}^6 \\ 0 & 0 & 0 & U_{56}^4 & -U_{46}^5 & -U_{45}^6 \end{bmatrix}. \tag{4.19}$$

This matrix has no linearly dependent rows $M_B^* = M = 4$, notice that one row in the non-resonant matrix (4.19) would be linearly dependent, if and only if the condition

$$row1\frac{U_{56}^4}{U_{12}^4} - row2\frac{U_{46}^5}{U_{13}^5} - row3\frac{U_{45}^6}{U_{23}^6} = row4, \tag{4.20}$$

is satisfied (in the generic case, the denominators in Eq. (4.20) are non-zero). It is easily seen by inspection of the first three columns of B that the condition (4.20) can not be satisfied, which means that the non-resonant matrix (4.19) has no linearly dependent rows $M_B^* = M$, as opposed to the scenario where this particular cluster would be in exact-resonance. Therefore the number of linearly independent quadratic invariants of this non-resonant cluster in the generic case is $J = N - M_B^* = 2$, and not $J = 3$ (resonant case). This yields that the real number of $d.o.f.$ of this cluster in the non-resonant case is then given by $\#d.o.f. = M_A^* + M_B^* = 7$, which is one degrees more than when the cluster is in exact-resonance $\#d.o.f. = 2M_A^* = 6$. We then conclude that in the regime of non-resonant clusters, the number of degrees of freedom of a given cluster is not the equal to the number of $d.o.f.$ for the same cluster in the regime of exact-resonance, even when the cluster is in the generic case.

Chapter 5

Conclusions and Future Work

In this thesis we have studied nonlinear dispersive waves described by the Charney-Hasegawa-Mima equation admitting both resonant and quasi-resonant interactions. In the first chapter we describe how nonlinear wave systems are analyzed nowadays, using the mathematical formalism of Wave Turbulence. Two different regimes are observed: small and finite nonlinearity. If nonlinearity is small (so that wave amplitudes are small) then energy is concentrated in triplets of waves which are grouped in so-called resonant triads. When these triads share common modes, they couple together to form bigger clusters. If nonlinearity is finite, strong transfers can occur towards non-resonant triads so the network of energy-exchanging connected triads is generally more complex than in the small-nonlinearity case. The so-called non-resonant clusters have not been studied in detail before this thesis.

The following chapters are dedicated to the study of these non-resonant clusters for the Charney-Hasegawa-Mima equation, a partial differential equation with applications in atmosphere and oceans, and in plasmas. We work in the context of spatial periodic boundary conditions which leads to discreteness of the allowed wave vectors in the Fourier representation of the underlying fields. In Chapter 2 we study in detail the systems of evolution equations arising in Fourier representation and show that energy and enstrophy are conserved. We construct the formalism for finding quadratic invariants of the dynamical systems that are obtained by truncating the set of interacting modes. This defines the clusters and we prove that the search for quadratic invariants can be replaced by a simple linear problem of finding the null space of a certain matrix: the cluster matrix. We show that the counting of degrees of freedom for a given cluster is done in terms of the properties of the cluster matrix. In Chapter 3 we consider two examples of the lowest-dimensional clusters: the triad and the kite. We apply the previous theory in order to analyze, for each cluster, the quadratic invariants and the number of degrees of freedom. Then we study non-generic situations where the wavenumbers are chosen so that extra invariants appear. In these non-generic cases the system of evolution equations typically become of lower dimensionality and sometimes can be integrated explicitly in terms of Floquet theory. In Chapter 4 we consider high-dimensional clusters as a starting point towards the long-term goal of understanding the mechanisms of turbulence in large clusters of many modes. The main result in this chapter is that non-generic cases (whereby the number of quadratic invariants is greater than in the generic case) are observed when some wave vectors have the same size. This situation, while non-generic, is nevertheless not too uncommon, particularly when waves are reflected by boundaries. Therefore our results are relevant in physically meaningful contexts.

Looking forward, the next steps of research include understanding the dynamical aspects of the evolution of large clusters, such as periodic orbits and chaos. On the other hand the aspect of changing the aspect ratio on the values of the interaction coefficients could be dramatic, namely the case of aspect ratio $\frac{L_x}{L_y}$ since it admits a triad with $|\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}_3|$, leading to the vanishing of all interaction coefficients. Such non-generic cases could be studied, leading perhaps to new exact solutions of the evolution equations.

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