Flight Control and Attitude Estimation of a Quadrotor

João Café
Instituto Superior Técnico – IST
jp_cafe@hotmail.com

Abstract—This work compares linear and nonlinear solutions for the flight control and attitude estimation of a quadrotor. A model of the aircraft QUAVIST, developed by UAVision®, is introduced as a basis for the analysis. Then, a cascade control design is defined. A low level controller stabilizes the quadrotor attitude and altitude; a high level controller acts on the low level one, controlling the horizontal position of the aircraft. Two approaches are presented for the controllers: a linear LQR solution and a Lyapunov based nonlinear solution. The attitude estimation problem is then considered. Two extended Kalman filters are designed to estimate the quadrotor attitude. Based on Lyapunov’s stability theory, a nonlinear estimator is proposed for the attitude and angular velocity of the aircraft. Finally, several closed loop simulations are performed. The Lyapunov low level controller presents better results than the LQR linear version, in simulations with high pitch and roll angles. Both high level controllers are robust to wind disturbances. All the three estimators proved to be good solutions to estimate the quadrotor attitude. The Lyapunov based estimation presented better simulation results.

Keywords: quadrotor, LQR, Lyapunov, extended Kalman filter

1 Introduction

The use of unmanned aerial vehicles (UAVs) in applications such as search and rescue operations, remote sensing or air surveillance has increased in the last decades. UAVs are acquiring an increased level of autonomy as more complex mission scenarios are envisioned. Thus, the technology is aiming to produce aircrafts with high maneuverability, capable of hover flight and vertical take-off and landing (VTOL). The quadrotor, a multicopter propelled and lifted by four rotors, holds some advantages over conventional rotorcrafts (helicopters). First, quadrotors do not require mechanical linkage to vary the rotor blade pitch angle. The propellers are in a fixed horizontal position, simplifying the design and the maintenance of the aircraft. Second, the use of four rotors allows a smaller diameter for each blade, allowing them to possess less kinetic energy during flight. For these reasons, the concept of the quadrotor has been a subject of interest in the scientific community.

This paper is structured as follows: the model of the quadrotor QUAVIST is presented in Chapters 2 and 3; Chapter 4 presents the low level controllers; the high level control is introduced in Chapter 5; the nonlinear estimation is presented in Chapter 6; several closed loop simulations are presented in Chapter 7; finally, Chapter 8 presents the conclusions and future work.

The main contribution of this work is the Lyapunov based estimation of the attitude and angular velocity of a quadrotor.

2 Dynamics and kinematics model

Consider a quadrotor aircraft illustrated in Figure 1. To model its behavior two frames are defined: an inertial reference frame (NED) and a body-fixed frame (xyz), centered in the quadrotor’s center of mass. In this work vectors with the superscripts 1 and q are relative to NED and xyz, respectively. The four rotors are grouped in two pairs, rotating in opposite directions. Varying the thrust produced by each rotor, the attitude of the quadrotor can be changed.
The Euler angles \( \Psi = [\phi, \theta, \psi]^T \) — roll, pitch and yaw — define the attitude of the aircraft and the rotation matrix
\[ S = S_x S_y S_z \] is used to express rotations from NED to xyz frame. The attitude can also be represented by the quaternion notation, \( q = [q_0, q_1, q_2, q_3]^T \). The dynamics and kinematics model of a quadrotor is given by \[1\]

\[
\dot{V} = \frac{1}{m} F - \Omega \times V + S q g^1
\]

\[
\dot{\Omega} = -J^{-1} (\Omega \times (J \Omega)) + J^{-1} M^q
\]

\[
\dot{q}_i = -\frac{1}{2} \omega_i q_i
\]

where \( V = [U, V, W]^T \), \( \Omega = [P, Q, R]^T \), \( P = [X, Y, Z]^T \) and \( q \) represent the linear velocity, angular velocity, position and attitude of the quadrotor. \( m \) is the mass, \( F \in \mathbb{R}^3 \) is the total thrust of the aircraft, \( M \in \mathbb{R}^3 \) is the total moment \( J \in \mathbb{R}^{3 \times 3} \) is the inertia matrix and \( g \in \mathbb{R}^3 \) is the gravity vector. \( \omega \) is the skew-symmetric matrix:

\[
\omega_i = \begin{bmatrix} 0 & -Q & P \\ P & 0 & -R \\ -Q & R & 0 \end{bmatrix}
\]

2 QUAVIST

QUAVIST (Figure 2) is a quadrotor vehicle developed by the portuguese company UAVision®. The system that controls the aircraft is based on the open source software Paparazzi and performs several tasks: receives and processes the information from the sensors, applies the control laws and acts on the aircraft, transmits telemetry information to a ground station and receives a remote control signal from the pilot. In manual mode the pilot can send \( [\theta, \phi, \psi, -F_z] \) commands to the system (\( F_z \) is the vertical thrust). It has PID attitude and position controllers and Kalman filters to estimate both attitude and altitude. Therefore, it is already considered a final product. The quadrotor is equipped with a wide range of sensors: a 3D accelerometer, a 3D gyroscope, a 3D magnetometer, a sonar, a barometer and a GPS.

Figure 2: QUAVIST

The noise affecting the readings of the sensors was measured; \( m = 2.93 \text{ kg} \) is the mass; \( d = 0.31 \text{ m} \) is the length of each arm; \( r_h = 0.1651 \text{ m} \) is the propeller radius. The actuators include, four brushless DC electric motors, four speed controllers and four propellers. A command is sent by the aircraft system to the speed controllers. The speed controllers control the input voltage, \( V_i \), of the motors. For each actuator, the induced rotation movement in the propeller by the motor \( (w_i) \) produce a correspondent thrust \( F_i \) and moment \( M_i \). The mathematical model of the motors is suggested in \[2\]. The quadrotor’s total thrust and moment is given by \[3\]:
4 Low level control

The cascade control design should be used in the presence of a fast dynamic process (secondary process) that has to be manipulated to control a slow dynamic process (primary process) [5]. Two sub-controllers are introduced:

- A low level controller, that controls the secondary process. The state $x_B = [W, P, Q, R, Z, \phi, \theta, \psi]^T$ is used to stabilize the quadrotor’s attitude and altitude. The controller acts directly on the quadrotor’s four inputs $\dot{V}_i$.
- An high level controller to control the primary process: the horizontal translational motion of the aircraft. Its state and output are $x_A = [U, V, X, Y]^T$ and $u_A = \left[\phi_{ref}, \theta_{ref}\right]^T$, the low level controller input.

According to [5], low level controller dynamics must be at least four times faster than the high level controller dynamics. It was chosen a sampling frequency of $f_A = 4\text{ Hz}$ for the high level controller and $f_B = 50\text{ Hz}$ for the low level controller. The control system is illustrated in Figure 3.

![Figure 3: Control system](image)

4.1 Optimal control

An optimal LQR controller was implemented. First the QUAVIST model introduced in Section 3 is linearized, using the Taylor’s series first order approximation:

$$\dot{x} = \frac{\partial x}{\partial x} \dot{x} + J_x(x_0, u_0) (x - x_0) + \frac{\partial x}{\partial u} (u - u_0)$$

This approximation is only valid in the vicinity of an operating point $(x_0, u_0)$. The hover flight condition was chosen as the operation point for the linearization. The control law is chosen to follow attitude and altitude references, $(x_{B,2})_{ref}$:

$$u = u_0 - K_B \left[ x_{B,1} - (x_{B,2})_{ref} \right]$$

where $x_{B,1} = [W, P, Q, R]^T$, $x_{B,2} = [Z, \phi, \theta, \psi]^T$ and $K_B$ is the LQR gain matrix.

4.2 Nonlinear control

The nonlinear controller is based in Lyapunov’s stability theory, presented in [7]. Consider a Lyapunov function $V(x)$ that represents the energy of a generic nonlinear system $f(x) = \dot{x}$. If $V(x) = 0$ the system is in equilibrium state; asymptotic stability implies that the energy is converging to $V(x) \rightarrow 0$; instability is related to the growth of $V(x)$ over time.

Theorem 1 Global stability [6]

Assume that there exists a scalar function $V$ of the state $x$, with continuous first order derivatives such that:

- $V(x) \rightarrow \infty$, $\|x\| \rightarrow \infty$
- $V$ is a positive definite function, $V(0) = 0 \land V(x) > 0$, $x \neq 0$
- The time derivative $\dot{V}(x) = \frac{dV}{dt}$ exists and is negative definite, $\dot{V}(0) = 0 \land \dot{V}(x) < 0$, $x \neq 0$

Then, the equilibrium point $x = 0$ is globally asymptotically stable.

The dynamic for the secondary process is defined as follows:

$$\dot{R} = R [\Omega] \times$$

$$\dot{\Omega}^g = - J^{-1} [\Omega] x J + J^{-1} M^q$$

$$\dot{W}^q = g_0 + \frac{1}{m} R (3, 3) F^q_z$$

where $R = S^T$ and $[\Omega]_\times$ represents a vector to skew-symmetric mapping: $\Omega \mapsto [\Omega]_\times$, so that $\forall v \in \mathbb{R}^3$, $[\Omega]_\times v = \Omega \times v$.

Now consider the following Lyapunov function $V$:

$$V = \frac{a}{2} \text{tr}(D (I_3 - R_{ref}^T R)) + \frac{b}{2} (Z - Z_{ref})^2 + \frac{1}{2} \Omega^T \Delta \Omega + \frac{1}{2} W^2$$

$D$ and $\Delta$ are $3 \times 3$ tuning positive definite matrices, $a$ and $b$ are positive constants. In [8] is proven that the first term in equation 15 is a positive definite representative function of the attitude error. This can be easily shown setting a reference attitude matrix $R_{ref} = I_3$.

$$\text{tr}(D (I_3 - I_3 R))$$ is then:

$$2d_1 (q_2^2 + q_3^2) + 2d_2 (q_1^2 + q_3^2) + 2d_3 (q_1^2 + q_2^2)$$

Equation 16 is always positive and only vanishes for $R = R_{ref}$, for which $q = [1, 0, 0, 0]^T$. The other terms are also positive definite, vanishing for $Z = Z_{ref} \land W = 0 \land \Omega = [0, 0, 0]^T$. The time derivative of $V$ is

$$\dot{V} = \left[ \begin{array}{c} W \\ \Omega^T \end{array} \right] \left[ \begin{array}{c} a \\ 2 \end{array} \right] \left( D R_{ref}^T R - R^T R_{ref} D \right) \times + \Delta \Omega$$

$\left[ M \right]_v$ is a skew-symmetric to vector mapping: $M \mapsto [M]_v$, so that $\forall v \in \mathbb{R}^3$, $[M]_v \times v = Mv$.

If the control input is chosen so that, with $\Lambda$ a $4 \times 4$ positive definite tuning matrix

$$\left[ \begin{array}{c} a \\ 2 \end{array} \right] \left( D R_{ref}^T R - R^T R_{ref} D \right) \times + \Delta \Omega = - \Lambda \left[ \begin{array}{c} W \\ \Omega \end{array} \right]$$
then $\dot{V}$ is negative definite and the system is asymptotically stable. Making $u_{\text{com}} = [W_{\text{com}}, \Omega_{\text{com}}]^T$ the control input of the system

$$u_{\text{com}} = -\Lambda \left[ \begin{array}{c} W \\ \Delta^{-1} \Omega \end{array} \right] - \frac{a}{2} \Delta^{-1} \left( b(Z - Z_{\text{ref}}) \right)$$

and knowing that

$$\begin{bmatrix} F_z \\ M \end{bmatrix}_{\text{com}} = \begin{bmatrix} m R(3,3)^{-1}(\dot{W}_{\text{com}} - g_0) \\ J\dot{\Omega}_{\text{com}} + [\Omega]_\times J\Omega \end{bmatrix}$$

the correspondent input of the quadrotor’s system, $u = [\dot{V}_1, \dot{V}_2, \dot{V}_3, \dot{V}_4]^T$, can be computed.

### 4.3 Simulation

Figures 4 and 5 present the plot response of the system for the two controllers implemented, with identical simulation conditions. As one can see both controllers are stable and their response is similar. However, if we excite the nonlinearities of the system by setting a reference vector $(x_{B,2})_{\text{ref}} = [-5, 45, 45, 0]^T$ far from the hover equilibrium point, we can clearly show the superior robustness of the nonlinear solution (Figures 6 and 7). The LQR response (Figure 6) shows a less stable control of the attitude; also the controller cannot control the quadrotor altitude.
4.4 Practical implementation

An altitude LQR controller was implemented in QUAVIST. The law control is given by

\[
RC = RC_0 - aK \begin{bmatrix} \bar{W}_b \\ \bar{Z}_s - Z_{ref} \end{bmatrix}
\]

(21)

where \(RC\) is a QUAVIST system variable proportional to \(F_z\), \(K\) is the Kalman gain matrix, \(\bar{Z}_s\) is the sonar measurement of the altitude, \(\bar{W}_b\) is a barometer based estimation of the vertical velocity (also provided by the system) and \(a\) is a proportional constant that relates \(RC\) to \(F_z\); \(RC_0\) is the equilibrium point input, for the hover flight. \(RC_0 = 6489\) and \(a = -129.663\) \(N^{-1}\) were determined using experimental data from a manual performed hover flight. The variables \(\bar{W}_b\) and \(\bar{Z}_s\) are given by QUAVIST system at a rate of \(f_z = 2.5\) Hz and \(f_w = 4\) Hz. It was chosen a control sampling period of \(T = \frac{1}{5}\) s. Figure 8 illustrates a performed flight; the controller was switched on and off at \(t = 32\) s and \(t = 70\) s. It was requested \(Z_{ref} = -1.5\) m to the controller. Clearly there is a static error in the control. This error can be caused by several factors. First, the control frequency and the measurements frequency, \(f_z\) and \(f_w\), are different. Second, the available control resolution is somehow low. Finally, the vertical velocity \(\bar{W}_b\) is based on barometric altitude and as we can see in Figure 8 there is an error between the two altitudes \(\bar{Z}_s\) and \(\bar{Z}_b\).

![Figure 8: QUAVIST’s altitude control. Controller is active in the interval [32,70] s](image)

5 High level control

To ensure a smooth horizontal movement of the aircraft two modes are defined for the high level control: horizontal velocity control mode and horizontal position control mode. Given an horizontal position reference \((x_{A.2})_{ref} = [x_{ref}, y_{ref}]^T\) and a desired linear velocity norm \(V_0\) the controller will navigate the aircraft at constant speed, in horizontal velocity control mode. The reference velocity vector is then defined, with \(\eta = [X - X_{ref}, Y - Y_{ref}]^T = [\eta_1, \eta_2]^T\)

\[
|U_{ref}| = \frac{V_0}{\sqrt{1 + \left(\frac{\eta_2}{\eta_1}\right)^2}} \quad (22)
\]

\[
|V_{ref}| = |U_{ref}| \left|\frac{\eta_2}{\eta_1}\right| \quad (23)
\]

so that \((x_{A.1})_{ref} = [U_{ref}, V_{ref}]^T\) and \(\eta\) are collinear. If \(|\eta| < \gamma\), the distance to the desired position, the controller will switch to horizontal position control mode. The distance \(\gamma\) is chosen by the user.

5.1 Optimal control

The state space formulation of the primary process is given by

\[
\dot{x}_A = \begin{bmatrix} 0_{2,2} & 0_{2,2} \\ I_{2,2} & 0_{2,2} \end{bmatrix} x_A + \begin{bmatrix} 0 & -g_0 \\ g_0 & 0 \end{bmatrix} u_A
\]

(24)

The control law for the horizontal position control mode is then

\[
u_A = -K_A S' \begin{bmatrix} x_{A,1} \\ \left(x_{A,2} - (x_{A,2})_{ref}\right) \end{bmatrix}
\]

(25)

with \(x_{A,1} = [U, V]^T\). \(K_A\) is the LQR gain matrix and \(S'\) is a rotation about \(xy\) plane

\[
S' = \begin{bmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{bmatrix}
\]

(26)

This rotation is needed because the system was linearized to a null yaw angle condition. For the velocity control mode, the state space formulation is reduced to

\[
\dot{x}_{A,1} = \begin{bmatrix} 0 & -g_0 \\ g_0 & 0 \end{bmatrix} u_A
\]

(27)

and thus resulting in the following control law

\[
u_A = -K_A S' \begin{bmatrix} x_{A,1} - (x_{A,1})_{ref} \end{bmatrix}
\]

(28)

Obviously the LQR gain matrix has to be redefined.

5.2 Nonlinear control

The high level nonlinear controller is based on Lyapunov’s stability theory and backstepping technique. Backstepping is a control technique that stabilizes a special class of nonlinear systems in a recursive way. Consider the system

\[
x = f_2(x) + g_2(x) \cdot z_1
\]

(29)

\[
\dot{z}_1 = f_1(x, z_1) + g_1(x, z_1) \cdot z_2
\]

(30)

\[
\vdots
\]

\[
\dot{z}_k = f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k) \cdot \bar{u}
\]

(31)
with virtual control inputs \( z_1, \ldots, z_k \). \( \mathbf{x} \) is the state vector and \( \bar{u} \) the real control input of the system. It is assumed that a Lyapunov function \( V_x \) and a control law \( z_1 = u_x \) are known for the subsystem 29.

The backstepping iterative method is now described. The process starts with the subsystem 29. As we already know \( V_x \) and \( u_x \), the next step will be extending the Lyapunov function \( V_x \) with a positive definite term that includes the control law \( u_x \). Let \( \varepsilon_1 = z_1 - (z_1)_{\text{d}} = z_1 - u_x \) be a fictitious variable and \( V_1 \) the extended Lyapunov function:

\[
V_1 = V_x + \frac{1}{2} (\varepsilon_1)^2
\]  

(32)

Using the function \( V_1 \) the fictitious control \( z_2 = u_1 \) can be found. The subsystem 30 now stabilizes \( z_1 \) in \( u_x \), which in turn is used to stabilize the subsystem 29. This way the asymptotic stability of the both subsystems is guaranteed. This process can be repeated until the real control input \( \bar{u} = u_k \) is known. The primary process can be rearranged in

\[
\mathbf{u}_A = \begin{bmatrix} 0 & \frac{1}{g_0} \\ -\frac{1}{g_0} & 0 \end{bmatrix} \mathbf{a}_h
\]

(33)

Then, with \( \mathbf{a}_h = [\dot{U}, \dot{V}]^T \), \( \mathbf{v}_h = [U, V]^T \) and \( \mathbf{x}_h = [X, Y]^T \)

\[
\mathbf{x}_h = \mathbf{v}_h
\]

(34)

\[
\mathbf{v}_h = \mathbf{a}_h
\]

(35)

is in the form of the system from equations 29 and 30. The subsystem 34 has the known Lyapunov function and control law:

\[
V_1 = \frac{1}{2} \eta^T \eta
\]

(36)

\[
\mathbf{v}_h = -\mathbf{a}_h \eta
\]

(37)

with \( \eta = \mathbf{x}_h - (\mathbf{x}_h)_{\text{ref}} \). As the backstepping method suggests, the fictitious variable \( \varepsilon = \mathbf{v}_h + \alpha \eta \) is added to \( V_1 \):

\[
V_2 = \frac{1}{2} (\mathbf{v}_h + \alpha \eta)^T (\mathbf{v}_h + \alpha \eta) + \frac{1}{2} \mathbf{v}_h^T \mathbf{v}_h
\]

(38)

\( V_2 \) is also a positive definite function and only vanishes for \( \mathbf{x}_h = (\mathbf{x}_h)_{\text{ref}} \wedge \mathbf{v}_h = [0, 0]^T \). The main result for the horizontal position control mode is now presented

\[
\mathbf{a}_h^N = -S_\psi (\alpha \mathbf{v}_h + \beta (\alpha \eta + 2 \mathbf{v}_h))
\]

(39)

with \( \mathbf{S}_\psi \) accounts for the changes in the yaw angle. If the horizontal position error is saturated, \( \eta = \mathbf{C}_0 = [C_x, C_y]^T \), the controller only acts on the subsystem 35 and therefore defining the horizontal velocity control mode. The controller will follow the velocity

\[
U_{\text{ref}} = \frac{\beta_1 \alpha_1 C_x}{\alpha_1 + 2 \beta_1}
\]

(40)

\[
V_{\text{ref}} = \frac{\beta_2 \alpha_2 C_y}{\alpha_2 + 2 \beta_2}
\]

(41)

Combining these results with equations 22 and 23, the constant vector \( \mathbf{C}_0 \) can be determined.

5.3 Simulation

A simulation using LQR high level controller is presented in Figure 9. The same simulation for the nonlinear high level controller is shown in Figure 10. Both controllers perform well; simulations were made with wind disturbances demonstrating that the controllers are robust solutions.
6 Nonlinear estimation

6.1 Extended Kalman filter

The extended Kalman filter is the linear version of the Kalman filter. Two solutions were design to estimate the attitude of the quadrotor based on the measurements of the 3D accelerometer, the 3D gyroscope and the 3D magnetometer. Consider the nonlinear discrete system with sample period $T$

$$x_k = f(x_{k-1}, u_{k-1}) + w_k \quad \text{Process model}$$

$$y_k = h(x_k, u_{k-1}) + v_k \quad \text{Observation model} \quad (42)$$

where $w_k$ and $v_k$ are Gaussian noises with covariance $Q_k$ and $R_k$, respectively. For each iteration, the extended Kalman filter estimate the state vector $x_k$ and the covariance error $P_{k|k}$, a estimation of the filter’s performance. Also, for a given $k$, the filter linearize the nonlinear system for the current state estimate: $F_{k-1} = \frac{\partial f}{\partial x} |_{x_{k-1}, u_{k-1}}$ and $H_k = \frac{\partial h}{\partial x} |_{x_{k-1}, u_{k-1}}$ are the Jacobian matrices of $f$ and $h$.

The extended Kalman filter algorithm is divided in two steps:

- Prediction — the filter predicts the state and the covariance error, based on the process model and the covariance process.

- Update — the filter corrects the previous step with the observation covariance and the measurements of the sensors.

The main result of the algorithm can be consulted in [9]. Consider the following nonlinear discrete model

$$x_k = I_3 x_{k-1} + T B(x) |_{x_0, u_{k-1}} \quad \text{Process model} \quad (43)$$

$$y_k = h(x_k, u_{k-1}) \quad \text{Observation model} \quad (44)$$

with $T = \frac{1}{50}$ s (the sampling period of the three sensors) and $B(x)$, from [1]:

$$B(x) = \begin{bmatrix} 1 & \tan \theta \sin \phi & \tan \theta \cos \phi \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \quad (45)$$

The state, output and input of the system are chosen as: $x_k = \Psi$ is the attitude, $y_k = [\vec{a}_n, \vec{\Omega}, \vec{N}]$ is the sensors measurements and $u_k = \vec{\Omega}$ is the gyroscope measurements.

6.2 Simplified extended Kalman filter

This solution is similar to the previous version, only the linearization point is different. Here, the linearization is performed for the equilibrium point $x_{\Phi}^0 = [0, 0, 0]^T$. Thus, the first order Taylor’s series is applied (equation 10) and the discrete linearized system is given by:

$$x_k = x_{k-1} + I_3 (x_{k-1} - x_{k-1}^0) + T I_3 u_{k-1}$$

$$y_k = y_{k}^0 + C_d (x_k - x_{k-1}^0) + D_d u_{k-1} \quad (46)$$

with

$$x_k^0 = \begin{bmatrix} 0_{3,1} \\ g_0 \\ \cos \psi_{k|k-1} \\ -\sin \psi_{k|k-1} \\ 0 \end{bmatrix}$$

$$y_k^0 = \begin{bmatrix} 0 \sin \hat{\vartheta}_{k|k-1} \\ \cos \hat{\vartheta}_{k|k-1} \end{bmatrix} \quad (47)$$

$$C_d = \begin{bmatrix} 0_{3,1} \\ 0_{4,1} \\ 0_{4,1} \\ 0 \end{bmatrix}$$

$$D_d = \begin{bmatrix} 0_{3,3} \\ I_{3,3} \\ 0_{3,3} \end{bmatrix} \quad (48)$$

$$y_k^0 = \begin{bmatrix} 0_{3,1} \\ g_0 \sin \hat{\vartheta}_{k|k-1} \\ \cos \hat{\vartheta}_{k|k-1} \end{bmatrix} \quad (49)$$

With these changes, the computational effort is reduced by a great amount.

6.3 Lyapunov based attitude and angular velocity estimation

Let define the Lyapunov tentative function

$$V = \frac{a}{2} \text{tr} \left( D \left( I_3 - R^T_m \hat{R} \right) \right) + \frac{1}{2} \text{tr} e^T_w e_w \quad (50)$$

where $\text{tr} (D (I_3 - R^T_m \hat{R}))$ represents the error between the estimated rotation matrix $\hat{R}$ and measurement rotation matrix $R_m$. $D$ is a positive definite $3 \times 3$ matrix and $a$ is a constant. They are both tuning parameters.

$$R_m = \begin{bmatrix} N_n & [\vec{a}_n]_x N_n & \vec{a}_n \end{bmatrix}^T \quad (51)$$

with $N_n$ and $a_n$ being the normalized accelerometer and magnetometer measurements. $e_w = \vec{w} - \hat{\vec{R}}^T R_m \vec{w}_m$ is the angular velocity error between the estimated angular velocity $\hat{\vec{w}}$ and the gyroscopic measurements $\vec{w}_m$. According to [10] the tangent vectors $\frac{d}{dt}(\hat{R}) \in T_{\hat{R}} SO(3)$ and $\frac{d}{dt}(R_m) \in T_{R_m} SO(3)$ lie in different tangent spaces and thus cannot be compared directly. For this reason $\frac{d}{dt}(R_m)$ is transformed in a vector in $T_{\hat{R}} SO(3)$ and $e_w$ results from the comparison $\frac{d}{dt}(\hat{R}) - \frac{d}{dt}(R_m) R_m^T \hat{R}$. The function $V$ from equation 50 is a positive definite function and only vanishes for the equilibrium state $V = 0 \Rightarrow (\vec{e}_w = 0 \land R_m^T \hat{R} = I_3) \Rightarrow (\hat{\vec{w}} = \vec{w}_m \land \hat{\vec{R}} = R_m)$. The main result for the time derivative of $V$ is then introduced

$$\dot{V} = a e_w^T \vec{e}_R + a^T e_w \vec{e}_w = e_w^T (ae_R + \vec{e}_w) \quad (52)$$
where the first term derivative can be computed as
\[
\frac{d}{dt} \frac{1}{2} \text{tr} \left( D \left( I_3 - R_T \hat{R}_m \right) \right) = \frac{1}{2} \text{tr} \left( -D \hat{R}_m^T \hat{R} \left[ \epsilon_w \right]_x \right) \quad (53)
\]
\[
\frac{d}{dt} \frac{1}{2} \text{tr} \left( D \left( I_3 - R_T \hat{R}_m \right) \right) = \frac{1}{2} e_x^T \left( D \hat{R}_m^T \hat{R} - \hat{R}^T R_m D \right) \quad (54)
\]
\[
\frac{d}{dt} \frac{1}{2} \text{tr} \left( D \left( I_3 - R_T \hat{R}_m \right) \right) = e_x^T \hat{R} \quad (55)
\]
with \( e_R = \frac{1}{2} \left( D \hat{R}_m^T \hat{R} - \hat{R}^T R_m D \right) \). Choosing an input, with \( \Delta \) positive definite, so that
\[
\dot{e}_w = -\Delta e_w - a e_R \quad (56)
\]
makes \( \dot{V} \) negative definite and the asymptotic stability for the system is achieved. Let us now define a discrete estimator with sampling period \( T \):
\[
e_{w,k+1} = e_{w,k} + T \dot{e}_{w,k} \quad (57)
\]
\[
\hat{\omega}_{k+1} = e_{w,k+1} + \hat{R}_k^T \hat{R}_{m,k} \omega_{m,k} \quad (58)
\]
\[
\hat{R}_{k+1} = \hat{R}_k \left( I_3 + T [\hat{\omega}_{k+1}]_x \right) \quad (59)
\]
At each time step the angular velocity error function is updated using the actual value for \( e_{w,k} \) and \( \dot{e}_{w,k} \), having its stability guaranteed by equation 56. Since \( e_{w,k+1} \) is converging to the equilibrium point, the angular velocity estimate and the rotation matrix estimate can now be determined by equations 58 and 59.

### 6.4 Simulation

Two simulations are presented (Figures 11 and 12). The references made to the low level control were \( \Psi_{ref} = [5, 5, 50] \) for the first simulation and \( \Psi_{ref} = [16, -16, 50] \) for the second (Euler angles, \( \Psi \), in degrees). For simplicity we now call KS the simplified Kalman filter, KE the extended Kalman filter and LP the nonlinear Lyapunov estimation. The error between the ideal and the estimated attitude is represented by \( x_{id} - x_{est} = [\epsilon_{\phi}, \epsilon_{\theta}, \epsilon_{\psi}] \). In Tables 1 and 2 the mean error \( |\bar{m}| \), the standard deviation \( \sigma \) and the rms value \( x_{rms} \) of the attitude error is presented. The angular velocity error characteristics from LP are presented in Table 3.

Figure 11: Nonlinear estimation with a reference of \( \Psi_{ref} = [5, 5, 50] \) for the low level LQR controller

Figure 12: Nonlinear estimation with a reference of \( \Psi_{ref} = [16, -16, 50] \) for the low level LQR controller

<table>
<thead>
<tr>
<th></th>
<th>( \epsilon_{\phi} )</th>
<th>( \epsilon_{\theta} )</th>
<th>( \epsilon_{\psi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>KS</td>
<td>0.0257</td>
<td>0.061</td>
<td>0.4231</td>
</tr>
<tr>
<td>KE</td>
<td>0.6769</td>
<td>0.5633</td>
<td>1.0543</td>
</tr>
<tr>
<td>LP</td>
<td>0.6733</td>
<td>0.5666</td>
<td>0.136</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( \sigma )</th>
<th>( x_{rms} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>KS</td>
<td>0.6457</td>
<td>0.5527</td>
</tr>
<tr>
<td>KE</td>
<td>0.6504</td>
<td>0.5538</td>
</tr>
<tr>
<td>LP</td>
<td>0.6504</td>
<td>0.5538</td>
</tr>
</tbody>
</table>

Table 1: Attitude error characteristics for Fig. 11
The LP estimation presents the better results: the standard deviation and rms value are smaller and works well for relatively high attitude angles. The KE filter appears to have a better mean error. Although, the standard deviation of the estimation is way higher than the LP estimator. The KS estimation presents similar results to the KE filter for small attitude angles. For relatively high attitude angles the mean error of the estimation increases considerably. This is understandable, because the KS model is linearized for near hover conditions. The angular velocity estimation of LP also provided good results. The three estimation solutions were tested with experimental data obtained from QUAVIST’s system (Figure 13).

### Table 2: Attitude error characteristics for Fig. 12

<table>
<thead>
<tr>
<th></th>
<th>$\bar{m}$ (°)</th>
<th>$m$ (°)</th>
<th>$\psi$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>KS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LP</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 3: Angular velocity error characteristics for Fig. 11

<table>
<thead>
<tr>
<th></th>
<th>$\bar{m}$ (°/s)</th>
<th>$m$ (°/s)</th>
<th>$\psi$ (°/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The LP estimation presents the better results: the standard deviation and rms value are smaller and works well for relatively high attitude angles. The KE filter appears to have a better mean error. Although, the standard deviation of the estimation is way higher than the LP estimator. The KS estimation presents similar results to the KE filter for small attitude angles. For relatively high attitude angles the mean error of the estimation increases considerably. This is understandable, because the KS model is linearized for near hover conditions. The angular velocity estimation of LP also provided good results. The three estimation solutions were tested with experimental data obtained from QUAVIST’s system (Figure 13).

### 7 Closed loop simulations

Given the flight control and estimation solutions from the previous Sections, we can now present some closed loop simulations. Four types of estimation are defined: the KS filter, the KE filter, the LP attitude estimation (here called $La$) and the LP attitude and angular velocity estimation (here called $Lv$). Naturally the LP attitude and angular velocity estimation is more suited to real implementation because it provides both attitude and angular velocity, needed by the low level controllers. Feedback control with each estimation is tested for each low level controller: Figures 14 and 15 for the low level LQR, 16 and 17 for the low level Lyapunov control.

### Figure 13: Estimation with real data

### Figure 14: Low level LQR with Kalman estimation. Reference of $\Psi_{ref} = [8, -8, 20]$.

### Figure 15: Low level LQR with Lyapunov estimation. Reference of $\Psi_{ref} = [8, -8, 20]$.
The output of the GPS sensor was used as feedback for the two high level controllers. The feedback attitude and angular velocity for the low level controllers is given by the Lyapunov estimation. The simulation for the high level Lyapunov controller is presented in Figure 18. This controller overall performs better than the linear version. The horizontal velocity control was better and the position control exhibited more stability.

8 Conclusions and future work

Comparing the low level control solutions, it was concluded that the low level LQR is a less robust solution outside the vicinity of the hover point. Both high level controllers perform well in the presence of wind disturbances. The linear altitude controller implemented on QUAVIST presented positive results. Despite the existence of static error in the control, the controller successfully stabilizes the aircraft altitude. In general, the nonlinear Lyapunov estimation provided better simulation results. There are three advantages over the other two solutions: it provides an estimation of the angular velocity of the aircraft, appears to be more robust to measurements noise and is independent of the observation model considered to model the sensors.

Aiming the continuation or improvement of this work, the following steps are defined:

- Perform a better experimental identification of QUAVIST’s actuators, mainly of the speed controllers. Determine the command sent by the system to the actuators so that it can be included in the computational model.

- Correct the static error of the implemented altitude controller. This can be achieved by selecting the same frequency for the controller and the measurements. A vertical altitude and velocity estimation can be done, based on both sonar and barometer.

- Implement the low and high level controllers in QUAVIST.

References


