Quadratic invariants for discrete non-resonant clusters of interacting waves in fully nonlinear regimes

Tiago Ferreira da Costa Faro

Under the supervision of Miguel Bustamante and Vasco Guerra
Mathematics department, UCD, Dublin, Republic of Ireland
Physics department, IST, Lisbon, Portugal

October 25, 2013

Abstract

In this work we study nonlinear dispersive waves such as atmospheric Rossby waves, and drift waves on plasmas modeled by the nonlinear equation of Charney-Hasegawa-Mima (CHM). This equation admits solutions of interacting triplets of waves called triads, and when sharing common modes, these triads couple together to form large clusters. In the limit of small nonlinearity, clusters only interact via exact-resonances, however for finite nonlinearity clusters can interact non-resonantly. We study in detail the construction of the evolution equation governing these non-resonant clusters, and develop a formalism to find a set of functionally independent quadratic invariants for these clusters, in terms of a basic linear algebra problem consisting of finding the null space of the so-called non-resonant matrix. We show how to count the number of effective degrees of freedom of the evolution equations using the non-resonant cluster matrix and the so-called resonant cluster matrix. The latter matrix was introduced in paper [1] by Bustamante and co-workers, but the non-resonant case had not been understood until this work. We apply this theory to discuss selected examples of non-resonant clusters arising in numerical simulations of the CHM equation. Clusters involving a few interaction modes can be integrable if the number of degrees of freedom is reduced. Such reduction is attained when wave vectors in the cluster satisfy certain relations so that the cluster matrices change their rank. The results of this work are staring point towards the modeling of turbulent cascades of energy and enstrophy in discrete and mesoscopic wave turbulence.

Keywords: Fluid Dynamics, Nonlinear Systems, Wave Turbulence, CHM equation, Quadratic Invariants.

1. Introduction

Dispersive waves play a crucial role in a vast range of physical applications, from quantum to astrophysical scales, such as geophysical Rossby waves in geophysical fluid dynamics [2, 3, 4], mode coupling in non-linear optics [5], drift waves in magnetized plasmas [6] and internal waves in stratified fluids [7, 8].

A mathematical differential equation describing a system of nonlinear waves is called dispersive if its linear part has wave solutions $\psi(\mathbf{x}, t)$, that depend on space and time as follows:

$$\psi(\mathbf{x}, t) = A_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t + \varphi), \tag{1}$$

where $A_{\mathbf{k}}$ is a constant wave amplitude, φ the constant wave phase, $\mathbf{k}$ the constant wave vector associated with the wave's wavelength λ by $k = |\mathbf{k}| = \frac{2\pi}{\lambda}$, and $\omega_{\mathbf{k}} \equiv \omega(\mathbf{k})$ the angular wave frequency which is a quantity determined by the particular wave system, commonly known as the wave dispersion relation [9]. The general solution of the differential equation will consist of the superposition of several modes, of the form (1) for different wave vectors. Waves are said to be dispersive if their phase velocity $\frac{\omega}{k}$, depends on k.

In bounded systems, the set of interacting wave vectors $\mathbf{k}$ are no longer continuous variables, but instead become discrete. For waves in a $d-$periodic box with all side lengths $L = 2\pi$, the normal modes are given by (1) with a discrete set of wavenumbers $\mathbf{k} = \frac{2\pi\mathbf{m}}{L}$ where $\mathbf{m} \in \mathbb{Z}^d$. Systems that are in a state where $\mathbf{k}$ are discrete variables, and intermediate states where both discrete and continuous $\mathbf{k}$ exist are called Discrete and Mesoscopic Wave Turbulence [10, 11, 12, 13]. In this work we focus on the aspects of these two regimes of wave turbulence.

It is often assumed in the literature that the most efficient energy transfers take place between three modes which are in exact resonance (limit of small amplitudes) known as resonant triad. Such that the wave vectors $\mathbf{k}$ obey:

$$\begin{aligned}\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 = 0,\\ |\omega_1 + \omega_2 - \omega_3| \leq \delta,\end{aligned} \tag{2}$$

when $\delta = 0$, where δ is the frequency mismatch of non-resonant triads. In discrete systems, it is often necessary to rely on approximate resonances to account for energy transfer, approximate resonance is possible due to the phenomenon known as nonlinear resonance broadening, such that triads which are not exactly in resonance can then interact at finite amplitude. Therefore, when $\delta \neq 0$ triads become quasi-resonants or non-resonants, the energy transfer between modes within the triad is less efficient so larger clusters form quickly (connected triads sharing common modes), however these are physically more sensible, since exact resonance conditions are hard to satisfy in real physical systems . Another usual assumption is that higher wave amplitudes give rise to more efficient energy transfers. It was shown in [14] that these assumptions do not hold in certain regimes, particularly in finite-sized systems. Therefore it is essential to study energy transfers between non-resonant triads, and non-resonant clusters.

Isolated triads are integrable systems where its motion is regular and periodic, but for large clusters the dynamics can be irregular and chaotic, where the systems are not integrable. We follow the principles of integrability in [15, 16] where it is stated that a n-dimensional system of autonomous evolution equations is usually integrable in many cases with the knowledge of only $(n-2)$ conservation laws. Such that a new conservation law can be constructed, which is functionally independent of the original ones, and therefore the system is integrable.

1.1. The CHM equation

Geophysical Rossby waves and drift waves in plasmas are often mentioned together because of the same simplified nonlinear PDE which was suggested for their description, namely the Charney-Hasegawa-Mima equation [17, 18]

$$\frac{\partial}{\partial t}(\Delta\psi - F\psi) + \beta\frac{\partial \psi}{\partial x} + J[\psi, \Delta\psi] = 0. \tag{3}$$

In the context of Rossby waves: the wave field $\psi(\mathbf{x}, t)$ with $\mathbf{x} \in \mathbb{R}^2$, is the streamfunction or the geopotential height, $F = \frac{1}{\rho^2}$ is the inverse square of the Rossby deformation radious ρ; and β is the gradient of the the Coriolis parameter f. In the plasma context ψ is the electrostatic potential for Drift waves; F is the inverse square of the Lamour radius; and β is the gradient of the logarithm of the plasma density. We introduced the notation for the Jacobian operator,

$$J[f, g] = (\frac{\partial f}{\partial x})(\frac{\partial g}{\partial y}) - (\frac{\partial f}{\partial y})(\frac{\partial g}{\partial x}). \tag{4}$$

Introducing the Fourier transform $\hat{\psi}_{\mathbf{k}} = \frac{1}{L^2}\int_{Box}\psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}$, and the interaction representation, which is a a set of complex amplitudes such that $b_{\mathbf{k}} = \psi_{\mathbf{k}}e^{i\omega_{\mathbf{k}}t}$, Eq. (3) is equivalent to the following:

$$\dot{b}_{\mathbf{k}} = \frac{1}{2}\sum_{1,2} U^{\mathbf{k}}_{12}\delta^{\mathbf{k}}_{12} b_1 b_2 e^{-i\omega^{\mathbf{k}}_{12}t}, \tag{5}$$

since ψ is a real wave function, in Fourier representation the two modes are equivalent $\psi_{-\mathbf{k}} = \psi^*_{\mathbf{k}}$, the sum is over all modes that satisfy Eq.(2), and $\omega^{\mathbf{k}}_{12} = \omega_{\mathbf{k}} - \omega_1 - \omega_2$ is the so-called detuning parameter of the triad, (where $\omega^{\mathbf{k}}_{12}$ corresponds to the frequency mismatch δ in Eq. (2)) such that $\omega(\mathbf{k}) \equiv \omega_{\mathbf{k}}$, and

$$\omega_{\mathbf{k}} = -\frac{k_x\beta}{k^2 + F}, \tag{6}$$

is the dispersion relation of the original PDE Eq. (3). The coefficients

$$U^{\mathbf{k}}_{12} = \frac{k_2^2 - k_1^2}{k^2 + F}(k_{1x}k_{2y} - k_{2x}k_{1y}), \tag{7}$$

are the so-called the interaction coefficients, where $\mathbf{k} = (k_x, k_y)$ and $k = |\mathbf{k}|$.

The CHM equation (3) conserves both energy E and enstrophy Ω for any system of Rossby and drift waves. In Fourier $\mathbf{k}$-space the conserved quantities read

$$E = \frac{1}{2}\sum_{\mathbf{k}}(|\mathbf{k}|^2 + F)|\hat{\psi}_{\mathbf{k}}|^2, \tag{8}$$

and

$$\Omega = \sum_{\mathbf{k}} |\mathbf{k}|^2\big(|\mathbf{k}|^2 + F\big)|\hat{\psi}_{\mathbf{k}}|^2, \tag{9}$$

where both E and Ω are characteristics of any system of turbulent nonlinear waves.

2. Clusters

As mentioned before, when resonant triads share common modes, they couple together to form a large network of connected triads called a resonant cluster. If the nonlinearity is finite, strong transfers can occur towards non-resonant triads so the network of energy-exchanging connected triads (i.e. the non-resonant clusters) is generally more complex than in the small-nonlinearity case [14]. Therefore non-resonant clusters of a system of interacting Rossby or Drift waves in the limit of finite amplitudes can be described by Eq. (5)

2.1. Conserved quantities in clusters general setting

We consider a conserved quantity I for any given non-resonant cluster with quadratic dependence on the modes' amplitudes (5),

$$I = \sum_{n=1}^{N} \varphi_n |b_n(t)|^2, \tag{10}$$

where $\varphi_n \equiv \varphi_{k_n}$ is a real function of the wave numbers (such as energy or enstrophy). The quantity I is a conserved quantity if its time derivative is equal to zero

$$\dot{I} = \sum_{n=1}^{N} \varphi_n (b_n^* \dot{b}_n + b_n \dot{b}_n^*) = 0, \tag{11}$$

we substitute for the amplitudes Eq. (5) in Eq. (11) and obtain that

$$\dot{I} = 2\sum_{123} (U_{12}^3 \varphi_3 - U_{23}^1 \varphi_1 - U_{13}^2 \varphi_2) Re(b_1 b_2 b_3^* e^{i\omega_{12}^3 t}) \\ = 0, \tag{12}$$

where the sum is over all triads in the cluster so in particular $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. Thus, I is a conserved quantity if, and only if, the corresponding linear system of equations of all triads in the cluster:

$$U_{12}^3 \varphi_3 - U_{23}^1 \varphi_1 - U_{13}^2 \varphi_2, \tag{13}$$

is equal to zero, where the factor $b_1 b_2 b_3^* e^{i\omega_{12}^3 t}$ is a function of time, typically independent of the corresponding factors for the other triads.

Energy (8) and enstrophy (9) are two functions of the wave modes that satisfy the linear system of Eqs. (13), and at the same time, are linearly independent. The proof of independence is given by contradiction: a given cluster of M triads and N modes has E and Ω linearly dependent if and only if all wave vectors $k_n = 1, ..., N$ have the same size. Therefore, in generic clusters with at least one wave vector of different size, there are at least two linearly independent invariants.

2.1.1 Connection with resonant clusters

In this section we consider the limit of exact-resonances $\omega_{12}^3 = 0$. Then it is possible to obtain a mapping between the linear system of equations Eq. (13), and the linear system of equations obtained in [1] for resonant clusters. Consider the transformation to normal variables (see. e.g. [10] for a better description of these variables):

$$\varphi_{\mathbf{k}} = \gamma_{\mathbf{k}} \frac{(k^2 + F)^2}{\beta k_x}, \tag{14}$$

where zonal modes $k_x = 0$ are discarded. Substituting Eq. (14) in Eq. (10), we get

$$I = \sum_{n=1}^{N} \gamma_n \frac{(k^2 + F)^2}{\beta k_x} |b_n|^2, \tag{15}$$

by taking the time derivative of I and making use of the properties of the resonance conditions of the frequencies Eq. (2):

$$\frac{k_{1x}}{k_1^2 + F} + \frac{k_{2x}}{k_2^2 + F} = \frac{k_{3x}}{k_3^2 + F}, \tag{16}$$

and using the resonance condition $k_{3x} = k_{1x} + k_{2x}$ terms can be manipulated and rearranged such that Eq. (16) becomes equivalent to

$$k_{1x}(k_1^2 - k_3^2)(k_2^2 + F) = k_{2x}(k_3^2 - k_2^2)(k_1^2 + F), \tag{17}$$

or

$$k_{1x}(k_2^2 - k_1^2)(k_3^2 + F) = k_{3x}(k_3^2 - k_2^2)(k_1^2 + F). \tag{18}$$

we obtain that

$$\dot{I} = \sum_{123} Z_{123} \Big(\gamma_1 + \gamma_2 - \gamma_3 \Big) (b_1 b_2 b_3^* + c.c.), \tag{19}$$

for simplicity of the equation we defined: $Z_{123} = 2(\mathbf{k}_1 \times \mathbf{k}_2)_z \frac{k_3^2 - k_2^2}{\beta k_{1x}} (k_1^2 + F)$, where $(\mathbf{k}_1 \times \mathbf{k}_2)_z = k_{1x} k_{2y} - k_{2x} k_{1y}$. Consequently, the linear system of equations in [1] is obtained: $(\gamma_1 + \gamma_2 - \gamma_3)$. This can be achieved by using the fact that it is not possible to find non-zero resonant triads with $\mathbf{k}_1 \parallel \mathbf{k}_2$ or $|\mathbf{k}_1|^2 = |\mathbf{k}_2|^2 = |\mathbf{k}_3|^2$, and if zonal modes are excluded.

2.1.2 Momentum conservation

Resonant cluster conserve both k_x and k_y. However, in the case of non-resonances only k_x is conserved[1], k_y on the other hand is not. Let us substitute $\gamma_n = k_{ny}$ in Eq. (15), again with the help of the relations of the resonance conditions in frequencies with a detuning:

$$k_{2x}(k_3^2 - k_2^2)(k_1^2 + F) = k_{1x}(k_1^2 - k_3^2)(k_2^2 + F) + \omega_{12}^3 \sum_{i}^{3} (k_i^2 + F), \tag{20}$$

we obtain that

$$\dot{I} = 2(\mathbf{k}_1 \times \mathbf{k}_2)_z \sum_{i}^{3} (k_i^2 + F) \omega_{12}^3 Re \Big[b_1 b_2 b_3^* e^{i\omega_{12}^3 t} \Big], \tag{21}$$

so that k_{ny} implies that

$$\dot{I} = 2 \sum_{triads} (\mathbf{k}_1 \times \mathbf{k}_2)_z \sum_{i}^{3} (k_i^2 + F) \omega_{12}^3 Re \Big[b_1 b_2 b_3^* e^{i\omega_{12}^3 t} \Big] = \\ = \frac{\partial}{\partial t} \sum_{triads} (k_1 \times \mathbf{k}_2)_z \sum_{i}^{3} (k_i^2 + F) Im \Big[e^{i\omega_{12}^3 t} b_1 b_2 b_3^* \Big], \tag{22}$$

where $\frac{\partial}{\partial t}$ means derivative with the explicit time dependence.

[1] it can be proved k_x is a linear combination of E and Ω.

2.2. Cluster matrices: resonant and non-resonant

Considering a cluster of M triads and N modes, the underlying problem of finding the solutions of the linear systems of equations can be reduced to a simple algebraic problem of finding the null space of a matrix. Thus solving linear system of equation in [1] for exact resonance clusters is equivalent to solving

$$\sum_{n=1}^{N} A_{mn}\gamma_n = 0, \qquad m = 1, ..., M, \tag{23}$$

where A_{mn} is a $N \times M$ matrix of N columns and M rows called the resonant matrix A. Similarly for non-resonant clusters, the solution of the linear system of equations Eq. (13) is equivalent to the solution of

$$\sum_{n=1}^{N} B_{mn}\varphi_n = 0, \qquad m = 1, ..., M, \tag{24}$$

where B_{mn} is a matrix of the same dimensions as A so-called the non-resonant matrix B.

2.2.1 Counting of degrees of freedom

Considering the same cluster introduced in Section 2.2, we prove a new way of counting the number of degrees of freedom $d.o.f.$ for any given cluster. We consider the wave amplitudes b_n of this cluster in the amplitude phase representation [19]:

$$b_n = c_n e^{i\theta_n}, \qquad n = 1, ..., N, \tag{25}$$

where $c_n = |b_n|$ are real amplitudes and θ_n individual phases of the interacting modes. In this representation the dimension of the system is reduced by elimination of the so-called "slave" variables, since these correspond to phases that can be obtained by quadratures [16, 19]. The effective number of degrees of freedom is then equal to $N + M_A^*$: N equations for the real amplitudes c_n and M_A^* equations for the linearly independent phases, so-called dynamical phases,

$$\varphi_m = \theta_1 + \theta_2 - \theta_3, \qquad m = 1, ..., M, \tag{26}$$

where M_A^* corresponds to the number of linearly independent rows of the resonant matrix A.

In the previous Section it was shown that the number of linearly independent quadratic invariants J of a non-resonant cluster is equal to the maximal set of linearly independent vectors in the null space B. From the algebraic properties of a matrix, these are equal to

$$J = N - M_B^*, \tag{27}$$

where M_B^* is the number of linearly independent rows of the B. This set of linearly independent quadratic invariants of the cluster reduce even further the degrees of freedom of the cluster such that the effective number of degrees of freedom of any given non-resonant cluster is equal to

$$\#d.o.f. = M_A^* + M_B^*. \tag{28}$$

This established that the effective number of $d.o.f.$ of non-resonant clusters is simply related to linearly independence of the rows of both resonant and non-resonant cluster matrices.

Note that a similar count of the degrees of freedom for resonant clusters can be deduced from Eq. (28), where in this case we obtain that

$$\#d.o.f. = 2M_A^*, \tag{29}$$

this is possible because of the mapping from A to B presented in Section 2.1.1 in terms of the respective linear system of equations. In other words, B is equal to A when the "non-resonant" clusters described by Eq. (5) are actually in exact resonance, consequently $M_B^* = M_A^*$.

3. Low Dimensional Clusters

In this section we analyze clusters up to connections of double triads.

3.1. Triad

The primary cluster is a simple isolated triad with a single 3-wave interaction.

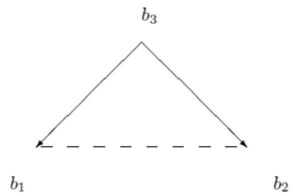

Figure 1: Isolated Triad - Arrows mean b_3 is active and b_1, b_2 are passive (see text for explanation).

The dynamical system of an Isolated triad is obtained from the projection of Eq. (2) in Eq. (5), and reads

$$\begin{aligned} \dot{b_1} &= -b_2^* b_3 U_{23}^1 e^{-i\omega_{12}^3 t} \\ \dot{b_2} &= -b_1^* b_3 U_{13}^2 e^{-i\omega_{12}^3 t} \\ \dot{b_3} &= b_1 b_2 U_{12}^3 e^{i\omega_{12}^3 t}. \end{aligned} \tag{30}$$

For a non-resonant triad there is an explicit time dependence $e^{-i\omega_{12}^3 t}$ due to the frequency detuning which is not seen in the resonant triad. The resonant matrix reads

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \tag{31}$$

whereas the non-resonant matrix B gives,

$$B = \begin{bmatrix} -U^1_{23} & -U^2_{13} & U^3_{12} \end{bmatrix}. \quad (32)$$

The number of quadratic invariants from Eq. (27) gives $J = 2$:

$$\begin{aligned} I_{13} &= U^1_{23}|b_3|^2 + U^3_{12}|b_1|^2, \\ I_{12} &= U^1_{23}|b_2|^2 - U^2_{13}|b_1|^2, \end{aligned} \quad (33)$$

where I_{13} and I_{12} correspond to the two Manley Rowe quadratic invariants (see, e.g.[20, 21]). The dynamical system Eq. (30) in the amplitude phase representation Eq. (25) reads

$$\begin{aligned} \dot{c}_1 &= -c_2 c_3 U^1_{23} \cos\varphi \\ \dot{c}_2 &= -c_1 c_3 U^2_{13} \cos\varphi \\ \dot{c}_3 &= c_1 c_2 U^3_{12} \cos\varphi \\ \dot{\varphi} &= c_1 c_2 c_3 \sin\varphi \left(\frac{U^1_{23}}{c_1^2} + \frac{U^2_{13}}{c_2^2} - \frac{U^3_{12}}{c_3^2} \right), \end{aligned} \quad (34)$$

thus the number of $\#d.o.f.$ of an isolated triad is equal to $M^*_A + M^*_B = 2$. The isolated triad is an Hamiltonian system, which means that the motion of the isolated triad is confined to a 1-dimensional manifold. Therefore the system of equations of the triad Eq. (30) is reduced to a single differential equation [11, 16]. The evolution of the triad is regular and periodic, and it can be seen in Figure 2.

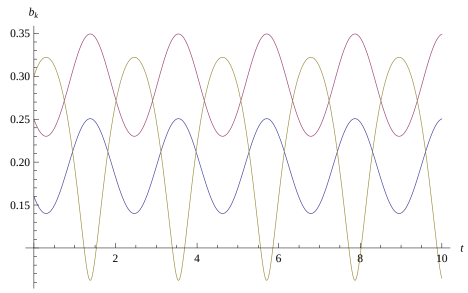

Figure 2: Complex wave amplitudes b_1, b_2, b_3 evolution of the non-resonant isolated triad for $0 \le t \le 10s$, with the initial conditions for the wave amplitudes, $b_1(0) = .16, b_2(0) = .25, b_3(0) = .3e^{i\pi/2}$, and the interaction coefficients $U^1_{23} = \frac{25}{9}, U^2_{13} = \frac{40}{9}, U^3_{12} = \frac{45}{7}$, $\beta = 10$ and $F = 1$.

3.2. Kite

There are two types of double triad clusters: butterfly and a kite. A butterfly is a cluster connected by one common mode, whereas in the kite both triads share two common modes. The kite is shown in Figure 3,

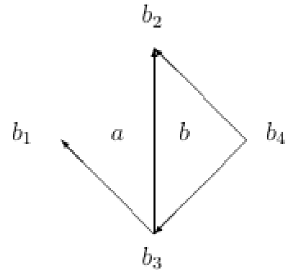

Figure 3: Kite: the only combination of modes for a kite is AP-PP because of the resonance conditions and the fact that for systems of Rossby and drift waves the wave function ψ is real (see more in [1].

Similarly to the isolated triad, the dynamical system of a kite is obtained from Eq. (5) but for two systems of 3-wave resonances with two common modes, and gives

$$\begin{aligned} \dot{b_1} &= -b_2^* b_3 U^1_{23} e^{-i\omega^3_{12} t} \\ \dot{b_2} &= -b_1^* b_3 U^2_{13} e^{-i\omega^3_{12} t} - b_3^* b_4 U^2_{34} e^{-i\omega^4_{23} t} \\ \dot{b_3} &= b_1 b_2 U^3_{12} e^{i\omega^3_{12} t} - b_1^* b_3 U^3_{24} e^{-i\omega^4_{23} t} \\ \dot{b_4} &= b_2 b_3 U^4_{23} e^{i\omega^4_{23} t}, \end{aligned} \quad (35)$$

The resonant matrix A reads

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \quad (36)$$

where $M^*_A = M = 2$. We consider two cases for the kite: a generic where all the interaction coefficients of B are non-zero; and a special case, where one ore more interaction coefficients can be zero.

3.2.1 Generic case

The non-resonant matrix B in the generic case reads

$$B = \begin{bmatrix} -U^1_{23} & -U^2_{13} & U^3_{12} & 0 \\ 0 & -U^2_{34} & -U^3_{24} & U^4_{23} \end{bmatrix}, \quad (37)$$

this gives $M^*_B = 2$, and according to Eq. (27) the kite has $J = 2$ independent invariants

$$\begin{aligned} I_{123} =& U^1_{23} U^2_{34} |b_3|^2 - U^1_{23} U^3_{24} |b_2|^2 + \\ &+ (U^2_{13} U^3_{24} + U^3_{12} U^2_{34}) |b_1|^2, \\ I_{124} =& U^2_{34} |b_4|^2 + U^1_{23} U^4_{23} |b_2|^2 - U^2_{13} U^4_{23} |b_1|^2, \end{aligned} \quad (38)$$

that can be obtained from the null space of B. Thus from Eq. (28), the kite in the generic case has,

$$\#d.o.f. = M^*_A + M^*_B = 4. \quad (39)$$

3.2.2 Particular case: $k_2^2 = k_3^2$

In order to understand which set of possible wave vectors of the non-resonant modes b_n with $n = 1, ..., 4$, satisfy the condition $k_2^2 = k_3^2$, we use the complex representation of 2-dimensional wave vectors as a tool. The resonance conditions of the kite (obtained from (36)) are written as

$$\begin{aligned} Re^{i\alpha_3} &= Re^{i\alpha_2} + |\mathbf{k}_1|e^{i\alpha_1} \\ |\mathbf{k}_4|e^{i\alpha_4} &= Re^{i\alpha_3} + Re^{i\alpha_2}, \end{aligned} \tag{40}$$

where $R = |\mathbf{k}_2| = |\mathbf{k}_3|$ is a natural number. This equation reduces to the equation

$$|\mathbf{k}_1|^2 + |\mathbf{k}_4|^2 = 4R^2, \tag{41}$$

of 4 integer variables (two components of $\mathbf{k}_1$ and two components of $\mathbf{k}_4$) and can be solved by using Lagrange's four square theorem [22]. The non-resonant cluster matrix reads,

$$B = \begin{bmatrix} 0 & -U_{13}^2 & U_{12}^3 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & 0 \end{bmatrix}, \tag{42}$$

where $U_{23}^1 = 0, U_{23}^4 = 0$, $U_{13}^2 = U_{12}^3$ and $U_{34}^2 = -U_{24}^3$. These conditions on the interaction coefficients imply that there is only one linearly independent row in B so that $M_B^* = 1$, therefore this cluster in the particular case of $k_2^2 = k_3^2$ has $J = 3$ quadratic invariants, given by

$$\begin{aligned} I_1 &= |b_1|^2, \\ I_{23} &= |b_3|^2 + |b_2|^2, \\ I_4 &= |b_4|^2. \end{aligned} \tag{43}$$

This is shows that the amplitudes $|b_2|$ and $|b_3|$ exchange energy periodically as $|b_3|^2 + |b_2|^2$=const., and that the amplitudes $|b_1|$ and $|b_4|$ are constants of motion. In other words, $|b_1|$ and $|b_4|$ are no longer interacting amplitudes, so that a consequent decoupling from modes b_2 and b_3 occurs in the cluster. Thus the number of $\#d.o.f.$ of the cluster in this particular case is equal to

$$\#d.o.f. = M_A^* + M_B^* = 3. \tag{44}$$

We might ask the question: is the dynamical system of the kite in the particular case of $k_2^2 = k_3^2$ integrable? As mentioned before not only the amplitudes $|b_1|$ and $|b_4|$ are constant but also the full complex functions b_1 and b_4 are constant. Therefore, equations for b_2 and b_3 become a linear system of ODEs with essentially time-dependent periodic coefficients, where

$$\begin{bmatrix} \dot{b}_2 \\ \dot{b}_2^* \\ \dot{b}_3 \\ \dot{b}_3^* \end{bmatrix} = M(t) \begin{bmatrix} b_2 \\ b_2^* \\ b_3 \\ b_3^* \end{bmatrix}. \tag{45}$$

The equations can be interpreted as a generalization of the swing equation [23]. The solutions are not trivial and can show the so-called parametric instability [24], however Floquet theory [25] ensures the construction of solutions. Moreover since $|b_2|^2 + |b_3|^2$ is bounded, there should be no instability.

4. High Dimensional Clusters

In this section we consider high-dimensional clusters as a starting point towards the long-term goal of understanding the mechanisms of turbulence in large clusters of many modes. Similarly to the analysis of the kite, we consider the two distinct cases: generic, where all $U_{12}^{\mathbf{k}} \neq 0$; particular, where specific $U_{12}^{\mathbf{k}} = 0$.

4.1. Five-Triad cluster

First we consider a fully connected non-resonant cluster of $N = 7$ modes and $M = 5$ triads.

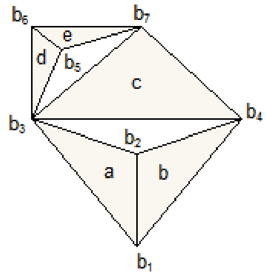

Figure 4: Five-Triad Cluster.

This cluster has the resonant matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \tag{46}$$

where there are no linearly dependent triads in the cluster, $M_A^* = M$.

4.1.1 Generic case

In the generic case the non-resonant cluster matrix reads,

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 & 0 \\ -U_{24}^1 & U_{14}^2 & 0 & -U_{12}^4 & 0 & 0 & 0 \\ 0 & 0 & -U_{47}^3 & -U_{37}^4 & 0 & 0 & U_{34}^7 \\ 0 & 0 & -U_{56}^3 & 0 & -U_{36}^5 & U_{35}^6 & 0 \\ 0 & 0 & 0 & 0 & -U_{67}^5 & -U_{57}^6 & U_{56}^7 \end{bmatrix}, \tag{47}$$

this matrix has $M_B^* = M = 5$ linearly independent rows, therefore this cluster will have $J = 2$ quadratic invariants that correspond to the energy Eq. (8) and enstrophy Eq. (9) of the cluster. Consequently by Eq. (28) we conclude that this cluster has, $M_A^* + M_B^* = 10$ degrees of freedom in the generic case. Note however (by the expression of Eq. (28)) that this large number of $\#d.o.f.$ is only related to large number of triads M and has no explicit dependence on the number of modes N.

In order to further analyze this cluster we form a smaller square 2×2 matrix B' located in the top left corner of (47). Where B' reads

$$B' = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 \\ -U_{24}^1 & U_{14}^2 \end{bmatrix}, \tag{48}$$

the determinant of this matrix gives

$$Det B' = (k_1^2 - k_2^2)(k_3^2 - k_4^2), \tag{49}$$

where two interesting particular cases must be considered:

- $k_1^2 = k_2^2$,
- $k_3^2 = k_4^2$,

such that in both cases, the determinant of B' is equal to zero.

4.1.2 Particular case: $k_1^2 = k_2^2$

In this case, the non-resonant matrix B reads,

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & 0 & 0 & 0 & 0 & 0 \\ -U_{24}^1 & U_{14}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -U_{47}^3 & -U_{37}^4 & 0 & 0 & U_{34}^7 \\ 0 & 0 & -U_{56}^3 & 0 & -U_{36}^5 & U_{35}^6 & 0 \\ 0 & 0 & 0 & 0 & -U_{67}^5 & -U_{57}^6 & U_{56}^7 \end{bmatrix}, \tag{50}$$

where: $U_{12}^3 = 0$, $U_{12}^4 = 0$, $U_{23}^1 = -U_{13}^2$ and $U_{24}^1 = U_{14}^2$. Therefore the non-resonant matrix B decomposes as a direct sum of B' with the bottom right 3×5 sub-matrix in (50). This implies that modes b_1 and b_2 no longer interact with the other modes of the cluster. In other words the two modes b_1 and b_2 get decoupled from the rest of the cluster, and the dynamics of the original cluster is determined by the dynamics of two separate smaller clusters. One containing modes b_1 and b_2 and a second with the remaining modes of the cluster b_3, b_4, b_5, b_6 and b_7. The smaller cluster of modes b_1 and b_2 contribute with one linearly independent quadratic invariant to the main cluster. Because $U_{23}^1 = -U_{13}^2$ and $U_{24}^1 = U_{14}^2$, both rows of B' become linearly dependent, therefore modes b_1 and b_2 contribute with an invariant given by,

$$I = |b_1|^2 + |b_2|^2. \tag{51}$$

The remaining part of B on the other hand has all its rows linearly independent. One way to detect the linearly dependence, can be done by eliminating linearly independent rows/columns until we are left with the two or more rows/columns linearly dependent. The fourth column of B has only one element $-U_{37}^4$, consequently the third row of B is necessarily linearly independent (because rows/columns with only one element non-zero necessarily are linearly independent). While the last two rows of B will be linearly dependent unless the cluster, in addition to satisfying the first condition $k_1^2 = k_2^2$, also satisfies a second condition $k_5^2 = k_6^2$. However, it is not possible to find such cluster that satisfies both conditions and the clusters resonance conditions[2]. This means that the remaining part of the cluster has always $J = 2$ quadratic invariants for the condition $k_1^2 = k_2^2$. Thus the total number of invariants of the original cluster in this case is $J = 3$, one more than in the generic case $J = 2$. Consequently, the number of $d.o.f.$ will also be less than the generic case, $\#d.o.f. = M_A^* + M_B^* = 9$.

4.1.3 Particular case: $k_3^2 = k_4^2$

The non-resonant cluster matrix B in this case reads:

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 & 0 \\ -U_{24}^1 & U_{14}^2 & 0 & -U_{12}^4 & 0 & 0 & 0 \\ 0 & 0 & -U_{47}^3 & -U_{37}^4 & 0 & 0 & 0 \\ 0 & 0 & -U_{56}^3 & 0 & -U_{36}^5 & U_{35}^6 & 0 \\ 0 & 0 & 0 & 0 & -U_{67}^5 & -U_{57}^6 & U_{56}^7 \end{bmatrix}. \tag{52}$$

When the cluster obeys the condition $k_3^2 = k_4^2$, there is only one interaction coefficient that becomes equal to zero: $U_{34}^7 = 0$. We count the number of independent rows in B in the same way as we did for the previous case (eliminating linearly independent rows/columns), and obtain that the$B$ has a linearly dependent row if,

$$B'' = \begin{bmatrix} -U_{13}^2 & U_{12}^3 & 0 \\ U_{14}^2 & 0 & -U_{12}^4 \\ 0 & -U_{47}^3 & -U_{37}^4 \end{bmatrix}, \tag{53}$$

has determinant equal to zero. Because of the condition $k_3^2 = k_4^2$, some interaction coefficients become equal: $U_{13}^2 = U_{14}^2$, $U_{12}^3 = U_{12}^4$ and $U_{47}^3 = -U_{37}^4$. This implies that in (53) $row1 + row2$ is proportional to $row3$. Therefore the non-resonant matrix (52) has one linearly dependent row, so the cluster in this particular case also has an extra invariant, i.e. $J = 3$. Thus, the cluster in the case of $k_3^2 = k_4^2$ has the same number of $d.o.f.$ as in the previous case $\#d.o.f. = 9$.

4.2. Five-Triad Triangle

In this section we consider a cluster of $N = 6$ modes and $M = 5$ triads.

[2] It can be showed from the resonance conditions given in (46), that the condition $k_1^2 = k_2^2$ implies that $k_6^2 = k_5^2 + k_3^2$.

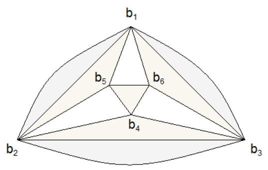

Figure 5: Five-Triad Triangle.

The resonant cluster matrix reads

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}. \tag{54}$$

The cluster has one linearly dependent resonant condition, that can be proved by showing that

$$row5 = row4 + row3 - row2, \tag{55}$$

so that, for example, row5 is linearly dependent $M_A^* = M - 1$.

4.2.1 Generic case

The non-resonant matrix of this cluster in the generic case reads

$$B = \begin{bmatrix} -U_{23}^1 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & U_{23}^4 & 0 & 0 \\ U_{25}^1 & -U_{15}^2 & 0 & 0 & -U_{12}^5 & 0 \\ -U_{36}^1 & 0 & -U_{16}^3 & 0 & 0 & U_{13}^6 \\ 0 & 0 & 0 & -U_{56}^4 & -U_{46}^5 & U_{45}^6 \end{bmatrix}. \tag{56}$$

The relation (55) is not observed in the non-resonant matrix (56), nevertheless row 5 of the non-resonant matrix B is linearly dependent $M_B^* = M - 1$. Explicitly, we have

$$\alpha\, row1 - \frac{U_{56}^4}{U_{23}^4} row2 + \frac{U_{46}^5}{U_{12}^5} row3 + \frac{U_{45}^6}{U_{13}^6} row4 = row5, \tag{57}$$

where α is a constant dependent on the wave vectors of the cluster. This means that despite the cluster being in the generic case, the minimum number of quadratic invariants is $J = N - M_B^* = 2$, where $M_B^* = M - 1$. This can be explained (and generalized for all clusters) because of the linear independence of the two quadratic invariants: energy (8) and enstrophy (9). The number of $d.o.f.$ in this cluster in the generic form is equal to:

$$\#d.o.f. = M_A^* + M_B^* = 8. \tag{58}$$

4.2.2 Particular case: $k_2^2 = k_3^2$

The non-resonant matrix in this case reads,

$$B = \begin{bmatrix} 0 & -U_{13}^2 & U_{12}^3 & 0 & 0 & 0 \\ 0 & -U_{34}^2 & -U_{24}^3 & 0 & 0 & 0 \\ U_{25}^1 & -U_{15}^2 & 0 & 0 & -U_{12}^5 & 0 \\ -U_{36}^1 & 0 & -U_{16}^3 & 0 & 0 & U_{13}^6 \\ 0 & 0 & 0 & 0 & -U_{46}^5 & U_{45}^6 \end{bmatrix}, \tag{59}$$

where $U_{23}^1 = 0$, $U_{23}^4 = 0$, $U_{56}^4 = 0$, $U_{46}^5 = U_{45}^6$, $U_{13}^2 = U_{12}^3$ and $U_{34}^2 = -U_{24}^3$. It can be deduced from the resonances conditions in (46), that the condition $k_1^2 = k_2^2$ implies that $k_6^2 = k_5^2$, thus the change in the interaction coefficients of the last row. Consequently the fourth column of (59) is equal to zero, this requires that mode b_4 is disconnected from the cluster, and treated separately, similarly to the particular case of the kite in Section 3.2 where $k_2^2 = k_3^2$, modes b_1 and b_4 become non-interacting modes of the cluster. Therefore it means that the amplitude $|b_4|$ will be a constant of motion.

Due to the equalities $U_{13}^2 = U_{12}^3$ and $U_{34}^2 = -U_{24}^3$ the first two rows of (59) are linearly dependent, therefore the sum of first two terms in Eq. (57) is equal to zero. Thus Eq. (57) gives

$$\frac{U_{46}^5}{U_{12}^5} row3 + \frac{U_{45}^6}{U_{13}^6} row4 = row5, \tag{60}$$

which means that the last row of (59) is also linearly dependent. Consequently, matrix (59) has $M_B^* = M - 2 = 3$ independent rows, that gives $J = 3$ linearly independent quadratic invariants, i.e. one more than the general case. Therefore the number of degrees of freedom is also reduced by one $\#d.o.f. = 7$, in comparison to the generic case where $\#d.o.f. = 8$.

4.3. Four-Triad Triangle

We consider the cluster presented in [1] of $N = 6$ modes and $M = 4$ triads.

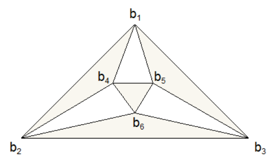

Figure 6: Four-Triad Triangle.

The resonant matrix of this cluster reads

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}, \tag{61}$$

it is easy to see that $row4 = row1 - row2 - row3$, therefore this cluster has $M_A^* = M - 1$. This means that in the resonant case, this cluster would have $J = 3$ independent quadratic invariants.

For non-resonant cluster and considering the cluster only in the generic case, the non-resonant matrix B reads

$$B = \begin{bmatrix} -U_{24}^1 & -U_{14}^2 & 0 & U_{12}^4 & 0 & 0 \\ -U_{35}^1 & 0 & -U_{15}^3 & 0 & U_{13}^5 & 0 \\ 0 & U_{36}^2 & -U_{26}^3 & 0 & 0 & -U_{23}^6 \\ 0 & 0 & 0 & U_{56}^4 & -U_{46}^5 & -U_{45}^6 \end{bmatrix}. \tag{62}$$

This matrix has no linearly dependent rows $M_B^* = M = 4$, where one of the rows in the non-resonant matrix (62) would be linearly dependent, if and only if the condition

$$row1\frac{U_{56}^4}{U_{12}^4} - row2\frac{U_{46}^5}{U_{13}^5} - row3\frac{U_{45}^6}{U_{23}^6} = row4, \qquad (63)$$

is satisfied (in the generic case, the denominators in Eq. (63) are non-zero). It is easily seen by inspection of the first three columns of B that the condition (63) can not be satisfied, which means that B has no linearly dependent rows $M_B^* = M$. Therefore the number of linearly independent quadratic invariants of this non-resonant cluster in the generic case is $J = N - M_B^* = 2$, and not $J = 3$ (resonant case). This yields that the real number of $d.o.f.$ of this cluster in the non-resonant case is then given by $\#d.o.f. = M_A^* + M_B^* = 7$, which is one degrees more than when the cluster is in exact-resonance $\#d.o.f. = 2M_A^* = 6$. We then conclude that in the regime of non-resonant clusters, the number of degrees of freedom of a given cluster is not the equal to the number of $d.o.f.$ for the same cluster in the regime of exact-resonance, even when the cluster is in the generic case.

5. Conclusions

In this work it was studied nonlinear dispersive waves described by the Charney-Hasegawa-Mima (CHM) equation admitting both resonant and quasi-resonant interactions. For different levels of nonlinearity in wave systems, two different regimes are observed: small nonlinearity, i.e. wave amplitudes are small, then energy is concentrated in triplets of waves which are grouped in so-called resonant triads. When these triads share common modes, they couple together to form bigger clusters. For finite nonlinearity, strong transfers can occur towards non-resonant triads so the network of energy-exchanging connected triads is generally more studied in detail before this work.

In a first part we study in detail the systems of evolution equations arising in Fourier representation of the CHM equation and show that both energy and enstrophy are conserved. We construct the formalism for finding quadratic invariants of the dynamical systems that are obtained by truncating the set of interacting modes. This defines the clusters and we prove that the search for quadratic invariants can be replaced by a simple linear problem of finding the null space of a certain matrix: the cluster matrix. We show that the counting of degrees of freedom for a given cluster is done in terms of the properties of the cluster matrix. In a second part we consider two examples of the lowest-dimensional clusters: the triad and the kite. We apply the previous theory in order to analyze, for each cluster, the quadratic invariants and the number of degrees of freedom. Then we study non-generic situations where the wavenumbers are chosen so that extra invariants appear. In these non-generic cases the system of evolution equations typically become of lower dimensionality and sometimes can be integrated explicitly in terms of Floquet theory. Finally, we consider high-dimensional clusters as a starting point towards the long-term goal of understanding the mechanisms of turbulence in large clusters of many modes. We conclude that non-generic cases (whereby the number of quadratic invariants is greater than in the generic case) are observed when some wave vectors have the same size. This situation, while non-generic, is nevertheless not too uncommon, particularly when waves are reflected by boundaries, this means that our results are relevant in physically meaningful contexts.

Future work would include understanding the dynamical aspects of the evolution of large clusters, such as periodic orbits and chaos. As well as changing the aspect ratio on the values of the interaction coefficients, leading to the vanishing of all interaction coefficients. Such non-generic cases could be studied, leading perhaps to new exact solutions of the evolution equations.

References

[1] K. L. Harper, M. D. Bustamante, and S. V. Nazarenko, Journal of Physics A Mathematical General **46**, 245501 (2013).

[2] A. Gill, *Atmosphere-Ocean Dynamics*, International Geophysics Series, Academic Press, 1982.

[3] C. P. Connaughton, B. T. Nadiga, S. V. Nazarenko, and B. E. Quinn, Journal of Fluid Mechanics **654**, 207 (2010).

[4] G. R. Bigg, *The Oceans and Climate*, Cambridge University Press, England, 266pp, 1996.

[5] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, Physical Review **127**, 1918 (1962).

[6] C. Horton and Y. Ichikawa, *Chaos and Structures in Nonlinear Plasmas*, World Scientific, 1996.

[7] C. H. McComas and F. P. Bretherton, Journal of Geophysical Research **82**, 1397 (1977).

[8] R. Ibragimov, Communications in Nonlinear Science and Numerical Simulations **13**, 593 (2008).

[9] G. Whitham, *Linear and Nonlinear Waves*, Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts, Wiley, 2011.

[10] S. Nazarenko, *Wave Turbulence*, Lecture Notes in Physics, Springer, 2011.

[11] E. Kartashova, EPL (Europhysics Letters) **87**, 44001 (2009).

[12] V. S. L'Vov and S. Nazarenko, Phys. Rev. E **82**, 056322 (2010).

[13] E.Kartashova, V. Lv́ov, S. Nazarenko, and I. Procaccia, Towards a Theory of Discrete and Mesoscopic Wave Turbulence, Technical report, RISC Report Series, University of Linz, Austria, 2010.

[14] M. D. Bustamante and B. Quinn, arXiv:1305.5517v1 (2013).

[15] E. Kartashova and M. D. Bustamante, arXiv:1002.4994 (2010).

[16] M. D. Bustamante and E. Kartashova, EPL (Europhysics Letters) **85**, 14004 (2009).

[17] A. Hasegawa and K. Mima, Physics of Fluids **21**, 87 (1978).

[18] J. Charney and J. Shukla, *Dynamics of Large-scale Atmospheric and Oceanic Processes: Selected Papers of Jule Gregory Charney*, A Deepak Publishing, 2001.

[19] M. D. Bustamante and E. Kartashova, EPL (Europhysics Letters) **85**, 34002 (2009).

[20] J. Manley and H. Rowe, Proceedings of the IRE **44**, 904 (1956).

[21] E. Kartashova and L. Tec, arXiv:1212.4295 (2012).

[22] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Graduate Texts in Mathematics, Springer, 2010.

[23] D. D. Holm and P. Lynch, Stepwise precession of the resonant swinging spring, SIAM J. Appl. Dynamical Systems, 2001.

[24] K. Kumar and L. S. Tuckerman, Journal of Fluid Mechanics **279**, 49 (1994).

[25] P. Kuchment, *Floquet theory for partial differential equations*, Springer, 1993.