Symplectic Embeddings

Manuel Araújo

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Contents

1 Introduction

In this thesis we study an embedding problem in symplectic geometry. It is often useful, in Mathematics, to view an object as a subobject of one of particularly simple type. For instance, it is a basic fact in Differential Topology that any manifold can be embedded into $\mathbb{R}^n$, for sufficiently large n (Whitney's embedding Theorem), and embeddings in projective space play a fundamental role in Algebraic Geometry.

If (M, α) and (N, β) are symplectic manifolds, then a smooth map $f : M \to N$ is a *symplectic embedding* if it is an embedding and $f^*\beta = \alpha$. In analogy with Whitney's embedding theorem, it is natural to ask whether there exists a "universal" family of symplectic manifolds, in which any compact symplectic manifold can be embedded. A first guess would be $\mathbb{R}^{2n}$ with the standard symplectic form $\lambda_n = \sum_i dx_i \wedge dy_i$, but this does not work, for topological reasons. Indeed, if M is a compact manifold and $f : M \to \mathbb{R}^{2n}$ is an embedding, then the cohomology class of the pullback is $[f^*\lambda_n] = f^*([\lambda_n]) = f^*(0) = 0$. However, a symplectic form on a compact manifold is never exact, because its top exterior power is a volume form, therefore $f^*\lambda_n$ can never be a symplectic form. The next natural guess would be $(\mathbb{C}P^n, \Omega_n)$, where Ω_n is the Fubini-Study form. In 1977, Tischler ([3]) proved the following Theorem:

Theorem 1.1. *Let (M, Ω) be a compact symplectic manifold, with $[\Omega]$ integral. Then, there exists $N \in \mathbb{N}$ and a symplectic embedding $(M, \Omega) \hookrightarrow (\mathbb{C}P^N, \Omega_N)$.*

This already followed from Gromov's previous work on the h-principle for symplectic embeddings ([1]), but Tischler gave a much more elementary proof. In this thesis, we correct a mistake in Tischler's proof, while also simplifying it. Furthermore, we refine this result, by proving the following Theorem:

Theorem 1.2. *Let M be a compact symplectic manifold and $\beta_1(M)$ the first Betti number of M. The space* $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ *is weakly homotopy equivalent to $(S^1)^{\beta_1(M)} \times \mathbb{C}P^\infty$.*

It should be possible to prove Theorem 1.2 using Gromov's h-principle for symplectic embeddings ([1]), but this is not done in the literature. We chose to prove it by more elementary methods, adapting Tischler's proof of Theorem 1.1.

2 The symplectic structure on $\mathbb{C}P^n$

Let $\omega_n \in \Omega^2(\mathbb{C}^n)$ be defined by

$$\omega_n = \frac{i}{2\pi}\partial\bar{\partial}\log(|z|^2 + 1). \tag{1}$$

This is a symplectic form on $\mathbb{C}^n$. Consider the standard charts $\phi_i^n : U_i^n \to \mathbb{C}^n$ given by

$$[z_0 : \cdots : z_n] \mapsto \frac{1}{z_i}(z_0, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n),$$

where

$$U_i^n = \{[z_0, \cdots, z_n] \in \mathbb{C}P^n : z_i \neq 0\}.$$

Definition 2.1. The Fubini-Study form Ω_n is the unique 2-form on $\mathbb{C}P^n$ such that $\Omega_n|_{U_i} = (\phi_i^n)^*(\omega_n)$.

The Fubini-Study form Ω_n is the standard symplectic form on $\mathbb{C}P^n$. For each $p \in \mathbb{C}P^n$, we choose an $x = x(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ which represents p. We make this choice in such a way that $j_{n+1}(x(p)) = x(i_n(p))$, where $i_n : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ and $j_{n+1} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+2}$ are the canonical inclusions. We let T_x^n be the complex hyperplane (in $\mathbb{C}^{n+1}$) that passes through x and is orthogonal to x. Let D_p^n be the subspace of $\mathbb{C}P^n$ consisting of those lines that intersect T_x^n and denote by $S_x^n : D_p^n \to T_x^n$ the map sending each line to its intersection point with T_x^n.

Choose unitary transformations $A_x^m : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ sending x to $(1, 0, \cdots, 0)$ such that $A_x^{m+1}|_{\mathbb{C}^{m+1}} = A_x^m$. If we identify $\{(1, x_1, \cdots, x_m) \in \mathbb{C}^{m+1} : x_i \in \mathbb{C}\}$ with $\mathbb{C}^m$ in the obvious way, then A_x^m maps $T_x^m \to \mathbb{C}^m$. Define $\phi_x^n = A_x^n \circ S_x^n : D_p^n \to \mathbb{C}^n$ and let $H^n : \mathbb{C}^n \to B^n(1)$ be given by

$$z \mapsto \frac{z}{(1 + |z|^2)^{1/2}},$$

where $B^n(1)$ is the open unit ball in $\mathbb{C}^n$.

Definition 2.2. The *standard coordinates* on D_p^n are the coordinates defined by

$$\varphi_x^n = H^n \circ \phi_x^n : D_p^n \to B^n(1),$$

where $x = x(p) \in S^{2n+1}$ is the chosen representative of p.

Proposition 2.3. *If $x_j + iy_j$ are the usual coordinates on $B^n(1) \subset \mathbb{C}^n$, then*

$$\Omega_n|_{D_p^n} = (\varphi_x^n)^*(\frac{1}{\pi}\sum_{j=1}^n dx_j \wedge dy_j).$$

Proposition 2.4. $i_n^*\Omega_{n+1} = \Omega_n$.

Proposition 2.5. *For any $p \in \mathbb{C}P^n$, consider $x = x(p) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ the chosen representative. Then, the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C}P^n \supset D_p^n \arrow[r, "\varphi_x^n"] \arrow[d, hook, "i_n"] & B^n(1) \arrow[d, hook, "j_n"] \\
\mathbb{C}P^{n+1} \supset D_p^{n+1} \arrow[r, "\varphi_x^{n+1}"] & B^{n+1}(1).
\end{tikzcd}$$

Note that $D_p^{n+1} \cap \mathbb{C}P^n = D_p^n$, for $p \in \mathbb{C}P^n$, because $T_x^{n+1} \cap \mathbb{C}^{n+1} = T_x^n$.

Definition 2.6. We say that $\alpha \in H^k(M;\mathbb{R})$ is *integral* if it is in the image of the canonical map $H^k(M;\mathbb{Z}) \to H^k(M;\mathbb{R})$. We say that $[\omega] \in H^k_{dR}(M)$ is integral if it corresponds to an integral cohomology class in $H^k(M;\mathbb{R})$ under the de Rham isomorphism. We say that $\omega \in \Omega^k(M)$ is integral if $[\omega]$ is integral.

Proposition 2.7. *A cohomology class $[\omega] \in H^k_{dR}(M)$ is integral if and only if it evaluates to an integer on every closed integral k-chain.*

In the case when $M = \mathbb{C}P^n$, the map $H^k(\mathbb{C}P^n;\mathbb{Z}) \to H^k(\mathbb{C}P^n;\mathbb{R}) = H^k_{dR}(\mathbb{C}P^n)$ is a monomorphism, so we think of $H^k(\mathbb{C}P^n;\mathbb{Z})$ as a subgroup of $H^k_{dR}(\mathbb{C}P^n)$.

Proposition 2.8. *$[\Omega_n]$ is integral and it is a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$.*

Consider
$$\mathbb{C}P^\infty = \operatorname{colim}(\cdots \to \mathbb{C}P^n \xrightarrow{i_n} \mathbb{C}P^{n+1} \to \cdots)$$
and let $\iota_n : \mathbb{C}P^n \to \mathbb{C}P^\infty$ be the canonical maps.

Proposition 2.9. *There exists a unique $\chi \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ such that*
$$\iota_n^*\chi = [\Omega_n] \text{ for all } n. \tag{2}$$

We leave this χ fixed for the rest of the text.

3 Existence of Symplectic Embeddings into $\mathbb{C}P^n$

In this section we sketch the proof of Theorem 1.1. The proof is divided into the two following results:

Proposition 3.1. *Let (M,Ω) be a compact symplectic manifold, with $[\Omega]$ integral. Then, there is $n \in \mathbb{N}$ and a smooth embedding $f_0 : M \hookrightarrow \mathbb{C}P^n$ such that $f_0^*[\Omega_n] = [\Omega]$.*

Sketch of proof. Since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$, there is $f : M \to \mathbb{C}P^\infty$ such that $f^*\chi = [\Omega]$. Since M is compact, there is $n \in \mathbb{N}$ such that $f(M) \subset \mathbb{C}P^n$ and then we have $f : M \to \mathbb{C}P^n$ with $f^*[\Omega_n] = [\Omega]$. Taking n large enough, we can approximate f by a smooth embedding f_0 homotopic to f.

□

Theorem 3.2. *Let $f_0 : M \to \mathbb{C}P^n$ be an embedding, such that $[f_0^*\Omega_n] = [\Omega]$. Then there is a $p \in \mathbb{N}$ and an isotopy $f_t : M \to \mathbb{C}P^{n+p}$ ($t \in [0,p]$) such that each f_t is C^0-close to f_0 and f_p is a symplectic embedding.*

Sketch of proof. We have $\Omega = f_0^*\Omega_n + d\omega$, for some 1-form ω. Let $0 < \epsilon < 1$ and consider an open cover $\{V(x_1, \epsilon), \cdots, V(x_q, \epsilon)\}$ of $\mathbb{C}P^n$, where $V(x_k, \epsilon) \subset D^n_{x_k}$ is an ϵ-ball centered at x_k, for the standard coordinates on $D^n_{x_k}$ and let $W_i = f_0^{-1}(V(x_i, \epsilon))$. We can write $d\omega = \frac{1}{\pi}\sum_{i=1}^p dh_i \wedge dt_i$ where $h_i, t_i : M \to \mathbb{R}$ are smooth functions such that for each k there is an $\ell = \mu(k)$ such that h_k and t_k are supported on W_ℓ. Moreover, one can take $\sup_{x\in M} |(h_i(x), t_i(x))|$ arbitrarily small. We construct $f_1 : M \to \mathbb{C}P^{n+1}$ by setting $f_1 = i_n \circ f_0$ outside $W_{\mu(1)}$ and on $W_{\mu(1)}$ we set $f_1(x) = (f_0(x), h_1(x) + it_1(x))$ in standard coordinates. We define f_t, for $t \in [0, 1]$, by using $t(h_1(x) + it_1(x))$ as the extra coordinate. Then we have $f_1^*\Omega_{n+1} = f_0^*\Omega_n + \frac{1}{\pi}dh_1 \wedge dt_1$. For $\sup_{x\in M} |(h_1(x), t_1(x))|$ small enough, we will have $f_1(W_k) \subset V(x_k, \epsilon_1)$ for some $\epsilon < \epsilon_1 < 1$ and so we can continue the process. In the end, we have $f_p^*\Omega_{n+p} = f_0^*\Omega_n + \frac{1}{\pi}\sum_i dh_i \wedge dt_i = \Omega$, so f_p is symplectic. It is easy to see that all the f_t are embeddings.

□

The mistake in Tischler's proof is in the method used to make $\sup_{x\in M} |(h_i(x), t_i(x))|$ arbitrarily small. Here we use the method of symplectic twisting:

Lemma 3.3. *Given $R > \epsilon > 0$, there is smooth map between disks $f : D^2_R \to D^2_\epsilon$ such that $f(0) = 0$ and $f^*(dx \wedge dy) = dx \wedge dy$.*

4 The space of symplectic embeddings into $\mathbb{C}P^\infty$

In this section we explain the statement and sketch the proof of Theorem 1.2.

Definition 4.1. Let (M, Ω) be a compact symplectic manifold. We define the space $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ by

$$\mathrm{SympEmb}(M, \mathbb{C}P^\infty) = \mathrm{colim}(\mathrm{SympEmb}(M, \mathbb{C}P^1) \hookrightarrow \mathrm{SympEmb}(M, \mathbb{C}P^2) \hookrightarrow \cdots),$$

with the colimit topology, induced by the C^∞ topology on $\mathrm{SympEmb}(M, \mathbb{C}P^n) \subset C^\infty(M, \mathbb{C}P^n)$ (see [2, page 34] for a definition of the C^∞ topology).

Note that an element in $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ is a symplectic embedding $M \hookrightarrow \mathbb{C}P^n$, for some n. We define the spaces $\mathrm{Emb}(M, \mathbb{C}P^\infty)$ and $C^\infty(M, \mathbb{C}P^\infty)$ in an analogous way.

Sketch of proof (Theorem 1.2). The computation of the weak homotopy type of $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ is done in four steps:

(i) Let $\overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty)$ be the subspace of $\mathrm{Emb}(M, \mathbb{C}P^\infty)$ consisting of embeddings $f : M \to \mathbb{C}P^\infty$ such that $f^*\chi = [\Omega]$. Then the inclusion

$$\mathrm{SympEmb}(M, \mathbb{C}P^\infty) \hookrightarrow \overline{\mathrm{Emb}}(M, \mathbb{C}P^\infty)$$

is a weak homotopy equivalence. This is the main step in the proof and will be explained below.

(ii) The inclusion

$$\mathrm{Emb}(M, \mathbb{C}P^\infty) \hookrightarrow C^\infty(M, \mathbb{C}P^\infty)$$

is a weak homotopy equivalence. This follows from standard results in Differential Topology, which deal with approximating smooth maps by smooth embeddings.

(iii) The inclusion

$$C^\infty(M, \mathbb{C}P^\infty) \hookrightarrow C(M, \mathbb{C}P^\infty)$$

is a weak homotopy equivalence. This follows from standard results in Differential Topology, which deal with approximating continuous maps by smooth maps.

(iv) The space $C(M, \mathbb{C}P^\infty)$ is homotopy equivalent to $H^2(M; \mathbb{Z}) \times (S^1)^{\beta_1(M)} \times \mathbb{C}P^\infty$. This follows from standard results in Algebraic Topology.

From (i), (ii) and (iii) it is easy to see that $\mathrm{SympEmb}(M, \mathbb{C}P^\infty)$ is weakly homotopy equivalent to a path component of $C(M, \mathbb{C}P^\infty)$ and then the result follows from (iv).

□

It remains to explain Step (i) in the previous proof. It follows (via some standard Algebraic Topology) from the following Theorem, which is a parametric version of Theorem 3.2:

Theorem 4.2. *Let (M, Ω) be a compact symplectic manifold and $f_0 : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^n)$ a continuous map (for the C^∞ topology), such that $f_0(z)$ is a symplectic embedding for z in a neighbourhood of ∂D^k. Then there is some $p \in \mathbb{N}$ and a C^0-small homotopy $f_t : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^{n+p})$, with $t \in [0, p]$, fixed on ∂D^k, such that $f_p(z) \in \mathrm{SympEmb}(M, \mathbb{C}P^{n+p})$ for all $z \in D^k$.*

Sketch of proof. We have $\Omega = f(z)^*\Omega_n + \alpha(z)$ where $\alpha : D^k \to \Omega^2(M)$ is a continuous family of exact 2-forms. Then we can find a continuous family of 1-forms $\omega : D^k \to \Omega^1(M)$ such that $d\omega(z) = \alpha(z)$. We take a finite open cover $\{V(x_k, \epsilon)\}$ of $\mathbb{C}P^n$, for some $0 < \epsilon < 1$ and consider the open cover $\{W_k = \tilde{f_0}^{-1}(V(x_k, \epsilon))\}$ of $D^k \times M$. We write $d\omega(z) = \frac{1}{\pi}\sum_{i=1}^p dh_i(z) \wedge dt_i(z)$, where the adjoint maps $\tilde{h}_i, \tilde{t}_i : D^k \times M \to \mathbb{R}$ have compact support contained in $W_{\mu(i)}$ and $h_i(z) = t_i(z) = 0$ for $z \in \partial D^k$. One can also take $\sup_{z \in D^k, x \in M} |(h_i(z)(x), t_i(z)(x))|$ arbitrarily small.

Now we inductively add new coordinates to $\tilde{f}_0$, just as in the non parametric case. Concretely, we define $\tilde{f}_1 : D^k \times M \to \mathbb{C}P^{n+1}$ by taking $\tilde{f}_1 = \tilde{f}_0$ outside $W_{\mu(1)}$. On $W_{\mu(1)}$, we use the standard coordinates on $V(x_{\mu(1)}, \epsilon) \subset D^n_{x_{\mu(1)}}$ to define $\tilde{f}_1(x) = (\tilde{f}_0(x), \tilde{h}_1(x) + i\tilde{t}_1(x))$. The adjoint map to $\tilde{f}_1$ is a well defined and continuous map $f_1 : D^k \to \mathrm{Emb}(M, \mathbb{C}P^{n+1})$.

We define $\tilde{f}_t$, for $t \in [0, 1]$, by using $t(h_1 + it_1)$ as the extra coordinate and so we get $f_t : D^k \to \mathrm{Emb}(M, \mathbb{C}P^{n+1})$. Since $h_i(z) = t_i(z) = 0$ for $z \in \partial D^k$, we get $f_t(z) = f_0(z)$ for all $t \in [0, 1]$ and $z \in \partial D^k$.

We have

$$f_1(z)^*\Omega_{n+1} = f_0(z)^*\Omega_n + \frac{1}{\pi} dh_1(z) \wedge dt_1(z),$$

for all $z \in D^k$. Furthermore, for $\sup_{z\in D^k, x\in M} |(h_i(z)(x), t_1(z)(x))|$ small enough, we have

$$\tilde{f}_1(W_k) \subset V(x_k, \epsilon_1)$$

for some $\epsilon < \epsilon_1 < 1$, so we can repeat the process.

Continuing in the same way, we obtain a homotopy $f_t : D^k \to \overline{\mathrm{Emb}}(M, \mathbb{C}P^{n+p})$ fixed on ∂D^k, with

$$f_p(z)^*\Omega_{n+p} = f_0(z)^*\Omega_n + \frac{1}{\pi}\sum_{k=1}^{p} dh_k(z) \wedge dt_k(z) = f_0^*\Omega_n + d\omega(z) = \Omega,$$

for all $z \in D^k$.

□

Note that in the previous proof we had a continuous map $\alpha : D^k \to d(\Omega^1(M))$ and we needed to find a continuous map $\omega : D^k \to \Omega^1(M)$ such that $d \circ \omega = \alpha$. The fact that this map ω exists is a consequence of the follwoing Theorem.

Theorem 4.3. *Let M be a closed manifold. Then there is a continuous linear map*

$$P_M : d(\Omega^*(M)) \to \Omega^*(M)$$

such that $dP_M = \mathrm{id}$.

Sketch of proof. The idea is to find an explicit formula for a primitive of any exact form $\omega \in \Omega^k(M)$. We consider a good cover $\{U_0, \cdots, U_n\}$ of M and write $\omega = \sum_i \omega_i$ with $\omega_i \in \Omega^k_c(U_i)$ a compactly supported exact form on U_i. Then we use the formula from the Poincaré Lemma for compactly supported forms on $\mathbb{R}^m$ to find an explicit primitive of each ω_i. To find explicit exact forms $\omega_i \in \Omega^k_c(U_i)$ such that $\omega = \sum_i \omega_i$, we start by taking a partition of unity $\{\rho_0, \cdots, \rho_n\}$ subordinate to $\{U_0, \cdots, U_n\}$. One could try to use the forms $\rho_i\omega$, but unfortunately there is no reason why these should be exact. Therefore, we need to use the Čech-de Rham double complex for compactly supported forms, associated to the open cover $\{U_0, \cdots, U_n\}$, to find "corrections" β_i such that $\sum_i \beta_i = 0$ and $\omega_i = \rho_i\omega + \beta_i$ is exact.

□

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