TÉCNICO LISBOA

Castelnuovo-Mumford regularity and Ulrich ideals

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Para a minha mãe

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Resumo

Introduzimos a noção de invólucro injectivo e caracterizamos os módulos injectivos sobre anéis Noetherianos. Definimos anel de Gorenstein, provamos que estes são Cohen-Macaulay e algumas condições necessárias e suficientes para que um anel Noetheriano local seja Gorenstein. Usando a teoria de invólucros injectivos, definimos cohomologia local, provamos que esta pode ser usada para medir depth e dimensão e definimos a regularidade de Castelnuovo-Mumford. Seguindo os resultados de [6], aplicamos estes conceitos à teoria de ideais de Ulrich e caracterizamo-los sobre anéis de Gorenstein.

Keywords: Invólucro injectivo, Dimensão injectiva, Anel de Gorenstein, Cohomologia local, Regularidade de Castelnuovo-Mumford, ideais de Ulrich

Abstract

We introduce the notion of injective envelope and characterize injective modules over Noetherian rings. We define Gorenstein rings, prove these are Cohen-Macaulay and some necessary and sufficient conditions for a Noetherian local ring to be Gorenstein. Using the theory of injective modules, we define local cohomology, prove it measures depth and dimension and define the Castelnuovo-Mumford regularity. Following [6], we apply these concepts to the theory of Ulrich ideals and characterize them over Gorenstein rings.

Keywords: Injective envelope, Injective dimension, Gorenstein ring, Local Cohomology, Castelnuovo-Mumford regularity, Ulrich ideals

Contents

Introduction

M. P. Brodmann and R. Y. Sharp wrote ([28]) that local cohomology is “an algebraic child of geometic parents.” First created by Grothendieck, originally to prove Lefschetz-type theorems in Algebraic Geometry, this theory soon proved to be useful in more algebraic contexts as well. In Commutative Algebra, local cohomology modules can be used to measure important algebraic invariants such as dimension and depth and to define other homological invariants such as the a_i-invariants and the Castelnuovo-Mumford regularity.

The study of local cohomology is directly linked to the study of injective envelopes, injective dimension and Gorenstein rings. The theory of injective envelopes was mainly developed by Eben Matlis ([18]) and relates to the study of Gorenstein rings. These rings were first introduced by Alexander Grothendieck ([8]), who named them that way because of their relation to a duality property of singular plane curves studied by Gorenstein ([5]), although Gorenstein himself claimed not to understand Gorenstein rings.

Local cohomology plays an important role in modern Commutative Algebra. Among its many applications, the recent topic of Ulrich ideals is particularly interesting as it arises from decades of study started with Northcott and Rees' reductions in the 1950s ([21]) and Sally's study of Abyankar's inequality in the 1970s ([25]). These topics are related to blow-up algebras, a class of graded rings that appears in many constructions in Commutative Algebra and Algebraic Geometry. This class includes polynomial rings, the Rees Algebra and the associated graded ring.

To define local cohomology we need injective modules. Every module M can be embedded in an injective module, and the injective envelope of M is the isomorphism class of the minimal such injective modules. Over a Noetherian ring, every injective module can be written as a direct sum of certain injective envelopes, namely those of $R/\mathfrak{p}$ for $\mathfrak{p}$ a prime ideal. Using injective envelopes we can build minimal injective resolutions and define the injective dimension. Rings of finite injective dimension form a special class of Cohen-Macaulay rings called Gorenstein rings.

Local cohomology modules are defined as the right derived functors of the following functor: fixing a ring R and an ideal I, we associate to each module M the submodule of elements annihilated by some power of I. This operation is not exact, and the local cohomology modules measure its failure to be exact.

For modules M over Noetherian rings almost all local cohomology modules of M vanish, except perhaps those in degrees i with grade$(I, M) \leqslant i \leqslant \dim(M)$. Moreover, if we consider graded rings and graded modules, the local cohomology modules inherit a graded structure. The study of graded local cohomology modules leads to the definition of the a_i-invariants and the Castelnuovo-Mumford regularity.

The purpose of this thesis is to organize existing results in the theory of local cohomology, understand the role local cohomology plays in modern Commutative Algebra and the necessary results to define the a_i-invariants and the Castelnuovo-Mumford regularity. It is intended that no knowledge of Homological Algebra is necessary except that which can be found in chapters 1, 2, 6 and 7 of [23].

We state all the necessary results and definitions from Commutative Algebra in Chapter 0, all of which can be found in [2], [20], [19], [23] or [4].

Chapters 1 and 2 were based on notes from a course taught by Wolmer Vasconcelos at Rutgers in the fall of 1993, [11] and [28]. Chapter 1 concerns injective modules and injective dimension. In section 1.1, we develop the theory of essential extensions, establish the existence of the injective envelope and prove a fundamental result concerning the structure of injective modules over Noetherian rings. Section 1.2 focuses on injective modules over Artinian local rings $(R, \mathfrak{m})$, characterizing the unique irreducible injective module $E(R/\mathfrak{m})$. Section 1.3 regards injective dimension and mainly functions as support to section 1.4. In 1.4, we define Gorenstein rings, prove that these are Cohen-Macaulay, establish necessary and sufficient conditions for a ring to be Gorenstein and prove that quotients of Gorenstein rings by regular sequences are Gorenstein. In section 1.6, we define the Bass numbers of a module and prove that the Bass numbers of finitely generated modules over Noetherian rings are finite.

In chapter 2 we study local cohomology. In section 2.1 we define local cohomology modules and prove that they measure grade and dimension. In section 2.2 we study the case of graded rings and graded modules and define the a_i-invariants and the Castelnuovo-Mumford regularity.

Chapter 3 follows chapter 8 of [13] and section 2 of [6] to study Ulrich ideals. In section 3.1 we establish the necessary preliminaries, starting with the definition of the necessary blow-up algebras: the Rees algebra, the associated graded ring and the fiber cone. We define reductions of ideals, proving the existence of minimal reductions and establishing results concerning the number of generators of minimal reductions. In section 3.2, we define Ulrich ideals over Cohen-Macaulay local rings, establish a bound for the e_0-multiplicity of an $\mathfrak{m}$-primary ideal with reduction number 1 and give necessary and sufficient conditions for an ideal to be Ulrich. Specializing to the case of Gorenstein rings, we define good ideals, prove that Ulrich ideals are always good and characterize Ulrich ideals over Gorenstein rings.

Chapter 0

Preliminaries

In this chapter we establish some notation and state several classical Commutative Algebra results which we will use later on, ommiting most proofs and rather giving references. We advise the reader to start with chapter 1 and use this chapter simply as reference.

All rings considered are commutative rings with identity. All modules considered are both left and right modules.

0.1 Classical Commutative Algebra

Notation 0.1.1. Let R be a ring. We denote the set of prime ideals in R by $\mathrm{Spec}(R)$.

Lemma 0.1.2 (Prime Avoidance)**.** *Let R be a ring, and $I_1, \ldots, I_n$ be ideals in R such that at most two of them are not prime ideals. If I is an ideal in R and $I \subseteq I_1 \cup \ldots \cup I_n$, then $I \subseteq I_i$ for some $1 \leqslant i \leqslant n$.*

Proof. See [4, Lemma 3.3]. □

Notation 0.1.3. Let R be a ring, M an R-module and U a multiplicative set in R. We denote the localization of M in U by M_U. If $\mathfrak{p}$ is a prime ideal in R, we denote the localization of M in the multiplicative set $R \setminus \mathfrak{p}$ by $M_{\mathfrak{p}}$.

Lemma 0.1.4. *Let R be a ring and S a multiplicative set in R. Every ideal in R_S is of the form I_S with I an ideal in R. Every prime ideal in R_S is of the form P_S with P a prime ideal in R disjoint from S.*

Proof. See [20, Theorem 4.1]. □

Lemma 0.1.5. *Let R be a ring, S a multiplicative set in R and M an R-module. If M is finitely generated R-module then M_S is a finitely generated R_S-module.*

Proof. Let $M = Ra_1 + \ldots + Ra_n$. It is an easy exercise to check that $\frac{a_1}{1}, \ldots, \frac{a_n}{1}$ generates M_U. □

Proposition 0.1.6. *Let R be a ring and M an R-module. If $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ in R, then $M = 0$.*

Proof. Obvious corollary of [20, Theorem 4.6]. □

Definition 0.1.7 (Flat module)**.** Let R be a ring and M an R-module. We say that M is a *flat* R-module if the functor $-\otimes_R M$ is exact, that is, if it preserves exact sequences.

Proposition 0.1.8. *Let R be a ring and S a multiplicative subset of R. Then R_S is a flat R-module. In particular, localization preserves exact sequences.*

Proof. See [19, 3D]. To see that localization preserves exact sequences, just note that for any R-module M, $M_S \cong M \otimes_R R_S$. □

Proposition 0.1.9. *Let R be a ring, I an ideal in R and M an R-module. Then $R/I \otimes M \cong M/IM$.*

Proof. See [16, XVI, 2.7]. □

Definition 0.1.10 (Minimal number of generators)**.** Let R be a Noetherian ring and I an ideal in R. We denote the *minimal number of generators* of I by $\mu(I)$, that is, $\mu(I)$ is the minimum integer $n \geqslant 0$ such that we can find $a_1, \ldots, a_n \in R$ with $I = (a_1, \ldots, a_n)$.

Proposition 0.1.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I any ideal in R. Then*

$$\mu(I) = \dim_{R/\mathfrak{m}} (I/\mathfrak{m}I)$$

and if $x_1, \ldots, x_n \in I$ are such that $\{x_1 + \mathfrak{m}I, \ldots, x_n + \mathfrak{m}I\}$ is a basis of $I/\mathfrak{m}I$ as a vector space over $R/\mathfrak{m}$, then $I = (x_1, \ldots, x_n)$.

Proof. See [20, Theorem 2.3]. □

Proposition 0.1.12. *Let R be a ring, $\{M_i\}_{i\in I}$ be a family of R-modules and N be an R-module. Then*

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$$

and

$$\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \cong \prod_{i\in I} \mathrm{Hom}_R(N, M_i).$$

In particular, if $I = \{1, \ldots, n\}$ for some integer $n \geqslant 1$, then

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right) \cong \bigoplus_{i\in I} \mathrm{Hom}_R(M_i, N)$$

and

$$\mathrm{Hom}_R\left(N, \bigoplus_{i\in I} M_i\right) \cong \bigoplus_{i\in I} \mathrm{Hom}_R(N, M_i).$$

Proof. See [23, Theorems 2.30 and 2.31]. □

Lemma 0.1.13. *Let R be a ring, I an ideal and M an R-module. Then*

$$\mathrm{Hom}_R(R/I, M) = \{a \in M : I\,a = 0\}.$$

Proof. Any $f \in \mathrm{Hom}_R(R/I, M)$ corresponds uniquely to a $g \in \mathrm{Hom}_R(R, M) \cong M$ such that $I \subseteq \ker(g)$, which in M corresponds to the condition $I\, g(1) = 0$. □

0.2 Dimension

Definition 0.2.1 (Height)**.** Let R be a ring and $\mathfrak{p}$ a prime ideal in R. The *height* of $\mathfrak{p}$ is defined as

$$\mathsf{ht}(\mathfrak{p}) := \sup\{n\colon\ \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n = \mathfrak{p},\ \mathfrak{p}_i \in \mathrm{Spec}(R)\}$$

so that this is either a non-negative integer or ∞. For I a general ideal in R, we define the height of I as

$$\mathsf{ht}(I) := \inf\{\mathsf{ht}\,(\mathfrak{p})\colon I \subseteq \mathfrak{p} \in \mathrm{Spec}(R)\}$$

Definition 0.2.2 (Krull dimension)**.** Let R be a ring. The *Krull dimension*, or simply *dimension*, of R is defined as

$$\dim(R) := \sup\{n:\ \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n,\ \mathfrak{p}_i \in \mathrm{Spec}(R)\}$$

so that $\dim(R)$ is either a non-negative integer or ∞. If M is an R-module, the Krull dimension of M is the Krull dimension of the ring $R/\mathrm{Ann}_R(M)$, and denoted by $\dim(M)$.

Proposition 0.2.3. *For any ring R and any $n \geqslant 1$, the polynomial ring $S = R[T_1, \ldots, T_n]$ has dimension*

$$\dim(S) = \dim(R) + n.$$

Proof. See [31, 1.1.9]. □

0.3 Chain Conditions

Definition 0.3.1. Let R be a ring and M an R-module. We say that M is

(1) a *Noetherian* module if all ascending chains of submodules of M stop, that is, if for every chain of submodules

$$M_0 \subseteq M_1 \subseteq M_2 \cdots$$

there exists $N \geqslant 0$ such that for all $n > N$, $M_n = M_N$.

(2) an *Artinian* module if all descending chains of submodules of M stop, that is, if for every chain of submodules

$$M_0 \supseteq M_1 \supseteq M_2 \cdots$$

there exists $N \geqslant 0$ such that for all $n > N$, $M_n = M_N$.

Proposition 0.3.2. *Let R be a ring. The following conditions are equivalent:*

(1) *R is a Noetherian ring*

(2) *Every non-empty set of ideals in R has a maximal element*

(3) *Every ideal of R is finitely generated.*

Moreover, an R-module M is a Noetherian module if and only if every submodule of M is finitely generated.

Proof. See [2, Propositions 6.1 and 6.2]. □

Proposition 0.3.3. *Let R be a ring.*

(1) *Finite direct sums of Noetherian (respectively, Artinian) R-modules are Noetherian (Artinian).*

(2) *Submodules of Noetherian (respectively, Artinian) R-modules are Noetherian (Artinian).*

(3) *Quotients of Noetherian (respectively, Artinian) R-modules are Noetherian (Artinian).*

Proof. Clear, once we understand that

(1) Submodules of $M_1 \oplus \cdots \oplus M_n$ are of the form $N_1 \oplus \cdots \oplus N_n$, with each N_i a submodule of M_i.

(2) For every R-modules $N \subseteq M$, every submodule of N is a submodule of M.

(3) For every R-modules $N \subseteq M$, the submodules of the quotient module M/N are of the form M'/N, with $M' \supseteq N$ a submodule of M.

□

Theorem 0.3.4 (Akizuki)**.** *Every Artinian ring is a Noetherian ring.*

Proof. See [20, Theorem 3.2]. □

Theorem 0.3.5. *Let R be an Artinian ring. Then $\dim(R) = 0$.*

Proof. See [2, Theorem 8.1]. □

Theorem 0.3.6. *Let R be a Noetherian ring. If $\dim(R) = 0$ then R is Artinian.*

Proof. See [2, Theorem 8.5]. □

Proposition 0.3.7. *Let R be a ring and M a finitely generated R-module.*

(1) *If R is a Noetherian ring, then M is a Noetherian R-module.*

(2) *If R is an Artinian ring, then M is an Artinian R-module.*

Proof. See [20, Theorem 3.1 (iii)]. □

Theorem 0.3.8 (Krull's Intersection Theorem)**.**

(1) *Let R be a Noetherian domain and $I \neq R$ a proper ideal in R. Then*

$$\bigcap_{n \geqslant 1} I^n = 0.$$

(2) *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$\bigcap_{n \geqslant 1} \mathfrak{m}^n = 0.$$

Proof. See [20, Theorems 8.9 and 8.10]. ☐

0.4 Radicals, Primary Ideals and Minimal Primes

Definition 0.4.1 (Radical)**.** Let R be a ring and I an ideal in R. The *radical* of I is the set

$$\sqrt{I} = \{r \in R \,|\, \exists\, n \geqslant 1 : \, r^n \in I\}.$$

The *Jacobson radical* of R is the intersection of all maximal ideals of R.

Remark 0.4.2. It can be shown that if I is an ideal in R, then so is $\sqrt{I}$. The Jacobson radical is also an ideal, as intersections of ideals are still ideals.

Proposition 0.4.3. *Let R be a ring. The nilradical of R, $\sqrt{0}$, coincides with the intersection of all the prime ideals in R. Moreover, if I is any ideal in R, then $\sqrt{I}$ coincides with the intersection of all the prime ideals containing I, or equivalently, with the intersection of all the minimal prime ideals over I.*

Proof. See [2, Proposition 1.8]. ☐

Lemma 0.4.4. *Let R be a Noetherian ring and I an ideal in R. There exists $s \geqslant 1$ such that $\left(\sqrt{I}\right)^s \subseteq I$.*

Proof. Let $\sqrt{I} = (a_1, \dots, a_n)$. For each $i = 1, \dots, n$, there exists $s_i \geqslant 1$ such that $a_i^{s_i} \in I$. Using the multinomial formula, we can show that $(r_1 a_1 + \ldots + r_n a_n)^s \in I$ for every $s \geqslant s_1 + \ldots + s_n$ and $r_1, \dots, r_n \in R$. ☐

Definition 0.4.5 (Primary ideal)**.** Let R be a ring and I an ideal in R. We say that I is a *primary ideal* if $I \neq R$ and for every $a, b \in R$, if $ab \in I$ then $a \in I$ or $b \in \sqrt{I}$.

Proposition 0.4.6. *Let R be a ring and Q a primary ideal in R. Then $\sqrt{Q}$ is a prime ideal. In particular, $\sqrt{Q}$ is the only minimal prime over Q.*

Proof. See [20, Ex. 4.1], and 0.4.3. ☐

Definition 0.4.7. Let R be a ring and Q a primary ideal in R. If $\sqrt{Q} = \mathfrak{p}$, we say that Q is a $\mathfrak{p}$-primary ideal.

Proposition 0.4.8. *Let R be a Noetherian ring and I an ideal in R. The set of minimal primes over I is a finite set.*

Proof. See [20, Theorem 6.5] and subsequent remark. □

Lemma 0.4.9 (Nakayama's Lemma)**.** *Let R be a ring, M a finitely generated R-module and $I \neq R$ an ideal contained in the Jacobson radical of R. If $IM = M$ then $M = 0$.*

Proof. See [2, Proposition 2.6]. □

Corollary 0.4.10. *Let R be a ring, M a finitely generated R-module, N an R-submodule of M and I an ideal contained in the Jacobson radical of R. If $M = IM + N$ then $M = N$.*

Proof. See [2, Corollary 2.7]. □

0.5 Length

Definition 0.5.1 (Composition series)**.** Let R be a ring and M an R-module. A *composition series* of M is a maximal chain of submodules of the form

$$M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n = 0$$

where maximal means that each quotient M_i/M_{i+1} is a simple R-module, so that the chain cannot be augmented.

Proposition 0.5.2. *Let R be a ring and M an R-module. If M has a composition series of length n, then every composition series of M has length n. Moreover, any chain in M can be extended to a composition series.*

Proof. See [2, Proposition 6.7]. □

Definition 0.5.3 (Length)**.** Let R be a ring and M an R-module. If M has a composition series of length n, we say that M has *length* n, and write $\lambda(M) = n$. Otherwise, we say that M has infinite length.

Proposition 0.5.4. *An R-module M has finite length if and only if M is both an Artinian and Noetherian R-module.*

Proof. See [2, Proposition 6.8]. □

Proposition 0.5.5. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of R-modules of finite length. Then

$$\lambda(B) = \lambda(A) + \lambda(C).$$

Proof. See [2, Proposition 6.9]. □

Proposition 0.5.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and consider a proper ideal $I \neq R$. Then $\lambda(R/I) < \infty$ if and only if I is an $\mathfrak{m}$-primary ideal.*

Proof. See [20, p. 98]. □

Proposition 0.5.7. *Let K be a field and V a K-vector space. Then the following conditions are equivalent:*

1. *K is a finite dimensional vector space over K*
2. *$\lambda(K)$ is finite*
3. *K is Noetherian*
4. *K is Artinian*

Moreover, when these conditions are satisfied then $\dim_K(V) = \lambda(V)$.

0.6 Regular Sequences, Grade and Depth

Definition 0.6.1. Let R be a ring and M an R-module. We say that $r \in R$ is a *zero-divisor* of M if there exists $0 \neq m \in M$ with $rm = 0$. Otherwise, we say that r is a *regular* element in M.

Definition 0.6.2 (Regular Sequence)**.** Let R be a ring and M an R-module. We say that $x_1, \ldots, x_n \in R$ is a *regular sequence* in M if

(i) $(x_1, \ldots, x_n)M \neq M$

(ii) x_1 is regular in M

(iii) x_2 is regular in $M/x_1 M$

(iv) x_n is regular in $M/(x_1, \ldots, x_n)M$.

Definition 0.6.3 (System of parameters)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring with $d = \dim(R)$. If $a_1, \ldots, a_d \in \mathfrak{m}$ generate an $\mathfrak{m}$-primary ideal, we say that $a_1, \ldots, a_d$ is a *system of parameters* of R.

Definition 0.6.4 (Parameter ideal)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal in R. We say that I is a *parameter ideal* of R if I is generated by a system of parameters.

Proposition 0.6.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. If $a_1, \ldots, a_r$ is a system of parameters in R, then it is a regular sequence. In particular, if I is any parameter ideal, then ht$(I) = \dim(R)$.*

Proof. See [20, Theorem 17.4, (iii)]. □

Theorem 0.6.6. *Let R be a Noetherian ring, $M \neq 0$ a finitely generated R-module and I an ideal in R such that $IM \neq M$. Any maximal regular sequence in M inside I has the same number of elements.*

Proof. See [20, Theorem 16.7]. □

Definition 0.6.7 (Grade)**.** Let R be a Noetherian ring, $M \neq 0$ be a finitely generated R-module and I an ideal in R such that $IM \neq M$. The *grade* of M with respect to I is the length of a maximal regular sequence in M inside I, which is a well-defined number by 0.6.6, and denoted by grade(I, M). In case $IM = M$, we write grade$(I, M) = \infty$.

Definition 0.6.8 (Depth)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring. We define the *depth* of and R-module M as depth$(M) :=$ grade$(\mathfrak{m}, M)$.

Theorem 0.6.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $x \in R$ a regular element. Then*

$$\dim(R/(x)) = \dim(R) - 1 \textit{ and depth}(R/(x)) = \textit{depth}(R) - 1.$$

Proof. See [20, Ex. 16.1]. □

Theorem 0.6.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$\textit{depth}(R) \leqslant \dim(R).$$

Proof. See [11, Proposition 1.2.12]. □

Definition 0.6.11 (Cohen-Macaulay ring)**.** We say that a Noetherian local ring R is a *Cohen-Macaulay* ring if depth$(R) = \dim(R)$. A Noetherian ring R is a Cohen-Macaulay ring if $R_\mathfrak{m}$ is a Cohen-Macaulay local ring for every maximal ideal $\mathfrak{m}$ in R.

Definition 0.6.12 (Cohen-Macaulay module)**.** Let $(R, \mathfrak{m})$ be a Noetherian local ring and M an R-module. We say that M is a *Cohen-Macaulay module* if depth$(M) = \dim(M)$.

Theorem 0.6.13. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $I \neq R$ any proper ideal in R. Then ht(I) = depth(I).*

Proof. See [20, Theorem 17.4 (i)]. □

Theorem 0.6.14. *Let R be a ring and M an R-module. If $x_1, \dots, x_n$ is a regular sequence in M, then ${x_1}^{t_1}, \dots, {x_n}^{t_n}$ is still a regular sequence, for any $t_1, \dots, t_n \geqslant 1$.*

Proof. See [20, Theorem 16.1]. □

Theorem 0.6.15. *Let R be a Noetherian ring, M a finitely generated R-module and I an ideal in R such that $IM \neq M$. Then*

$$\textit{grade}(I, M) = \min\left\{i\colon \operatorname{Ext}^i_R(R/I, M) \neq 0\right\}$$

Proof. See [11, Theorem 1.2.5]. □

0.7 Associated Prime Ideals

Definition 0.7.1 (Annihilator)**.** Let R be a ring and M an R-module. The annihilator of M is the set

$$\mathrm{Ann}_R(M) = \{r \in R : rM = 0\}$$

Similarly, we define the annihilator of $m \in M$ as

$$\mathrm{Ann}_R(m) = \{r \in R : \ rm = 0\}$$

If M and N are two submodules of the same R-module, we define

$$(N :_R M) := \{r \in R : \ rM \subseteq N\}$$

which can be shown to be an ideal in R. We can see $\mathrm{Ann}_R(M)$ as $(0 :_R M)$.

Lemma 0.7.2. *Let R be a ring, M a finitely generated R-module, and N and P submodules of M. Consider a multiplicative set S in R and I an ideal in R. Then*

(1) $(N :_R P)_S \cong (N_S :_{R_S} P_S)$.

(2) $\sqrt{I_S} = \left(\sqrt{I}\right)_S$.

Proof.

(1) The proof is an easy exercise, and consists of showing that $(\mathrm{Ann}_R(M))_S = \mathrm{Ann}_{R_S}(M_S)$ and that $(N :_R P) = \mathrm{Ann}_R\left(\frac{N+P}{N}\right)$.

(2) By 0.4.3, $\sqrt{I_S}$ is the intersection of the minimal primes over I_S, and by 0.1.4, the minimal primes over I_S are the localization of the minimal primes Q over I with $Q \cap S = \emptyset$.

□

Definition 0.7.3 (Associated prime ideal)**.** Let R be a ring and M an R-module. A prime ideal $\mathfrak{p}$ of R is said to be an *associated prime* of M if $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some $0 \neq m \in M$. We denote the set of associated primes of M by $\mathrm{Ass}(M)$.

Remark 0.7.4. Let M be an R-module. A prime ideal $\mathfrak{p}$ is an associated prime of M if an only if there exists a monomorphism $R/\mathfrak{p} \hookrightarrow M$. If $m \in M$ is a non-zero element such that $\mathfrak{p} = \mathrm{Ann}_R(m)$, the image of such a monomorphism is the submodule of M generated by m.

Lemma 0.7.5. *Let R be a ring and consider R-modules $M \subseteq N$. Then $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(N)$.*

Proof. Obvious, as if $\mathfrak{p}$ is the annihilator of an element of M, it is also the annihilator of an element in N (the same element). □

Proposition 0.7.6. *Let R be a ring and $\mathfrak{p}$ a prime ideal in R. Then $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$.*

Proof. The non-zero elements in $R/\mathfrak{p}$ are of the form $r+\mathfrak{p}$, for some $r \in R$ with $r \notin \mathfrak{p}$. Clearly, $\mathfrak{p}(r+\mathfrak{p}) = \mathfrak{p}$, meaning that $\mathfrak{p} \subseteq \operatorname{Ann}_R(r+\mathfrak{p})$. On the other hand, if $s \in R$ is an element in $\operatorname{Ann}_R(r+\mathfrak{p})$, then $s(r+\mathfrak{p}) = 0$, meaning that $sr \in \mathfrak{p}$. Considering that $r \notin \mathfrak{p}$ and that $\mathfrak{p}$ is a prime ideal, we conclude that $s \in \mathfrak{p}$. Thus, $\mathfrak{p}$ is the only associated prime of $R/\mathfrak{p}$, and moreover the annihilator of every non-zero element. ☐

Theorem 0.7.7. *Let R be a Noetherian ring and $M \neq 0$ an R-module. Then*

(1) *Every maximal element of the family of ideals*

$$\{\operatorname{Ann}_R(x) : 0 \neq x \in M\}$$

is an associated prime of M, and in particular $\operatorname{Ass}(M) \neq \emptyset$.

(2) *The set of zero-divisors of M coincides with the union of the associated prime ideals of M.*

Proof. See [20, Theorem 6.1]. ☐

Definition 0.7.8 (Support)**.** Let R be a ring and M an R-module. The *support* of M is the set

$$\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) :\ M_{\mathfrak{p}} \neq 0\}\,.$$

Notation 0.7.9. Let R be a ring and I an ideal in R. The set of prime ideals containing I is

$$V(I) := \{P \in \operatorname{Spec}(R) :\ I \subseteq P\}\,.$$

Theorem 0.7.10. *Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then*

$$\operatorname{Ass}(M) \subseteq \operatorname{Supp}(M),$$

and the minimal elements on these two sets coincide. Moreover, $\operatorname{Ass}(M)$ is a finite set.

Proof. See [20, Theorem 6.5]. ☐

Remark 0.7.11. Notice that the minimal primes in $\operatorname{Supp}(M)$ are the minimal primes over $\operatorname{Ann}_R(M)$, so 0.7.10 states that the minimal primes over $\operatorname{Ann}_R(M)$ are also associated primes of M.

Lemma 0.7.12. *Let R be a ring and M a finitely generated R-module. Then $\operatorname{Supp}(M) = V(\operatorname{Ann}_R(M))$.*

Proof. See [20, p. 26]. ☐

Theorem 0.7.13. *Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then there exists a chain*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

of submodules of M such that for each $1 \leqslant i \leqslant n$, $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ in R. Moreover, $\dim(R/\mathfrak{p}_i) \leqslant \dim(M)$.

Proof. See [20, Theorem 6.4]. In the proof, $\mathfrak{p}_1 \in \operatorname{Ass}(M)$, $\mathfrak{p}_2 \in \operatorname{Ass}(M/M_1)$, $\mathfrak{p}_3 \in \operatorname{Ass}(M_1/M_2)$, and so on. By 0.7.5, $\operatorname{Ass}(M) \subseteq \operatorname{Supp}(M) = V(\operatorname{Ann}_R(M))$. Then $\operatorname{Ann}_R(M) \subseteq \mathfrak{p}_1$. Moreover, it is clear that $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(M_i/M_{i+1}) \subseteq \mathfrak{p}_i$, so that $\operatorname{Ann}_R(M) \subseteq \mathfrak{p}_i$ for every $i = 1, \ldots, n$. Then

$$\dim\left(\frac{R}{\mathfrak{p}_i}\right) \leqslant \dim\left(\frac{R}{\operatorname{Ann}_R(M)}\right) = \dim(M).$$

□

0.8 Injective and Projective Modules

Proposition 0.8.1. *Every free module is projective.*

Proof. See [17, Lemma 5.4]. □

Proposition 0.8.2. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Then the following conditions are equivalent:*

(1) *M is free*

(2) *M is projective*

(3) *M is flat.*

Proof. See [19, (3G)]. □

Theorem 0.8.3. *Let M be an R-module. There exists an injective module I such that $M \subseteq I$.*

Proof. See [16, XX, Theorem 4.1]. □

Proposition 0.8.4. *A module I over a ring R is injective if and only if the functor $\operatorname{Hom}_R(-, I)$ is exact.*

Proof. See [23, Proposition 3.25]. □

Proposition 0.8.5. *Let R be a ring and consider a family of injective R-modules $\{I_i\}_{i\in I}$. Then $\prod\limits_{i\in I} I_I$ is an injective module.*

Proof. See [23, Proposition 3.28]. □

Theorem 0.8.6 (Baer Criterion)**.** *An R-module E is injective if and only if for every ideal I of R and every $f\colon I \longrightarrow E$, f can be extended to a homomorphism $\overline{f}\colon R \longrightarrow E$.*

Proof. See Rotman, Theorem 3.30. □

Proposition 0.8.7. *Let $I = A \oplus B$ be an injective module. Then A and B are injective modules.*

Proof. See [23, Proposition 3.28]. □

Proposition 0.8.8. *Let R be a Noetherian ring and $\{I_\lambda\}_{\lambda \in \Lambda}$ a family of injective R-modules. Then*

$$\bigoplus_{\lambda \in \Lambda} I_\lambda$$

is an injective module.

Proof. See [11, Theorem 3.1.3]. □

Proposition 0.8.9. *An R-module I is injective if and only if I is a direct summand of every R-module M with $I \subseteq M$.*

Proof. See [11, Theorem 3.1.2]. □

0.9 Injective and Projective Dimension

Definition 0.9.1 (Injective Resolution)**.** Let M be an R-module. An *injective resolution* of M is an exact sequence of the form

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_2 \longrightarrow \cdots$$

where each I_i is a projective module.

Every R-module M has an injective resolution. We start by choosing an injective module I_0 and a monomorphism $\alpha_0 \colon M \to I_0$. Let $\pi_0 \colon I_0 \to \operatorname{coker} \alpha_0$ be the canonical projection. This yields an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha_0} I_0 \xrightarrow{\pi_0} \operatorname{coker} \alpha_0 \longrightarrow 0 \ .$$

Now find an injective module I_1 and a monomorphism β_0: $\operatorname{coker} \alpha_0 \to I_1$. Define α_1: $I_0 \to I_1$ as the composition $\alpha_1 = \beta_0 \circ \pi_0$. Since β_1 is injective, $\ker \alpha_1 = \ker \pi_0 = \alpha_0(I_0)$, so that the sequence is indeed exact.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 \ . \\
 & & & \operatorname{coker} \alpha_0 \arrow[ur] \arrow[dr, "\beta_1"] & \\
0 \arrow[r] & M \arrow[r, "\alpha_0"'] & I_0 \arrow[ur, "\pi_0"] \arrow[rr, "\alpha_1"'] & & I_1
\end{tikzcd}$$

Now repeat the procedure. Assume $\alpha_n \colon I_{n-1} \longrightarrow I_n$ is defined. Let $\pi_n \colon I_n \to \operatorname{coker} \alpha_n$ be the canonical projection and find an injective module I_{n+1} and a monomorphism $\beta_n \colon \operatorname{coker}\alpha_n \longrightarrow I_{n+1}$.

Then define $\alpha_{n+1} = \beta_n \circ \pi_n$. By the same argument as before, we have exactness at I_n.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & \operatorname{coker}\alpha_0 \arrow[ur] \arrow[dr, "\beta_1"] & & & & \operatorname{coker}\alpha_2 \arrow[ur] & \\
0 \arrow[r] & M \arrow[r, "\alpha_0"'] & I_0 \arrow[ur, "\pi_0"] \arrow[rr, "\alpha_1"'] & & I_1 \arrow[rr, "\alpha_2"'] \arrow[dr, "\pi_1"'] & & I_2 \arrow[ur, "\pi_2"'] \arrow[rr, "\alpha_3"'] & & \cdots \\
 & & & & & \operatorname{coker}\alpha_1 \arrow[ur, "\beta_2"] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

This yields the desired injective resolution.

Definition 0.9.2 (Projective resolution)**.** Let M be an R-module. A *projective resolution* of M is an exact sequence of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_n is a projective module for all $n \geqslant 0$.

Definition 0.9.3 (Injective dimension)**.** Let M be an R-module. If M admits a finite injective resolution, meaning an injective resolution $M \longrightarrow \mathbf{I}_\bullet$ with $I_n = 0$ for all $n \geqslant m$ for some m, then the minimal length m amongst such injective resolutions is called the *injective dimension* of M, and denoted inj $\dim_R(M)$ or simply inj $\dim(M)$. If M has no finite injective resolution, then we define inj $\dim(M) = \infty$.

Definition 0.9.4 (Minimal injective resolution)**.** An injective resolution of M of minimal length is said to be a *minimal injective resolution*.

Definition 0.9.5 (Projective dimension)**.** Let M be an R-module. If M admits a finite projective resolution, meaning a projective resolution $\mathbf{P}_\bullet \longrightarrow \mathbf{M}$ with $P_n = 0$ for all $n \geqslant m$ for some m, then the minimal length m among such projective resolutions is called the *projective dimension* of M, and denoted proj $\dim_R(M)$ or simply proj $\dim(M)$. If M has no finite projective resolution, then we define proj $\dim(M) = \infty$.

Proposition 0.9.6. *Let R be a ring and M an R-module. Then*

$$\textit{inj dim}(M) = \sup\left\{ i \colon \operatorname{Ext}^i_R(N, M) \neq 0, N\ R\textit{-module} \right\}.$$

Proof. See [32, Lemma 4.1.8]. □

Proposition 0.9.7. *Let R be a ring and M an R-module. Then*

$$\textit{proj dim}(M) = \sup\left\{ i \colon \operatorname{Ext}^i_R(M, N) \neq 0, N\ R\textit{-module} \right\}.$$

Proof. See [23, Proposition 8.6]. □

Proposition 0.9.8. *Let R be a ring and M an R-module. Then*

$$\textit{inj dim}(M) = \sup\left\{i\colon \operatorname{Ext}^i_R(R/I, M) \neq 0, I \textit{ ideal in } R\right\}$$

Proof. See [20, Chapter 18, Lemma 1]. □

Proposition 0.9.9. *Let R be a ring and M an R-module. Then*

$$\textit{inj dim}(M) = \sup\left\{i\colon \operatorname{Ext}^i_R(N, M) \neq 0, N \textit{ finitely generated } R\textit{-module } \right\}$$

Proof. Obvious corollary of 0.9.6 and 0.9.8. □

Theorem 0.9.10 (Auslander-Buchsbaum Formula)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of finite projective dimension. Then*

$$\textit{proj dim}(M) + \textit{depth}(M) = \textit{depth}(R).$$

Proof. See [4, Theorem 19.9]. □

0.10 Ext and Tor

Proposition 0.10.1. *Homology commutes with direct sums: for all n and every family of complexes $\{C_\alpha\}_{\alpha\in A}$,*

$$\mathrm{H}_n\left(\bigoplus_{\alpha\in A} C_\alpha\right) = \bigoplus_{\alpha\in A} \mathrm{H}_n\left(C_\alpha\right)$$

Proof. See [23, Exercise 6.9 (i)]. □

Theorem 0.10.2. *Given a short exact sequence of chain complexes and chain maps*

$$0 \longrightarrow \mathbf{A} \xrightarrow{i} \mathbf{B} \xrightarrow{p} \mathbf{C} \longrightarrow 0\ ,$$

there exists a long exact sequence

$$\cdots \xrightarrow{\partial_{n+1}} \mathrm{H}_n(\mathbf{A}) \xrightarrow{i_*} \mathrm{H}_n(\mathbf{B}) \xrightarrow{p_*} \mathrm{H}_n(\mathbf{C}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\mathbf{A}) \longrightarrow \cdots.$$

Moreover, given a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{A} \arrow[r] \arrow[d, "f"'] & \mathbf{B} \arrow[r, "p"] \arrow[d, "g"'] & \mathbf{C} \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & \mathbf{A}' \arrow[r, "i"] & \mathbf{B}' \arrow[r, "p"] & \mathbf{C}' \arrow[r] & 0
\end{tikzcd}$$

there exists a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_n(\mathbf{A}) \arrow[r, "i_*"] \arrow[d, "f_*"] & \mathrm{H}_n(\mathbf{B}) \arrow[r, "p_*"] \arrow[d, "g_*"] & \mathrm{H}_n(\mathbf{C}) \arrow[r, "\partial_n"] \arrow[d, "h_*"] & \mathrm{H}_{n-1}(\mathbf{A}) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_n(\mathbf{A}') \arrow[r, "i_*"] & \mathrm{H}_n(\mathbf{B}') \arrow[r, "p_*"] & \mathrm{H}_n(\mathbf{C}') \arrow[r, "\partial_n"] & \mathrm{H}_{n-1}(\mathbf{A}) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. See [23, Theorem 6.10]. For naturality, see [23, Theorem 6.13]. □

Theorem 0.10.3 (Long Exact Sequences for Ext)**.** *For every short exact sequence of R-modules and R-module homomorphisms*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\ ,$$

there exists a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow \mathrm{Ext}^1_R(C, M) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_R(B, M) \longrightarrow \mathrm{Ext}^1_R(A, M) \longrightarrow \mathrm{Ext}^2_R(C, M) \longrightarrow \mathrm{Ext}^2_R(B, M) \longrightarrow \dots$$

and a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow \mathrm{Ext}^1_R(M, A) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_R(M, B) \longrightarrow \mathrm{Ext}^1_R(M, C) \longrightarrow \mathrm{Ext}^2_R(M, A) \longrightarrow \mathrm{Ext}^2_R(M, B) \longrightarrow \dots$$

so that $\mathrm{Ext}_R(-, M)$ and $\mathrm{Ext}_R(M, -)$ measure the failure of $\mathrm{Hom}_R(-, M)$ and $\mathrm{Hom}_R(M, -)$ to be exact.

Proof. See [23, Theorem 6.46 and 6.59]. For naturality, see [23, Theorem 6.13]. □

Proposition 0.10.4. *Let S be a multiplicative set in the ring R. For every $i \geqslant 0$ and every R-modules A and B,*

$$\left[\mathrm{Tor}^R_i(A, B)\right]_S \cong \mathrm{Tor}^{R_S}_i(A_S, B_S).$$

If, moreover, A is finitely generated, then

$$\left[\mathrm{Ext}^i_R(A, B)\right]_S \cong \mathrm{Ext}^i_{R_S}(A_S, B_S).$$

Proof. See [23, Propositions 7.17 and 7.39]. □

Proposition 0.10.5. *Let R be a Noetherian ring and consider finitely generated R-modules A and B. For every $i \geqslant 0$, $\mathrm{Tor}^R_i(A, B)$ and $\mathrm{Ext}^i_R(A, B)$ are finitely generated R-modules.*

Proof. See [23, Propositions 7.20 and 7.36]. □

Proposition 0.10.6. *Let R be a ring, A and B be R-modules. Consider a projective resolution of A*

$$\cdots \longrightarrow P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha} A \longrightarrow 0$$

and an injective resolution of B

$$0 \longrightarrow B \xrightarrow{\beta} I_0 \xrightarrow{\beta_0} I_1 \xrightarrow{\beta_1} I_2 \xrightarrow{\beta_2} \cdots$$

and let $K_0 = \ker(\alpha)$, $V_0 =$ im(β), $K_n =$ im(α_n) and $V_n =$ im(β_{n-1}) for $n \geqslant 1$. Then, for all $n \geqslant 0$,

$$\mathrm{Ext}_R^{n+1}(A,B) \cong \mathrm{Ext}_R^{n}(K_0,B) \cong \cdots \cong \mathrm{Ext}_R^{1}(K_n,B)$$

and

$$\mathrm{Ext}_R^{n+1}(A,B) \cong \mathrm{Ext}_R^{n}(A,V_0) \cong \cdots \cong \mathrm{Ext}_R^{1}(A,V_n).$$

Proof. See [23, Corollary 6.42]. □

Proposition 0.10.7. *In M is a projective R-module, then for all R-modules N and all $n \geqslant 1$,*

$$\mathrm{Ext}_R^{n}(M,N) \cong 0.$$

Proof. See [23, Corollary 6.58]. □

Proposition 0.10.8. *If N is an injective R-module, then for every R-module M and all $n \geqslant 1$,*

$$\mathrm{Ext}_R^{n}(M,N) \cong 0.$$

Proof. See [23, Corollary 6.41]. □

Lemma 0.10.9 (Five Lemma)**.** *Consider the commutative diagram with exact rows*

$$\begin{tikzcd}
A \arrow[r, "g_1"] \arrow[d, "f_1"'] & B \arrow[r, "g_2"] \arrow[d, "f_2"'] & C \arrow[r, "g_3"] \arrow[d, "f_3"] & D \arrow[r, "g_4"] \arrow[d, "f_4"] & E \arrow[d, "f_5"] \\
A' \arrow[r, "{g_1}'"'] & B' \arrow[r, "{g_2}'"'] & C' \arrow[r, "{g_3}'"'] & D' \arrow[r, "{g_4}'"'] & E'
\end{tikzcd}$$

1. *If f_2 and f_4 are epimorphisms and f_5 is a monomorphism, then f_3 is an epimorphism.*
2. *If f_2 and f_4 are monomorphisms and f_1 is an epimorphism, then f_3 is a monomorphism.*

Proposition 0.10.10. *Direct limit commutes with tensor product, that is, for any ring R, any R-module A and any directed system of R-modules $\{M_n\}$,*

$$A \otimes \left(\varinjlim M_n\right) \cong \varinjlim M_n \,(\otimes A)$$

Proof. See [23, Theorem 5.27]. □

Proposition 0.10.11. *Let R be a ring and A and B be R-module. For any directed system $\{M_n\}$ of R-modules,*

$$\mathrm{Hom}_R(\varinjlim M_n, B) \cong \varprojlim \mathrm{Hom}_R(M_n, B)$$

and for any inverse system of R-modules $\{M_n\}$,

$$\operatorname{Hom}_R(A, \varprojlim M_n) \cong \varprojlim \operatorname{Hom}_R(A, M_n).$$

Proof. See [23, Propositions 5.21 and 5.26]. □

Chapter 1

Injective Envelope and Gorenstein Rings

Introduction

It is a well-known fact that any module M can be embedded in an injective module. This allows us to define injective resolutions for any module, and moreover to find the minimal injective module containing M, known as the injective envelope or the injective hull of M. Understanding injective envelopes is a step towards understanding injective modules. Indeed, to study injective modules over a Noetherian ring R, we only need to know certain injective envelopes, namely those of $R/\mathfrak{p}$ for $\mathfrak{p}$ a prime ideal of R. Any injective module is a direct sum of some copies of these.

The injective envelope was first defined in 1953, by Eckmann and Schopf ([3]), but most of the theory of injective envelopes we will study was established a few years later by Eben Matlis for his PhD thesis ([18]), including the theorem known as Matlis Duality. Sections 1.1 and 1.2 deal with injective envelopes, and section 1.3 with injective dimension.

The study of injective modules and injective dimension leads to Gorenstein rings, first defined by Alexander Grothendieck ([8]) and named that way because of their relation to a duality property of singular plane curves studied by Daniel Gorenstein ([5]), although Gorenstein himself claimed not to understand Gorenstein rings. These rings, which we will deal with in section 1.4, are special cases of Cohen-Macaulay rings, and play an important role in several theories across Commutative Algebra. We will see some examples of this in chapters 2 and 3.

We prove Matlis Duality in section 1.5 for the case of Artinian local rings. In section 1.6, we prove a theorem about the structure of injective modules over Notherian local rings that completes the description of injective envelopes of finitely generated modules.

1.1 Essential Extensions and Injective Envelope

Using the theory of essential extensions, we will prove that we can find the smallest injective module containing a module M, which is known as the injective hull or injective envelope of M. Once we have established the existence and uniqueness up to isomorphism of the injective hull, we will prove some basic properties of this object and proceed to study injective modules over Noetherian rings. Using the theory of injective envelopes, we will be able to characterize every injective module over a Noetherian ring as a direct sum of certain injective envelopes, reducing the study of injective modules over Noetherian rings to the study of the injective envelope of $R/\mathfrak{p}$, for $\mathfrak{p}$ a prime ideal in R.

In this section we shall consider a fixed ring R.

Remark 1.1.1. Let $f\colon E \longrightarrow F$ be a monomorphism. We may write $E \cap G$ instead of $f(E) \cap G$. We will resort to this abuse of notation often.

Definition 1.1.2 (Essential Extension)**.** Let E and F be R-modules. We say that F is an *essential extension* of E if

(1) We have a monomorphism $E \hookrightarrow F$

(2) For every non-zero submodule $G \subseteq F$, $E \cap G \neq 0$.

Remark 1.1.3. If $E \subseteq F$ are two R-modules, then F is an essential extension of E if and only if

$$\{G\colon 0 \neq G \text{ is a submodule of } F \text{ and } G \cap E = 0\} = \emptyset.$$

Remark 1.1.4. If $E = F \neq 0$, both conditions in the previous definition are satisfied, so that every module has at least one essential extension.

Example 1.1.5. $\mathbb{Q}$ is an essential extension of $\mathbb{Z}$. Indeed, if $G \neq 0$ is any subgroup of $\mathbb{Q}$, there must be some element $\frac{p}{q} \in G$, with $p, q \in \mathbb{Z}$, $p, q \neq 0$, and therefore $p = q\frac{p}{q} \in \mathbb{Z}$.

Proposition 1.1.6. *Let $E \hookrightarrow M$, and $M \hookrightarrow N$ be R-modules. Then N is an essential extension of E if and only if N is an essential extension of M and M is an essential extension of E.*

Proof.

$(\Rightarrow)$ Consider any submodule $G \neq 0$ of N. As $0 \neq G \cap E \subseteq G \cap M$, then $G \cap M \neq 0$. This shows that N is an essential extension of M.

Now consider any submodule $G' \neq 0$ of M. Since G' is also a submodule of N, we must have $G' \cap E \neq 0$. Then M is an essential extension of E.

$(\Leftarrow)$ Let $G \neq 0$ be a submodule of N. As N is an essential extension of M, $G \cap M \neq 0$. Also, $G \cap M$ is a non-zero submodule of M, which is an essential extension of E, so that $0 \neq (G \cap M) \cap E$. As $(G \cap M) \cap E \subseteq G \cap E$, we must have $G \cap E \neq 0$. Then N is an essential extension of E. □

Lemma 1.1.7. *Let R be a Noetherian ring, E an injective R-module and S a multiplicative set of R. Then E_S is an injective R_S-module.*

Proof. Since E is an injective R-module, $\mathrm{Ext}^i_R(R/I, E) = 0$ for every ideal I in R and $i \geqslant 1$, by 0.9.8. Notice that R/I is a finitely generated R-module. By 0.10.4

$$\mathrm{Ext}^i_{R_S}(R_S/I_S, E_S) \cong [\mathrm{Ext}^i_R(R/I, E)]_S = 0_S = 0$$

for every $i \geqslant 1$. By 0.1.4, every ideal in R_S is of the form I_S. Since, by 0.9.8,

$$\text{inj dim}_{R_S}(E_S) = \sup\left\{i \colon \mathrm{Ext}^i_{R_S}(R_S/I_S, E_S) \neq 0, I \text{ an ideal in } R\right\},$$

we must therefore have inj $\dim_{R_S}(E_S) = 0$, that is, E_S is an injective R_S-module. □

Proposition 1.1.8. *Let R be a Noetherian ring, S a multiplicative set in R and $N \hookrightarrow E$ an essential extension of R-modules. Then $N_S \hookrightarrow E_S$ is an essential extension of R_S-modules.*

Proof. It is enough to show that for every $x \in E_S$, $x \neq 0$, we have $R_S x \cap N_S \neq 0$. So write $x = \frac{e}{s}$, where $e \in E$ and $s \in S$. Consider the family of ideals of R

$$A = \{\mathrm{Ann}_R(te) \colon t \in S\},$$

which is non-empty. By 0.3.2, A must have a maximal element, since R is a Noetherian ring. Choose $t \in S$ such that $\mathrm{Ann}_R(te)$ is maximal in A.

If $\mathrm{Ann}_R(te) = R$, then we would have $1 \in \mathrm{Ann}_R(te)$ and so

$$te = 0 \Rightarrow \frac{te}{1} = 0 \Rightarrow \frac{1}{st}\frac{te}{1} = 0 \Rightarrow x = 0,$$

which is a contradiction to $x \neq 0$. Then $te \neq 0$ and $\mathrm{Ann}_R(te) \subsetneq R$.

Since $N \hookrightarrow E$ is an essential extension, then $(te) \cap N \neq 0$. Suppose that $(R_S x) \cap N_S = 0$. Since t and s are invertible in R_S, then $(te)_S = R_S x$, and thus

$$((te) \cap N)_S \subseteq (te)_S \cap N_S = (R_S x) \cap N_S = 0.$$

Then $((te) \cap N)_S = 0$. Consider

$$I = \{r \in R : rte \in N\} = (N :_R te),$$

which is an ideal in R. Since R is Noetherian, then I is finitely generated. Let $I = (a_1, \ldots, a_n)$. Clearly, $Ite = (te) \cap N = 0$. Then $\frac{a_i te}{1} = 0$ for each $i = 1, \ldots, n$, and thus we can find $u_i \in S$ such that $u_i(a_i te) = 0$. Let $u = u_1 \cdots u_n \in S$. Since

$$I(ute) = uIte = 0$$

then $I \subseteq \mathrm{Ann}_R(ute)$. Moreover, we always have $\mathrm{Ann}_R(te) \subseteq \mathrm{Ann}_R(ute)$, which by maximality of $\mathrm{Ann}_R(te)$ implies $\mathrm{Ann}_R(te) = \mathrm{Ann}_R(ute)$. Then $I \subseteq \mathrm{Ann}_R(te)$, and thus $Ite = 0$. But this is impossible, considering that $Ite = (te) \cap N \neq 0$.

So indeed we must have $(R_S x) \cap N_S \neq 0$, as desired. $\square$

Proposition 1.1.9. *Let $E \subseteq H$ be R-modules. Then E has an essential extension $M \subseteq H$ that is maximal with respect to this property.*

Proof. All we have to check is that Zorn's Lemma applies to the set Γ of essential extensions F of E with $F \subseteq H$. Since $E \in \Gamma$, then $\Gamma \neq \emptyset$. If $\{F_\lambda\}_{\lambda \in \Lambda}$ is any family of R-modules in Γ totally ordered by inclusion, then

$$F := \bigcup_{\lambda \in \Lambda} F_\lambda \in \Gamma.$$

The fact that F is an R-module is true because the union of a chain of R-modules is always an R-module. Clearly, $F \subseteq H$. Moreover, let $G \neq 0$ be a submodule of F and consider a non-zero element $x \in G$. There must be some $\lambda \in \Lambda$ for which $x \in F_\lambda$, so that $F_\lambda \cap G \neq 0$. Now $F_\lambda \cap G$ is a non-zero submodule of F_λ, and since $F_\lambda \in \Gamma$ is an essential extension of E, then $(F_\lambda \cap G) \cap E \neq 0$. Then $G \cap E \neq 0$, so that F is indeed an essential extension of E. As $F \subseteq H$, this completes the proof that $F \in \Gamma$. Therefore, the conditions of Zorn's Lemma apply. $\square$

Proposition 1.1.10. *An R-module E is injective if and only if it has no proper essential extensions.*

Proof.

($\Rightarrow$) Let F be a proper essential extension of E. Since E is injective, the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow E/F \longrightarrow 0$$

splits, so that $F = E \oplus E'$ for some submodule $E' \neq 0$ of F. By definition of direct sum, $E \cap E' = 0$, but this cannot happen unless $E' = 0$, since F is an essential extension of E.

($\Leftarrow$) Suppose F is an injective module with $E \subseteq F$. If $E \neq F$, then F cannot be an essential extension of E, so that

$$\Gamma = \{G \colon 0 \neq G \text{ is a submodule of } F \text{ and } G \cap E = 0\} \neq \emptyset,$$

as noted in 1.1.3 . If $\{F_\lambda\}_\lambda$ is a family of R-modules in Γ totally ordered by inclusion, the union

$$U := \bigcup_{\lambda \in \Lambda} F_\lambda$$

is also a submodule of F. Also, for every $x \in U$, we must have $x \in F_\lambda$ for some $\lambda \in \Lambda$, and therefore either $x = 0$ or $x \notin E$. This means that $U \cap E = 0$, and thus $U \in \Gamma$. This guarantees that Zorn's Lemma applies. Therefore, there exists a maximal element K in Γ, that is, there exists a non-zero submodule $K \subseteq F$ with $K \cap E = 0$ that is maximal with respect to this property. Notice that $K \neq F$, since $F \cap E = E \neq 0$.

Consider the composition $\varphi\colon E \longrightarrow F/K$ of the natural embedding with the canonical projection that sends $e \in E$ to $\varphi(e) = e + K \in F/K$. We will prove that F/K is an essential extension of E. First note that φ is a monomorphism. Then, consider a non-zero submodule H/K of F/K. The fact that H/K is a non-zero submodule of F/K means that $K \subsetneq H$, so that by maximality of K in Γ we must have $H \cap E \neq 0$. Then $E \cap H/K \neq 0$, so that F/K is an essential extension of E. Given that E has no proper essential extensions, the map $\varphi\colon E \longrightarrow F/K$ must be an isomorphism, so that every element $f \in F$ can be written as $f = e + k$, with $e \in E$ and $k \in K$. Also, $K \cap E = 0$. This means that $F = E \oplus K$, and since by 0.8.7 a direct summand of an injective module is injective, then E must be injective. $\square$

Theorem 1.1.11. *Let E be an R-module. If I is an injective R-module with $E \hookrightarrow I$ and G is a maximal essential extension of E contained in I, then*

(1) *G is an injective module.*

(2) *G is unique up to isomorphism and independent of the choice of I.*

Proof.

(1) By 1.1.10, all we have to show is that G has no proper essential extensions. So let H be an essential extension of G.

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r, "j"] \arrow[d, "i"'] & H \arrow[dl, dashed, "k"] \\ & I & \end{tikzcd}$$

As I is injective, we can find k such that the diagram above commutes. Consider $K := \ker k$. If $K \neq 0$, then $K \cap G \neq 0$, since H is an essential extension of G. But $K \cap G \neq 0$ is impossible, as i is mono. Therefore, we must have $K = 0$, and thus that k is injective. Then we have

$$G \cong j(G) \subseteq H \cong k(H) \subseteq I.$$

By 1.1.6, H is an essential extension of E, which implies that $G = H$, by the maximality of G.

Therefore, G has no proper essential extensions, and as we have seen, this completes the proof.

(2) Let I, I' be two injective modules with $E \subseteq I$ and $E \subseteq I'$. Let G be a maximal essential extension of E in I and G' a maximal essential extension of E in I', and consider the natural embeddings $E \hookrightarrow G$ and $E \hookrightarrow G'$. Since G' is an injective module, there exists $\varphi\colon G \longrightarrow G'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d] & G \arrow[dl, dashed, "\varphi"] \\ & G' & \end{tikzcd}$$

commutes. Since the vertical map is injective, then for every $x \in E$, $x \neq 0$, we must have $\varphi(x) \neq 0$, so that $\ker \varphi \cap E = 0$. But G is an essential extension of E, so that we must have $\ker \varphi = 0$. Therefore, φ is an injective homomorphism.

Also, since G is an injective module, the exact sequence

$$0 \longrightarrow G \xrightarrow{\varphi} G' \longrightarrow G'/\varphi(G) \longrightarrow 0$$

splits, so that $G' = \varphi(G) \oplus H$ for some submodule $H \cong G'/\varphi(G)$ of G'.

Notice that $E \subseteq \varphi(G)$ and $\varphi(G) \cap H = 0$, so that $E \cap H = 0$. But H is a submodule of G', which is an essential extension of E, and therefore $H = 0$. Then $\varphi(G) = G'$, so that φ is also surjective, and hence an isomorphism.

□

Definition 1.1.12 (Injective Envelope)**.** Let M be an R-module. The *injective envelope* of M, denoted by $E(M)$, is the isomorphism class of modules defined by 1.1.11.

Remark 1.1.13. Let M be any R-module. By 0.8.3, there exists an injective R-module I with $M \subseteq I$. Let G be a maximal essential extension of M contained in I, which exists by 1.1.9. Then G is an injective envelope of M. Notice that the injective envelope is defined only up to isomorphism, but 1.1.11 guarantees that this is well-defined, not depending on the choice of I or G. Thus, although this is an abuse of notation, we may write $E(M) = G$. This shows that every module has an injective envelope.

Corollary 1.1.14. *Let R be a ring and M an R-module. If I is an injective R-module such that $M \subseteq I$ is an essential extension, then $E(M) = I$. In particular, $E(I) = I$ for every injective R-module I.*

Proof. Clearly, I is the maximal essential extension of M inside I. By 1.1.11, $I = E(M)$. □

Lemma 1.1.15. *Let R be a ring and $M \subseteq N$ be R-modules. If N is an essential extension of M, then $N \hookrightarrow E(M)$ and $E(M) = E(N)$.*

Proof. By 1.1.6 applied to $M \subseteq N \subseteq E(N)$, $M \subseteq E(N)$ is an essential extension. But $E(N)$ is injective, and thus, by 1.1.14, $E(M) = E(N)$. □

Remark 1.1.16. In the case $M \cong N$, the previous lemma also guarantees that $E(M) = E(N)$.

Example 1.1.17. Consider $\mathbb{Z} \subseteq \mathbb{Q}$, which we have seen to be an essential extension, in 1.1.5. Injective $\mathbb{Z}$-modules are the same as divisible abelian groups,[1] and $\mathbb{Q}$ is obviously a divisible abelian group. By 1.1.14, $E(\mathbb{Z}) = \mathbb{Q}$.

Example 1.1.18. Consider the ring $\mathbb{Z}$, a prime number p and $\mathbb{Z}[p^\infty]$, the direct limit of

$$\mathbb{Z}/p \subseteq \mathbb{Z}/p^2 \subseteq \mathbb{Z}/p^3 \subseteq \mathbb{Z}/p^4 \subseteq \dots$$

which can be seen as the submodule of $\mathbb{Q}/\mathbb{Z}$ whose elements are of the form $\frac{a}{p^n} + \mathbb{Z}$ for some $a \in \mathbb{Z}$ and $n \geqslant 1$. Similarly, for each n we can identify $\mathbb{Z}/p^n$ with the submodule of $\mathbb{Q}/\mathbb{Z}$ whose elements are of the form $\frac{a}{p^n} + \mathbb{Z}$ with $a \in \mathbb{Z}$.

[1] For a proof, see [16, Chapter IV, Lemma 3.9].

First, let us show that $\mathbb{Z}[p^\infty]$ is a divisible abelian group. Given any $y \in \mathbb{Z}[p^\infty]$ and any $0 \neq n \in \mathbb{Z}$, consider $1 \leqslant a \leqslant p^m - 1$ and $m \geqslant 1$ such that $y = \frac{a}{p^m} + \mathbb{Z}$. Then

$$a\left(\frac{1}{p^m} + \mathbb{Z}\right) = \frac{a}{p^m} + \mathbb{Z} = y.$$

Therefore, $\mathbb{Z}[p^\infty]$ is injective. Fix $n \geqslant 1$ and consider the canonical inclusion

$$\mathbb{Z}/p^n \subseteq \mathbb{Z}[p^\infty].$$

Let $\frac{a}{p^m} + \mathbb{Z} \in \mathbb{Z}[p^\infty]$. If $m \leqslant n$, then

$$\frac{a}{p^m} + \mathbb{Z} = \frac{p^{n-m}a}{p^n} + \mathbb{Z} \in \mathbb{Z}/p^n.$$

If $m \geqslant n$, then

$$p^{m-n}\left(\frac{a}{p^m} + \mathbb{Z}\right) = \frac{a}{p^n} + \mathbb{Z} \in \mathbb{Z}/p^n.$$

Either way, $\left(\frac{a}{p^m} + \mathbb{Z}\right) \cap \mathbb{Z}/p \neq 0$. This shows that

$$\mathbb{Z}/p^n \subseteq \mathbb{Z}[p^\infty]$$

is an essential extension. By 1.1.14, $E(R/p^n) = \mathbb{Z}[p^\infty]$.

Lemma 1.1.19. *Let R be a Noetherian ring, M an R-module and S a multiplicative set in R. Then $E(M_S) = E(M)_S$.*

Proof. By 1.1.8, $M_S \subseteq E(M)_S$ is an essential extension. By 1.1.7, $E(M)_S$ is an injective R_S-module, and by 1.1.14, this finishes the proof. ☐

Lemma 1.1.20. *Let A, B be R-modules. Then*

$$E(A \oplus B) = E(A) \oplus E(B).$$

Proof. Since $A \subseteq E(A)$ and $B \subseteq E(B)$, then $A \oplus B \subseteq E(A) \oplus E(B)$, and this is an essential extension. Indeed, if $M \subseteq E(A) \oplus E(B)$ is a non-zero submodule, then there exists $a \in E(A)$ and $b \in E(B)$ such that $(a, b) \in M$ and $a \neq 0$ or $b \neq 0$. Let us assume without loss of generality that $a \neq 0$. Then since $A \subseteq E(A)$ is an essential extension, $Ra \cap A \neq 0$, and therefore $ra \in A$. If $b = 0$ then $r(a, b) = (ra, 0) \in M \cap (A \oplus B)$ and we are done. If $b \neq 0$, then by the same argument we can find $s \in R$ such that $sb \in B$. Clearly, $rsa \in A$ and $rsb \in B$. Then $rs(a, b) \in M \cap (A \oplus B)$. Therefore, $A \oplus B \subseteq E(A) \oplus E(B)$ is an essential extension.

But $E(A) \oplus E(B)$ is a direct sum of two injective modules, and therefore injective, by 0.8.5. By 1.1.14, $E(A \oplus B) = E(A) \oplus E(B)$. ☐

Example 1.1.21. Now we can compute the injective envelope of any finitely generated abelian group. A finitely generated abelian group is of the form

$$M \cong \mathbb{Z}^n \oplus \mathbb{Z}/p_1^{a_1} \oplus \cdots \oplus \mathbb{Z}/p_m^{a_m}$$

with $n \geqslant 0$, $p_1, \ldots, p_m$ primes, and $a_1, \ldots, a_m \geqslant 1$. Using 1.1.20,

$$E\left(\mathbb{Z}^n \oplus \mathbb{Z}/p_1^{a_1} \oplus \cdots \oplus \mathbb{Z}/p_m^{a_m}\right) = E(\mathbb{Z})^n \oplus \left(\mathbb{Z}/p_1^{a_1}\right) \oplus \cdots \oplus \left(\mathbb{Z}/p_m^{a_m}\right) = \mathbb{Q}^n \oplus \left(\mathbb{Z}[p_1^{\infty}]\right) \oplus \cdots \oplus \left(\mathbb{Z}[p_m^{\infty}]\right).$$

Theorem 1.1.22. *Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then*

(1) $\operatorname{Ass}(M) = \operatorname{Ass}(E(M))$.

(2) *For every prime ideal $\mathfrak{p}$ in R,* $\operatorname{Ass}(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$.

Moreover, $E(R/\mathfrak{p}) \ncong E(R/\mathfrak{q})$ whenever $\mathfrak{p} \neq \mathfrak{q}$ are prime ideals in R.

Proof.

(1) As $M \subseteq E(M)$, then by 0.7.5 we always have $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(E(M))$. On the other hand, consider $\mathfrak{q} \in \operatorname{Ass}(E(M))$. Then $\mathfrak{q} = \operatorname{Ann}_R(x)$ for some non-zero $x \in E(M)$, so that $R/\mathfrak{q} \cong Rx \subseteq E(M)$. Since $M \subseteq E(M)$ is an essential extension, we must have $Rx \cap M \neq 0$. Let $rx \in Rx \cap M$ be a non-zero element, which in particular means that $r \notin \mathfrak{q}$. As $\mathfrak{q}(rx) = r(\mathfrak{q}x) = 0$, then $\mathfrak{q} \subseteq \operatorname{Ann}_R(rx)$. Let $y \in \operatorname{Ann}_R(rx)$. As $(yr)x = y(rx) = 0$, then in particular $yr \in \operatorname{Ann}_R(x) = \mathfrak{q}$. Since $\mathfrak{q}$ is prime and $r \notin \mathfrak{q}$, we conclude that $y \in \mathfrak{q}$. Thus, $\mathfrak{q} = \operatorname{Ann}_R(rx)$. Since $rx \in Rx \cap M \subseteq M$, then $\mathfrak{q} \in \operatorname{Ass}(M)$, as desired.

(2) Follows from (1), since $\operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$, by 0.7.6.

The fact that $E(R/\mathfrak{p}) \ncong E(R/\mathfrak{q})$ whenever $\mathfrak{p} \neq \mathfrak{q}$ are prime ideals in R is an obvious corollary, as $\operatorname{Ass}(E(R/\mathfrak{p})) \neq \operatorname{Ass}(E(R/\mathfrak{q}))$. □

Definition 1.1.23 (Indecomposable)**.** An R-module I is said to be *indecomposable* if it cannot be written as the direct sum of two non-zero submodules.

Remark 1.1.24. Let R be a Noetherian ring. For an injective R-module I, I is indecomposable if and only if I has no proper injective submodules. Indeed, any injective submodule of I is a direct summand of I by 0.8.9, and any direct summand of I is injective by 0.8.7.

Proposition 1.1.25. *Let R be a Noetherian ring. An injective R-module E is an indecomposable injective module if and only if $E = E(R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof.

($\Rightarrow$) Let $\mathfrak{p} \in \operatorname{Ass}(E)$, so that there exists some $0 \neq e \in E$ with $\mathfrak{p} = \operatorname{Ann}_R(e)$, and thus $R/\mathfrak{p} \cong Re \subseteq E$. As E is injective, then by definition of injective envelope we have $Re \subseteq E(Re) = E(R/\mathfrak{p}) \subseteq E$. But by 0.8.9, $E(R/\mathfrak{p})$ must be a direct summand of E, and since E is an indecomposable injective module, then $E = E(R/\mathfrak{p})$.

($\Leftarrow$) We will show that $E = E(R/\mathfrak{p})$ is an indecomposable injective module. We already know that E is injective, so we just have to show that E is indecomposable. If not, then there exist some non-zero injective R-modules $E_1, E_2 \subseteq E$ such that $E = E(R/\mathfrak{p}) = E_1 \oplus E_2$. Consider the inclusion $i\colon R/\mathfrak{p} \hookrightarrow E_1 \oplus E_2 = E(R/\mathfrak{p})$ and the canonical projections

$$\pi\colon R \longrightarrow R/\mathfrak{p},\ \pi_1\colon E_1 \oplus E_2 \longrightarrow E_1 \text{ and } \pi_2\colon E_1 \oplus E_2 \longrightarrow E_2.$$

The compositions

$$\begin{tikzcd} & & & E_1 \\ R \arrow[r, "\pi"] \arrow[urrr, bend left, "\varphi_1"] \arrow[drrr, bend right, "\varphi_2"'] & R/\mathfrak{p} \arrow[r, "i"] & E_1 \oplus E_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & & E_2 \end{tikzcd}$$

give us two maps, φ_1: $R \longrightarrow E_1$ and φ_2: $R \longrightarrow E_2$. Let $I_i = \ker(\varphi_i)$ for $i = 1, 2$.

By 1.1.22, $\mathrm{Ass}(E) = \{\mathfrak{p}\}$. Consider a non-zero $a \in E$ with $\mathfrak{p} = \mathrm{Ann}_R(a)$. There is a unique way of writing $a = a_1 + a_2$ with $a_i \in E_i$. Notice that the maps φ_i are given by

$$\begin{tikzcd} R \arrow[r] & R/\mathfrak{p} \cong Ra \arrow[r, hook] & E_1 \oplus E_2 \arrow[r] & E_i \\ r \arrow[r, mapsto] & r + \mathfrak{p} \leftrightarrow ra \arrow[r, mapsto] & ra \arrow[r] & ra_i \end{tikzcd}$$

and thus $I_i = \{r \in R : ra_i = 0\}$. Then $I_1 I_2 = I_1 \cap I_2 = \{r \in R : r\,a_1 = 0 = r\,a_2\} = \mathfrak{p}$. As $\mathfrak{p}$ is a prime ideal, this implies that $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$. Assume $I_1 \subseteq \mathfrak{p}$. As $\mathfrak{p} \subseteq I_1$, we must have $\mathfrak{p} = I_1$. Then

$$R/\mathfrak{p} = R/I_1 = R/\ker\varphi_1 \cong \operatorname{im}\varphi_1 \subseteq E_1$$

so that $R/\mathfrak{p} \cap E_2 = 0$. Since E is an essential extension of $R/\mathfrak{p}$, this can only mean that $E_2 = 0$. This contradicts the assumption that $E_1, E_2 \neq 0$. Therefore, E is an indecomposable injective R-module, as desired. $\square$

Theorem 1.1.26. *Let R be a Noetherian ring. Every injective R-module is a direct sum of indecomposable injective R-modules.*

Proof. Let I be an injective R-module. Consider the set T of all indecomposable injective submodules of I, and the set S of all subsets of T with the following property: if $F \in S$, then the sum of all modules in F is direct. The set S is partially ordered by inclusion.

First let us check that S is non-empty. Pick $\mathfrak{p} \in \mathrm{Ass}(I)$, which must be the annihilator of some non-zero $a \in I$. Then $E(R/\mathfrak{p})$ is an indecomposable injective submodule of I, by 1.1.25. So $\{E(R/\mathfrak{p})\} \in S$, and therefore S is non-empty.

Also, if $\{F_\lambda\}_{\lambda \in \Lambda}$ is a family in S totally ordered by inclusion, then

$$F := \bigcup_{\lambda \in \Lambda} F_\lambda \in S.$$

Indeed, if $a_{\lambda_1} + \ldots + a_{\lambda_n} = 0$ for a_{λ_j} in some module in F_{λ_j}, with $\lambda_j \in \Lambda$ all distinct, then there exists $\lambda \in \Lambda$ such that all the a_{λ_j} are elements of modules in F_λ, and therefore, since the sum of modules in F_λ is direct, we get that all the a_{λ_j} must be zero. This shows that the sum of the modules in F is direct, and therefore that $F \in S$. It is clear that it is an upper bound of the chain we were considering.

By Zorn's Lemma, S has a maximal element F. Let E be the direct sum of all modules in F. Being a direct sum of injective modules over a Noetherian ring, then E is injective, by 0.8.8. Also, as $E \subseteq I$, then E is a direct summand of I, by 0.8.9. Consider H such that $I = E \oplus H$. We will show that $H = 0$ and thus that $I = E$.

If $H \neq 0$, then by 0.7.7 there exists some $\mathfrak{q} \in \operatorname{Ass}(H)$. Consider $x \in H$ such that $\mathfrak{q} = \operatorname{Ann}_R(Rx)$. As $Rx \cong R/\mathfrak{q}$ and H is injective, then $E(R/\mathfrak{q}) = E(Rx) \subseteq H \subseteq E$. As $E(R/\mathfrak{q})$ is injective and R is Noetherian, then $E(R/\mathfrak{q})$ is a direct summand of H and of E, by 0.8.9. Thus, we may enlarge F by $E(Rx) = E(R/\mathfrak{q})$, which cannot be by the maximality of F. Then $H = 0$ and $I = E$, which is a direct sum of indecomposable injective modules. $\square$

Corollary 1.1.27. *Let R be a Noetherian ring and E an injective R-module Then E is of the form*

$$E = \bigoplus_{\mathfrak{p} \in \operatorname{Ass}(E)} (E(R/\mathfrak{p}))^{\mu_\mathfrak{p}}$$

where μ_p are cardinals.

Proof. By 1.1.26, E is an arbitrary direct sum of indecomposable injective R-modules, and by 1.1.25, those are of the form $E(R/\mathfrak{p})$ for $\mathfrak{p}$ some prime ideal in R. If E has a submodule isomorphic to $E(R/\mathfrak{p})$, then $\mathfrak{p} \in \operatorname{Ass}(R/\mathfrak{p}) \subseteq \operatorname{Ass}(E)$. $\square$

1.2 The Case of Artinian Local Rings

In the previous section, we proved that every injective module over a Noetherian ring R is a direct sum of modules of the form $E(R/\mathfrak{p})$, for $\mathfrak{p}$ a prime ideal in R. In this section we will focus on the special case of Artinian local rings, over which every injective module is simply a direct sum of copies of $E(R/\mathfrak{m})$, for $\mathfrak{m}$ the unique maximal ideal. The results in this section will be useful later on, both when we study the case of Gorenstein rings (section 1.4) and in the proof of Matlis Duality (section 1.5).

Lemma 1.2.1. *If $(R, \mathfrak{m})$ is a local Artinian ring, then $\operatorname{Spec}(R) = \{\mathfrak{m}\}$ and for every R-module $M \neq 0$, $\operatorname{Ass}(M) = \{\mathfrak{m}\}$.*

Proof. Since R is Artinian, the descending chain of ideals

$$\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \dots$$

stops, so that $\mathfrak{m}^r = \mathfrak{m}^{r+1}$ for some $r \geqslant 1$. Since R is Artinian, it is also Noetherian, by 0.3.4, so that $\mathfrak{m}^r$ is finitely generated. Moreover, $\mathfrak{m}$ is the Jacobson radical of R. Then, by 0.4.9, we have $\mathfrak{m}^r = (0)$. This implies that $\mathfrak{m}$ is the only prime ideal in R: if $\mathfrak{p}$ is a prime ideal then $\mathfrak{m}^r = (0) \subseteq \mathfrak{p} \subseteq \mathfrak{m} \Rightarrow \mathfrak{m} \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} = \mathfrak{m}$.

As $M \neq 0$ and R is Noetherian, then $\mathrm{Ass}(M) \neq \emptyset$, by 0.7.7. As $\mathrm{Ass}(M) \subseteq \mathrm{Spec}(R)$, the result follows from 1.2.1. $\square$

Corollary 1.2.2. *Let $(R, \mathfrak{m})$ be an Artinian local ring and $M \neq 0$ an R-module. Then $E(M)$ is a direct sum of copies of $E(R/\mathfrak{m})$.*

Proof. Since $\mathrm{Ass}(E(M)) = \{\mathfrak{m}\}$, by 1.2.1, then by 1.1.27, $E(M)$ must be a direct sum of copies of $E(R/\mathfrak{m})$. $\square$

Definition 1.2.3. We will denote the category of finitely generated R-modules by $\mathcal{M}(R)$, the category of Noetherian R-modules by $\mathcal{N}(R)$ and the category of Artinian R-modules by $\mathcal{A}(R)$.

Lemma 1.2.4. *Let R be an Artinian ring. Then $\mathcal{M}(R) = \mathcal{N}(R)$.*

Proof. Noetherian modules are always finitely generated, and so $\mathcal{N}(R) \subseteq \mathcal{M}(R)$ for any general ring R. So all we need to check is that if R is Artinian and M is finitely generated as an R-module, then M is Noetherian. By 0.3.4, R is a Noetherian ring. By 0.3.7, M is a Noetherian R-module. $\square$

Remark 1.2.5. We will see that in fact if R is an Artinian local ring, then $\mathcal{M}(R) = \mathcal{N}(R) = \mathcal{A}(R)$. The containment $\mathcal{M}(R) = \mathcal{N}(R) \subseteq \mathcal{A}(R)$ is clear, from 0.3.7. However, we will only prove $\mathcal{A}(R) \subseteq \mathcal{N}(R)$ in section 1.5.

Lemma 1.2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

(1) $\mathrm{Hom}_R(R, E(R/\mathfrak{m})) \cong E(R/\mathfrak{m})$.

(2) $\mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong R/\mathfrak{m}$.

Proof.

(1) Obvious.

(2) Define $\varphi\colon \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \longrightarrow E(R/\mathfrak{m})$ the following way: for each $f \in \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}))$, $\varphi(f) = f(1+\mathfrak{m})$. Clearly, φ is a homomorphism of R-modules. Moreover, φ is injective:

$$\varphi(f) = 0 \Rightarrow f(1+\mathfrak{m}) = 0 \Rightarrow \underset{r\in R}{\forall}\ f(r+\mathfrak{m}) = rf(1+\mathfrak{m}) = 0 \Rightarrow f = 0.$$

As $\mathrm{im}\,(\varphi) \subseteq E(R/\mathfrak{m})$ and for any $f \in \mathrm{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}))$,

$$\mathfrak{m}\,\varphi(f) = \mathfrak{m}\, f(1+\mathfrak{m}) = f(\mathfrak{m}) = f(0) = 0$$

then $\mathfrak{m}\,(\text{im}(\varphi)) = 0$. Therefore, im($\varphi$) is also a module over $R/\mathfrak{m}$, and non-zero, as it is the image of a non-zero module by an injective homomorphism. Thus, $\dim_{R/\mathfrak{m}}(\text{im}(\varphi)) \geqslant 1$.

Suppose $\dim_{R/\mathfrak{m}}(\text{im}(\varphi)) \geqslant 2$. Then im($\varphi$) has a submodule $N \cong R/\mathfrak{m} \oplus R/\mathfrak{m}$ of dimension 2. We can assume that the first copy is the canonical inclusion $R/\mathfrak{m} \subseteq E(R/\mathfrak{m})$, which is an essential extension. But $0 \oplus R/\mathfrak{m}$ is a non-zero submodule of $E(R/\mathfrak{m})$ with

$$(0 \oplus R/\mathfrak{m}) \cap (R/\mathfrak{m} \oplus 0) = (0)$$

which is clearly impossible. Therefore, $\dim_{R/\mathfrak{m}}(\text{im}(\varphi)) = 1$, and thus

$$\operatorname{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong R/\mathfrak{m}.$$

□

Lemma 1.2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $Q \subsetneq \mathfrak{m}$ a prime ideal. Then*

$$\operatorname{Hom}_R(R/\mathfrak{m}, E(R/Q)) = 0.$$

Proof. By 0.1.13,

$$\operatorname{Hom}_R(R/\mathfrak{m}, E(R/Q)) = \{a \in E(R/Q) \, : \, \mathfrak{m}a = 0\}.$$

If this is non-zero then $\mathfrak{m}$ is an associated prime ideal of $E(R/Q)$, which is impossible, by 1.1.22. □

Proposition 1.2.8. *Let $(R, \mathfrak{m})$ be an Artinian local ring. Let M be a finitely generated R-module. Then,*

$$\lambda(M) = \lambda\left(\operatorname{Hom}_R(M, E(R/\mathfrak{m})\right).$$

Proof. R is an Artinian and thus Noetherian ring, by 0.3.4. As M is a finitely generated R-module, this implies that M is both an Artinian and a Noetherian R-module, by 0.3.7. Therefore, $\lambda(M) < \infty$. We will prove the statement by induction on $\lambda(M)$.

(i) $\lambda(M) = 0$

In this case, $M = 0$, and thus $\operatorname{Hom}_R(M, E(R/\mathfrak{m}) = 0$, so that the statement is obviously true.

(ii) $\lambda(M) = 1$

In this case, $M \neq 0$, and so, by 0.7.7, $\operatorname{Ass}(M) \neq \emptyset$. By 1.2.1, $\operatorname{Ass}(M) = \{\mathfrak{m}\}$. So consider a non-zero element $a \in M$ with $\mathfrak{m} = \operatorname{Ann}_R(a)$. As $Ra \cong R/\mathfrak{m} \neq 0$, we have a chain

$$(0) \subsetneq Ra \subseteq M.$$

As $\lambda(M) = 1$, we must have $R/\mathfrak{m} \cong Ra = M$. Now

$$\operatorname{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m})) \cong R/\mathfrak{m},$$

by 1.2.6, and clearly

$$\lambda\left(\operatorname{Hom}_R(R/\mathfrak{m}, E(R/\mathfrak{m}))\right) = \lambda\left(R/\mathfrak{m}\right).$$

(iii) $\lambda(M) \geqslant 2$, assuming the statement is true for finitely generated R-modules N with $\lambda(N) < \lambda(M)$.

There exists some R-submodule N of M satisfying

$$0 \subsetneq N \subseteq M$$

and a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0$$

with $C \cong M/N$. Notice that $\lambda(N) < \infty$ and $\lambda(C) < \infty$, and by 0.5.5,

$$\lambda(M) = \lambda(N) + \lambda(C).$$

Moreover, $\lambda(N) < \lambda(M)$ and $\lambda(C) < \lambda(M)$.

Since $E(R/\mathfrak{m})$ is an injective R-module, then $\operatorname{Hom}_R(-, E(R/\mathfrak{m}))$ is exact, by 0.8.4, and so we also have a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(C, E(R/\mathfrak{m})) \longrightarrow \operatorname{Hom}_R(M, E(R/\mathfrak{m})) \longrightarrow \operatorname{Hom}_R(N, E(R/\mathfrak{m})) \longrightarrow 0\,.$$

By the induction hypothesis,

$$\lambda\left(\operatorname{Hom}_R(N, E(R/\mathfrak{m}))\right) = \lambda(N)$$

and

$$\lambda\left(\operatorname{Hom}_R(C, E(R/\mathfrak{m}))\right) = \lambda(C).$$

By 0.5.5,

$$\lambda\left(\operatorname{Hom}_R(M, E(R/\mathfrak{m}))\right) = \lambda\left(\operatorname{Hom}_R(N, E(R/\mathfrak{m}))\right) + \lambda\left(\operatorname{Hom}_R(C, E(R/\mathfrak{m}))\right) = \lambda(N) + \lambda(C) = \lambda(M).$$

as desired.

□

Corollary 1.2.9. *Let* $(R, \mathfrak{m})$ *be an Artinian local ring. Then*

$$\lambda\left(E(R/\mathfrak{m})\right) = \lambda(R) < \infty.$$

In particular, $E(R/\mathfrak{m})$ *is a Noetherian and Artinian* R*-module.*

Proof. The fact that $\lambda(R) < \infty$ is obvious, by 0.5.4, considering that R is an Artinian ring and therefore a Noetherian ring, by 0.3.4, and thus a Noetherian and Artinian R-module. Also, using 1.2.6 and 1.2.8,

$$\lambda\left(E(R/\mathfrak{m})\right) = \lambda\left(\mathrm{Hom}_R(R, E(R/\mathfrak{m}))\right) = \lambda(R).$$

Then

$$\lambda\left(E(R/\mathfrak{m})\right) < \infty,$$

which implies that $E(R/\mathfrak{m})$ is a Noetherian and Artinian R-module, and in particular finitely generated, by 0.3.2. □

Proposition 1.2.10. *Let* $(R, \mathfrak{m})$ *be a local Artinian ring. Then*

$$\mathrm{Hom}_R\left(E(R/\mathfrak{m}), E(R/\mathfrak{m})\right) \cong R.$$

Proof. First, notice that $\lambda(R) < \infty$. Indeed, Artinian rings are Noetherian, by 0.3.4, and thus R is both a Noetherian and an Artinian R-module, which means precisely that $\lambda(R) < \infty$.

Consider the map

$$\begin{array}{rccc} \varphi\colon & R & \longrightarrow & \mathrm{Hom}_R(E(R/\mathfrak{m}), E(R/\mathfrak{m})) \\ & r & \longmapsto & (a \longmapsto ra) \end{array}$$

If $\varphi(r) = 0$, then $ra = 0$ for every $a \in E(R/\mathfrak{m})$, so that $r \in \mathrm{Ann}_R(E(R/\mathfrak{m})$, and $E(R/\mathfrak{m})$ is actually an injective $R/(r)$-module. Write $\overline{R} := R/(r)$ and $\overline{\mathfrak{m}} := \mathfrak{m}/(r)$. Since $R/\mathfrak{m} \cong \dfrac{R/(r)}{\mathfrak{m}/(r)} \cong \overline{R}/\overline{\mathfrak{m}}$, then $E(R/\mathfrak{m})$ is an essential extension of $\overline{R}/\overline{\mathfrak{m}}$. Notice that $(\overline{R}, \overline{\mathfrak{m}})$ is still an Artinian local ring, and $E(R/\mathfrak{m}) = E(\overline{R}/\overline{\mathfrak{m}})$, so that, by 1.2.9, $\lambda(R) = \lambda(E(R/\mathfrak{m})) = \lambda(\overline{R})$. But this cannot be true for $r \neq 0$, since in that case $\lambda(\overline{R})$ must be one less than $\lambda(R)$. Then $r = 0$, so that φ is injective.

Now consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{\varphi} \mathrm{Hom}_R(E(R/\mathfrak{m}), E(R/\mathfrak{m})) \longrightarrow \text{coker}(\varphi) \longrightarrow 0\ .$$

By 1.2.9, $\lambda(R) = \lambda(E(R/\mathfrak{m}))$. By 1.2.9, $E(R/\mathfrak{m})$ is a finitely generated R-module, and thus, by 1.2.8, $\lambda(E(R/\mathfrak{m})) = \lambda(\mathrm{Hom}(E(R/\mathfrak{m}), E(R/\mathfrak{m})))$. Quotients of Artinian and Noetherian modules are Artinian and Noetherian, by 0.3.3, so that coker(φ) is Artinian and Noetherian too, and thus $\lambda(\text{coker}(\varphi)) < \infty$.

Now we can apply 0.5.5 to the previous short exact sequence, obtaining

$$\lambda(\mathrm{Hom}_R(E(R/\mathfrak{m}), E(R/\mathfrak{m}))) = \lambda(R) + \lambda(\text{coker}(\varphi)),$$

and thus $\lambda(\text{coker}(\varphi)) = 0$, which is equivalent to coker(φ) $= 0$. Therefore, φ is surjective, and thus an isomorphism. □

1.3 Injective Dimension

In this section, we will characterize the injective dimension of an R-module M. We will use this characterization in 1.4 for the particular case of $M = R$ when we study Gorenstein rings. In particular, we will relate the injective dimension with Krull dimension and depth, two invariants that will play a very important role in the study of local cohomology modules in chapter 2.

Lemma 1.3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely generated R-module. Then $\lambda(M) < \infty$ if and only if $\dim(M) = 0$.*

Proof.

($\Rightarrow$) As $\lambda(M) < \infty$ is equivalent to M being an Artinian and Noetherian R-module, by 0.5.4, then the descending chain

$$M \supseteq \mathfrak{m}\,M \supseteq \mathfrak{m}^2 M \supseteq \dots$$

must stop, so that there exists i with $\mathfrak{m}^i M = \mathfrak{m}^{i+1} M$. Since $\mathfrak{m}$ is the Jacobson radical of R and $\mathfrak{m}^i M$ is a finitely generated R-module, by Nakayama's Lemma we have $\mathfrak{m}^i M = 0$. Therefore, $\mathfrak{m}^i \subseteq \mathrm{Ann}_R(M)$. Suppose there exists a prime ideal $\mathfrak{p}$ with $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$. Then $\mathfrak{m}^i \subseteq \mathfrak{p}$, and since $\mathfrak{p}$ is prime this implies that $\mathfrak{m} \subseteq \mathfrak{p}$. By maximality, $\mathfrak{m} = \mathfrak{p}$. Therefore, $\mathfrak{m}$ is the only prime ideal containing $\mathrm{Ann}_R(M)$, so that the ring $R/\mathrm{Ann}_R(M)$ has only one prime ideal, $\mathfrak{m}/\mathrm{Ann}_R(M)$, and thus dimension 0.

($\Leftarrow$) As quotients of Noetherian rings are Noetherian, by 0.3.3, then $R/\mathrm{Ann}_R(M)$ is a Noetherian ring. Also,

$$\dim(R/\mathrm{Ann}_R(M)) = \dim(M) = 0.$$

Every Noetherian ring of dimension 0 is Artinian, by 0.3.6. Therefore, $R/\mathrm{Ann}_R(M)$ is both Artinian and Noetherian as an R-module, and so $\lambda(R/\mathrm{Ann}_R(M)) < \infty$, by 0.5.4. The structure of M as an R-module coincides with its structure as a module over $R/\mathrm{Ann}_R(M)$. Therefore, M is a finitely generated module over the Artinian and Noetherian ring $R/\mathrm{Ann}_R(M)$, and thus an Artinian and Noetherian module over this ring, by 0.3.7. This means that $\lambda(M) < \infty$ as a module over $R/\mathrm{Ann}_R(M)$ and over R. □

Proposition 1.3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely generated R-module. Then*

$$\textit{inj dim}(M) = \sup\left\{i \colon \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\right\}.$$

Proof. Using 0.9.9, all we need to show is that

$$\sup\left\{i \colon \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\right\} = \sup\left\{i \colon \mathrm{Ext}^i_R(N, M) \neq 0, N \text{ finitely generated } R\text{-module } \right\}.$$

Considering that $R/\mathfrak{m}$ is a finitely generated R-module, as R is a Noetherian ring, ($\leqslant$) is obvious. Thus, we just have to show ($\geqslant$). Write

$$r_0 := \sup\left\{i \colon \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\right\}.$$

If $r_0 = \infty$, we are done, so we shall assume that $r_0 < \infty$. We will show that $\operatorname{Ext}^i_R(N,M) = 0$ for every finitely generated R-module N and every $i > r_0$, which leads to the conclusion that inj dim$(M) \leqslant r_0$.

Let N be a finitely generated R-module. We will use induction on $\dim(N)$.

(i) $\dim(N) = 0$, or equivalently, $\lambda(N) < \infty$

We will use induction on $\lambda(N)$. If $\lambda(N) = 0$, then $N = 0$. If $\lambda(N) = 1$, then $N \cong R/\mathfrak{m}$. In both cases, the statement is obvious.

So assume that $\lambda(N) \geqslant 2$ and that the result is true for any finitely generated R-module N' such that $\lambda(N') < \lambda(N)$. There exists a chain of finitely generated R-modules of the form $0 \subsetneq B \subsetneq N$ and a corresponding exact sequence

$$0 \longrightarrow B \longrightarrow N \longrightarrow N/B \longrightarrow 0\ ,$$

with the usual inclusion and projection, so that

$$\lambda(N) = \lambda(B) + \lambda(N/B),$$

by 0.5.5, and both $\lambda(B) < \lambda(N)$ and $\lambda(N/B) < \lambda(N)$, so that the result is true for both N and N/B. Applying $\operatorname{Ext}^i_R(-,M)$ to the exact sequence yields, by 0.10.3, the following long exact sequence:

$$\cdots \longrightarrow \operatorname{Ext}^i_R(N/B,M) \longrightarrow \operatorname{Ext}^i_R(N,M) \longrightarrow \operatorname{Ext}^i_R(B,M) \longrightarrow \cdots,$$

and for $i > r_0$, $\operatorname{Ext}^i_R(B,M) = 0$ and $\operatorname{Ext}^i_R(N/B,M) = 0$, so that $\operatorname{Ext}^i_R(N,M) = 0$ as well.

(ii) $\dim N = s > 0$

Assume that $\operatorname{Ext}^i_R(C,M) = 0$ for every finitely generated R-module C with $\dim(C) < s$ and $i > r_0$.

If $N = R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$, as $\dim(R/\mathfrak{p}) = \dim(N) = s \neq 0$, then $\mathfrak{p}$ is not the maximal ideal $\mathfrak{m}$. So pick $x \in \mathfrak{m}$, $x \notin \mathfrak{p}$. We have an exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{.x} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p},x) \longrightarrow 0\ .$$

Notice that, since $x \notin \mathfrak{p}$, then x is regular in $R/\mathfrak{p}$, and thus

$$\dim(R/(\mathfrak{p},x)) \leqslant s-1 < s = \dim(R/\mathfrak{p}),$$

by 0.6.9. By the induction hypothesis $Ext^i_R(R/(\mathfrak{p},x),M) = 0$ for $i > r_0$. This short exact sequence yields a long exact sequence, by 0.10.3:

$$\ldots \longrightarrow \operatorname{Ext}^i_R(R/(\mathfrak{p},x),M) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{p},M) \xrightarrow{.x} \operatorname{Ext}^i_R(R/\mathfrak{p},M) \longrightarrow \ldots,$$

which in fact, for $i > r_0$, is

$$\ldots \longrightarrow 0 \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{p}\,, M) \xrightarrow{.x} \operatorname{Ext}^i_R(R/\mathfrak{p}\,, M) \longrightarrow 0 \longrightarrow \ldots\,,$$

so that the map $\operatorname{Ext}^i_R(R/\mathfrak{p}\,, M) \xrightarrow{.x} \operatorname{Ext}^i_R(R/\mathfrak{p}\,, M)$ is actually an isomorphism. Then

$$\operatorname{Ext}^i_R(R/\mathfrak{p}\,, M) = x \operatorname{Ext}^i_R(R/\mathfrak{p}\,, M)$$

which by Nakayama's Lemma (0.4.9) implies that $\operatorname{Ext}^i_R(R/\mathfrak{p}\,, M) = 0$. Nakayama's Lemma applies given that $x \in \mathfrak{m}$, which is the Jacobson Radical of R, and that $\operatorname{Ext}^i_R(R/\mathfrak{p}\,, M)$ is a finitely generated R-module, by 0.10.5, because so are M and $R/\mathfrak{p}$.

This proves that for every prime ideal $\mathfrak{p}$ in R and $i > r_0$, $\operatorname{Ext}^i_R(R/\mathfrak{p}\,, M) = 0$.

When $N \neq 0$ is any finitely generated R-module, there exists, by 0.7.13, a chain of submodules

$$0 = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_t = N$$

with $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i$ in R with $\dim(R/\mathfrak{p}_i) \leqslant \dim(M)$. With this chain we get the short exact sequence

$$0 \longrightarrow N_1 \cong R/\mathfrak{p}_1 \longrightarrow N_2 \longrightarrow N_2/N_1 \cong R/\mathfrak{p}_2 \longrightarrow 0$$

and by the same argument as before, the long exact sequence for $\operatorname{Ext}_R(-, M)$ (0.10.3) and the fact that

$$\operatorname{Ext}^i_R(R/\mathfrak{p}_1\,, M) = 0 = \operatorname{Ext}^i_R(R/\mathfrak{p}_2\,, M)$$

for $i > r_0$, we conclude that $\operatorname{Ext}^i_R(N_2, M) = 0$ for $i > r_0$.

Now repeating the argument for the short exact sequence

$$0 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow N_3/N_2 \cong R/\mathfrak{p}_3 \longrightarrow 0\,,$$

where again

$$\operatorname{Ext}^i_R(N_2, M) = 0 = \operatorname{Ext}^i_R(R/\mathfrak{p}_3\,, M)$$

for $i > r_0$, we prove that $\operatorname{Ext}^i_R(N_3, M) = 0$ for $i > r_0$. Repeating for each prime ideal $\mathfrak{p}_i$ leads to $\operatorname{Ext}^i_R(N, M) = \operatorname{Ext}^i_R(N_t, M) = 0$ for $i > r_0$, as desired.

$\square$

Proposition 1.3.3. *Let $(R, \mathfrak{m})$ be Noetherian local ring, $M \neq 0$ a finitely generated R-module and $A \neq 0$ a finitely generated R-module of finite injective dimension. Let*

$$r_0 := \sup\left\{i : \operatorname{Ext}^i_R(M, A) \neq 0\right\}.$$

Then

$$r_0 + \textit{depth}(M) = \textit{inj dim}(A)$$

and $r_0 < \infty$*.*

Proof. Let $r :=$ inj dim(A). First, notice that

$$r_0 = \sup\left\{i : \mathrm{Ext}^i_R(M, A) \neq 0\right\} \leqslant \sup\left\{i : \mathrm{Ext}^i_R(N, A) \neq 0, N \text{ finitely generated } R\text{-module } \right\} = r < \infty.$$

We will prove the claim by induction on depth(M).

(i) depth$(M) = 0$

Since there are no regular elements on M, then $\mathfrak{m}$ must be contained in the set of zero divisors of M, which is also the union of the primes in $\mathrm{Ass}(M)$, and therefore $\mathfrak{m} \in \mathrm{Ass}(M)$, by 0.1.2. Let $a \in M$ be a non-zero element such that $\mathfrak{m} = \mathrm{Ann}_R(a)$, and thus $R/\mathfrak{m} \cong Ra$. We have a short exact sequence

$$0 \longrightarrow Ra \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0\ ,$$

where $\overline{M} = M/Ra$. The long exact sequence in 0.10.3 applied to this exact sequence is

$$\ldots \longrightarrow \mathrm{Ext}^i_R(\overline{M}, A) \longrightarrow \mathrm{Ext}^i_R(M, A) \longrightarrow \mathrm{Ext}^i_R(Ra, A) \longrightarrow \ldots .$$

By 1.3.2, $\mathrm{Ext}^r_R(R/\mathfrak{m}, A) \neq 0$, and therefore $\mathrm{Ext}^r_R(Ra, A) \neq 0$. Also, $\mathrm{Ext}^{r+1}_R(\overline{M}, A) = 0$, since r is the injective dimension of A, and $\overline{M}$ is a finitely generated R-module (0.9.6). The exactness of the long exact sequence

$$\ldots \longrightarrow \mathrm{Ext}^r_R(M, A) \longrightarrow \mathrm{Ext}^r_R(Ra, A) \neq 0 \longrightarrow \mathrm{Ext}^{r+1}_R(\overline{M}, A) = 0 \longrightarrow \ldots$$

implies that $\mathrm{Ext}^r_R(M, A) \neq 0$, so that $r_0 \geqslant r$. We already knew that $r_0 \leqslant r$, and thus

$$r_0 + \text{depth}(M) = r_0 = r.$$

(ii) depth$(M) = s > 0$

Assume that the result is true for all finitely generated R-modules N with depth$(N) \leqslant s - 1$. Since depth$(M) > 0$, we can find a maximal regular sequence in M inside $\mathfrak{m}$. Let $x \in \mathfrak{m}$ be regular in M be the first element in a maximal regular sequence in M inside $\mathfrak{m}$. Write $\overline{M} := M/xM$. We have a short exact sequence

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow \overline{M} \longrightarrow 0\ .$$

As depth$(\overline{M}) = s - 1$, by 0.6.9, by the induction hypothesis we have

$$\sup\left\{i : \mathrm{Ext}^i_R(\overline{M}, A) \neq 0\right\} = r - (s - 1) = r - s + 1,$$

which in particular implies that $\mathrm{Ext}_R^{r-s+1}(\overline{M}, A) \neq 0$ and $\mathrm{Ext}_R^{r-s+2}(\overline{M}, A) = 0$.

The long exact sequence for $\mathrm{Ext}_R(-, A)$ in 0.10.3 applied to the previous short exact sequence is

$$\cdots \longrightarrow \mathrm{Ext}_R^{r-s+1}(\overline{M}, A) \longrightarrow \mathrm{Ext}_R^{r-s+1}(M, A) \xrightarrow{.x} \mathrm{Ext}_R^{r-s+1}(M, A) \longrightarrow 0 \; .$$

Since the map $\mathrm{Ext}_R^{r-s+1}(M, A) \xrightarrow{.x} \mathrm{Ext}_R^{r-s+1}(M, A)$ is onto, then

$$\mathrm{Ext}_R^{r-s+1}(M, A) = x \; \mathrm{Ext}_R^{r-s+1}(M, A),$$

and since $x \in \mathfrak{m}$ and $\mathrm{Ext}_R^{r-s+1}(M, A)$ is a finitely generated R-module, Nakayama's Lemma (0.4.9) implies that

$$\mathrm{Ext}_R^{r-s+1}(M, A) = 0.$$

Then

$$\ldots \longrightarrow \mathrm{Ext}_R^{r-s}(M, A) \longrightarrow \mathrm{Ext}_R^{r-s+1}(\overline{M}, A) \neq 0 \longrightarrow \mathrm{Ext}_R^{r-s+1}(M, A) = 0$$

is exact and therefore $\mathrm{Ext}_R^{r-s}(M, A) \neq 0$. Therefore, $r_0 \geqslant r - s$.

Notice also that for $i \geqslant r - s + 1$,

$$\mathrm{Ext}_R^{i}(M, A) = 0.$$

As before, the argument uses

$$\sup \left\{ i : \mathrm{Ext}_R^{i}(\overline{M}, A) \neq 0 \right\} = r - (s - 1) = r - s + 1,$$

Nakayama's Lemma (0.4.9) and the exactness of the complex below:

$$\ldots \longrightarrow \mathrm{Ext}_R^{i}(M, A) \xrightarrow{.x} \mathrm{Ext}_R^{i}(M, A) \longrightarrow \mathrm{Ext}_R^{i+1}(\overline{M}, A) = 0 \; .$$

Therefore, $r_0 \leqslant r - s$. This concludes the proof that $r_0 = r - s$.

□

Corollary 1.3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $A \neq 0$ a finitely generated R-module of finite injective dimension. Then inj dim$(A) =$ depth(R).*

Proof. Let $a \in A$ be a non-zero element. Define a non-zero homomorphism $f \colon R \longrightarrow A$ by $f(r) = ra$. Then $\mathrm{Hom}_R(R, A) \neq 0$. As R is a projective R-module, by 0.10.7 we have $\mathrm{Ext}_R^{i}(R, A) = 0$, for $i \geqslant 1$. Therefore,

$$r_0 = \sup \left\{ i : \mathrm{Ext}_R^{i}(R, A) \neq 0 \right\} = 0.$$

Applying 1.3.3, we get the desired equality. □

Proposition 1.3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $A \neq 0$ a finitely generated R-module of finite injective dimension. Then*

$$\textit{inj dim}(A) \leqslant \dim(R).$$

Proof. By 1.3.4, inj dim$(A) =$ depth(R), but we always have depth$(R) \leqslant \dim(R)$, by 0.6.10. □

1.4 Gorenstein Rings

A Noetherian local ring is a Gorenstein ring if inj dim$(R) < \infty$. This simple restriction carries several good properties, which we will study in this section. Using results from 1.3, we will show that Gorenstein rings are always Cohen-Macaulay and relate injective dimension and Krull dimension over such rings. Moreover, we will show that quotients of Gorenstein rings by ideals generated by regular elements are still Gorenstein, and that such quotients make the injective dimension decrease by the number of regular elements considered. This will allow us to reduce the study of local Gorenstein rings to the study of Gorenstein rings of dimension 0, and we will see that over such rings every module of finite injective dimension is actually injective. We will close the section with further characterization of these rings, which we will use in chapters 2 and 3 when we study local cohomology and Ulrich ideals over Gorenstein rings.

Definition 1.4.1 (Gorenstein ring)**.** A Noetherian local ring $(R, \mathfrak{m})$ is said to be a *Gorenstein ring* if inj dim$_R(R) < \infty$. A Noetherian ring R is said to be *Gorenstein* if $R_\mathfrak{m}$ is a local Gorenstein ring for every maximal ideal $\mathfrak{m}$ in R.

Example 1.4.2. Every field is a Gorenstein ring.

Example 1.4.3. Let K be a field. Then $R = K[[x_1, \dots, x_n]]$ is a Gorenstein ring of injective dimension n. K is a Gorenstein ring with inj dim$(K) = 0$, and we will show in 1.4.7 that since $x_1, \dots, x_n$ is a regular sequence in R, then

$$\text{inj dim}(R) = \text{inj dim}\left(\frac{R}{(x_1, \dots, x_d)}\right) + n = \text{inj dim}(K) + n = n.$$

Example 1.4.4. Let K be a field, and consider the polynomial ring $R = \frac{K[[x,y,z]]}{(x^3-y^2, z^2-x^2y)}$. We will see in 1.4.7 that quotients of Gorenstein rings by ideals generated by regular sequences are Gorenstein rings. As $x^3 - y^2, z^2 - x^2y$ is a regular sequence in $K[[x, y, z]]$ and $K[[x, y, x]]$ is a Gorenstein ring, then R is a Gorenstein ring. Moreover, inj dim$(R) = 1$.

Lemma 1.4.5. *Let R be a local Gorenstein ring and P a prime ideal in R. Then R_P is a Gorenstein local ring.*

Proof. Consider a finite injective resolution of R

$$0 \longrightarrow R \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots.$$

Since localization preserves injectives modules (1.1.7) and localization is exact (0.1.8), then

$$0 \longrightarrow R_P \longrightarrow (I_0)_P \longrightarrow (I_1)_P \longrightarrow \cdots$$

is a finite injective resolution of R_P. $\square$

Lemma 1.4.6 (Rees). *Let A and B be R-modules and $x \in R$ be such that $xB = 0$, and x is regular on R and A. Write $\overline{A} := A/xA$ and $\overline{R} := R/(x)$. Then for all $i \geqslant 0$,*

$$\operatorname{Ext}^i_R(B, A) \cong \operatorname{Ext}^{i-1}_{\overline{R}}(B, \overline{A}).$$

Proof. We will show this by induction on i.

(i) $i = 0$

We want to show that $\operatorname{Hom}_R(B, A) = 0$. So consider $\varphi \in \operatorname{Hom}_R(B, A)$. For each $b \in B$,

$$x\,\varphi(b) = \varphi(xb) = \varphi(0) = 0,$$

which implies, as x is regular in A, that $\varphi(b) = 0$, so that $\varphi = 0$. Then $\operatorname{Hom}_R(B, A) = 0$, as desired.

(ii) $i = 1$

Since x is regular in A, multiplication by x is injective, so that we get a short exact sequence

$$0 \longrightarrow A \xrightarrow{.x} A \longrightarrow \overline{A} \longrightarrow 0\ ,$$

and therefore a long exact sequence (by 0.10.3)

$$0 \longrightarrow \operatorname{Hom}_R(B, A) \longrightarrow \operatorname{Hom}_R(B, A) \longrightarrow \operatorname{Hom}_R(B, \overline{A}) \longrightarrow \operatorname{Ext}^1_R(B, A) \longrightarrow \dots .$$

By the case $i = 0$, $\operatorname{Hom}_R(B, A) = 0$. Since $xB = 0$, B is also an $\overline{R}$-module with the same structure, so that $\operatorname{Hom}_R(B, \overline{A}) = \operatorname{Hom}_{\overline{R}}(B, \overline{A})$. Also, since $xB = 0$, then the map $\operatorname{Ext}^1_R(B, A) \longrightarrow \operatorname{Ext}^1_R(B, A)$ induced by multiplication by x must be 0. Then we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\overline{R}}(B, \overline{A}) \longrightarrow \operatorname{Ext}^1_R(B, A) \longrightarrow 0\ ,$$

so that $\operatorname{Hom}_{\overline{R}}(B, \overline{A}) \cong \operatorname{Ext}^1_R(B, A)$, as desired.

(iii) $i \geqslant 2$ (assuming the statement is true for $i - 1$)

Consider a free R-module F such and an epimorphism $F \to B$. We have an exact sequence of R-modules of the form

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\alpha} B \longrightarrow 0\ ,$$

where $K = \ker(\alpha)$.

Tensoring this exact sequence with $R/(x)$, we obtain the following exact sequence of R-modules and $R/(x)$-modules

$$\overline{F} \xrightarrow{\theta} B \longrightarrow 0\ ,$$

where $\overline{F} = F/xF \cong R/(x) \otimes F$. Since $xB = 0$, then $B = B/xB \cong R/(x) \otimes B$. Let $C = \ker(\theta)$, which is both an R-submodule and an $R/(x)$-submodule of $\overline{F}$, considering $x\,C \subseteq x\,\overline{F} = 0$. We have thus an exact sequence of R-modules and $R/(x)$-modules

$$0 \longrightarrow C \longrightarrow \overline{F} \xrightarrow{\theta} B \longrightarrow 0\ .$$

Since F is a free R-module, then F is of the form

$$F \cong \bigoplus_{j\in J} R.$$

Therefore,

$$\overline{F} = F/xF \cong F \otimes R/(x) \cong \left(\bigoplus_{j\in J} R\right) \otimes R/(x) \cong \bigoplus_{j\in J} R/(x) = \bigoplus_{j\in J} \overline{R},$$

so that $\overline{F}$ is a free and hence projective $\overline{R}$-module. By 0.10.7,

$$\mathrm{Ext}^i_{\overline{R}}(\overline{F}, \overline{A}) = 0$$

for $i \geqslant 1$.

Moreover, since x is regular in R, then x is also regular in F, and thus we have a short exact sequence of R-modules

$$0 \longrightarrow F \xrightarrow{.x} F \longrightarrow \overline{F} \longrightarrow 0\ .$$

Therefore, this short exact sequence is a projective resolution of $\overline{F}$, and thus proj dim$_R(\overline{F}) \leqslant 1$, so that

$$\mathrm{Ext}^i_R(\overline{F}, A) = 0$$

for $i \geqslant 2$, by 0.9.7.

Now look at the long exact sequences induced by

$$0 \longrightarrow C \longrightarrow \overline{F} \longrightarrow B \longrightarrow 0$$

as in 0.10.3:

$$\begin{tikzcd}
\mathrm{Ext}^{i-1}_R(B,A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{i-1}_R(\overline{F},A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{i-1}_R(C,A) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^{i}_R(B,A) \arrow[r] \arrow[d] & \mathrm{Ext}^{i}_R(\overline{F},A) = 0 \arrow[d, equal] \\
\mathrm{Ext}^{i-2}_{\overline{R}}(B,\overline{A}) \arrow[r] & \mathrm{Ext}^{i-2}_{\overline{R}}(\overline{F},\overline{A}) \arrow[r] & \mathrm{Ext}^{i-2}_{\overline{R}}(C,\overline{A}) \arrow[r] & \mathrm{Ext}^{i-1}_{\overline{R}}(B,\overline{A}) \arrow[r] & \mathrm{Ext}^{i-1}_{\overline{R}}(\overline{F},\overline{A}) = 0
\end{tikzcd}\ .$$

The naturality of the long exact sequence for Ext guarantees that diagram commutes. By induction, we get:

(a) $\operatorname{Ext}_R^{i-1}(B, A) \cong \operatorname{Ext}_{\overline{R}}^{i-2}(B, \overline{A})$ because $xB = 0$,

(b) $\operatorname{Ext}_R^{i-1}(\overline{F}, A) \cong \operatorname{Ext}_{\overline{R}}^{i-2}(\overline{F}, \overline{A})$ because $x\overline{F} = 0$,

(c) $\operatorname{Ext}_R^{i-1}(C, A) \cong \operatorname{Ext}_{\overline{R}}^{i-2}(C, \overline{A})$ because $xC = 0$.

Moreover, $\operatorname{Ext}_R^{i}(\overline{F}, A) = 0$ and $\operatorname{Ext}_{\overline{R}}^{i-1}(\overline{F}, \overline{A}) = 0$ because $i, i-1 \geqslant 1$, by 0.8.1 and 0.10.7.

Therefore, we must have

$$\operatorname{Ext}_R^{i}(B, A) \cong \operatorname{Ext}_{\overline{R}}^{i-1}(B, \overline{A}).$$

□

Corollary 1.4.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $x \in \mathfrak{m}$ a regular element and $\overline{R} := R/(x)$. Then*

$$\textit{inj dim}(\overline{R}) = \textit{inj dim}(R) - 1.$$

In particular, R is Gorenstein if and only if $\overline{R}$ is Gorenstein.

Proof. Using 1.3.2 and 1.4.6, we obtain

$$\text{inj dim}(R) = \sup\left\{i \colon \operatorname{Ext}_R^{i}(R/\mathfrak{m}, R) \neq 0\right\} = \sup\left\{i \colon \operatorname{Ext}_{\overline{R}}^{i-1}(R/\mathfrak{m}, \overline{R}) \neq 0\right\}$$

$$= \sup\left\{i \colon \operatorname{Ext}_{\overline{R}}^{i-1}(\overline{R}/\overline{\mathfrak{m}}, \overline{R}) \neq 0\right\} = \sup\left\{j \colon \operatorname{Ext}_{\overline{R}}^{j}(\overline{R}/\overline{\mathfrak{m}}, \overline{R}) \neq 0\right\} + 1 = \text{inj dim}(\overline{R}) + 1.$$

□

Theorem 1.4.8. *Every Gorenstein ring is Cohen-Macaulay.*

Proof. It is enough to consider the case of local rings, as a Noetherian ring is Cohen-Macaulay (respectively, Gorenstein) if and only if its localization at every prime ideal is Cohen-Macaulay (Gorenstein).

(1) depth$(R) = 0$

We want to show that for every Noetherian local ring $(R, \mathfrak{m})$,

$$\text{depth}(R) = 0 \text{ and inj dim}(R) < \infty \Rightarrow \dim(R) = 0.$$

By 1.3.4, inj dim$(R) = 0$. We want to show that $\dim(R) = 0$. If $\dim(R) > 0$, there exists at least one minimal prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$ in R. By 0.7.10, $\mathfrak{p} \in \operatorname{Ass}(R/(0)) = \operatorname{Ass}(R)$. This implies that we have a monomorphism $R/\mathfrak{p} \hookrightarrow R$, and in particular, $\operatorname{Hom}_R(R/\mathfrak{p}, R) \neq 0$. Also, as $\mathfrak{p} \subsetneq \mathfrak{m}$, we can pick $x \in \mathfrak{m}$ with $x \notin \mathfrak{p}$, and for such x we have a short exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{.x} R/\mathfrak{p} \longrightarrow E \longrightarrow 0\ ,$$

which by 0.10.3 gives rise to a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(E,R) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{p},R) \xrightarrow{.x} \mathrm{Hom}_R(R/\mathfrak{p},R) \longrightarrow \mathrm{Ext}^1_R(E,R) \longrightarrow \dots,$$

where $\mathrm{Ext}^1_R(E,R) = 0$ because R is an injective R-module (0.10.8). Hence multiplication by x is onto, and thus $\mathrm{Hom}_R(R/\mathfrak{p},R) = x\ \mathrm{Hom}_R(R/\mathfrak{p},R)$. As $x \in \mathfrak{m}$ and both R and $R/\mathfrak{p}$ are finitely generated R-modules, then $\mathrm{Hom}_R(R/\mathfrak{p},R)$ is a finitely generated R-module, by 0.10.5. By Nakayama's Lemma (0.4.9), we must have $\mathrm{Hom}_R(R/\mathfrak{p},R) = 0$. But we have seen that $\mathrm{Hom}_R(R/\mathfrak{p},R) \neq 0$, so this is a contradiction. Then such $\mathfrak{p}$ cannot exist, so that $\dim(R) = 0$.

(2) depth$(R) = n > 0$

Pick a regular sequence $x_1,\dots,x_n \in \mathfrak{m}$. Applying 1.4.7 n times, we get

$$\text{inj dim}(R) = n + \text{inj dim}(\overline{R}),$$

where $\overline{R} = R/(x_1,\dots,x_n)$. So $\overline{R}$ is still a Gorenstein ring, and depth$(\overline{R}) = 0$ by 0.6.9. We have shown that this implies $\dim(\overline{R}) = 0$. By 0.6.9, $\dim(R) = n =$ depth(R), so that R is a Cohen-Macaulay ring.

□

Definition 1.4.9 (Self-injective)**.** A ring R is said to be *self-injective* if it is Artinian and an injective R-module.

Remark 1.4.10. A local ring is self-injective if and only if it is an Artinian Gorenstein ring.

Corollary 1.4.11. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring. If $\dim(R) = d$ and $x_1,\dots,x_d$ is a maximal regular sequence in $\mathfrak{m}$, then $\overline{R} = R/(x_1,\dots,x_d)$ is a self-injective Artinian local ring.*

Proof. By 0.6.9, $\dim(\overline{R}) = 0$, and since $\overline{R}$ is Noetherian, by 0.3.3, then $\overline{R}$ is an Artinian ring, by 0.3.6. Also, by 1.4.7 and 1.3.5, inj dim$(\overline{R}) =$ inj dim$(R) - d \leqslant \dim(\overline{R}) - d = 0$, so that $\overline{R}$ is a self-injective ring. □

Proposition 1.4.12. *Let $(R,\mathfrak{m})$ be an Artinian local ring and $M \neq 0$ a finitely generated R-module of finite injective dimension. Then M is an injective R-module.*

Proof. Since R is Artinian, $\dim(R) = 0$, by 0.3.5, and by 1.3.5, inj dim$(M) \leqslant \dim(R) = 0$, so that inj dim$(M) = 0$. Then M is an injective R-module. □

Proposition 1.4.13. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring, $x_1,\dots,x_n \in \mathfrak{m}$ a maximal regular sequence in R, $\overline{R} = R/(x_1,\dots,x_n)$ and $\overline{\mathfrak{m}} = \mathfrak{m}/(x_1,\dots,x_n)$. Then $E(\overline{R}/\overline{\mathfrak{m}}) = \overline{R}$. In particular, if $(R,\mathfrak{m})$ is an Artinian Gorenstein local ring, then $R = E(R/\mathfrak{m})$.*

Proof. Since $\overline{R}$ is an Artinian local ring, by 1.4.11, then $\mathrm{Ass}(\overline{R}) = \{\overline{\mathfrak{m}}\}$ by 1.2.1. Therefore we have a monomorphism $\overline{R}/\overline{\mathfrak{m}} \hookrightarrow \overline{R}$. But $\overline{R}$ is an injective module, by 1.4.11, so that $\overline{R}/\overline{\mathfrak{m}} \subseteq E(\overline{R}/\overline{\mathfrak{m}}) \subseteq \overline{R}$.

Since, by 1.2.9, the last two modules have the same length, they must be the same. □

Definition 1.4.14 (Type)**.** Let $(R,\mathfrak{m})$ be a Noetherian local ring, $d=$ depth(R). We define the *type* of R by

$$\mathrm{type}(R) := \dim_{R/\mathfrak{m}} \mathrm{Ext}^d_R(R/\mathfrak{m}, R).$$

Remark 1.4.15. Let M be an R-module. Consider an injective module E such that $M \subseteq E$ and let K be the cokernel of the canonical inclusion. From the long exact sequence in 0.10.3 obtained by applying $\mathrm{Hom}_R(R/\mathfrak{m}, -)$ to the short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow C \longrightarrow 0\ ,$$

we get an exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{m}, C) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{m}, M) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{m}, M) = 0$$

so that $\mathrm{Hom}_R(R/\mathfrak{m}, C) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{m}, M)$ is an epimorphism.

If $f \in \mathrm{Hom}_R(R/\mathfrak{m}, M)$ and $a \in \mathfrak{m}$, $b \in R/\mathfrak{m}$, then $af(b) = f(ab) = f(0) = 0$, and thus $\mathfrak{m}f = 0$. This shows that $\mathfrak{m}\,\mathrm{Hom}_R(R/\mathfrak{m}, M) = 0$. Moreover, we can always find an R-module V such that $\mathrm{Ext}^1_R(R/\mathfrak{m}, V) \cong \mathrm{Ext}^n_R(R/\mathfrak{m}, M)$, by 0.10.6, and for such V we can find an R-module C and an epimorphism $\mathrm{Hom}_R(R/\mathfrak{m}, C) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{m}, V)$. As $\mathfrak{m}\,\mathrm{Hom}_R(R/\mathfrak{m}, C) = 0$, then $\mathfrak{m}\,\mathrm{Ext}^n_R(R/\mathfrak{m}, M) = 0$.

Therefore, $\mathrm{Ext}^d_R(R/\mathfrak{m}, R)$ has the same structure as a module over R and over $R/\mathfrak{m}$. As $R/\mathfrak{m}$ is a field, then $\dim_{R/\mathfrak{m}} \mathrm{Ext}^d_R(R/\mathfrak{m}, R)$ is defined.

Theorem 1.4.16. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $\dim(R) = d$. The following conditions are equivalent:*

(1) *inj dim*$(R) < \infty$.

(2) *inj dim*$(R) = d$.

(3) R *is Cohen-Macaulay and* $\mathrm{Ext}^d_R(R/\mathfrak{m}, R) \cong R/\mathfrak{m}$.

(4) R *is Cohen-Macaulay and* $\mathrm{type}(R) = 1$.

Proof. $(3) \Leftrightarrow (4)$ is obvious from the definition of type, and $(2) \Rightarrow (1)$ is also obvious.

For $(1) \Rightarrow (2)$, we know by 1.3.4 that inj dim$(R) =$ depth(R). But by 1.4.8, R is Cohen-Macaulay, as it is Gorenstein, and then depth$(R) = \dim(R) = d$, so that inj dim$(R) = d$.

Let us now prove $(1) \Rightarrow (3)$. Since R is Gorenstein, then R is Cohen-Macaulay, by 1.4.8, and

$$\text{depth}(R) = \dim(R) = d.$$

Consider a regular sequence $x_1, \ldots, x_d$ in R and let

$$\overline{R} := R/(x_1, \ldots, x_d) \text{ and } \overline{\mathfrak{m}} := \mathfrak{m}/(x_1, \ldots, x_d)\,.$$

Notice that

$$\overline{R}/\overline{\mathfrak{m}} \cong \frac{R/(x_1,\ldots,x_d)}{\mathfrak{m}/(x_1,\ldots,x_d)}.$$

Now $(\overline{R},\overline{\mathfrak{m}})$ is an Artinian local ring. By 1.4.7,

$$\text{inj dim}(R) = \text{inj dim}(\overline{R}) + d.$$

As we have seen in $(1) \Rightarrow (2)$, inj dim$(R) = d$, and therefore inj dim$(\overline{R}) = 0$, i.e., $\overline{R}$ is a self-injective $\overline{R}$-module. As we saw in 1.4.13, $\overline{R} = E(\overline{R}/\overline{\mathfrak{m}})$.

Applying 1.4.6 d times and then 1.2.6, we get

$$\mathrm{Ext}^d_R\left(R/\mathfrak{m},R\right) \cong \mathrm{Hom}_{\overline{R}}\left(R/\mathfrak{m},\overline{R}\right) \cong \mathrm{Hom}_{\overline{R}}\left(\overline{R}/\overline{\mathfrak{m}},E(\overline{R}/\overline{\mathfrak{m}})\right) \cong E(\overline{R}/\overline{\mathfrak{m}}) \cong \overline{R}/\overline{\mathfrak{m}} \cong R/\mathfrak{m}.$$

Now all that remains to prove is $(3) \Rightarrow (2)$. Since R is Cohen-Macaulay and $\dim(R) = d$, then depth$(R) = d$. Consider a maximal regular sequence $x_1,\ldots,x_d$ in R, and let $\overline{R} := R/(x_1,\ldots,x_d)$. By 1.4.7, inj dim$(R) = d +$ inj dim$(\overline{R})$. To show that inj dim$(R) = d$, it is enough to show that inj dim$(\overline{R}) = 0$.

We know that $\mathrm{Ext}^d_R(R/\mathfrak{m},R) \cong R/\mathfrak{m}$ and that $\overline{R}$ is a Cohen Macaulay Artinian ring, and we want to show that $\overline{R}$ is an injective $\overline{R}$-module. By 1.2.1, $\mathrm{Ass}(\overline{R}) = \{\overline{\mathfrak{m}}\}$. Then $\overline{\mathfrak{m}} = \mathrm{Ann}_{\overline{R}}(a)$ for some non-zero $a \in \overline{R}$, or equivalently, there exists a monomorphism $\overline{R}/\overline{\mathfrak{m}} \cong \overline{R}a \hookrightarrow \overline{R}$. If we show that $\overline{R}/\overline{\mathfrak{m}} \hookrightarrow \overline{R}$ is actually an essential extension, then by 1.1.15

$$\overline{R}/\overline{\mathfrak{m}} \subseteq \overline{R} \subseteq E(\overline{R}/\overline{\mathfrak{m}}).$$

But since $\lambda(\overline{R}) = \lambda(E(\overline{R}/\overline{\mathfrak{m}}))$, by 1.2.9, we must have $\overline{R} = E(\overline{R}/\overline{\mathfrak{m}})$, and, in particular, $\overline{R}$ is an injective $\overline{R}$-module. So we will prove that $\overline{R}/\overline{\mathfrak{m}} \cong \overline{R}a \subseteq \overline{R}$ is an essential extension and this will conclude the proof.

Consider an ideal $B \neq 0$ of $\overline{R}$. Since $\overline{R}$ is an Artinian local ring, by 1.4.11, then $\mathrm{Ass}(B) = \{\overline{\mathfrak{m}}\}$, by 1.2.1. Then there exists non-zero $b \in B$ such that $\overline{\mathfrak{m}} = \mathrm{Ann}_{\overline{R}}(b)$, and a monomorphism $\overline{R}/\overline{\mathfrak{m}} \cong \overline{R}b \hookrightarrow B$.

Consider

$$(0 :_{\overline{R}} \overline{\mathfrak{m}}) = \left\{r \in \overline{R} \colon r\,\overline{\mathfrak{m}} = 0\right\},$$

which is an $\overline{R}$-module annihilated by $\overline{\mathfrak{m}}$ and thus a vector space over the field $\overline{R}/\overline{\mathfrak{m}}$. This is the largest $\overline{R}/\overline{\mathfrak{m}}$-vector space contained in $\overline{R}$. Indeed, if $L \subseteq \overline{R}$ is an $\overline{R}/\overline{\mathfrak{m}}$-vector space, then for every $l \in L$, $\overline{\mathfrak{m}}\,l = 0$, so that $L \subseteq (0 :_{\overline{R}} \overline{\mathfrak{m}})$.

By 0.1.13, $(0 :_{\overline{R}} \overline{\mathfrak{m}})$ and $\mathrm{Hom}_{\overline{R}}(\overline{R}/\overline{\mathfrak{m}},\overline{R})$ are isomorphic $\overline{R}$-modules, and by 1.4.15, as $\overline{R}/\overline{\mathfrak{m}}$-modules. Using 1.4.6,

$$(0 :_{\overline{R}} \overline{\mathfrak{m}}) \cong \mathrm{Hom}_{\overline{R}}(\overline{R}/\overline{\mathfrak{m}},\overline{R}) \cong \mathrm{Hom}_{\overline{R}}(R/\mathfrak{m},\overline{R}) \cong \mathrm{Ext}^d_R(R/\mathfrak{m},R) \cong R/\mathfrak{m} \cong \overline{R}/\overline{\mathfrak{m}}.$$

As $\overline{R}a$ and $\overline{R}b$ are non-zero $\overline{R}/\overline{\mathfrak{m}}$-vector spaces contained in $\overline{R}$, then

$$\overline{R}a \subseteq (0 :_{\overline{R}} \overline{\mathfrak{m}}) \text{ and } \overline{R}b \subseteq (0 :_{\overline{R}} \overline{\mathfrak{m}}).$$

Since

$$1 = \dim_{\overline{R}/\overline{\mathfrak{m}}}(\overline{R}\,a) = \dim_{\overline{R}/\overline{\mathfrak{m}}}(\overline{R}\,b) = \dim_{\overline{R}/\overline{\mathfrak{m}}}(0 :_{\overline{R}} \overline{\mathfrak{m}}),$$

we conclude that

$$\overline{R}a = (0 :_{\overline{R}} \overline{\mathfrak{m}}) = \overline{R}\,b.$$

Therefore, $0 \neq b \in \overline{R}\,a \cap B$, and $\overline{R}/\overline{\mathfrak{m}} \cong \overline{R}a \subseteq \overline{R}$ is an essential extension, and as we have seen, this completes the proof. $\square$

Example 1.4.17. Not every Cohen-Macaulay ring is Gorenstein. Consider a field K and the polynomial ring $R = \frac{K[x,y]}{(x^2,xy,y^2)}$. This is an Artinian local ring, with unique maximal ideal $\mathfrak{m} = \frac{(x,y)}{(x^2,xy,y^2)}$, and thus Cohen-Macaulay. Now notice that in $R[x,y]$, $(x,y)^2 = (x^2,xy,y^2)$, and thus $(0 :_R \mathfrak{m}) = \mathfrak{m}$. Moreover, $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \mu(\mathfrak{m}) = 2$ by 0.1.11. Thus

$$\operatorname{type}(R) = \dim_{R/\mathfrak{m}}(\operatorname{Hom}_R(R/\mathfrak{m}, R)) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 2,$$

and by 1.4.16 that this implies that R is not a Gorenstein ring.

1.5 Matlis Duality

Eben Matlis proved ([18]) a duality between Noetherian and Artinian modules over complete Noetherian rings $(R, \mathfrak{m})$ involving the functor $\operatorname{Hom}_R(-, E(R/\mathfrak{m}))$. Matlis Duality states that this functor takes Noetherian (respectively, Artinian) modules into Artinian (respectively, Noetherian) modules, and that applying this functor twice yields the identity. We will prove this duality when R is an Artinian local ring. In this case, however, Noetherian and Artinian modules coincide, a fact which we will prove using part of Matlis Duality. We still present a complete proof, considering it can easily be extended to a proof of the general case, by extending some of the results in section 1.2 to complete Noetherian rings. For a proof of the general case, see [11, Section 3.2].

Proposition 1.5.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathfrak{p}$ be a prime ideal in R. Then $E(R/\mathfrak{p})$ is an $R_{\mathfrak{p}}$-module.*

Proof. Let $x \notin \mathfrak{p}$. We want to show that $(E(R/\mathfrak{p}))_{\mathfrak{p}} = E(R/\mathfrak{p})$.

$$\psi \colon E(R/\mathfrak{p}) \xrightarrow{.x} E(R/\mathfrak{p})$$

is an isomorphism. First, suppose that $r \in R$ is such that $x(r+\mathfrak{p}) = 0$. Then $xr \in \mathfrak{p}$, but since $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$, we must have $r \in \mathfrak{p} \Leftrightarrow r + \mathfrak{p} = 0$. Then the map $\varphi \colon R/\mathfrak{p} \xrightarrow{.x} R/\mathfrak{p}$ is injective, and thus

$\ker(\psi) \cap R/\mathfrak{p} = \ker(\varphi) = 0$. Since $R/\mathfrak{p} \hookrightarrow E(R/\mathfrak{p})$ is an essential extension, we must have $\ker(\psi) = 0$, so that ψ is injective. Let C be the cokernel of ψ. We have a short exact sequence

$$0 \longrightarrow E(R/\mathfrak{p}) \xrightarrow{.x} E(R/\mathfrak{p}) \longrightarrow C \longrightarrow 0\ ,$$

which must split since $E(R/\mathfrak{p})$ is an injective module. Then,

$$E(R/\mathfrak{p}) \cong E(R/\mathfrak{p}) \oplus C$$

but $E(R/\mathfrak{p})$ is an indecomposable injective module, by 1.1.25, and thus $C = 0$ and ψ is iso, as desired. $\square$

Remark 1.5.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $E = E(R/\mathfrak{m})$. Consider

$$E_n := \{e \in E \colon \mathfrak{m}^n e = 0\}\,,$$

and $R_n = R/\mathfrak{m}^n$ for each n. Every R_n-module A can be seen as an R-module, defining

$$ra := (r + \mathfrak{m}^n)\, a$$

for each $r \in R$ and $a \in A$. In particular, if $A \subseteq B$ is an essential extension of R_n-modules, it is also an essential extension of R-modules, since B is annihilated by $\mathfrak{m}^n$, and therefore the R-submodules of B and the R_n-submodules of B coincide.

Proposition 1.5.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $E = E(R/\mathfrak{m})$. Let*

$$E_n := \{e \in E \colon \mathfrak{m}^n e = 0\}$$

and $R_n = R/\mathfrak{m}^n$. Then

$$E_n = E_{R_n}(R/\mathfrak{m}).$$

Proof. First, we will see that $R/\mathfrak{m} \subseteq E_n$, and that this is an essential extension as both R-modules and R_n-modules. Surely $E_n \neq 0$, as $\mathfrak{m}^n(1 + \mathfrak{m}) = 0$ for every n, so that $1 + \mathfrak{m} \in E_n$ is a non-zero element of E_n. Since $R/\mathfrak{m} \subseteq E$ is an essential extension, then $E_n \cap R/\mathfrak{m} \neq 0$. But $R/\mathfrak{m}$ is a simple R-module, so that $0 \neq E_n \cap R/\mathfrak{m} \subseteq R/\mathfrak{m}$ implies that $E_n \cap R/\mathfrak{m} = R/\mathfrak{m}$. By 1.1.6, $R/\mathfrak{m} \subseteq E_n$ is an essential extension of R-modules. But $R/\mathfrak{m}$ and E_n are also R_n-modules, since they are both annihilated by $\mathfrak{m}^n$. Therefore, $R/\mathfrak{m} \subseteq E_n$ is also an essential extension of R_n-modules.

Also, E_n is an injective R_n-module. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B\ , \\ & E_n & \end{tikzcd}$$

where A and B are R_n-modules and f is a monomorphism. There exists a map φ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[d, dashed, "\varphi"] \\ & E_n \arrow[r, hook] & E \end{tikzcd}$$

commute, since E is an injective R-module, and A and B are R-modules as well. But since B is an R_n-module, we must have $\mathfrak{m}^n B = 0$, and therefore $\mathfrak{m}^n(\mathrm{im}(\varphi)) = 0$. Then $\mathrm{im}(\varphi) \subseteq E_n$, meaning that $\varphi\colon B \longrightarrow E_n$, which shows that E_n is injective, by 0.8.6.

So $R/\mathfrak{m} \subseteq E_n$ is an essential extension, both as R-modules and as R_n-modules. As E_n is an injective R_n-module, we must have $E_n = E_{R_n}(R/\mathfrak{m})$. $\square$

Proposition 1.5.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $E = E(R/\mathfrak{m})$. Let*

$$E_n := \{e \in E \colon \mathfrak{m}^n\, e = 0\}$$

and $R_n = R/\mathfrak{m}^n$. Then

$$E = \bigcup_n E_n.$$

Proof. Since each E_n is defined as an submodule of E, all we have to prove is that $E \subseteq \bigcup_n E_n$.

Let $x \in E$, and consider the submodules of E of the form $\mathfrak{m}^n x \subseteq (x)$. Since R is Noetherian, and (x) is a finitely generated module, then by Krull's Intersection Theorem, 0.3.8,

$$\bigcap_n \mathfrak{m}^n = 0,$$

and thus

$$\left(\bigcap_n \mathfrak{m}^n\right) x = \bigcap_n \mathfrak{m}^n\, x = 0.$$

Suppose $\mathfrak{m}^n\, x \neq 0$ for all n. Then $\mathfrak{m}^n\, x$ are non-zero submodules of E, which is an essential extension of $R/\mathfrak{m}$, and therefore $R/\mathfrak{m} \cap \mathfrak{m}^n\, x \neq 0$. Since $R/\mathfrak{m}$ is a simple module, and $0 \neq R/\mathfrak{m} \cap \mathfrak{m}^n\, x \subseteq R/\mathfrak{m}$, we must have $R/\mathfrak{m} \cap \mathfrak{m}^n\, x = R/\mathfrak{m}$. This means that

$$R/\mathfrak{m} \subseteq \bigcap_n \mathfrak{m}^n\, x = 0,$$

which cannot be because $R/\mathfrak{m} \neq 0$. Then $\mathfrak{m}^n\, x = 0$ for some n, and by definition, this means that $x \in E_n$. $\square$

Theorem 1.5.5 (Matlis Duality)**.** *Let $(R, \mathfrak{m})$ be an complete local ring. Consider the functor*

$$T := \mathrm{Hom}_R(-, E(R/\mathfrak{m}))\colon \textit{Mod-}R \longrightarrow \textit{Mod-}R.$$

Let $M \in \mathcal{N}(R)$ and $N \in \mathcal{A}(R)$. Then $T(M) \in \mathcal{A}(R)$ and $T(N) \in \mathcal{N}(R)$. Moreover, $T(T(M)) \cong M$ and

$T(T(N)) \cong N$.

Remark 1.5.6. For a proof, see [11, Section 3.2]. We will show this theorem for the particular case of Artinian local rings. The proof we follow here can be easily extended to a proof of the general case, by extending results from section 1.2. For this particular case, however, Noetherian and Artinian R-modules coincide, which we will show in 1.5.7. But to prove 1.5.7 we will use the fact that $T(N) \in \mathcal{N}(R)$ and $T(T(N)) \cong N$ for $N \in \mathcal{A}(R)$. A proof of 1.5.5 without using 1.5.7 is presented considering it can easily be extended to the general case of complete Noetherian local rings, where 1.5.7 does not hold.

Proof. For simplicity, let us write $E := E(R/\mathfrak{m})$. Recall that $\mathcal{N}(R) = \mathcal{M}(R)$, by 1.2.4, and thus Noetherian R-modules are the same as finitely generated R-modules.

(1) For every finitely generated R-module M, $\mathrm{Hom}_R(M, E)$ is an Artinian R-module

For $M = R$ this is clearly true, since $\mathrm{Hom}_R(R, E) \cong E$ by 1.2.6, and E is Artinian by 1.2.9. Now let M be any finitely generated R-module, generated by n elements, and pick a surjective homomorphism $R^n \longrightarrow M$. The exact sequence

$$R^n \xrightarrow{\varphi} M \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(R^n, E) \ ,$$

i.e., $\mathrm{Hom}_R(M, E)$ is a submodule of $\mathrm{Hom}_R(R^n, E)$. Using 0.1.12,

$$\mathrm{Hom}_R(R^n, M) \cong (\mathrm{Hom}_R(R, M))^n \cong E^n,$$

so that $\mathrm{Hom}_R(M, E)$ is isomorphic to some submodule of E^n.

By 1.2.9, E is an Artinian R-module, and therefore E^n is also an Artinian R-module by 0.3.3. Since submodules of Artinian modules are Artinian, by 0.3.3, then $\mathrm{Hom}_R(M, E)$ is an Artinian R-module.

(2) For every Artinian R-module N, $\mathrm{Hom}_R(N, E)$ is a Noetherian R-module

Consider

$$N_0 := \{x \in N \mid \mathfrak{m}\, x = 0\} .$$

If $N \neq 0$, let $n \neq 0$ be an element in N. The descending chain of submodules

$$\mathfrak{m}n \supseteq \mathfrak{m}^2 n \supseteq \ldots$$

must stop and therefore $\mathfrak{m}^k n = \mathfrak{m}^{k+1} n$ for some $k \geqslant 1$. Notice that $\mathfrak{m}^k n$ is a finitely generated R-module, since Rn is a Noetherian R-module. By 0.4.9, $\mathfrak{m}^k n = 0$. Choose $k \geqslant 1$ to be the smallest integer such that $\mathfrak{m}^k n = 0$, so that $\mathfrak{m}^{k-1} n \neq 0$. Since $\mathfrak{m}^{k-1} n \subseteq N_0$, we must have $N_0 \neq 0$.

Since $\mathfrak{m}\, N_0 = 0$, then N_0 is an R-module and an $R/\mathfrak{m}$-module. Notice that N_0 cannot be an infinite dimensional $R/\mathfrak{m}$-vector space, since that would yield an infinite descending chain of submodules of N_0, and being a submodule of an Artinian R-module, by 0.3.3, it must also be Artinian. Then $N_0 \cong (R/\mathfrak{m})^r$, for some $r \geqslant 1$.

Also, $(R/\mathfrak{m})^r \cong N_0 \subseteq N$ is an essential extension. Let $B \neq 0$ be an R-submodule of N. Following the same technique we used to prove that $N_0 \neq 0$, we can show that $\mathfrak{m} = \mathrm{Ann}_R(x)$ for some element $x \neq 0$ of B. Then $\mathfrak{m}\, x = 0$, so that $x \in B \cap N_0$. Then $B \cap N_0 \neq 0$, and this proves that $(R/\mathfrak{m})^r \cong N_0 \subseteq N$ is an essential extension.

By 1.1.15 and 1.1.20, we have

$$N \subseteq E(N_0) \cong E((R/\mathfrak{m})^r) \cong (E(R/\mathfrak{m}))^r = E^r.$$

Applying the functor $\mathrm{Hom}_R(-, E)$ to the exact sequence

$$0 \longrightarrow N \hookrightarrow E^r \ ,$$

we get an exact sequence

$$\mathrm{Hom}_R(E^r, E) \longrightarrow \mathrm{Hom}_R(N, E) \longrightarrow 0 \ .$$

Since, by 1.2.10,

$$\mathrm{Hom}_R(E^r, E) \cong (\mathrm{Hom}_R(E, E))^r = R^r,$$

the last exact sequence is actually

$$R^r \longrightarrow \mathrm{Hom}_R(N, E) \longrightarrow 0 \ .$$

Then $\mathrm{Hom}_R(N, E)$ is a Noetherian R-module, by 0.3.3, being the quotient of a Noetherian R-module. So indeed, $T(N) = \mathrm{Hom}_R(N, E) \in \mathcal{M}(R)$ for $N \in \mathcal{A}(R)$.

(3) For each $M \in \mathcal{M}(R)$, $\mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) \cong M$, and the isomorphism is natural

It is clear that applying T to M twice induces the map

$$\begin{aligned} M &\longmapsto \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) \\ m &\longmapsto (\varphi \mapsto \varphi(m)) \end{aligned}$$

which is clearly a natural homomorphism.

If $M = R$, by 1.2.10 we have an isomorphism

$$\begin{array}{ccccc} R & \xrightarrow{\cong} & (\mathrm{Hom}_R(E, E)) & \xrightarrow{\cong} & \mathrm{Hom}_R((\mathrm{Hom}_R(R, E)), E) \ , \\ r & \longmapsto & (e \mapsto re) & \longmapsto & ((1 \mapsto e) \mapsto re) \end{array}$$

which is the map above.

Similarly, for $M = R^n$, we have

$$R^n \cong (\mathrm{Hom}_R(E,E))^n \cong \mathrm{Hom}_R(E^n,E) \cong \mathrm{Hom}_R((\mathrm{Hom}_R(R,E))^n,E) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R^n,E),E),$$

by 0.1.12 and 1.2.10, and this isomorphism is the map above.

Now let M be any finitely generated R-module and consider an epimorphism $\varphi\colon R^n \longrightarrow M$. Since its kernel, $\ker(\varphi) \subseteq R^n$, must also be finitely generated, consider an epimorphism $R^m \longrightarrow \ker(\varphi)$ and the composition $\psi\colon R^m \longrightarrow \ker(\varphi) \hookrightarrow R^n$. We have an exact sequence

$$R^m \xrightarrow{\psi} R^n \xrightarrow{\varphi} M \longrightarrow 0\ ,$$

and since $\mathrm{Hom}_R(-,E)$ is exact, we get a commutative diagram

$$\begin{tikzcd}
R^m \arrow[r, "\psi"] \arrow[d, "\cong"'] & R^n \arrow[r, "\varphi"] \arrow[d, "\cong"] & M \arrow[r] \arrow[d] & 0\ , \\
T(T(R^m)) \arrow[r] & T(T(R^n)) \arrow[r] & T(T(M)) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are given by applying the functor twice, and commutativity follows from naturality. The 5-lemma (0.10.9) guarantees that the right vertical arrow is also an isomorphism, so that indeed $\mathrm{Hom}_R(\mathrm{Hom}_R(M,E),E) \cong M$.

(4) For each $N \in \mathcal{A}(R)$, $\mathrm{Hom}_R(\mathrm{Hom}_R(N,E),E) \cong N$, and the isomorphism is natural

Let $N \in \mathcal{A}(R)$. We want to show that $\mathrm{Hom}_R(\mathrm{Hom}_R(N,E),E) \cong N$, and we have shown that $N \subseteq (E(R/\mathfrak{m}))^r$ for some r, so consider the cokernel N_1 of the inclusion. This is a quotient of an Artinian module and therefore Artinian, so that for the same reason $N_1 \hookrightarrow (E(R/\mathfrak{m}))^s$ for some s. Denote by N_2 the cokernel of $N_1 \hookrightarrow (E(R/\mathfrak{m}))^s$. We have short exact sequences

$$0 \longrightarrow N \longrightarrow E^r \longrightarrow N_1 \longrightarrow 0$$

and

$$0 \longrightarrow N_1 \longrightarrow E^s \longrightarrow N_2 \longrightarrow 0\ .$$

With the composition $E^r \longrightarrow N_1 \longrightarrow E^s$, we get an exact sequence

$$0 \longrightarrow N \longrightarrow E^r \longrightarrow E^s\ ,$$

and applying the functor two times to this sequence we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & E^r \arrow[r] \arrow[d, "\cong"'] & E^s\ , \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_R(\mathrm{Hom}_R(N,E),E) \arrow[r] & E^r \arrow[r] & E^s
\end{tikzcd}$$

so that $N \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(N,E),E)$ must also be an isomorphism, by 0.10.9.

$\square$

Theorem 1.5.7. *Let $(R,\mathfrak{m})$ be an Artinian local ring and M an R-module. The following conditions are equivalent:*

(1) *M is finitely generated*

(2) *M is Noetherian*

(3) *M is Artinian*

Proof. We have seen in 1.2.4 that $(1) \Leftrightarrow (2)$. Since R is Artinian, $(1) \Rightarrow (3)$, by 0.3.7. For $(3) \Rightarrow (1)$, let M be an Artinian R-module. By 1.5.5, $N := \mathrm{Hom}_R(M, E(R/\mathfrak{m}))$ is a Noetherian R-module, and thus finitely generated. Moreover, $M := \mathrm{Hom}_R(N, E(R/\mathfrak{m}))$. By 1.2.9, $E(R/\mathfrak{m})$ is a Noetherian R-module, and thus finitely generated. By 0.10.5, N is a finitely generated R-module. $\square$

Corollary 1.5.8. *Let $(R,\mathfrak{m})$ be an Artinian local ring. If $M \neq N$ are two non-isomorphic finitely generated R-modules, then $\mathrm{Hom}_R(M, E(R/\mathfrak{m}))$ and $\mathrm{Hom}_R(N, E(R/\mathfrak{m}))$ are non-isomorphic R-modules.*

Proof. By 1.5.5,

$$T(\mathrm{Hom}_R(M, E(R/\mathfrak{m}))) \cong M \ncong N \cong T(\mathrm{Hom}_R(N, E(R/\mathfrak{m}))),$$

and thus $\mathrm{Hom}_R(M, E(R/\mathfrak{m}))$ and $\mathrm{Hom}_R(N, E(R/\mathfrak{m}))$ cannot be isomorphic. $\square$

1.6 Bass Numbers

We have seen that injective modules over Noetherian rings R are direct sums of injective hulls of quotients of R by prime ideals. We can construct minimal injective resolutions using injective envelopes. The number of copies of $E(R/P)$ that appear in the i-th component of a minimal injective resolution of M depends only on M, i and P, and we will call them the Bass numbers of M with respect to P. In this section, we will show that the Bass numbers of M are finite. We will also give another characterization of Gorenstein rings via the Bass numbers of R.

Definition 1.6.1 (Bass Numbers)**.** Let R be a Noetherian ring and M an R-module. There exists an injective resolution of M of the form

$$M \xrightarrow{\alpha_0} E^0(M) \xrightarrow{\alpha_1} E^1(M) \xrightarrow{\alpha_2} E^2(M) \xrightarrow{\alpha_3} \cdots$$

where $E^0(M) = E(M)$ and $E^{i+1}(M) = E(\text{coker}(\alpha_i))$. By 1.1.27, we have

$$E^0(M) = E(M) = \bigoplus_{P \in \mathrm{Ass}(M)} (E(R/P))^{\mu_0(M,P)}$$

and

$$E^i(M) \;=\; \bigoplus_{P\in\mathrm{Ass}(\mathsf{coker}(\alpha_i))} (E(R/P))^{\mu_i(M,P)}$$

where the sum is taken over all prime ideals $P \in \mathrm{Ass}(M)$. The cardinals $\mu_i(M,P)$ are called the *Bass numbers* of the module M.

Remark 1.6.2. In fact, given any module M, the injective resolution

$$M \xrightarrow{\alpha_0} E^0(M) \xrightarrow{\alpha_1} E^1(M) \xrightarrow{\alpha_2} E^2(M) \xrightarrow{\alpha_3} \cdots$$

is minimal.

Theorem 1.6.3. *Let R be a Noetherian ring, M a finitely generated R-module and P a prime ideal in R. Then*

$$\mu_i(M,P) = \dim_{R_P} \mathrm{Ext}^i_{R_P}(R_P/PM_P\,, M_P)$$

and, in particular, $\mu_i(M,P)$ is finite.

Proof. Consider the injective resolution of M

$$M \xrightarrow{\alpha_0} E^0(M) \xrightarrow{\alpha_1} E^1(M) \xrightarrow{\alpha_2} E^2(M) \xrightarrow{\alpha_3} \cdots.$$

Since localization is exact, by 0.1.8, then

$$M \xrightarrow{\alpha_0} \left(E^0(M)\right)_P \xrightarrow{\alpha_1} \left(E^1(M)\right)_P \xrightarrow{\alpha_2} \left(E^2(M)\right)_P \xrightarrow{\alpha_3} \cdots$$

is also an exact sequence. Moreover, $E(N)_P = E(N_P)$ for any R-module N, by 1.1.19. We conclude that $\left(E^i(M)\right)_P = E^i(M_P)$. Thus, the previous resolution is an injective resolution of M_P. Applying $\mathrm{Hom}_{R_P}(R_P/PR_P\,, -)$ to this resolution, the i-th cohomology module of the resulting complex is, by definition, the module $\mathrm{Ext}^i_{R_P}(R_P/PR_P\,, M)$.

Consider

$$C^i = \left\{x \in \left(E^i(M)\right)_P :\ PR_P\, x = 0\right\}.$$

By 0.1.13,

$$C^i \cong \mathrm{Hom}_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right),$$

and thus $\mathrm{Ext}^i_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right)$ can be computed as the cohomology modules of the complex

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots.$$

Consider the following diagram with exact diagonals (see 0.9.1):

$$\begin{tikzcd}
& & 0 \arrow[dr] & & 0 & & & & 0 \\
& & & \text{coker}\,\alpha_0 \arrow[ur] \arrow[dr, "\beta_1"] & & & & \text{coker}\,\alpha_2 \arrow[ur] & \\
0 \arrow[r] & M \arrow[r, "\alpha_0"'] & E^0(M) \arrow[ur, "\pi_0"] \arrow[rr, "\alpha_1"'] & & E^1(M) \arrow[dr, "\pi_1"'] \arrow[rr, "\alpha_2"'] & & E^2(M) \arrow[ur, "\pi_2"] \arrow[rr, "\alpha_3"'] & & \cdots \\
& & & & & \text{coker}\,\alpha_1 \arrow[ur, "\beta_2"'] \arrow[dr] & & & \\
& & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

with im $(\alpha_{i+1}) =$ im $(\beta_{i+1}) \cong$ coker (α_i). We have $\alpha_{n+1} = \beta_n \circ \pi_n$. Therefore, for every $i \geqslant 0$,

$$\text{im}\,(\alpha_{i+1}) = \text{im}\,(\beta_{i+1}) \cong \text{coker}\,(\alpha_i) \subseteq E^{i+1}(M)$$

is an essential extension. We will denote by α_{i+1} the map induced on $E^i(M_P) \longrightarrow E^{i+1}(M_P)$ by α_{i+1}: $E^i(M) \longrightarrow E^{i+1}(M)$. Let d_{i+1} be the induced map on

$$\operatorname{Hom}_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right) \xrightarrow{d_{i+1}} \operatorname{Hom}_{R_P}\left(R_P/PR_P\,, E^{i+1}(M_P)\right)\,,$$

and thus on

$$C^i \xrightarrow{d_{i+1}} C^{i+1}\,.$$

Since essential extensions are preserved by localization, by 1.1.8, then

$$\text{im}\,(\alpha_i) \subseteq E^i(M_P)$$

is an essential extension of R_P-modules. If $C^i \subseteq E^i(M)$ is a non-zero submodule, then for each $x \in C^i$ with $x \neq 0$,

$$R_P\, x \cap \text{im}\,(\alpha_i) \neq 0.$$

Let $x \in C_i$, $x \neq 0$, and consider $a \in R_P$ such that $ax \neq 0$ and $ax \in$ im (α_i). By definition of C^i, $PR_P\, x = 0$, which implies that $a \notin PR_P$, considering that PR_P is a prime ideal. Since PR_P is the unique maximal ideal in R_P, this implies that a is an invertible element. But then $x \in$ im $(\alpha_i) \subseteq \ker(\alpha_{i+1})$. Thus,

$$\alpha_{i+1}(x) = 0$$

for all $x \in C^i$.

Also, with the identifications

$$C^i \cong \operatorname{Hom}_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right),$$

and given that $C^i \subseteq E^i(M)$, the map

$$C^i \xrightarrow{d_{i+1}} C^{i+1}$$

is the restriction of α_{i+1} to C^i. Also, the modules $\operatorname{Ext}^i_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right)$ can be computed as the cohomology modules of the complex

$$0 \longrightarrow C^0 \xrightarrow{\alpha_{1|\,C_0}} C^1 \xrightarrow{\alpha_{2|\,C_1}} C^2 \xrightarrow{\alpha_{3|\,C_2}} \cdots,$$

and thus

$$\operatorname{Ext}^i_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right) = \frac{\ker\left(\alpha_{i+1|\,C_i}\right)}{\mathrm{im}\left(\alpha_{i|\,C_{i-1}}\right)} = \frac{C^i}{(0)} \cong C^i.$$

Therefore,

$$\operatorname{Ext}^i_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right) \cong \operatorname{Hom}_{R_P}\left(R_P/PR_P\,, E^i(M_P)\right).$$

As stated in 1.6.1,

$$E^i(M) = \bigoplus_{Q\in \operatorname{Spec}(R)} E(R/Q)^{\mu_i(M,Q)},$$

and using 1.1.19,

$$E^i(M_P) = \left(E^i(M)\right)_P = \bigoplus_{Q\in(R)} \left(E(R/Q)\right)_P{}^{\mu_i(M,Q)} = \bigoplus_{Q\in \operatorname{Spec}(R)} \left(E_{R_P}\left((R/Q)_P\right)\right)^{\mu_i(M,Q)}.$$

If $Q \nsubseteq P$, then $(R/Q)_P = 0$. Otherwise, $(R/Q)_P = R_P/Q\,R_P$. Thus,

$$E^i(M_P) = \bigoplus_{\substack{Q\subseteq P\\ Q\in \operatorname{Spec}(R)}} \left(E_{R_P}\left(R_P/Q\,R_P\right)\right)^{\mu_i(M,Q)},$$

and therefore

$$\operatorname{Ext}^i_{R_P}\left(\frac{R_P}{PR_P}, E^i(M_P)\right) \cong \operatorname{Hom}_{R_P}\left(\frac{R_P}{PR_P}, \bigoplus_{\substack{Q\subseteq P\\ Q\in \operatorname{Spec}(R)}} \left(E_{R_P}\left(\frac{R_P}{Q\,R_P}\right)\right)^{\mu_i(M,Q)}\right).$$

Using 0.1.12, we get

$$\operatorname{Ext}^i_{R_P}\left(\frac{R_P}{PR_P}, E^i(M_P)\right) \cong \bigoplus_{\substack{Q\subseteq P\\ Q\in \operatorname{Spec}(R)}} \left[\operatorname{Hom}_{R_P}\left(\frac{R_P}{PR_P}, E_{R_P}\left(\frac{R_P}{Q\,R_P}\right)\right)\right]^{\mu_i(M,Q)}.$$

By 1.2.7, considering that (R_P, PR_P) is a Noetherian local ring, $\operatorname{Hom}_{R_P}\left(\frac{R_P}{PR_P}, E_{R_P}\left(\frac{R_P}{QR_P}\right)\right) = 0$ for any $Q \subsetneq P$. By 1.2.6,

$$\operatorname{Hom}_{R_P}\left(\frac{R_P}{PR_P}, E_{R_P}\left(\frac{R_P}{P\,R_P}\right)\right) \cong \frac{R_P}{PR_P}.$$

We now conclude that

$$\operatorname{Ext}^i_{R_P}\left(\frac{R_P}{PR_P}, E^i(M_P)\right) \cong \left(\frac{R_P}{PR_P}\right)^{\mu_i(M,P)},$$

so that $\operatorname{Ext}^i_{R_P}\left(\frac{R_P}{PR_P}, E^i(M_P)\right)$ is a vector space over the field $\frac{R_P}{PR_P}$ and

$$\mu_i(M,P) = \dim_{R_P} \operatorname{Ext}^i_{R_P}(R_P/PM_P, M_P).$$

By the same reasoning applied in 1.4.15, the structures of $\operatorname{Ext}^i_{R_P}(R_P/PM_P, M_P)$ as a module over R_P and as a module over $\frac{R_P}{PR_P}$ coincide, and since it is a finitely generated R_P-module, by 0.1.5 applied to the finitely generated R-module M and 0.10.5, then it is also a finite dimensional vector space over $\frac{R_P}{PR_P}$, so that $\mu_i(M,P)$ is finite. □

We can also characterize Gorenstein rings via the Bass numbers of R:

Proposition 1.6.4. *A Noetherian ring R with $d := \dim(R) < \infty$ is Gorenstein if and only if for all $i \geqslant 0$ and all prime ideals $\mathfrak{p}$ in R,*

$$\mu_i(R,\mathfrak{p}) = \begin{cases} 1, & \textit{if ht}(\mathfrak{p}) = i \\ 0, & \textit{if ht}(\mathfrak{p}) \neq i \end{cases}$$

Proof. Notice that given 1.6.3, all we have to show to prove $(\Rightarrow)$ is that for each prime ideal P,

$$\dim_{R_P} \operatorname{Ext}^i_{R_P}(R_P/PM_P, M_P) = \begin{cases} 1, & \text{if ht}(\mathfrak{p}) = i \\ 0, & \text{if ht}(\mathfrak{p}) \neq i \end{cases}$$

It is enough to show that the stament holds for the case when $(R,\mathfrak{m})$ is a Noetherian local ring. For $(\Leftarrow)$, since ht$(\mathfrak{m}) = \dim(R) = d$, then

$$\mu_d(R,\mathfrak{p}) = \dim_{R/\mathfrak{m}}\left(\operatorname{Ext}^d_R(R/\mathfrak{m}, R)\right) = 1,$$

which by 1.4.16 implies that R is a Gorenstein local ring.

Now assume that R is a Gorenstein local ring, and let P be any prime ideal. By 1.4.5, R_P is a Gorenstein local ring. Since ht$(P) = \dim(R_P)$, then by 1.4.16

$$\dim_{R_P}\left(\operatorname{Ext}^{\text{ht}(P)}_{R_P}(R_P/PM_P, M_P)\right) = \operatorname{type}(R_P) = 1.$$

To show that the remaining bass numbers are 0, see [20, Theorem 18.8]. □

We can now give a more complete description of Gorestein rings:

Theorem 1.6.5. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $\dim(R) = d$. The following conditions are equivalent:*

(1) *inj dim*$(R) < \infty$.

(2) *inj dim*$(R) = d$.

(3) R *is Cohen-Macaulay and* $\operatorname{Ext}^d_R(R/\mathfrak{m}, R) \cong R/\mathfrak{m}$.

(4) R *is Cohen-Macaulay and* $\operatorname{type}(R) = 1$.

(5) *For all* $i \geqslant 0$ *and all prime ideals* $\mathfrak{p}$ *in* R*,*

$$\mu_i(R, \mathfrak{p}) = \begin{cases} 1, & \textit{if } \textit{ht}(\mathfrak{p}) = i \\ 0, & \textit{if } \textit{ht}(\mathfrak{p}) \neq i \end{cases} .$$

Chapter 2

Local Cohomology

Introduction

In the words of M. P. Brodmann and R. Y. Sharp ([28]), local cohomology is "an algebraic child of geometric parents". Indeed, local cohomology was first defined by Grothendieck in the context of algebraic geometry.

Fixing a ring R and an ideal I, we associate to each module M the submodule of elements annihilated by some power of I. This operation is not exact, and the local cohomology modules measure its failure to be exact. Using local cohomology, we can measure dimension and depth.

In this chapter we study local cohomology and define the a_i-invariants and the Castelnuovo-Mumford regularity. Section 2.1 deals with the general theory of local cohomology, while in section 2.2 we focus on the case of graded rings and graded modules.

2.1 Local Cohomology

We will start with two constructions for the local cohomology modules of an R-module M with respect to an ideal I in R, and prove the necessary lemmas to show that the two constructions describe the same modules. We will show some basic properties of these objects, including their behavior with localization and direct sums. We will then focus on Noetherian local rings and finitely generated modules and show that under such conditions, the local cohomology modules with respect to the maximal ideal are Artinian. We will also see that local cohomology can be used as a tool to measure depth and dimension. Indeed, almost all local cohomology modules vanish, except those in degrees between depth (with respect to the ideal considered) and dimension.

Let R be a Noetherian ring, $I \subseteq R$ an ideal in R and let $\mathcal{M}$ denote the category of R-modules. We will study the covariant functor

$$
\begin{array}{rccl}
\Gamma_I\colon & \mathcal{M} & \longrightarrow & \mathcal{M} \\
& M & \longmapsto & \Gamma_I(M) = \{e \in M \mid \exists\, s \geqslant 1 : I^s e = 0\}
\end{array}
$$

Proposition 2.1.1. *For every ring R and every ideal I, Γ_I is a left exact functor.*

Proof. For a homomorphism of R-modules $\varphi\colon M \longrightarrow N$, $\Gamma_I(M)$ and $\Gamma_I(N)$ are submodules of M and N, and so $\Gamma_I(\varphi)$ is simply given by restriction, that is, $\Gamma_I(\varphi) = \varphi|_{\Gamma_I(M)}\colon \Gamma_I(M) \longrightarrow \Gamma_I(N)$. To see this makes sense, all we have to check is that for every $e \in \Gamma_I(M)$, $\varphi(e) \in \Gamma_I(N)$. Indeed, there must be some s with $I^s e = 0$, and thus

$$
I^s \varphi(e) = \varphi(I^s e) = \varphi(0) = 0
$$

so that $\varphi(e) \in \Gamma_I(N)$.

Therefore, Γ_I is a subfunctor of the identity functor, since it assigns to each R-module an R-submodule of it and to each homomorphism a restriction to the corresponding submodules.

Now let us check that this functor is indeed left exact. Consider an exact sequence of R-modules and R-homomorphisms

$$
0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P\ .
$$

We need to see that

$$
0 \longrightarrow \Gamma_I(M) \xrightarrow{\varphi|_{\Gamma_I(M)}} \Gamma_I(N) \xrightarrow{\psi|_{\Gamma_I(N)}} \Gamma_I(P)
$$

is also exact.

Being a subfunctor of the identity functor, Γ_I preserves inclusions, and this guarantees exactness at the beginning. Since $\Gamma_I(\varphi)$ and $\Gamma_I(\psi)$ are just restrictions of φ and ψ, and $\mathrm{im}(\varphi) = \ker(\psi)$, then

$$
\mathrm{im}\left(\Gamma_I(\varphi)\right) \subseteq \mathrm{im}(\varphi) \cap \Gamma_I(N) = \ker(\psi) \cap \Gamma_I(N) = \ker\left(\Gamma_I(\psi)\right).
$$

Let $e \in \Gamma_I(N)$ be an element in $\ker\left(\Gamma_I(\psi)\right)$, which means that $\psi(e) = 0$. By exactness of the original sequence, there exists some $e' \in M$ with $\varphi(e') = e$. Pick s such that $I^s\, e = 0$, which exists because $e \in \Gamma_I(N)$. Then $\varphi(I^s e') = I^s \varphi(e') = I^s e = 0$, and so $I^s e' \subseteq \ker\, \varphi = 0$. Therefore, $I^s\, e' = 0$, meaning that $e' \in \Gamma_I(M)$, and so $e \in \mathrm{im}\left(\Gamma_I(\varphi)\right)$. Then Γ_I is a left exact functor. □

Proposition 2.1.2. *Let R be a ring, I an ideal in R and M an R-module. Then*

$$
\Gamma_I(M) = \varinjlim_n \mathrm{Hom}(R/I^n, M) = \varinjlim_n (0 :_M I^n).
$$

Proof. Notice that to define an element in $\mathrm{Hom}(R/I^n, M)$, we just need to know what the image of $1 + I^n$ is, and that must be an element of M that is annihilated by a power of I (in this case, the n-th power), which is exactly $(0 :_M I^n)$.

For a directed system $\{M_i, f_{i,j}\}$ of R-modules M_i and R-homomorphisms $f_{i,j}\colon M_i \longrightarrow M_j$, the direct limit $\varinjlim_i M_i$ is just the quotient of the disjoint sum $\bigoplus_i M_i$ by the identification $x \equiv y$ whenever $x \in M_i$, $y \in M_j$, and there exists $k \geqslant i,j$ with $f_{k,i}(x) = f_{k,j}(y)$. So in our specific case, we have $M_i = \mathrm{Hom}(R/I, M) = (0 :_M I)$, and for $i \leqslant j$, $f_{j,i} : M_i \hookrightarrow M_j$ is the inclusion, and so the direct limit is the union, which is indeed the set of elements of M that are annihilated by some power of I. $\square$

Lemma 2.1.3. *Let I be an ideal in a ring R, and let $\{A_j\}_{j\in J}$ be a family of R-modules. Then,*

$$\Gamma_I\left(\bigoplus_{j\in J} A_j\right) = \bigoplus_{j\in J}\Gamma_I(A_j).$$

Proof.

($\subseteq$) If $(a_j)_{j\in J} \in \Gamma_I(\bigoplus_{j\in J} A_j)$, then there exists some $s \geqslant 1$ such that

$$I^s\left((a_j)_{j\in J}\right) = 0 \Leftrightarrow \underset{j\in J}{\forall} I^s a_j = 0 \Leftrightarrow \underset{j\in J}{\forall} a_j \in \Gamma_I(A_j) \Leftrightarrow (a_j)_{j\in J} \in \underset{j\in J}{\oplus}\Gamma_I(A_j).$$

($\supseteq$) Consider $(a_j)_{j\in J} \in \bigoplus_{j\in J}\Gamma_I(A_j)$, and let

$$\{j_1, \dots, j_n\} = \{j \in J \,|\, a_j \neq 0\}.$$

For each $1 \leqslant i \leqslant n$, there exists $s_i \geqslant 1$ such that $I^{s_i} a_{j_i} = 0$. Let $s = \max\{s_1, \dots, s_n\}$. Clearly, $I^s a_j = 0$ for every $j \in J$. Therefore,

$$I^s\left((a_j)_{j\in J}\right) = 0,$$

and thus $(a_j)_{j\in J} \in \Gamma_I(\bigoplus_{j\in J} A_j)$. $\square$

Definition 2.1.4. Fix an ideal I in R. Let M be an R-module and consider an injective resolution of M

$$0 \longrightarrow M \xrightarrow{\alpha_0} E^0 \xrightarrow{\alpha_1} E^1 \xrightarrow{\alpha_2} \cdots$$

Applying Γ_I to the complex

$$0 \longrightarrow E^0 \xrightarrow{\alpha_1} E^1 \xrightarrow{\alpha_2} E^2 \xrightarrow{\alpha_3} \cdots$$

yields a cochain complex

$$0 \longrightarrow \Gamma_I(E^0) \xrightarrow{G_1} \Gamma_I(E^1) \xrightarrow{G_2} \cdots$$

the cohomology of which we denote by $\mathrm{H}^i_I(M)$. We call the module $\mathrm{H}^i_I(M)$ the i-th *cohomology module* of M.

Remark 2.1.5. It is clear that $\mathrm{H}^i_I(M)$ is the i-th right derived functor of Γ_I applied to M. As it is always the case for right derived functors, the left exactness of Γ_I implies that there exists a natural isomorphism $\mathrm{H}^0_I(M) = \ker(G_1) = \Gamma_I(M)$.

Remark 2.1.6. Since $\varinjlim_n$ is exact, 2.1.2 shows that $\mathrm{H}^i_I(M) = \varinjlim_n \mathrm{Ext}^i(R/I^n, M)$.

Lemma 2.1.7. *Let R be a Noetherian ring and M a finitely generated R-module which has a unique minimal associated prime ideal, $\mathfrak{p}$. Then there exists some $s \geqslant 1$ such that $\mathfrak{p}^s M = 0$.*

Proof. Using 0.7.10, 0.7.12 and 0.4.3, we get

$$\sqrt{\operatorname{Ann}(M)} = \bigcap_{\substack{\mathfrak{q} \supseteq \operatorname{Ann}(M) \\ \mathfrak{q} \text{ prime}}} \mathfrak{q} = \bigcap_{\mathfrak{q} \in V(\operatorname{Ann}(M))} \mathfrak{q} = \bigcap_{\mathfrak{q} \in \operatorname{Supp}(M)} \mathfrak{q} = \bigcap_{\substack{\mathfrak{q} \text{ minimal} \\ \text{in } \operatorname{Supp}(M)}} \mathfrak{q} = \bigcap_{\substack{\mathfrak{q} \text{ minimal} \\ \text{in } \operatorname{Ass}(M)}} \mathfrak{q} = \mathfrak{p}.$$

By 0.4.4, there exists $s \geqslant 1$ such that $\mathfrak{p}^s \subseteq \operatorname{Ann}_R(M)$. Then $\mathfrak{p}^s M = 0$. □

Theorem 2.1.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I any ideal in R and $\mathfrak{p}$ a prime ideal. Then*

$$\Gamma_I(E(R/\mathfrak{p})) = \begin{cases} 0, & \text{if } I \nsubseteq \mathfrak{p} \\ E(R/\mathfrak{p}), & \text{if } I \subseteq \mathfrak{p} \end{cases}$$

Proof.

(1) $I \nsubseteq \mathfrak{p}$

Let $e \in \Gamma_I(E(R/\mathfrak{p}))$. We want to show that $e = 0$. By definition, $e \in E(R/\mathfrak{p})$, and there exists some $s \geqslant 1$ with $I^s e = 0$. Since $I \nsubseteq \mathfrak{p}$, there must be some $a \in I$ with $a \notin \mathfrak{p}$, so that $\frac{1}{a}$ is an invertible element in $R_{\mathfrak{p}}$. By 1.5.1, $E(R/\mathfrak{p})$ is an $R_{\mathfrak{p}}$-module, and so

$$I^s e = 0 \Rightarrow a^s e = 0 \Rightarrow \left(\frac{a}{1}\right)^s e = 0 \Rightarrow \left(\frac{1}{a}\right)^s \left(\frac{a}{1}\right)^s e = 0 \Rightarrow e = 0.$$

(2) $I \subseteq \mathfrak{p}$

By 1.1.22, $\operatorname{Ass}(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$. In particular, $\mathfrak{p}$ is the only minimal prime in $\operatorname{Ass}(E(R/\mathfrak{p}))$.

We will show that for every non-zero $x \in E(R/\mathfrak{p})$, there exists $s \geqslant 1$ with $\mathfrak{p}^s x = 0$. Consider $J = \operatorname{Ann}_R(x)$, which we know to be a proper ideal in R because $x \neq 0$. We know that $Rx \cong R/J$, and by 0.7.7, $\operatorname{Ass}(R/J) \neq \emptyset$. But. by 0.7.5,

$$\operatorname{Ass}(R/J) = \operatorname{Ass}(Rx) \subseteq \operatorname{Ass}(E(R/\mathfrak{p})) = \{\mathfrak{p}\},$$

and thus $\sqrt{J} = \mathfrak{p}$, by 0.4.3. Thus, we can find $s \geqslant 1$ with $\mathfrak{p}^s \subseteq J$, by 0.4.4, given that $\mathfrak{p}$ is finitely generated, and therefore $\mathfrak{p}^s x = 0$.

By assumption, $I \subseteq \mathfrak{p}$, and so $I^s x = 0$ and $x \in \Gamma_I(E(R/\mathfrak{p}))$. This guarantees that

$$E(R/\mathfrak{p}) \subseteq \Gamma_I(E(R/\mathfrak{p})).$$

By definition, $\Gamma_I(E(R/\mathfrak{p}))$ is always a submodule of $E(R/\mathfrak{p})$, and therefore we must have

$$E(R/\mathfrak{p}) = \Gamma_I(E(R/\mathfrak{p})).$$

□

Remark 2.1.9. As $\mathrm{H}^i_I(-)$ is the i-th right derived functor of Γ_I, we know that $\mathrm{H}^i(E) = 0$ for any $i \geqslant 1$ whenever E is an injective R-module.

Theorem 2.1.10. *Let $(R, \mathfrak{m})$ be an Artinian local ring, M an R-module and I any ideal in R. For every $i \geqslant 0$,*

$$\Gamma_I(\mathrm{H}^i_I(M)) = \mathrm{H}^i_I(M).$$

Proof. Consider an injective resolution of M,

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots .$$

By 1.1.27, each E^i is a direct sum of of indecomposable injective modules $E(R/\mathfrak{p})$, so for each i we can write $E^i = E^i_1 \oplus E^i_2$, where E^i_1 is a direct sum of terms of the form $E(R/\mathfrak{p})$ with $I \not\subseteq \mathfrak{p}$, and E^i_2 is a direct sum of terms of the form $E(R/\mathfrak{p})$ with $I \subseteq \mathfrak{p}$. By 2.1.3 and 2.1.8, we have

$$\Gamma_I(E^i) = \Gamma_I(E^i_1) \oplus \Gamma_I(E^i_2) = E^i_2.$$

Therefore, H^i_I is the i-th cohomology module of the complex

$$\cdots \longrightarrow E^{i-1}_2 \xrightarrow{\varphi} E^i_2 \xrightarrow{\psi} E^{i+1}_2 \longrightarrow \cdots ,$$

so that $\mathrm{H}^i_I(M) = \frac{\ker(\psi)}{\mathrm{im}(\varphi)}$. We want to show that $\Gamma_I(\mathrm{H}^i_I(M)) = \mathrm{H}^i_I(M)$. In other words, we want to show that every element in $\mathrm{H}^i_I(M)$ is annihilated by some power of I. Indeed, let $x + \mathrm{im}(\varphi) \in \mathrm{H}^i_I(M)$. Then $x \in E^i_2 = \Gamma_I(E^i_2)$ and thus $I^s x = 0$ for some $s \geqslant 1$. Then

$$I^s\left(x + \mathrm{im}(\varphi)\right) = I^s x + \mathrm{im}(\varphi) = 0.$$

□

Proposition 2.1.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. For every $i \geqslant 0$, $\mathrm{H}^i_{\mathfrak{m}}(M)$ are Artinian modules.*

Proof. Consider a minimal injective resolution of M,

$$0 \longrightarrow M \longrightarrow E^0(M) \longrightarrow E^1(M) \longrightarrow \cdots .$$

Notice that since $\mathfrak{m}$ is the maximal ideal of R, $\mathfrak{m} \subseteq \mathfrak{p} \Rightarrow \mathfrak{m} = \mathfrak{p}$, and therefore using 1.1.27, 2.1.3 and 2.1.8 we get

$$\Gamma_{\mathfrak{m}}\left(E^i(M)\right) = \Gamma_{\mathfrak{m}}\left(E(R/\mathfrak{m})^{\mu_i(M,\mathfrak{m})}\right) = \left(\Gamma_{\mathfrak{m}}(E(R/\mathfrak{m}))\right)^{\mu_i(M,\mathfrak{m})} = \left(E(R/\mathfrak{m})\right)^{\mu_i(M,\mathfrak{m})}.$$

By 1.6.3, $\mu_i(M)$ are all finite, so that, by 0.3.3 and 2.1.11, $E(R/\mathfrak{m})^{\mu_i(M,\mathfrak{m})}$ are all Artinian modules.

Therefore, in order to compute $\mathrm{H}^i_{\mathfrak{m}}(M)$ we just need to compute the cohomology of the complex

$$E(R/\mathfrak{m})^{\mu_0(M,\mathfrak{m})} \longrightarrow E(R/\mathfrak{m})^{\mu_1(M,\mathfrak{m})} \longrightarrow E(R/\mathfrak{m})^{\mu_2(M,\mathfrak{m})} \longrightarrow \cdots,$$

and this is a complex of Artinian modules, so its cohomology modules must also be Artinian, by 0.3.3. □

Theorem 2.1.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I any ideal in R and M a finitely generated R-module. Then*

$$\textit{grade}(I, M) = \inf \left\{i : \mathrm{H}^i_I(M) \neq 0\right\}.$$

Proof. We will use induction on

$$r = \inf \left\{i : \mathrm{H}^i_I(M) \neq 0\right\}.$$

(i) $r = 0$

Since $\mathrm{H}^0_I(M) = \Gamma_I(M)$, by 2.1.5, $r = 0$ means that $\Gamma_I(M) \neq 0$ and thus we can find $0 \neq m \in M$ such that $I^s m = 0$ for some $s \geqslant 1$. Choose the smallest such s, so that $I^{s-1} m \neq 0$. We can pick $a \in I^{s-1}$ with $Iu = 0$ for $u := am \neq 0$. This implies that we cannot find M-regular elements in I, and therefore grade$(I, M) = 0 = r$, as desired.

(ii) $r > 0$

First of all, notice that $r > 0$ implies that $\mathrm{H}^0_I(M) = \Gamma_I(M) = 0$, so that for all $n \geqslant 1$ and all non-zero $m \in M$, $I^n m \neq 0$, and in particular $Im \neq 0$. Using this, we will show that grade$(I, M) > 0$. Suppose that grade$(I, M) = 0$. Then every element in I is a zero divisor of M, and so I is contained in the union of the associated primes of M, by 0.7.7. By 0.1.2, this implies that $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Let $x \in M$ be such that $\mathfrak{p} = \mathrm{Ann}_R(x)$. Then $Ix \subseteq \mathfrak{p}x = 0$, which is a contradiction.

We now know that grade$(I, M) > 0$. Pick an element $x \in I$ that is regular in M, and consider $\overline{M} = M/xM$. From the short exact sequence

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow \overline{M} \longrightarrow 0,$$

by 0.10.3, we get a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}_I(M) \longrightarrow \mathrm{H}^{i-1}_I(\overline{M}) \longrightarrow \mathrm{H}^i_I(M) \longrightarrow \mathrm{H}^i_I(M) \longrightarrow \cdots.$$

By definition of r, $\mathrm{H}^{i-1}_I(M) = 0$ and $\mathrm{H}^i_I(M) = 0$ for $i \leqslant r-1$, and thus $\mathrm{H}^{i-1}_I(\overline{M}) = 0$ for $i \leqslant r-1$. Using that $\mathrm{H}^r_I(M) \neq 0$ and the exact sequence

$$0 \longrightarrow \mathrm{H}^{r-1}_I(\overline{M}) \xrightarrow{\varphi} \mathrm{H}^r_I(M) \xrightarrow{\psi} \mathrm{H}^r_I(M),$$

we can also show that $\mathrm{H}^{r-1}_I(\overline{M}) \neq 0$. Indeed, suppose $\mathrm{H}^{r-1}_I(\overline{M}) = 0$. Then ψ, which is given by multiplication by x, is injective. By 2.1.10, $\Gamma_I(\mathrm{H}^r_I(M)) = \mathrm{H}^r_I(M)$, so that every element in $\mathrm{H}^r_I(M)$

is annihilated by some power of I. In particular, if we pick a non-zero element $a \in \mathrm{H}^r_I(M)$, which exists by definition of r, $x^s a = 0$ for some $s \geqslant 1$, since $x \in I$, and choosing minimal such s we get $x\,(x^{s-1}a) = 0$, $x^{s-1}a \neq 0$, which is a contradiction to multiplication by x being injective. Then $\mathrm{H}^{r-1}_I(\overline{M}) \neq 0$. This shows that

$$r - 1 = \inf\left\{i : \mathrm{H}^i_I(\overline{M}) \neq 0\right\}.$$

By the induction hypothesis, $r-1 = \text{grade}(I, \overline{M})$. Since $\text{grade}(I, M) = \text{grade}(I, \overline{M}) + 1$, we get the desired equality,

$$r = \text{grade}(I, M).$$

□

Definition 2.1.13 (Čech Complex)**.** Let R be a Noetherian ring. For each $x \in R$, denote by R_x and M_x the localizations by the multiplicative set of powers of x. Consider $x_1, \ldots, x_n \in R$, $\underline{x} := x_1, \ldots, x_n$ and the complexes

$$\begin{array}{c} \check{\mathsf{C}}(x_i)\colon\ 0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0 \\ r \longmapsto \frac{r}{1} \end{array}$$

The *Čech complex* is the chain complex[1]

$$\check{\mathsf{C}}(\underline{x}) := \check{\mathsf{C}}(x_1, \ldots, x_n) = \left(\bigotimes_{i=1}^{n} (\ 0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0\ )\right) = \bigotimes_{i=1}^{n} \check{\mathsf{C}}(x_i),$$

that is,

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j}^{n} R_{x_i x_j} \longrightarrow \ldots \longrightarrow R_{x_1 \ldots x_n} \longrightarrow 0\ .$$

For an R-module M, $\check{\mathsf{C}}(\underline{x}, M) := \check{\mathsf{C}}(\underline{x}) \otimes M$. We will write

$$\mathrm{H}^i_{\underline{x}}(M) := \mathrm{H}^i(\check{\mathsf{C}}(\underline{x}, M)).$$

Remark 2.1.14. There is another good way of thinking about the Čech complex $\check{\mathsf{C}}(x)$. Consider the direct limit of the following horizontal Koszul Complexes:

$$\begin{tikzcd}
K_1^\bullet \arrow[d] & 0 \arrow[r] & R \arrow[r, ".x"] \arrow[d, equal] & R \arrow[r] \arrow[d, ".x"] & 0 \\
K_2^\bullet \arrow[d] & 0 \arrow[r] & R \arrow[r, ".x^2"] \arrow[d, equal] & R \arrow[r] \arrow[d, ".x"] & 0 \\
K_3^\bullet \arrow[d] & 0 \arrow[r] & R \arrow[r, ".x^3"] & R \arrow[r] & 0 \\
\vdots & & \vdots & \vdots &
\end{tikzcd}$$

[1] For a definition of tensor product of complexes, see [23, pp. 614-615].

The direct limit of this direct system of complexes is the complex

$$\check{\mathrm{C}}(\underline{x}) = \begin{tikzcd} 0 \arrow[r] & R \arrow[r] & R_x \arrow[r] & 0 \end{tikzcd}.$$

$$r \longmapsto \tfrac{r}{1}$$

Now denoting by

$$K_t^\bullet(x_1,\dots,x_n) = \bigotimes_{i=1}^n \Big(0 \longrightarrow R \xrightarrow{x_i^t} R \longrightarrow 0 \Big) = \bigotimes_{i=1}^n K_t^\bullet(x_i),$$

and since the direct limit commutes with tensor product, by 0.10.10, we have that

$$\varinjlim K_t^\bullet(x_1,\dots,x_n) = \varinjlim \left[\bigotimes_{i=1}^n K_t^\bullet(x_i)\right] = \bigotimes_{i=1}^n \left[\varinjlim K_t^\bullet(x_i)\right] = \bigotimes_{i=1}^n \check{\mathrm{C}}(x_i) = \check{\mathrm{C}}(x_1,\dots,x_n).$$

Lemma 2.1.15. *Let R be a ring, M a finitely generated R-module and $x_1,\dots,x_n \in R$ a regular sequence on M. Write $\underline{x} = x_1,\dots,x_n$. Then for every $i < n$,*

$$\mathrm{H}^i_{\underline{x}}(M) = 0$$

and

$$\mathrm{H}^n_{\underline{x}}(M) = M/(x_1,\dots,x_n)M \neq 0.$$

Proof. See [4, Corollary 17.5]. □

Lemma 2.1.16. *Let R be a Noetherian ring, M an R-module, $I = (x_1,\dots,x_n)$ an ideal in R and write $\underline{x} = x_1,\dots,x_n$. There is a natural isomorphism*

$$\mathrm{H}^0_I(M) \cong \mathrm{H}^0(\check{C}(\underline{x}, M)).$$

Proof. Notice that $\mathrm{H}^0(\check{\mathrm{C}}(\underline{x}, M))$ is the kernel of the map φ as follows:

$$M \xrightarrow{\varphi} M_{x_1} \oplus \cdots \oplus M_{x_n}.$$

$$m \longmapsto \left(\tfrac{m}{1},\dots,\tfrac{m}{1}\right)$$

Let us show that $\ker(\varphi) \subseteq \Gamma_I(M) = \mathrm{H}^0_I(M)$. Indeed, if $\varphi(m) = 0$, then $\frac{m}{1} = 0$ in R_{x_i} for each $i = 1,\dots,n$, so that ${x_i}^{s_i} m = 0$ for some $s_i \geqslant 1$. Choosing s sufficiently large, $I^s m = 0$, so that $m \in \Gamma_I(M)$.

On the other hand, if $m \in \Gamma_I(M)$, then $I^s m = 0$ for some $s \geqslant 1$, and thus $x_i^s m = 0$ for $i = 1,\dots,n$. Therefore, $m \in \ker(\varphi)$. □

Definition 2.1.17 (Nilpotent elements). If R is a ring and M is an R-module, we say that $r \in R$ is *nilpotent* in M if for every $m \in M$ there exists $n \geqslant 1$ such that $r^n m = 0$.

Remark 2.1.18. Let R be a ring, $x_1, \ldots, x_n \in R$ be nilpotent on M and $I = (x_1, \ldots, x_n)$. Every element of M is annihilated by a power of I, and thus $\Gamma_I(M) = M$.

Lemma 2.1.19. *Let R be a Noetherian ring, M an R-module and $x_1, \ldots, x_n \in R$ nilpotent on M. Write $\underline{x} = x_1, \ldots, x_n$, and let $I = (x_1, \ldots, x_n)$. Then for all $i \geqslant 1$,*

$$\mathrm{H}^i_{\underline{x}}(M) = 0.$$

Proof. We want to compute the cohomology modules of the complex

$$0 \longrightarrow M \longrightarrow M_{x_1} \oplus \ldots \oplus M_{x_n} \longrightarrow \bigoplus_{\substack{i,j=1 \\ i<j}}^{n} M_{x_i x_j} \longrightarrow \ldots \longrightarrow M_{x_1 \ldots x_n} \longrightarrow 0$$

As every element in M is annihilated by a power of x_i, then $M_{x_i} = 0$. Indeed, given $m \in M$, choose $k \geqslant 1$ such that ${x_i}^k m = 0$. Then

$$\frac{m}{{x_i}^a} = \frac{{x_i}^k}{{x_i}^k} \frac{m}{{x_i}^a} = \frac{0}{{x_i}^{k+a}} = 0.$$

So we conclude that $M_{x_1} \oplus \ldots \oplus M_{x_n} = 0$.

Similarly, we can show that all the other modules in the complex but M are 0. Thus, we are actually computing the cohomology of the complex

$$0 \longrightarrow M \longrightarrow 0 \ ,$$

so that

$$\mathrm{H}^i_{\underline{x}}(M) = \begin{cases} M, & \text{if } i = 0 \\ 0, & \text{if } i \geqslant 1 \end{cases}$$

as desired. $\square$

Lemma 2.1.20. *Let R be a Noetherian ring, $x_1, \ldots, x_n \in R$, M an R-module and $\underline{x} = x_1, \ldots, x_n$. If there exists i such that $M \xrightarrow{.x_i} M$ is an isomorphism, then for all $i \geqslant 0$,*

$$\mathrm{H}^i_{\underline{x}}(M) = 0.$$

Proof. Without loss of generality, we can assume that $M \xrightarrow{.x_1} M$ is an isomorphism.

Consider the polynomial ring $S := R[T_1, \ldots, T_n]$. The sequence $\underline{T} = T_1, \ldots, T_n$ is a maximal regular sequence in S. By 2.1.15,

$$\mathrm{H}^i_{\underline{T}}(R) = 0$$

for $i < n$, and thus the Čech complex in $\underline{T}$ is exact except possibly in degree n, that is, on $S_{T_1 \cdots T_n}$. To make this complex exact, we can add coker(α), where α is the map in the complex going from degree

$n-1$ to degree n. So we are now considering the following exact sequence:

$$\cdots \longrightarrow \bigoplus_{i=1}^{n} S_{T_1\cdots \hat{T}_i \cdots T_n} \xrightarrow{\alpha} S_{T_1\cdots T_n} \longrightarrow \text{coker}(\alpha) \longrightarrow 0\ ,$$

with

$$\begin{aligned} &\alpha\left(\frac{s_1}{(T_2\cdots T_n)^{a_1}}, \frac{s_2}{(T_1T_3\cdots T_n)^{a_2}}, \ldots, \frac{s_n}{(T_1\cdots T_{n-1})^{a_n}}\right) \\ &= \frac{s_1}{(T_2\cdots T_n)^{a_1}} - \frac{s_2}{(T_1T_3\cdots T_n)^{a_2}} + \ldots + (-1)^{n+1}\frac{s_n}{(T_1\cdots T_{n-1})^{a_n}} \\ &= \frac{s_1 {T_1}^{a_1}}{(T_1\cdots T_n)^{a_1}} - \frac{s_2 {T_2}^{a_2}}{(T_1\cdots T_n)^{a_2}} + \ldots + (-1)^{n+1}\frac{s_n {T_n}^{a_n}}{(T_1\cdots T_n)^{a_n}}. \end{aligned}$$

Consider $x = \frac{s}{(T_1\cdots T_n)^a} + \text{im}(\alpha) \in \text{coker}(\alpha) = \frac{S_{T_1\cdots T_n}}{\text{im}(\alpha)}$. We will show that x is annihilated by a power of T_1. Indeed,

$${T_1}^a\, x = \frac{{T_1}^a\, s}{(T_1\cdots T_n)^a} = \alpha\left(\frac{s}{(T_2\cdots T_n)^a}, 0, \ldots, 0\right) \in \text{im}(\alpha),$$

so that ${T_1}^a x = 0$. As x is a general element in coker(α), we just proved that every element in coker(α) is annihilated by a power of T_1. Therefore, $(\text{coker}(\alpha))_{T_1} = 0$.

We will now show that

$$\mathrm{H}^j_{\underline{x}}(M) \xrightarrow{.x_1} \mathrm{H}^j_{\underline{x}}(M)$$

is an isomorphism for every $j \geqslant 0$.

Pick $\{x_{i_1}, \ldots, x_{i_l}\} \subseteq \{x_1, \ldots, x_n\}$. We know that $R_{x_{i_1}\cdots x_{i_l}}$ is a flat R-module, by 0.1.8. We also know that

$$0 \longrightarrow M \xrightarrow{.x_1} M \longrightarrow 0$$

is exact and that $M \otimes R_{x_{i_1}\cdots x_{i_l}} \cong M_{x_{i_1}\cdots x_{i_l}}$, and thus

$$0 \longrightarrow M_{x_{i_1}\cdots x_{i_l}} \xrightarrow{.x_1} M_{x_{i_1}\cdots x_{i_l}} \longrightarrow 0$$

is exact. This means that the graded map induced by the multiplication by x_1 on the Čech complex is an isomorphism on each level.

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus\limits_{i=1}^{n} M_{x_i} \arrow[r, "\delta_1"] \arrow[d, ".x_1"] & \bigoplus\limits_{\substack{i,j=1\\ i<j}}^{n} M_{x_ix_j} \arrow[r, "\delta_2"] \arrow[d, ".x_1"] & \cdots \arrow[r] & M_{x_1\ldots x_n} \arrow[r, "\delta_n"] \arrow[d, ".x_1"] & 0 \\ 0 \arrow[r] & \bigoplus\limits_{i=1}^{n} M_{x_i} \arrow[r, "\delta_1"] & \bigoplus\limits_{\substack{i,j=1\\ i<j}}^{n} M_{x_ix_j} \arrow[r, "\delta_2"] & \cdots \arrow[r] & M_{x_1\ldots x_n} \arrow[r, "\delta_n"] & 0 \end{tikzcd}$$

This shows that

$$\mathrm{H}^j_{\underline{x}}(M) \xrightarrow{.x_1} \mathrm{H}^j_{\underline{x}}(M)$$

is an isomorphism for every $j \geqslant 0$, as cohomology is an additive functor.

Consider M as an S-module, with the following structure:

$$\begin{array}{lcl} R[T_1, \ldots, T_n] \times M & \longrightarrow & M \\ (T_i, a) & \longmapsto & T_i a := x_i a \end{array}$$

where each variable T_i corresponds to x_i. With this structure, it is clear that

$$\check{\mathsf{C}}(\underline{T}, M) = \check{\mathsf{C}}(\underline{x}, M)$$

and therefore

$$\mathrm{H}^i_{\underline{T}}(M) = \mathrm{H}^i(\check{\mathsf{C}}(\underline{T}, M)) = \mathrm{H}^i(\check{\mathsf{C}}(\underline{x}, M)) = \mathrm{H}^i_{\underline{x}}(M).$$

As multiplication by x_1 on $\mathrm{H}^i_{\underline{x}}(M)$ is an isomorphism, and T_1 corresponds to x_1, then multiplication by T_1 on $\mathrm{H}^i_{\underline{T}}(M)$ is also an isomorphism.

As multiplication by T_1 is an isomorphism on $\mathrm{H}^i_{\underline{T}}(M)$, we conclude that

$$\mathrm{H}^i_{\underline{T}}(M) = \left[\mathrm{H}^i_{\underline{T}}(M)\right]_{T_1}.$$

Now notice that

$$\cdots \longrightarrow \bigoplus_{i=1}^n S_{T_1 \cdots \hat{T}_i \cdots T_n} \xrightarrow{\alpha} S_{T_1 \cdots T_n} \longrightarrow \operatorname{coker} \alpha \longrightarrow 0$$

is a flat resolution of $\operatorname{coker} \alpha$. If we apply $- \otimes_S M$ and then take cohomology, the cohomology modules we get are $\operatorname{Tor}^S_{-}(\operatorname{coker} \alpha, M)$. On the other hand, they are also the modules $\mathrm{H}^i_{\underline{T}}(M)$, in such a way that

$$\mathrm{H}^i_{\underline{T}}(M) = \operatorname{Tor}^S_{n-i}(\operatorname{coker} \alpha, M).$$

for each $i \leqslant n$.

Therefore,

$$\mathrm{H}^i_{\underline{T}}(M) = \left[\mathrm{H}^i_{\underline{T}}(M)\right]_{T_1} = \left[\operatorname{Tor}^S_{n-i}(\operatorname{coker} \alpha, M)\right]_{T_1},$$

and by 0.10.4,

$$\left[\operatorname{Tor}^S_{n-i}(\operatorname{coker} \alpha, M)\right]_{T_1} \cong \operatorname{Tor}^{S_{T_1}}_{n-i}((\operatorname{coker} \alpha)_{T_1}, M_{T_1}) = \operatorname{Tor}^{S_{T_1}}_{n-i}(0, M_{T_1}) = 0.$$

Therefore,

$$\mathrm{H}^i_{\underline{x}}(M) = 0$$

for all $i \geqslant 0$, as desired. □

Notice that we also proved the following:

Lemma 2.1.21. *Let R be a Noetherian ring, M an R-module and $\underline{x} = x_1, \ldots, x_n$ elements in R. Consider* *the polynomial ring $T = R[X_1, \dots, X_n]$. Then there exists an S-module D such that*

$$\mathrm{H}^i_{\underline{x}}(M) \cong \mathrm{Tor}^T_{n-i}(D, M),$$

where we consider M as an S-module with the following structure:

$$\begin{array}{lll} R[X_1, \dots, X_n] \times M & \longrightarrow & M \\ (X_i, a) & \longmapsto & T_i a := x_i a \end{array}.$$

Lemma 2.1.22. *Let R be a Noetherian ring, $\mathfrak{p}$ a prime ideal in R and $x_1, \dots, x_n \in R$. Write $\underline{x} = x_1, \dots, x_n$ and $E = E(R/\mathfrak{p})$. Then for every $i \geqslant 1$,*

$$\mathrm{H}^i_{\underline{x}}(E) = 0.$$

Proof. First recall that $\{\mathfrak{p}\} = \mathrm{Ass}(E(R/\mathfrak{p}))$, by 1.1.22, so that $\mathfrak{p}$ coincides with the set of zero divisors in $E(R/\mathfrak{p})$. We will consider two cases:

(1) $x_1, \dots, x_n \in \mathfrak{p}$

Let $i \in \{1, \dots, n\}$. By 2.1.8, $\Gamma_{(x_i)}(E) = E$, so that every element in E is annihilated by a power of (x_i), and thus x_i is nilpotent on M. The statement follows by 2.1.19.

(2) $x_i \notin \mathfrak{p}$ for some i

Without loss of generality, assume $x_1 \notin \mathfrak{p}$. Then $E(R/\mathfrak{p}) \xrightarrow{.x_1} E(R/\mathfrak{p})$ is injective, because x_1 is not a zero divisor in $E(R/\mathfrak{p})$. Also, let C be its cokernel, and consider the short exact sequence

$$0 \longrightarrow E(R/\mathfrak{p}) \xrightarrow{x_1} E(R/\mathfrak{p}) \longrightarrow C \longrightarrow 0\ .$$

As $E(R/\mathfrak{p})$ is an injective module, then the sequence splits, so that $E(R/\mathfrak{p}) = E(R/\mathfrak{p}) \oplus C$. Considering that $E(R/\mathfrak{p})$ is an indecomposable injective module, by 1.1.25, we must have $C = 0$. Then the map $E(R/\mathfrak{p}) \xrightarrow{.x_1} E(R/\mathfrak{p})$ is an isomorphism. By 2.1.20, $\mathrm{H}^i_{\underline{x}}(M) = 0$, for every $i \geqslant 0$.

□

Corollary 2.1.23. *Let R be a Noetherian ring, E an injective R-module and $x_1, \dots, x_n \in R$. Write $\underline{x} = x_1, \dots, x_n$. Then for every $i \geqslant 1$,*

$$\mathrm{H}^i_{\underline{x}}(E) = 0.$$

Proof. By 1.1.27,

$$E \ = \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(E)} (E(R/\mathfrak{p}))^{\mu_{\mathfrak{p}}}\ .$$

Then

$$\mathrm{H}^i_{\underline{x}}(E) = \mathrm{H}^i(\check{\mathrm{C}}(\underline{x}) \otimes E) = \mathrm{H}^i\left(\check{\mathrm{C}}(\underline{x}) \otimes \left(\bigoplus_{\mathfrak{p}} (E(R/\mathfrak{p}))\right)^{\mu_{\mathfrak{p}}}\right) = \mathrm{H}^i\left(\left(\bigoplus_{\mathfrak{p}} \check{\mathrm{C}}(\underline{x}) \otimes (E(R/\mathfrak{p}))\right)^{\mu_{\mathfrak{p}}}\right),$$

and by 0.10.1 and 2.1.22, we get

$$\mathrm{H}^i_{\underline{x}}(E) = \bigoplus_{\mathfrak{p}} \left(\mathrm{H}^i\left(\check{\mathsf{C}}(\underline{x}) \otimes E(R/\mathfrak{p})\right)\right)^{\mu_{\mathfrak{p}}} = \bigoplus_{\mathfrak{p}} \left(\mathrm{H}^i_{\underline{x}}(E(R/\mathfrak{p})\right)^{\mu_{\mathfrak{p}}} = 0.$$

□

Corollary 2.1.24. *Let R be a Noetherian ring, M an R-module, $I = (x_1, \dots, x_n)$ and $\underline{x} = x_1, \dots, x_n$. For every $i \geqslant 0$, there is a natural isomorphism*

$$\mathrm{H}^i_I(M) = \mathrm{H}^i_{\underline{x}}(M).$$

Proof. The case $i = 0$ is 2.1.16. For $i \geqslant 1$,

$$\mathrm{H}^i_I(E) = 0 = \mathrm{H}^i_{\underline{x}}(E)$$

holds whenever E is injective. Indeed, $\mathrm{H}^i_{\underline{x}}(E) = 0$ by 2.1.23 and $\mathrm{H}^i_I(E) = 0$ by 2.1.9.

For $i = 1$, consider an injective R-module E and a monomorphism $M \hookrightarrow E$ with cokernel K, so that we have a short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow K \longrightarrow 0\ .$$

By 2.1.16, $\mathrm{H}^0_{\underline{x}} = \mathrm{H}^0_I(E)$ and $\mathrm{H}^0_{\underline{x}} = \mathrm{H}^0_I(K)$. By 0.10.2, we get long exact sequences

$$\mathrm{H}^0_{\underline{x}}(E) \xrightarrow{a} \mathrm{H}^0_{\underline{x}}(K) \longrightarrow \mathrm{H}^1_{\underline{x}}(M) \longrightarrow \mathrm{H}^1_{\underline{x}}(E) = 0$$

and

$$\mathrm{H}^0_I(E) \xrightarrow{a} \mathrm{H}^0_I(K) \longrightarrow \mathrm{H}^1_I(M) \longrightarrow \mathrm{H}^1_I(E) = 0\ .$$

As

$$\mathrm{H}^1_{\underline{x}}(M) = \mathsf{coker}(a) = \mathrm{H}^1_I(M),$$

then $\mathrm{H}^1_{\underline{x}}(M)$ and $\mathrm{H}^1_I(M)$ are naturally isomorphic.

Let $i \geqslant 1$. Assume that we have shown the result for every $j \leqslant i$ and every R-module. Consider an R-module M, an injective R-module E and a monomorphism $M \hookrightarrow E$ with cokernel K, so that we have a short exact sequence

$$0 \longrightarrow M \hookrightarrow E \longrightarrow K \longrightarrow 0\ .$$

By 0.10.2, we get long exact sequences

$$0 = \mathrm{H}^i_{\underline{x}}(E) \longrightarrow \mathrm{H}^i_{\underline{x}}(K) \longrightarrow \mathrm{H}^{i+1}_{\underline{x}}(M) \longrightarrow \mathrm{H}^{i+1}_{\underline{x}}(E) = 0$$

and

$$0 = \mathrm{H}^i_I(E) \longrightarrow \mathrm{H}^i_I(K) \longrightarrow \mathrm{H}^{i+1}_I(M) \longrightarrow \mathrm{H}^{i+1}_I(E) = 0\ .$$

By the exactness of the first complex, $\mathrm{H}^i_{\underline{x}}(K) \cong \mathrm{H}^{i+1}_{\underline{x}}(M)$. By the exactness of the second complex, $\mathrm{H}^i_I(K) \cong \mathrm{H}^{i+1}_I(M)$. By the induction hypothesis, $\mathrm{H}^i_I(K) \cong \mathrm{H}^i_{\underline{x}}(K)$. Then,

$$\mathrm{H}^{i+1}_I(M) \cong \mathrm{H}^{i+1}_{\underline{x}}(M).$$

Naturality follows by induction. $\square$

Corollary 2.1.25. *Let R be a Noetherian ring, $I = (x_1, \ldots, x_n)$ an ideal in R, and M an R-module. Then for every $i > n$,*

$$\mathrm{H}^i_I(M) = 0.$$

Proof. By 2.1.24, we are computing the cohomology of the Čech complex, and the complex is zero in degree above n. $\square$

Proposition 2.1.26. *Let R be a Noetherian ring, $I = (x_1, \ldots, x_n)$ be an ideal in R, and M an R-module. Then for all $i \geqslant 0$,*

$$\mathrm{H}^i_I(M) = \mathrm{H}^i_{\sqrt{I}}(M)$$

Proof. We will show that Γ_I and $\Gamma_{\sqrt{I}}$ are the same functor, so that their derived functors are the same.

Consider an R-module N. We will see that $\Gamma_I(N) = \Gamma_{\sqrt{I}}(N)$:

($\subseteq$) Let $n \in \Gamma_I(N)$, which means that $n \in N$ and there exists $t > 0$ with $I^t n = 0$. By 0.4.4, there exists $l \geqslant 1$ such that $\left(\sqrt{I}\right)^l \subseteq I$. Then $\left(\sqrt{I}\right)^{lt} n = 0$, so that $n \in \Gamma_{\sqrt{I}}(M)$.

($\supseteq$) Let $n \in \Gamma_{\sqrt{I}}(N)$, which means that $n \in N$ and there exists a $t > 0$ with $(\sqrt{I})^t n = 0$. As $I \subseteq \sqrt{I}$, then $I^t n = 0$, so that $n \in \Gamma_I(N)$. $\square$

Lemma 2.1.27. *Local cohomology commutes with localization, that is, if R is a Noetherian ring, M an R-module, I an ideal in R and S a multiplicative set on R, then*

$$\left[\mathrm{H}^i_I(M)\right]_S = \mathrm{H}^i_{I_S}(M_S).$$

Proof. To compute $\mathrm{H}^i_{I_S}(M_S)$, we take an injective resolution of M_S, apply Γ_{I_S} and take cohomology. As localization is flat, by 0.1.8, it commutes with taking cohomology, and an injective resolution of M_S can be obtained by localizing an injective resolution of M. Then, it is enough to show that

$$(\Gamma_I(M))_S = \Gamma_{I_S}(M_S).$$

($\subseteq$) is clear. Let $\frac{m}{s} \in \Gamma_{I_S}(M_S)$ and consider $t \geqslant 1$ such that $(I_S)^t \frac{m}{s} = 0$. As I^t is finitely generated, we can find $s' \in S$ such that $s' I^t m = 0$, so that $s' m \in \Gamma_I(M)$. Then $\frac{m}{s} \in (\Gamma_I(M))_S$. $\square$

Proposition 2.1.28. *Let R be a Noetherian ring and S a Noetherian R-algebra. Let I be an ideal in R and M an S-module. Then*

$$\mathrm{H}^i_I(M) \cong \mathrm{H}^i_{IS}(M)$$

where we consider M as an R-module on the left and as an S-module on the right.

Proof. Let $I = (x_1, \ldots, x_n)$ and write $\overline{x} = x_1, \ldots, x_n$. Then

$$\check{\mathsf{C}}(\overline{x}, M) = \check{\mathsf{C}}(\overline{x}, R) \otimes_R M = \check{\mathsf{C}}(\overline{x}, R) \otimes_R (S \otimes_S M) = \check{\mathsf{C}}(\overline{x}, S) \otimes_S M.$$

By 2.1.24,

$$\mathrm{H}^i_I(M) \cong \mathrm{H}^i_{\underline{x}}(M) \cong \mathrm{H}^i_{\underline{x}S}(M) \cong \mathrm{H}^i_{IS}(M).$$

$\square$

Definition 2.1.29 (ara)**.** Let I be an ideal in a Noetherian ring R. We set ara(I) to be the least integer n such that there exists an ideal J generated by n elements with $\sqrt{J} = \sqrt{I}$.

Lemma 2.1.30. *Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal of R. Then for every $i >$ ara(I),*

$$\mathrm{H}^i_I(M) = 0.$$

Proof. Let J be an ideal of R generated by ara(I) elements, say

$$J = (a_1, \ldots, a_{\text{ara}(I)})$$

with $\sqrt{J} = \sqrt{I}$. By 2.1.26,

$$\mathrm{H}^i_I(M) = \mathrm{H}^i_{\sqrt{I}}(M) = \mathrm{H}^i_{\sqrt{J}}(M) = \mathrm{H}^i_{\underline{a}}(M) = \mathrm{H}^i_J(M).$$

Now the result follows by 2.1.25. $\square$

Theorem 2.1.31. *Let R be a Noetherian ring, M a finitely generated R-module and $d := \dim(R)$. Then for every ideal I of R and every $i > d$,*

$$\mathrm{H}^i_I(M) = 0.$$

Proof. Let us start by reducing to the case where R is a local ring. Suppose that the statement is true for every Noetherian local ring. In particular, using 2.1.27, for every $i > d$ and every prime ideal $\mathfrak{p}$ of R,

$$\left[\mathrm{H}^i_I(M)\right]_{\mathfrak{p}} = \mathrm{H}^i_{I_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$$

as $\dim(R_{\mathfrak{p}}) \leqslant d$ and $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module. By 0.1.6, if $\left[\mathrm{H}^i_I(M)\right]_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$, then $\mathrm{H}^i_I(M) = 0$.

So let us prove that the statement holds for Noetherian local rings. Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim(R) = d$, and I an ideal in R. We will use induction on d to show that ara$(I) \leqslant d$, and by 2.1.30, this proves the claim.

(i) $d = 0$.

As $\mathfrak{m}$ is the only prime ideal in R, then $\sqrt{I} = \mathfrak{m}$, as $\sqrt{I}$ is the intersection of all primes containing I, by 0.4.3. In particular, $\sqrt{(0)} = \mathfrak{m} = \sqrt{I}$, and thus ara$(I) = 0 \leqslant d$.

(ii) $d > 0$

Suppose that for all Noetherian local rings R of dimension $d-1$ and every ideal I of R, we have ara$(I) \leqslant d-1$.

If I is contained in every minimal prime of R, then by 0.4.3

$$\sqrt{I} = \bigcap_{P \in \operatorname{Spec}(R)} P = \sqrt{(0)}$$

and thus ara$(I) = 0 \leqslant d$.

If I is not contained in any minimal prime, let $\{P_1, \ldots, P_p, Q_1, \ldots, Q_q\}$ be the set of minimal primes in R, which we know to be finite by 0.4.8. Suppose $I \nsubseteq P_k$ for all $1 \leqslant k \leqslant p$ and $I \subseteq Q_j$ for all $1 \leqslant j \leqslant q$. Then by 0.1.2, we can find an element $x \in I$ such that $x \notin P_1 \cup \ldots \cup P_p$. Consider the ideal

$$J = (\sqrt{(x)} :_R I).$$

We will show that J is not contained in any minimal prime. Indeed, as

$$I\,x \subseteq R\,x = (x) \subseteq \sqrt{(x)}$$

then $x \in J$, and thus $J \nsubseteq P_k$ for all k. So assume that $J \subseteq Q_k$, for some k. To simplify notation, write $Q := Q_k$. As all the ideals are finitely generated, since R is Noetherian, then localization commutes with radicals and $(- :_R -)$, by 0.7.2, so that

$$J_Q = \left(\sqrt{(x)_Q} :_{R_Q} I_Q\right).$$

As $x \in J \subseteq Q$, then $(x)_Q \subseteq Q_Q$. As Q is a minimal prime of R, then Q_Q is a minimal prime of R_Q, but it is also the unique maximal ideal, and therefore it is the unique prime ideal in R_Q. Thus $\sqrt{(x)_Q} = Q_Q$. Therefore, since $I_Q \subseteq Q_Q$,

$$J_Q = (Q_Q :_{R_Q} I_Q) = R_Q.$$

But this is impossible, considering that $J \subseteq Q$, which would imply $J_Q \subseteq Q_Q \subsetneq R_Q$. So the assumption that $J \subseteq Q$ must be false.

Then $\dim(R/J) \leqslant d-1$, because J is not contained in any minimal prime in R. By the induction hypothesis, $n :=$ ara$((I+J)/J) \leqslant d-1$, so that there exist $a_1, \ldots, a_n \in I$ such that $a_1+J, \ldots, a_n+J$ generate an ideal whose radical coincides with the radical of $(I+J)/J$. But

$$(a_1+J, \ldots, a_n+J) = \frac{(a_1, \ldots, a_n)+J}{J}$$

and thus

$$\frac{\sqrt{I+J}}{J} = \sqrt{\frac{I+J}{J}} = \sqrt{\frac{(a_1, \ldots, a_n)+J}{J}} = \frac{\sqrt{(a_1, \ldots, a_n)+J}}{J}.$$

This implies that $\sqrt{(a_1, \ldots, a_n) + J} = \sqrt{I + J}$. Therefore,

$$I \subseteq I + J \subseteq \sqrt{I + J} = \sqrt{(a_1, \ldots, a_n) + J}$$

and since I is finitely generated, we can find N such that $I^N \subseteq (a_1, \ldots, a_n) + J$, by 0.4.4. Moreover, $I^{N+1} \subseteq (a_1, \ldots, a_n) + JI$. By definition of J, $JI \subseteq \sqrt{(x)}$.

This is enough to show that $\sqrt{I} = \sqrt{(a_1, \ldots, a_n, x)}$:

$(\subseteq)$

Let $a \in \sqrt{I}$. Pick t such that $a^t \in I$. Then $a^{t(N+1)} \in (a_1, \ldots, a_n) + JI$. Hence there exists l such that $\left(a^{t(N+1)}\right)^l = a^{t(N+1)l} \in (a_1, \ldots, a_n, x)$. Then $a \in \sqrt{(a_1, \ldots, a_n, x)}$.

$(\supseteq)$

As $a_1, \ldots, a_n, x \in I$, then $(a_1, \ldots, a_n, x) \subseteq I \Rightarrow \sqrt{(a_1, \ldots, a_n, x)} \subseteq \sqrt{I}$.

So as $\sqrt{I} = \sqrt{(a_1, \ldots, a_n, x)}$, then ara$(I) \leqslant n + 1 \leqslant (d - 1) + 1 = d$.

$\square$

Theorem 2.1.32. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely generated R-module. Then*

$$\dim(M) = \sup\left\{i \colon \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\right\}.$$

Proof. By 2.1.31, we already know that $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i \geqslant d + 1$. So all that remains is to show that $\mathrm{H}^d_{\mathfrak{m}}(M) \neq 0$. A proof of this can be found in [28, Theorem 6.1.4]. $\square$

2.2 Local Cohomology of Graded Rings

In this section, we will study local cohomology modules in the graded case. We will start with some basic definitions from the theory of graded rings and modules, and prove some similar results to the ones in chapter 2.

We will prove that when we consider a graded module over a graded ring and a homogeneous ideal, then the local cohomology modules inherit a graded structure. As exact sequences of graded modules factor in exact sequences in each degree, we will be able to study local cohomology modules in each degree separately, and prove that the graded components are also almost all zero. This allows us to define the a-invariants and the Castelnuovo-Mumford regularity of a module.

Definition 2.2.1 (Graded ring)**.** A *graded ring* is a ring R that can be written as a direct sum of abelian groups

$$R = \bigoplus_{n \in \mathbb{Z}} R_n$$

with $R_i R_j \subseteq R_{i+j}$ for every $i, j \in \mathbb{Z}$.

Remark 2.2.2. Notice that R_0 is therefore a ring, as $R_0 R_0 \subseteq R_0$, and that $1 \in R_0$.

Definition 2.2.3 (Homogeneous elements)**.** We refer to the elements in R_i for some $i \geqslant 0$ as *homogeneous elements*, and denote the set of all such elements by U. If $r \in R_i$ is a homogeneous element, we say that the *degree* of r is i, and write $\deg(r) = i$. Furthermore, U_n will denote the set of homogeneous elements of degree n.

Remark 2.2.4. The set U of homogeneous non-zero elements in R is a multiplicative set, since $1 \in R_0$ is the identity element in R and $R_i R_j \subseteq R_{i+j}$.

Definition 2.2.5 (Graded module)**.** Let R be a graded ring. A *graded R-module* is an R-module M that can be written as a direct sum of abelian groups

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

with $R_i M_j \subseteq M_{i+j}$ for every $i, j \in \mathbb{Z}$. We say that M_i is the i-th homogeneous component of M, and the elements $m \in M_i$ are said to be *homogeneous* of degree i, which we write as $\deg(m) = i$.

Moreover, we will use the notation $M(d)_n := M_{n+d}$.

Definition 2.2.6 (Homogeneous ideal)**.** Let R be a graded ring. A *homogeneous ideal* in R is an ideal that is also a graded R-module (with the grading induced by R).

Remark 2.2.7. Let

$$R = \bigoplus_{n \in \mathbb{Z}} R_n$$

be a graded ring. As $R_0 R_n \subseteq R_n$ for every $n \in \mathbb{Z}$, then R_n is an R_0-module, and so is R. In particular, we can think of R_0 as a graded ring where every non-zero element has degree zero, and this case R is a graded R_0-module.

Proposition 2.2.8. *A graded ring $R = \bigoplus_{n \geqslant 0} R_n$ is Noetherian if and only if R_0 is a Noetherian ring and R is a finitely generated R_0-algebra.*

Proof. See [20, Theorem 13.1]. $\square$

Proposition 2.2.9. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring and $M = \bigoplus_{n \geqslant 0} M_n$ be a finitely generated graded R-module. Then each M_n is a finitely generated R_0-module.*

Proof. See [20, p. 94]. $\square$

Remark 2.2.10. Let R be a graded ring and M a graded R-module. Notice that every non-zero element $m \in M$ has a unique representation

$$m = \sum_{n \in \mathbb{Z}} m_n$$

as a sum of non-zero homogeneous elements, where $\{n : m_n \neq 0\}$ is finite.

Definition 2.2.11 (Graded homomorphism)**.** Let R be a graded ring. A homomorphism $f\colon M \longrightarrow N$ of graded R-modules is said to be *graded* if $f(M_i) \subseteq N_{i+d}$ for every i, and in this case f is said to be of *degree* d.

Lemma 2.2.12. *Let R be a graded ring and M a graded R-module. Setting*

$$\deg\left(\frac{m}{u}\right) = \deg(m) - \deg(u)$$

for each $m \in M_n$ and $u \in U$ gives M_U the structure of a graded R-module.

Proof. As M_U is an R-module, we just need to see that

$$M_U = \bigoplus_{n\in\mathbb{Z}} (M_U)_n$$

where $(M_U)_n$ is the set of elements of the form $\frac{m}{u}$ with $m \in M$ an homogeneous element and $u \in U$ such that $\deg(m) - \deg(u) = n$, and that $R_i(M_U)_j \subseteq (M_U)_{i+j}$ for every $i, j \geqslant 0$.

Let $\frac{m}{u} \in M_U$ be any element. We can write $m = \sum_{n\in\mathbb{Z}} m_n$ as a sum of homogeneous elements, where only finitely many m_n are non-zero, and thus

$$\frac{m}{u} = \frac{\sum_{n\in\mathbb{Z}} m_n}{u} = \sum_{n\in\mathbb{Z}} \frac{m_n}{u}$$

and for each $n \in \mathbb{Z}$, we have $\frac{m_n}{u}$ homogeneous of degree $n - \deg(u)$.

Now we must check that it is well-defined. That is, that if $\frac{m}{u} = \frac{m'}{u'}$ for some m', u' homogeneous elements in M and R respectively, then $\deg(m) - \deg(u) = \deg(m') - \deg(u')$. But $\frac{m}{u} = \frac{m'}{u'}$ means that there exists some homogeneous element $h \in U$ such that $h(u'm - um') = 0$, and thus $hu'm = hum'$. In particular, $hu'm$ and hum' have the same degree, which means that

$$\deg(h) + \deg(u') + \deg(m) = \deg(h) + \deg(u) + \deg(m) \Leftrightarrow \deg(m) - \deg(u) = \deg(m') - \deg(u')$$

Also, if $\frac{r}{u} \in (R_U)_i$ and $\frac{m'}{u'} \in (M_U)_j$, then

$$\frac{r}{u}\frac{m'}{u'} = \frac{rm'}{uu'}$$

and thus

$$\deg\left(\frac{rm'}{uu'}\right) = \deg(rm') - \deg(uu') = \deg(r) + \deg(m') - \deg(u) - \deg(u')$$

$$(\deg(r) - \deg(u)) + (\deg(m') - \deg(u')) = i + j$$

which proves that $(R_U)_i(M_U)_j \subseteq (M_U)_{i+j}$. □

Lemma 2.2.13. *Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring, M a graded R-module, and I an homogeneous ideal in R. Then $\mathrm{H}^i_I(M)$ are graded R-modules.*

Proof. We can choose homogeneous generators $x_1, \ldots, x_n$ for I, and thus

$$\check{\mathrm{C}}(x_1, \ldots, x_n)\colon\ R \longrightarrow \oplus_{i=1}^n R_{x_i} \longrightarrow \underset{i<j}{\overset{n}{\oplus}} R_{x_i x_j} \longrightarrow \ldots \longrightarrow R_{x_1 \ldots x_n} \longrightarrow 0$$

is a complex of graded modules, where all the maps have degree zero. Therefore, the kernels and images of all maps are graded R-modules, so that the cohomology modules are also graded, and those are precisely the local cohomology modules $\mathrm{H}^i_I(M)$. ☐

Remark 2.2.14. Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring such $(R_0, \mathfrak{m}_0)$ a Noetherian local ring, and $R_+ = \bigoplus_{n \geqslant 1} R_n$. Consider the ideal $N := (\mathfrak{m}_0, R_+)$. This is the unique homogeneous maximal ideal in R. The behavior of R with respect to this ideal is very similar to the behavior of a local ring with respect to its unique maximal ideal. One can show that most properties of a Noetherian local ring can be generalized for this special graded case. However, the purpose of this thesis was not to study graded rings and modules, and so we will skip the proofs of some technical results that would assure us of this local behavior of graded rings. In particular, we will use results from section 2.1 that we only showed for Noetherian local rings in this special graded case. For a more complete study of local cohomology in the graded case, see [28] or [10].

Proposition 2.2.15. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring such that $(R_0, \mathfrak{m}_0)$ is a Noetherian local ring and $R_+ = \bigoplus_{n \geqslant 1} R_n$. Consider the unique maximal homogeneous ideal $N := (\mathfrak{m}_0, R_+)$. Let M be a finitely generated graded R-module. Then $\mathrm{H}^i_N(M)$ are Artinian R-modules.*

Proof. See [10, Lemma 36.1]. ☐

Proposition 2.2.16. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring such that $(R_0, \mathfrak{m}_0)$ a local ring and $R_+ = \bigoplus_{n \geqslant 1} R_n$. Consider the unique maximal homogeneous ideal $N := (\mathfrak{m}_0, R_+)$. Let $M \neq 0$ be a finitely generated graded R-module. Then:*

(1) *For all $i > \dim(R)$, $\mathrm{H}^i_N(M) = 0$, and for $i = \dim(M)$, $\mathrm{H}^i_N(M) \neq 0$.*

(2) *For all $i <$ grade(M, N), $\mathrm{H}^i_N(M) = 0$, and for $i =$ grade(M, N), $\mathrm{H}^i_N(M) \neq 0$.*

In particular, M is Cohen-Macaulay if and only if for all $i < \dim(M)$, $\mathrm{H}^i_N(M) = 0$.

Proof. See [10, Lemmas 36.2 and 36.3 and Theorem 36.7]. ☐

Lemma 2.2.17. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a graded ring and L a graded Artinian R-module. Then there exists $N \geqslant 0$ such that $L_n = 0$ for all $n \geqslant N$.*

Proof. For each $n \geqslant 0$, let

$$L_{\geqslant n} = \bigoplus_{k \geqslant n} L_k.$$

By definition of graded R-module, we have $RL_{\geqslant n} \subseteq L_{\geqslant n}$, and thus $L_{\geqslant n}$ is an R-submodule of L for each $n \geqslant 0$. As L is Artinian, the descending chain of submodules

$$L \supseteq L_{\geqslant n} \supseteq L_{\geqslant n+1} \supseteq \cdots$$

must stop, and thus there exists $N \geqslant 0$ such that $L_{\geqslant N} = L_{\geqslant n}$ for all $n \geqslant N$. Therefore, for every $n \geqslant N$, the quotient R-module $L_{\geqslant n}/L_{\geqslant n+1}$ is actually the zero module. But the elements in this module can be identified with the elements of the set L_n. Therefore, $L_n = 0$ for $n \geqslant N$. □

Proposition 2.2.18. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring for which $(R_0, \mathfrak{m}_0)$ is a Noetherian local ring and consider the maximal ideal $N := (\mathfrak{m}_0, R_+)$. Let M be a graded R-module. Then, for every $i \geqslant 0$, there exists $N \geqslant 0$ such that for all $n \geqslant N$,*

$$\left(\mathrm{H}^i_N(M)\right)_n = 0.$$

Proof. By 2.2.15, $\mathrm{H}^i_N(M)$ is an Artinian R-module, and thus 2.2.17 applies. □

Definition 2.2.19 (a_i-invariants)**.** Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring such that $(R_0, \mathfrak{m}_0)$ is a Noetherian local ring. Consider the maximal ideal $N := (\mathfrak{m}_0, R_+)$. Let M be a graded R-module. For each i, the *a_i-invariant* of M, denoted by $a_i(M)$, is

$$a_i(M) := \sup\left\{j \colon \mathrm{H}^i_N(M)_j \neq 0\right\}$$

In case M is a Cohen-Macaulay R-module, we denote $a_d(M)$ by $a(M)$ for $d = \dim(R)$.

Remark 2.2.20. Notice that if i is such that $\mathrm{H}^i_N(M) = 0$, then $a_i(M) = -\infty$. By 2.1.31, we know that $a_i(M) = -\infty$ for $i > \dim(M)$. By 2.1.12, $a_i(M) = -\infty$ for $i <$ depth(M).

Remark 2.2.21. Let K be a field and $R = K[x_1, \dots, x_n]$. Let $I = (x_1, \dots, x_n)$. We can think of R as a graded ring with following grading:

$$R = \bigoplus_{i \geqslant 0} R_i$$

where R_i is the K-submodule of R generated by the monomials of degree i. Notice that $R_0 = K$ is a Noetherian local ring with maximal ideal $\mathfrak{m}_0 = 0$, and $N = (\mathfrak{m}_0, R_+)$ coincides with I. As R is a Cohen-Macaulay ring, then 2.2.16 guarantees that $\mathrm{H}^i_N(R) = 0$ for $i \neq n = \dim(R) =$ depth(R), and $\mathrm{H}^n_N(R) \neq 0$, so that the only a_i-invariant that carries non-trivial information is $a_n(R) =: a(R)$.

Theorem 2.2.22. *Let K be a field, $R = K[x_1, \dots, x_n]$ and $I = (x_1, \dots, x_n)$. Then $a(R) = a_n(R) = -n$.*

Proof. We will show this by induction on n. Recall that $n = \dim(R) =$ depth(R), by 0.2.3.

(1) Let $n = 0$.

In this case, $R = K$ has only elements of degree 0, and the same goes for $\mathrm{H}^0_N(R) \neq 0$. Therefore, $a(R) = 0$.

(2) Suppose the result is true for n.

Let $R = K[x_1, \dots, x_{n+1}]$. Consider the short exact sequence

$$0 \longrightarrow R(-1) \xrightarrow{.x_{n+1}} R \longrightarrow R' \longrightarrow 0\ .$$

The map $R(-1) \xrightarrow{.x_{n+1}} R$ has degree 0.

Let $N = (x_1, \dots, x_{n+1})$. The cokernel R' is isomorphic to $R/(x_{n+1}) \cong K[x_1, \dots, x_n]$. We can see R' as an algebra over R, by defining $x_{n+1}r := r$ for each $r \in R'$. As $(x_1, \dots, x_{n+1})R' = (x_1, \dots, x_n)$, 2.1.28 guarantees that

$$\mathrm{H}^i_N(M) \cong \mathrm{H}^i_{(x_1,\dots,x_n)}(M)$$

for each $i \geqslant 0$ and every R-module M.

As noted in 2.2.21, $\mathrm{H}^n_N(R') \neq 0$ and $\mathrm{H}^i_N(R') = 0$ for $i \neq n$. From the short exact sequence above, we get a long exact sequence, by 0.10.2:

$$0 = \mathrm{H}^n_N(R) \longrightarrow \mathrm{H}^n_N(R') \longrightarrow \mathrm{H}^{n+1}_N(R(-1)) \longrightarrow \mathrm{H}^{n+1}_N(R) \longrightarrow \mathrm{H}^{n+1}_N(R') = 0$$

As all the maps have degree 0, we can break this exact sequence into exact sequences on each degree:

$$0 \longrightarrow \mathrm{H}^n_N(R')_j \longrightarrow \mathrm{H}^{n+1}_N(R(-1))_j \longrightarrow \mathrm{H}^{n+1}_N(R)_j \longrightarrow 0$$

that is,

$$0 \longrightarrow \mathrm{H}^n_N(R')_j \longrightarrow \mathrm{H}^{n+1}_N(R)_{j-1} \longrightarrow \mathrm{H}^{n+1}_N(R)_j \longrightarrow 0 \; .$$

By induction hypothesis, $a_n(R') = -n$, so that the sequence

$$0 \longrightarrow \mathrm{H}^n_N(R')_{-n} \neq 0 \longrightarrow \mathrm{H}^{n+1}_N(R)_{-n-1} \longrightarrow \mathrm{H}^{n+1}_N(R)_{-n} \longrightarrow 0 \; .$$

is exact, and this implies that $\mathrm{H}^{n+1}_N(R)_{-n-1} \neq 0$. Thus,

$$a_n(R) \geqslant -(n+1).$$

Moreover, the following sequence is also exact:

$$0 = \mathrm{H}^n_N(R')_{-n+1} \longrightarrow \mathrm{H}^{n+1}_N(R)_{-n} \longrightarrow \mathrm{H}^{n+1}_N(R)_{-n+1} \longrightarrow 0 \; .$$

Suppose that $\mathrm{H}^{n+1}_N(R)_{-n} \neq 0$. As this implies that $\mathrm{H}^{n+1}_N(R)_{-n} \cong \mathrm{H}^{n+1}_N(R)_{-(n-1)} \neq 0$, we can use the same sequence in one degree higher and conclude that $\mathrm{H}^{n+1}_N(R)_{-n} \cong \mathrm{H}^{n+1}_N(R)_{-n+2}$. Repeating the process an infinite number of times, we can show that

$$\mathrm{H}^{n+1}_N(R)_{-n} \cong \mathrm{H}^{n+1}_N(R)_{-n+k}$$

for every $k \geqslant 0$, which is impossible because we know that $\mathrm{H}^{n+1}_N(R)_k$ is eventually 0, by 2.2.18. Therefore, $\mathrm{H}^{n+1}_N(R)_{-n} = 0$. We also proved that $\mathrm{H}^{n+1}_N(R)_k = 0$ for every $k \geqslant -n$, and thus

$$a_n(R) = -(n+1).$$

$\square$

Definition 2.2.23 (Castelnuovo-Mumford regularity)**.** Let $R = \bigoplus_{n \geqslant 0} R_n$ be a Noetherian graded ring such that $(R_0, \mathfrak{m}_0)$ is a Noetherian local ring. Consider the maximal ideal $N := (\mathfrak{m}_0, R_+)$. Let M be a graded R-module. The *Castelnuovo-Mumford regularity* of M is the number

$$\text{reg}(M) := \max \{a_i(M) + i \mid i \geqslant 0\}.$$

Remark 2.2.24. By 2.2.20, the set

$$\{i \colon a_i(M) > -\infty\}$$

is finite, and thus the set

$$\{a_i(M) + i \mid i \geqslant 0\}$$

has a maximum, so that the definition of the Castelnuovo-Mumford regularity makes sense.

Chapter 3

Ulrich Ideals

Introduction

In this chapter we will study Ulrich ideals. These relate to the a-invariant, blow-up algebras and reductions of ideals.

Reductions were first defined by Douglas Northcott and David Rees in 1954 ([21]). For two ideals $J \subseteq I$, J is a reduction of I if $I^{n+1} = JI^n$ for some integer $n \geqslant 1$.

There is a close connection between reductions of ideals and the theory of blow-up algebras, a class of graded rings that appears in many constructions in Commutative Algebra and Algebraic Geometry and that includes polynomial rings, the associated graded ring or the fiber cone. The Krull dimension of the fiber cone of a certain ideal, known as the analytic spread of that ideal, coincides with the minimal number of generators of minimal reductions. This also relates to Hilbert Functions and the e_0-multiplicity.

For Cohen-Macaulay local rings $(R, \mathfrak{m})$, Abhyankar proved an inequality that relates the minimal number of generators $\mathfrak{m}$, $\mu(\mathfrak{m})$, the dimension of R and the e_0-multiplicity of $\mathfrak{m}$, $e_0(\mathfrak{m})$, which measures the length of $\mathfrak{m}/\mathfrak{m}^2$:

$$e_0(\mathfrak{m}) \geqslant \mu(\mathfrak{m}) - \dim(R) + 1.$$

Sally showed ([25]) that in the case of equality, the associated graded ring of $\mathfrak{m}$ is Cohen-Macaulay, and that for any minimal reduction J of $\mathfrak{m}$, $\mathfrak{m}^2 = J\mathfrak{m}$. Ulrich ideals appear as a result of the attempts to obtain similar results for $\mathfrak{m}$-primary ideals.

In section 3.1, we will introduce reductions, and in section 3.2, we will study the results in [6, Section 2].

3.1 Preliminaries

In this section, we introduce Reductions of ideals and prove some basic results we will need in section 3.2 to study Ulrich ideals. We also introduce the concepts of analytic spread, fiber cone, associaded graded ring and Rees algebra, which will be crucial to understand reductions of ideals and consequently

Ulrich ideals. This section may be skipped if the reader is already familiar with the subject of reductions.

Definition 3.1.1 (Filtration)**.** Let R be a ring. A sequence $\mathcal{F} = (I_n)_{n\geqslant 0}$ of ideals in R is said to be a *filtration* if

$$R = I_0 \supsetneq I_1 \supseteq I_2 \supseteq \cdots$$

and $I_iI_j \subseteq I_{i+j}$ for all $i, j \geqslant 0$.

Definition 3.1.2 (I-adic filtration)**.** Given an ideal $I \neq R$,

$$R \supsetneq I \supseteq I^2 \supseteq I^3 \cdots$$

is a filtration, called the *I-adic filtration*.

Theorem 3.1.3 (Noether Normalization Theorem (Graded Version))**.** *Let K be a field and R a finitely generated graded K-algebra such that $R_0 = K$.*[1] *There exist algebraically independent elements $x_1, \ldots, x_m \in R$, homogeneous of the same degree, such that R is integral over $K[x_1, \ldots, x_m]$. If K is infinite and R is generated over K by elements of degree 1, then the x_i's may be taken to be of degree 1.*

Proof. See [13, pp. 58-59]. □

Definition 3.1.4 (Rees Algebra)**.** Let R be a ring, $\mathcal{F} = (I_n)_{n\geqslant 0}$ a filtration in R and t a variable over R. The *Rees algebra* of $\mathcal{F}$ is the graded ring

$$R(\mathcal{F}) := R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots = \bigoplus_{n\geqslant 0} I_n t^n.$$

If the filtration is I-adic, we say that $R(\mathcal{F})$ is the *Rees algebra* of I, and denote it by $R[It]$.

Definition 3.1.5 (Associated Graded Ring)**.** Let R be a ring and consider a filtration $\mathcal{F} = (I_n)_{n\geqslant 0}$ in R. The *associated graded ring* of $\mathcal{F}$ is the graded ring

$$\mathrm{gr}_{\mathcal{F}}(R) := R/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \cdots = \bigoplus_{n\geqslant 0} I_n/I_{n+1}\,.$$

In case the filtration is I-adic, we say that $\mathrm{gr}_{\mathcal{F}}(R)$ is the *associated graded ring* of R with respect to I, and denote it by $\mathrm{gr}_I(R)$.

Remark 3.1.6. It can be shown that if R is a Noetherian ring and I is any ideal in R, then $\mathrm{gr}_I(R)$ is Noetherian. By 2.2.8, this implies that $\mathrm{gr}_I(R)$ must be a finitely generated algebra over R/I. Moreover,

$$\mathrm{gr}_I(R) = \frac{R}{I}\left[a_1 + I^2, \ldots, a_n + I^2\right]$$

where $I = (a_1, \ldots, a_n)$, and this is a minimal set of generators.

[1] We represent by R_i the homogeneous elements of R of degree i, so that in this case we mean that K corresponds to the homogeneous elements of R of degree 0.

Theorem 3.1.7. *Let R be a ring, M an R-module and $x_1, \ldots, x_n \in I$ a regular sequence in M such that $I = (x_1, \ldots, x_n)$. Then*

$$R/I\,[X_1, \ldots, X_n] \cong gr_I(R).$$

Proof. Easy corollary of [11, Theorem 1.18]. □

Definition 3.1.8 (Fiber cone)**.** Let $(R, \mathfrak{m})$ be a local Noetherian ring and $\mathcal{F}$ a filtration in R. The *fiber cone* of $\mathcal{F}$ is the ring

$$F(\mathcal{F}) := \frac{R(\mathcal{F})}{\mathfrak{m}\,R(\mathcal{F})} = \frac{R}{\mathfrak{m}} \oplus \frac{I_1}{\mathfrak{m}\,I_1} \oplus \frac{I_2}{\mathfrak{m}\,I_2} \oplus \cdots .$$

In case $\mathcal{F}$ is an I-adic filtration, we simply write $F(\mathcal{F})$ as $F(I)$, and call it the *fiber cone* of I.

Definition 3.1.9 (Analytic Spread)**.** Let R be a local Noetherian ring and $\mathcal{F}$ a filtration in R. The *analytic spread* of $\mathcal{F}$, denoted by $\ell(\mathcal{F})$, is the Krull dimension of the fiber cone of $\mathcal{F}$, that is,

$$\ell(\mathcal{F}) = \dim(F(\mathcal{F})).$$

If $\mathcal{F}$ is an I-adic filtration, we write $\ell(I)$ for $\ell(\mathcal{F})$ and call it the *analytic spread* of I.

Theorem 3.1.10. *Given a local ring $(R, \mathfrak{m})$ and an ideal I in R,*

$$\ell(I) = \dim F(I) \leqslant \dim(gr_I(R)) = \dim(R).$$

Moreover, if R is a Noetherian local ring then ht$(I) \leqslant \ell(I) \leqslant \dim(R)$ and $\ell(I) \leqslant \mu(I)$.

Proof. See [13, 5.1.6 and 8.4.3]. □

Definition 3.1.11 (Reduction)**.** Let R be a ring and $\mathcal{F} = (I_n)_{n \geqslant 0}$ a filtration in R. A *reduction* of $\mathcal{F}$ is an ideal $J \subseteq I_1$ such that $JI_n = I_{n+1}$ for every n sufficiently large. If $\mathcal{F}$ is an I-adic filtration in R, we simply say that J is a reduction of I.

Definition 3.1.12 (Minimal Reduction)**.** Let R be a ring, $\mathcal{F}$ a filtration in R and J a reduction of $\mathcal{F}$. We say that J is a *minimal reduction* of $\mathcal{F}$ if it is minimal with respect to containment. In other words, for every ideal K in R such that K is a reduction of $\mathcal{F}$, $K \subseteq J \Rightarrow K = J$.

Definition 3.1.13 (Reduction Number)**.** Let R be a ring, $\mathcal{F} = (I_n)_{n \geqslant 0}$ a filtration in R and J an ideal in R. If J is a reduction of $\mathcal{F}$, the least integer n such that $JI_n = I_{n+1}$ is said to be the *reduction number of $\mathcal{F}$ with respect to J*, and it is denoted by $r_J(\mathcal{F})$. The (absolute) *reduction number* of $\mathcal{F}$, denoted by $r(\mathcal{F})$, is given by

$$r(\mathcal{F}) := \min\left\{r_J(\mathcal{F}) \mid J \text{ is a minimal reduction of } \mathcal{F}\right\}.$$

Lemma 3.1.14. *Let R be a ring and $J \subseteq I$ ideals in R. The following conditions are equivalent:*

(1) $I^{n+1} = JI^n$.

(2) $I^{n+1} \subseteq JI^n$.

(3) $\underset{m \geqslant n}{\forall} I^{m+1} = JI^m$.

(4) $(JI^n : I^{n+1}) = R$.

(5) $\underset{m \geqslant 0}{\forall} I^{n+m} = J^m I^n$.

Lemma 3.1.15. *Let R be a ring and $J \subseteq I$ be ideals in R such that J is a reduction of I. Then $\sqrt{I} = \sqrt{J}$.*

Proof. We will prove that the primes ideals containing I and J coincide, which is enough. Indeed, that implies that the intersections of the minimal prime ideals over J and I coincide, and by 0.4.3, those intersections are precisely $\sqrt{J}$ and $\sqrt{I}$.

Let P be a prime ideal in R. If $I \subseteq P$ then $J \subseteq P$, as $J \subseteq I$. Now suppose $J \subseteq P$. As J is a reduction of I, the $I^{n+1} \subseteq JI^n$ for some $n \geqslant 1$, and then $I^{n+1} \subseteq PI^n \subseteq P$. As P is a prime ideal, this implies that $I \subseteq P$. □

Theorem 3.1.16. *Let $K \subseteq J \subseteq I$ be ideals in R.*

(1) *If K is a reduction of J and J is a reduction of I, then K is a reduction of I.*

(2) *If K is a reduction of I, then J is a reduction of I.*

(3) *Suppose I is finitely generated and $J = K + (r_1, \dots, r_k)$ for some $r_1, \dots, r_k \in R$. If K is a reduction of I, then K is a reduction of J.*

Proof.

(1) Let m, n be such that $J^{n+1} = KJ^n$ and $I^{m+1} = JI^m$. Then

$$I^{mn+m+n+1} = \left(I^{m+1}\right)^{n+1} = (JI^m)^{n+1} = J^{n+1} I^{nm} I^m$$

$$= KJ^n I^{nm} I^m = K(JI^m)^n I^m = K\left(I^{m+1}\right)^n I^m = KI^{n(m+1)+m} = KI^{nm+n+m}$$

and so K is a reduction of I.

(2) Assuming $I^{n+1} = KI^n$,

$$K \subseteq J \Rightarrow KI^n \subseteq JI^n \Leftrightarrow I^{n+1} \subseteq JI^n \Leftrightarrow I^{n+1} = JI^n.$$

(3) See [13, Proposition 1.2.4]. The proof is a simple proof by induction, but uses the fact that $J \subseteq J + (r)$ is a reduction if and only if r is integral over J, and that would require a study of integral closure of ideals, which goes beyond the scope of this thesis.

□

Corollary 3.1.17. *Let R be a Noetherian ring and $K \subseteq J \subseteq I$ be ideals in R. If K is a reduction of I then K is a reduction of J.*

Proof. As R is Noetherian, I and J are finitely generated. Let $a_1, \dots, a_n$ be such that $J = (a_1, \dots, a_n)$. Then $J = K + (a_1, \dots, a_n)$. By 3.1.16 (3), K is a reduction of J. □

Theorem 3.1.18. *Let R be a Noetherian ring and $\mathcal{J}$ its Jacobson radical. Consider ideals J, J', L and I in R such that $J, J' \subseteq I$ and $L \subseteq \mathcal{J}I$. If $J + L = J' + L$, then J is a reduction of I if and only if J' is a reduction of I.*

Proof. Suppose J is a reduction of I, with $I^{n+1} = JI^n$. As $J' \subseteq I$ and $\mathcal{J}I \subseteq I$, then $J'I^n + \mathcal{J}I^{n+1} \subseteq I^{n+1}$. Also

$$I^{n+1} = JI^n \subseteq (J + L)I^n = (J' + L)I^n \subseteq (J' + \mathcal{J}I)I^n = J'I^n + \mathcal{J}I^{n+1},$$

so that $I^{n+1} = J'I^n + \mathcal{J}I^{n+1}$. By Nakayama's Lemma (0.4.9), this implies $I^{n+1} = J'I^n$, that is, J' is a reduction of I.

Replacing J' by J and J by J', we get the other implication. □

Theorem 3.1.19. *Let R be a Noetherian ring and $K \subseteq J \subseteq I$ ideals in R. If K is a minimal reduction of I then K is also a minimal reduction of J.*

Proof. As R is Noetherian and K is a reduction of I, then K is a reduction of J by 3.1.17. Also, by 3.1.16 (2), J is a reduction of I. Suppose $K' \subseteq K$ is also a reduction of J. Then K' is a reduction of J and J is a reduction of I, which implies, by 3.1.16 (1), that K' is a reduction of I. By the minimality of K, we must have $K' = K$. □

Theorem 3.1.20. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $J \subseteq I$ ideals in R. If J is a reduction of I then there exists $K \subseteq J$ that is a minimal reduction of I.*

Proof. Let Σ denote the set of all $K \subseteq J \subseteq I$ such that K is a reduction of I. In Σ, consider the order $\leqslant$ defined as follows:

$$\frac{K + \mathfrak{m}I}{\mathfrak{m}I} \subseteq \frac{K' + \mathfrak{m}I}{\mathfrak{m}I} \Leftrightarrow K' \leqslant K.$$

Notice that $\Sigma \neq \emptyset$ because J is a reduction of I and so $J \in \Sigma$. Since R is Noetherian, $I/\mathfrak{m}I$ is a vector space over $R/\mathfrak{m}$ with finite dimension. For every $K \in \Sigma$, $\frac{K+\mathfrak{m}I}{\mathfrak{m}I} \subseteq \frac{I}{\mathfrak{m}I}$ is a subspace of $I/\mathfrak{m}I$, which implies that its dimension as a vector space over $R/\mathfrak{m}$ is finite. As this is true for all $K \in \Sigma$, there exists $K \in \Sigma$ such that $\frac{K+\mathfrak{m}I}{\mathfrak{m}I}$ has minimal dimension as a vector space over $R/\mathfrak{m}$, say n, and K is therefore maximal in $(\Sigma, \leqslant)$.

Consider a basis $V = \{v_1, \ldots, v_n\}$ for the vector space $\frac{K+\mathfrak{m}I}{\mathfrak{m}I}$ and take $k_1, \ldots, k_n \in K$ such that $k_i + \mathfrak{m}I = v_i$ for each $i = 1, \ldots, n$. Let $K_0 = (k_1, \ldots, k_n)$. As K is a reduction of I and $K + \mathfrak{m}I = K_0 + \mathfrak{m}I$, then K_0 is also a reduction of I, by 3.1.18. As $K_0 \subseteq K$ implies $\frac{K_0+\mathfrak{m}I}{\mathfrak{m}I} \subseteq \frac{K+\mathfrak{m}I}{\mathfrak{m}I} \Leftrightarrow K \leqslant K_0$, then $\frac{K_0+\mathfrak{m}I}{\mathfrak{m}I} = \frac{K+\mathfrak{m}I}{\mathfrak{m}I}$, by the maximality of K, so without loss of generality we can take $K_0 = K$. In particular, $\mu(K) = n$.

Then, by 0.1.11,

$$\dim\left(\frac{K}{\mathfrak{m}K}\right) = \mu(K) = n = \dim\left(\frac{K + \mathfrak{m}I}{\mathfrak{m}I}\right)$$

so that $\frac{K}{\mathfrak{m}K}$ and $\frac{K+\mathfrak{m}I}{\mathfrak{m}I}$ are both n-dimensional vector spaces.

Given that $\mathfrak{m}K \subseteq K \cap \mathfrak{m}I$, we must have $\frac{K}{\mathfrak{m}K} \cong \frac{K}{K \cap \mathfrak{m}I}$, which implies that $\mathfrak{m}K = K \cap \mathfrak{m}I$.

Suppose $L \subseteq K$ is a reduction of I. As $\frac{L+\mathfrak{m}I}{\mathfrak{m}I} \subseteq \frac{K+\mathfrak{m}I}{\mathfrak{m}I}$, by the maximality of K in Σ, we must have $\frac{L+\mathfrak{m}I}{\mathfrak{m}I} = \frac{K+\mathfrak{m}I}{\mathfrak{m}I}$, which implies $K + \mathfrak{m}I = L + \mathfrak{m}I$. Then, using $L \cap K = L$ and $K \cap \mathfrak{m}I = \mathfrak{m}K$,

$$K \subseteq (L + \mathfrak{m}I) \cap K = L + (K \cap \mathfrak{m}I) = L + \mathfrak{m}K$$

that is, $K \subseteq L + \mathfrak{m}K$. Also, $L \subseteq K, \mathfrak{m}K \subseteq K \Rightarrow L + \mathfrak{m}K \subseteq K$, which proves $L + \mathfrak{m}K = K$. By Nakayama's Lemma, $L = K$.

Then there is no reduction of I strictly contained in K, that is, K is a minimal reduction of I contained in J. □

Theorem 3.1.21. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and J, I ideals such that J is a minimal reduction of I. Then*

(1) $J \cap \mathfrak{m}I = \mathfrak{m}J$

(2) *If K is an ideal in R such that $J \subseteq K \subseteq I$, every minimal set of generators of J can be extended to a minimal set of generators of K.*

Proof.

(1) $\mathfrak{m}J \subseteq \mathfrak{m}I \cap J$ is obvious, as $J \subseteq I \Rightarrow \mathfrak{m}J \subseteq \mathfrak{m}I$ and $\mathfrak{m}J \subseteq J$.

Since R is a Noetherian ring, J is finitely generated, and $\frac{J}{J \cap \mathfrak{m}I}$ is also a finitely generated R-module. Since $\mathfrak{m}\,\frac{J}{J \cap \mathfrak{m}I} = 0$, then $\frac{J}{J \cap \mathfrak{m}I}$ can also be seen as an $R/\mathfrak{m}$-module, with the same structure, so that $\frac{J}{J \cap \mathfrak{m}I}$ is a finite dimensional vector space over $R/\mathfrak{m}$. Suppose its dimension as a vector space over $R/\mathfrak{m}$ is n, $(\frac{J}{J \cap \mathfrak{m}I}) \cong (\frac{R}{\mathfrak{m}})^n$. Thus there exist $\overline{a}_1, \ldots, \overline{a}_n \in \frac{J}{J \cap \mathfrak{m}\, I}$ such that $\frac{J}{J \cap \mathfrak{m}\, I} = \frac{R}{\mathfrak{m}}\overline{a}_1 + \ldots + \frac{R}{\mathfrak{m}}\overline{a}_n$. If $a_1, \ldots, a_n \in J$ are such that $a_i + (J \cap \mathfrak{m}\, I) = \overline{a}_i$ for each $1 \leqslant i \leqslant n$, then $J = (a_1, \ldots, a_n) + (J \cap \mathfrak{m}\, I)$.

$$\begin{tikzcd}[arrows=dash]
J \arrow[dr, "n"] \arrow[ddr, bend right, "\mu(J)=n"'] & & \mathfrak{m}\, I \arrow[dl] \\
& J \cap \mathfrak{m}\, I \arrow[d] & \\
& \mathfrak{m}\, J &
\end{tikzcd}$$

Now notice that $J, (a_1, \ldots, a_n) \subseteq I$, $J \cap \mathfrak{m}I \subseteq \mathfrak{m}I$ and

$$J + (J \cap I\mathfrak{m}) = J = (a_1, \ldots, a_n) + (J \cap \mathfrak{m}I).$$

This is enough to apply 3.1.18 to $J' = (a_1, \ldots, a_n)$ and $L = J \cap \mathfrak{m}I$. As J is a reduction of I, 3.1.18 implies that $(a_1, \ldots, a_n)$ is also a reduction of I. But $(a_1, \ldots, a_n) \subseteq J$, so we must have $J = (a_1, \ldots, a_n)$. Now recall that n is the minimal number of generators of $\frac{J}{J \cap \mathfrak{m}I}$, and so n is indeed the minimal number of generators of J. Then,

$$\dim_{\frac{R}{\mathfrak{m}}} \left(\frac{J}{J \cap \mathfrak{m}I}\right) = n = \mu(J) = \dim_{\frac{R}{\mathfrak{m}}} \left(\frac{J}{\mathfrak{m}J}\right)$$

which as $\mathfrak{m}J \subseteq J \cap \mathfrak{m}I$ implies that $\mathfrak{m}J = J \cap \mathfrak{m}I$.

(2) By 3.1.19, J is a minimal reduction of K, which by (1) implies $J \cap \mathfrak{m}K = \mathfrak{m}J$. By 0.1.11, $\{k_1, \ldots, k_s\}$ is a minimal generating set of K if and only if $\{k_1 + \mathfrak{m}K, \ldots, k_s + \mathfrak{m}K\}$ is a basis of $\frac{K}{\mathfrak{m}K}$ over $\frac{R}{\mathfrak{m}}$.

Consider a minimal set of generators of J, $\{a_1, \ldots, a_n\}$, which corresponds to a basis

$$\{a_1 + \mathfrak{m}J, \ldots, a_n + \mathfrak{m}J\}$$

of the n-dimensional $\frac{R}{\mathfrak{m}}$-vector space $\frac{J}{\mathfrak{m}J}$. We will now show that the set $\{a_1 + \mathfrak{m}K, \ldots, a_n + \mathfrak{m}K\}$ is linearly independent in $\frac{K}{\mathfrak{m}K}$ over $\frac{R}{\mathfrak{m}}$. Let $r_1, \ldots, r_n \in R$ be such that

$$0 = (r_1 + \mathfrak{m})(a_1 + \mathfrak{m}K) + \ldots + (r_n + \mathfrak{m})(a_1 + \mathfrak{m}K) = (r_1 a_1 + \ldots + r_n a_n) + \mathfrak{m}K.$$

Then

$$r_1 a_1 + \ldots + r_n a_n \in \mathfrak{m}K,$$

and given that $a_1, \ldots, a_n \in J$,

$$r_1 a_1 + \ldots + r_n a_n \in J \cap \mathfrak{m}K = \mathfrak{m}J.$$

Then

$$(r_1 + \mathfrak{m})(a_1 + \mathfrak{m}J) + \ldots + (r_n + \mathfrak{m})(a_1 + \mathfrak{m}J) = (r_1 a_1 + \ldots + r_n a_n) + \mathfrak{m}J = 0,$$

which, as $\{a_1 + \mathfrak{m}J, \ldots, a_n + \mathfrak{m}J\}$ is a basis for $\frac{J}{\mathfrak{m}J}$ over $\frac{R}{\mathfrak{m}}$, implies

$$r_1 + \mathfrak{m} = \ldots = r_n + \mathfrak{m} = 0$$

Then $\{a_1 + \mathfrak{m}K, \ldots, a_s + \mathfrak{m}K\}$ is a linearly independent set in $\frac{K}{\mathfrak{m}K}$ over $\frac{R}{\mathfrak{m}}$, which therefore can be extended to a basis

$$\{a_1 + \mathfrak{m}K, \ldots, a_s + \mathfrak{m}K, a_{s+1} + \mathfrak{m}K, \ldots, a_m + \mathfrak{m}K\}$$

so that $\{a_1, \ldots, a_s, a_{s+1}, \ldots, a_m\}$ is a minimal generating set of K.

□

Theorem 3.1.22. *Let n be a positive integer and $(R, \mathfrak{m})$ a Noetherian local ring. Consider ideals I and J in R such that $J \subseteq I^n$ and $B = \frac{R}{\mathfrak{m}}\left[\frac{J + \mathfrak{m}I^n}{\mathfrak{m}I^n}\right]$. Then J is a reduction of I^n if and only if the fiber cone $F(I)$ is a finitely generated module over B. If these conditions hold, then the reduction number of I^n with respect to J is the largest degree of an element in a minimal set of generators of $F(I^n)$ as a B-module containing only homogeneous elements.*

Proof. See [13, Proposition 8.2.4]. □

Corollary 3.1.23. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and $J \subseteq I$ ideals such that J is a reduction of I and $\mu(J) = \ell(I)$. Then*

(1) *J is a minimal reduction of I.*

(2) *$F(J)$ is isomorphic to $\frac{R}{\mathfrak{m}}\left[\frac{J+\mathfrak{m}I}{\mathfrak{m}I}\right]$ and to a polynomial ring in $l(I)$ variables over $R/\mathfrak{m}$.*

(3) *For all positive integers k, $J^k \cap \mathfrak{m}I^k = \mathfrak{m}J^k$.*

Proof. See [13, Proposition 8.3.6]. □

Theorem 3.1.24. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with infinite residue field and I an ideal in R. Then every minimal reduction of I is minimally generated by exactly $\ell(I)$ elements. In particular, every reduction of I contains a reduction generated by $\ell(I)$ elements.*

Proof. We will prove that given any $J \subseteq I$ such that J is a reduction of I, there exists a minimal reduction $K \subseteq J$ of I such that K is generated by precisely $\ell(I)$ elements. Given this, any minimal reduction of I is generated by $\ell(I)$ elements. So let us show how to construct a minimal reduction $K \subseteq J$ of I generated by $\ell(I)$ elements, and that concludes the proof:

Write $l := \ell(I)$. Let J be a reduction of I and $B = \frac{R}{\mathfrak{m}}\left[\frac{J+\mathfrak{m}I}{\mathfrak{m}I}\right] \subseteq F(I)$, which by 3.1.22 is a finite extension of modules. As B is a finitely generated graded $R/\mathfrak{m}$ algebra and $R/\mathfrak{m}$ is a field, by 3.1.3 there exist $\overline{a_1},\ldots,\overline{a_l} \in B$ such that $A := \frac{R}{\mathfrak{m}}[\overline{a_1},\ldots,\overline{a_l}]$ is a polynomial subring of B such that B is integral over A, and thus finitely generated over A. $F(I)$ is finitely generated over B, and so $F(I)$ is also finitely generated over A. Notice that since $R/\mathfrak{m}$ is infinite and B is generated over $R/\mathfrak{m}$ by elements of degree one, we may take $\overline{a}_i$ all of degree one, that is, $\overline{a}_i \in \frac{J+\mathfrak{m}I}{\mathfrak{m}I}$. Also, notice that we have exactly l such elements, because $\dim(A) = \dim(B) = \dim(F(I)) = l$.

Let $f\colon J \to (J+\mathfrak{m}I)/\mathfrak{m}I$ be the canonical projection. For each i, consider $a_i \in J$ such that $f(a_i) = \overline{a_i}$, and set $K := (a_1,\ldots,a_l)$. By definition, $K \subseteq J \subseteq I$ and $A = \frac{R}{\mathfrak{m}}[\overline{a_1},\ldots,\overline{a_l}] \cong \frac{R}{\mathfrak{m}}\left[\frac{K+\mathfrak{m}I}{\mathfrak{m}I}\right]$. Given that $\mathcal{F}_I(R)$ is finitely generated over A, by 3.1.22, then K is a reduction of I. As K is generated by $\ell(I)$ elements, then K is a minimal reduction of I, by 3.1.23. □

Definition 3.1.25 (Hilbert Function)**.** Let $R = \bigoplus_{n\geqslant 0} R_n$ be a Noetherian graded ring with R_0 an Artinian ring and $M = \bigoplus_{n\geqslant 0} M_n$ be a graded finitely generated R-module. The *Hilbert function* of M is the function $H_M(n) = \lambda_{R_0}(M_n)$.

Remark 3.1.26. The previous definition makes sense, that is, $\lambda_{R_0}(M_n) < \infty$ for each $n \geqslant 0$. We know by 2.2.8 that R_0 is a Noetherian ring. By 0.3.7 and 2.2.9, M_n is a Noetherian and Artinian module over R_0, and thus $\lambda_{R_0}(M_n) < \infty$.

Proposition 3.1.27. *Let $R = \bigoplus_{n\geqslant 0} R_n$ be a Noetherian graded ring with R_0 an Artinian ring, and $M = \bigoplus_{n\geqslant 0} M_n$ a graded finitely generated R-module. Suppose that R is generated by r elements of degree 1 over R_0. Then there exists a polynomial h_M with integer coefficients such that for every n sufficiently large,*

$$h_M(n) = H_M(n).$$

Proof. See [20, pp. 94-97]. □

Remark 3.1.28. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $d := \dim(R)$ and I an $\mathfrak{m}$-primary ideal. For n sufficiently large, the function $\lambda(R/I^{n+1})$ coincides with a polynomial in n with integer coefficients and degree d. A proof of this can be found in [20, Chapter 13].

Definition 3.1.29. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $d := \dim(R)$ and I an $\mathfrak{m}$-primary ideal. The *multiplicity* of I with respect to M is the integer number $e_0(I)$ such that

$$\lambda(R/I^{n+1}) = \frac{e_0(I)}{d!} n^d + \text{ terms of lower order .}$$

Theorem 3.1.30. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field, and let I be any $\mathfrak{m}$-primary ideal. If J is a minimal reduction of I, then $\lambda(R/J) = e_0(I)$.*

Proof. See [13, Proposition 11.2.2]. □

3.2 Ulrich Ideals

In this section, we define Ulrich ideals, prove some basic results and characterize Ulrich ideals over Gorenstein rings, following section 2 of [6]. We finish the section with an example.

Remark 3.2.1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and I an ideal in R. By 3.1.20, I has a minimal reduction Q. By 3.1.10,

$$\text{ht}\,(I) \leqslant \ell(I) \leqslant \dim\,(R) := d.$$

In particular, if I is an $\mathfrak{m}$-primary ideal of R, then Q is also an $\mathfrak{m}$-primary ideal of R, by 3.1.15, and since $\mathfrak{m}$ is the only minimal prime over Q and over I, by 0.4.6, then

$$\text{ht}\,(Q) = \text{ht}\,(I) = \text{ht}\,(\mathfrak{m}) = \dim\,(R) = d.$$

In general, $d = \text{ht}(Q) \leqslant \mu(Q)$ and $d = \text{ht}(I) \leqslant \mu(I)$. If $R/\mathfrak{m}$ is infinite, then by 3.1.24, we know that such Q is generated by $\ell(I)$ elements, and thus $\mu\,(Q) = d$. Therefore, if $R/\mathfrak{m}$ is infinite, then every minimal reduction of I is a parameter ideal. For this reason, we will always assume $R/\mathfrak{m}$ infinite.

Definition 3.2.2 (Ulrich ideal)**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $R/\mathfrak{m}$ infinite and I be an $\mathfrak{m}$-primary ideal in R. We say that I is an *Ulrich ideal* if the following conditions are satisfied:

(1) For some minimal reduction Q of I, $I^2 = QI$.

(2) The R/I-module I/I^2 is free.

Remark 3.2.3. Condition (1) says that some minimal reduction of I has reduction number 1. It can be shown (see [7]) that this definition is equivalent to

(a) The associated graded ring of I, $\text{gr}_I(R)$, is Cohen-Macaulay with $a(\text{gr}_I(R)) \leqslant 1 - \dim(R)$.

(b) I/I^2 is a free R/I-module.

When the associated graded ring is Cohen-Macaulay, the reduction number does not depend on the minimal reduction chosen (see [12]). Therefore, whenever I is an Ulrich ideal, $I^2 = QI$ for every minimal reduction Q of I. This clearly implies condition (1) in the definition. Therefore, we can also use the following definition: an $\mathfrak{m}$-primary ideal is an Ulrich ideal if

(1) For every minimal reduction Q of I, $I^2 = QI$.

(2) The R/I-module I/I^2 is free.

Lemma 3.2.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring with $R/\mathfrak{m}$ infinite, $\dim(R) = d$ and I an $\mathfrak{m}$-primary ideal of R. If I is a parameter ideal then I is an Ulrich ideal.*

Proof. Since I is a parameter ideal, then

$$I = (a_1, \ldots, a_d),$$

where $a_1, \ldots, a_d$ is a regular sequence in R, by 0.6.5. Since I is $\mathfrak{m}$-primary, $d = \text{ht}(I) \leqslant \mu(I)$. Therefore, $\mu(I) = d$.

By 3.1.10,

$$d = \text{ht}(I) \leqslant \ell(I) \leqslant d,$$

so that $d = \ell(I)$. Since I is trivially a reduction of I, by 3.1.23 we conclude that I must be a minimal reduction of I, and thus the unique reduction of I.

Moreover, the R/I-module I/I^2 is free. To see this, just note that by 3.1.7,

$$\text{gr}_I(R) \cong \frac{R}{I}\left[X_1, \ldots, X_d\right],$$

and that I/I^2 is the homogeneous component in degree 1 of the polynomial ring in d variables, which is isomorphic to $(R/I)^d$. □

Lemma 3.2.5. *[6, Lemma 2.3] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field, $d := \dim(R)$, and I an $\mathfrak{m}$-primary ideal in R. Let Q be a minimal reduction of I and assume that $I^2 = QI$. Then*

$$e_0(I) \leqslant (\mu(I) - d + 1)\lambda(R/I).$$

Moreover, the following statements are equivalent:

(1) $e_0(I) = (\mu(I) - d + 1)\lambda(R/I)$.

(2) *I is an Ulrich ideal*

(3) *I/Q is a free R/I-module.*

Proof.

$$\begin{tikzcd}
 & R \arrow[d, dash] \arrow[ddl, dash, bend right] \arrow[ddr, dash, bend left] & \\
 & \mathfrak{m} \arrow[dl, dash] \arrow[dr, dash] & \\
Q \arrow[dr, dash] & & I \arrow[dl, dash] \\
 & I^2 = IQ &
\end{tikzcd}$$

First, notice that $\lambda(R/I) < \infty$ and $\lambda(R/Q) < \infty$, by 0.5.6. Also,

$$\lambda(R/I) + \lambda(I/I^2) = \lambda(R/Q) + \lambda(Q/QI).$$

By 3.1.30, $\lambda(R/Q) = e_0(I)$. Thus,

$$e_0(I) = \lambda(I/I^2) + \lambda(R/I) - \lambda(Q/QI).$$

We will show that $\lambda(Q/QI) = d\,\lambda(R/I)$ and that $\lambda\,(I/I^2) \leqslant \mu(I)\,\lambda\,(R/I)$ with equality if and only if I/I^2 is a free R/I-module. This proves that

$$e_0(I) \leqslant (\mu(I) - d + 1)\,\lambda\,(R/I)\,.$$

and that equality holds if and only if I/I^2 is a free R/I-module. This will show the inequality we want and also $(1) \Leftrightarrow (2)$.

(a) $\lambda(Q/QI) = d\,\lambda(R/I)$

Using 0.1.9,
$$R/I \otimes R/Q \cong \frac{R/Q}{I\,R/Q} \cong \frac{R/Q}{I/Q} \cong R/I\,.$$

Moreover,
$$I\,\mathsf{gr}_Q(R) := I/Q \oplus IQ/Q^2 \oplus IQ^2/Q^3 \oplus \cdots$$

and thus
$$\frac{\mathsf{gr}_Q(R)}{I\,\mathsf{gr}_Q(R)} = \frac{R/Q \oplus Q/Q^2 \oplus Q^2/Q^3 \oplus \dots}{I/Q \oplus IQ/Q^2 \oplus IQ^2/Q^3 \oplus \dots} = R/I \oplus Q/IQ \oplus Q^2/IQ^2 \oplus \cdots\,.$$

By 0.1.9, using the same reasoning as before, $Q^m/IQ^m = Q^m/Q^{m+1} \otimes R/I$, and thus
$$\frac{\mathsf{gr}_Q(R)}{I\,\mathsf{gr}_Q(R)} \cong \mathsf{gr}_Q(R) \otimes R/I\,.$$

We know, by 3.1.7 and 3.2.1, that $\mathsf{gr}_Q(R)$ is isomorphic to a polynomial ring in d variables over R/Q, that is, $\mathrm{gr}_Q(R) = R/Q\,[T_1,\ldots,T_d]$. Therefore,

$$\frac{\mathrm{gr}_Q(R)}{I\,\mathrm{gr}_Q(R)} \cong R/I\,[T_1,\ldots,T_d]$$

and since Q/QI is the component in degree 1,

$$\frac{Q}{QI} = \frac{R}{I}T_1 \oplus \cdots \oplus \frac{R}{I}T_d \cong \left(\frac{R}{I}\right)^d$$

so that

$$\lambda\left(\frac{Q}{QI}\right) = d\,\lambda(R/I).$$

(b) $\lambda\,(I/I^2) \leqslant \mu(I)\,\lambda\,(R/I)$

As $I\,\frac{I}{I^2} = 0$, the R-module I/I^2 is also a module over R/I, and with the same structure. Let $n = \mu(I)$ and let $x_1,\ldots,x_n \in I$ be such that $\{x_1 + \mathfrak{m}I,\ldots,x_n + \mathfrak{m}I\}$ is a basis of $I/\mathfrak{m}I$ as a vector space over $R/\mathfrak{m}$, so that $I = (x_1,\ldots,x_n)$, by 0.1.11. Clearly, $x_1 + I^2,\ldots,x_n + I^2$ generate I/I^2. As $I^2 \subseteq \mathfrak{m}I$ and $x_i \notin \mathfrak{m}I$ for each i, then $x_i \notin I^2$. This shows that $\mu\,(I/I^2) = \mu(I)$. Consider the map $f\colon (R/I)^{\mu(I)} \longrightarrow I/I^2$ given by

$$f(a_1 + I,\ldots,a_n + I) = (a_1x_1 + \ldots + a_nx_n) + I^2.$$

This map is well-defined: indeed, if $a_1,\ldots,a_n,b_1,\ldots,b_n \in R$ are such that

$$(a_1 + I,\ldots,a_n + I) = (b_1 + I,\ldots,b_n + I),$$

then $a_i - b_i \in I$ for each $i \leqslant n$, and thus $(a_i - b_i)x_i \in I^2$, so that

$$((a_1 - b_1)x_1 + \ldots + (a_n - b_n)x_n) + I^2 = 0,$$

and therefore

$$(a_1x_1 + \ldots + a_nx_n) + I^2 = (b_1x_1 + \ldots + b_nx_n) + I^2.$$

It is also clear that f is a homomorphism of R-modules. Moreover, f is surjective, considering that $x_1 + I^2,\ldots,x_n + I^2$ generate I/I^2. Thus, we have a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow (R/I)^{\mu(I)} \longrightarrow I/I^2 \longrightarrow 0\ ,$$

so that, by 0.5.5,

$$\lambda\,(I/I^2) + \lambda(\ker(f)) = \lambda\left((R/I)^{\mu(I)}\right) = \mu(I)\,\lambda\,(R/I)\,.$$

Since $\lambda(\ker(f)) \geqslant 0$ and $\lambda(\ker(f)) = 0$ if and only if $\ker(f) = 0$, we have

$$\lambda\,(I/I^2) \leqslant \mu(I)\,\lambda\,(R/I)$$

with equality if and only if f is an isomorphism.

(c) Equality holds if and only if I/I^2 is a free R/I-module

We have seen that equality holds if and only if f is an isomorphism, that is,

$$I/I^2 \cong (R/I)^{\mu(I)}$$

and in that case I/I^2 is a free R/I-module.

On the other hand, if I/I^2 is a free R/I-module, then

$$I/I^2 \cong (R/I)^{\mu(I)}$$

considering that I/I^2 is minimally generated by $\mu(I)$ elements. Then

$$\lambda(I/I^2) = \mu(I)\,\lambda(R/I)\,.$$

Now let us show $(2) \Leftrightarrow (3)$.

As $I^2 = IQ \subseteq Q$, we have a canonical inclusion $Q/I^2 \subseteq I/I^2$ with cokernel

$$\frac{I/I^2}{Q/I^2} \cong I/Q\,.$$

Then we have a short exact sequence of R/I-modules

$$0 \longrightarrow Q/I^2 \longrightarrow I/I^2 \longrightarrow I/Q \longrightarrow 0\ .$$

As we have seen in (a),

$$\frac{Q}{I^2} \cong \frac{Q}{IQ} \cong (R/I)^d$$

is a free R/I-module.

$(\Leftarrow)$ Assume that I/Q is a free R/I-module, and thus a projective R/I-module. Then the sequence splits, and thus

$$\frac{I}{I^2} \cong \frac{I}{Q} \oplus \frac{Q}{I^2}$$

is a free R/I-module, being the direct sum of two free R/I-modules.

$(\Rightarrow)$ Now assume that I/I^2 is a free R/I-module. Then

$$0 \longrightarrow Q/I^2 \longrightarrow I/I^2 \longrightarrow I/Q \longrightarrow 0$$

is a projective resolution of I/Q, so that proj dim $(I/Q) \leqslant 1$.

As R/I is Noetherian, by 0.3.3, then I/Q is a finitely generated R/I-module. By 0.9.10,

$$\text{proj dim}\left(\frac{I}{Q}\right) + \text{depth}\left(\frac{I}{Q}\right) = \text{depth}\left(\frac{R}{I}\right).$$

But $\dim\left(\frac{R}{I}\right) = 0$, as I is an $\mathfrak{m}$-primary ideal and so

$$\text{depth}\left(\frac{R}{I}\right) \leqslant \dim\left(\frac{R}{I}\right) = 0.$$

Therefore, proj dim $\left(\frac{I}{Q}\right) = 0$, so that $\frac{I}{Q}$ is a projective module. By 0.8.2, $\frac{I}{Q}$ is a free module over R/I. □

Proposition 3.2.6. *[6, Proposition 2.4] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field $R/\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary ideal. Then I is an Ulrich ideal if and only if for every minimal reduction Q of I, $I^2 \subseteq Q$ and the R/I-module I/Q is free.*

Proof.

($\Rightarrow$) Let Q be a minimal reduction of I. If I is an Ulrich ideal, then $I^2 = IQ \subseteq Q$, by 3.2.3, and I/Q is a free R/I-module by 3.2.5.

($\Leftarrow$) Let $n = \mu(I)$ and $d = \dim(R)$. If $n = d$, then I is a parameter ideal and thus an Ulrich ideal by 3.2.4. Now assume that $n > d$. Pick generators $x_1, \ldots, x_n \in I$ such that $I = (x_1, \ldots, x_n)$ and for every choice of $1 \leqslant i_1 < \cdots < i_d \leqslant n$, $(x_{i_1}, \ldots, x_{i_d})$ is a reduction of I.

Let us show that I/I^2 is a free R/I-module by showing that

$$\left\{x_1 + I^2, \ldots, x_n + I^2\right\}$$

is a basis for I/I^2 over R/I. It is clear that it is a generating set, so all we have to do is show that it is a linearly independent set. So consider any $c_1, \ldots, c_n \in R$ such that

$$(c_1 + I)(x_1 + I^2) + \ldots + (c_n + I)(x_n + I^2) = 0.$$

We want to show that $c_1, \ldots, c_n \in I$. We have

$$c_1 x_1 + \ldots + c_n x_n \in I^2.$$

Fix $1 \leqslant i \leqslant n$ and let $\Lambda \subseteq \{1, \ldots, n\}$ be such that $\#\Lambda = d$ and $i \notin \Lambda$. Let $\overline{\Lambda} = \{1, \ldots, n\} \setminus \Lambda$. Consider the reduction $Q = (x_j \mid j \in \Lambda)$, which is minimal by 3.1.23, because it is generated by $d = \ell(I)$ elements. As $I = (x_1, \ldots, x_n)$ and $Q = (x_j \mid j \in \Lambda)$, then

$$I/Q = \left(x_j + Q \mid j \in \overline{\Lambda}\right).$$

Since, by 3.1.21,

$$\mu\left(I/Q\right) = \mu(I) - \mu(Q) = n - d,$$

then $\left\{x_j + Q \mid j \in \overline{\Lambda}\right\}$ is a basis for I/Q as a free module over R/I. As $x_j + Q = 0$ for every $j \in \Lambda$, then

$$\sum_{j \in \Lambda} (c_j + I)(x_j + Q) = 0 \Rightarrow \sum_{j \in \overline{\Lambda}} (c_j + I)(x_j + Q) = 0.$$

Then $c_j \in I$ for every $j \in \overline{\Lambda}$, considering that $\{x_j + Q \mid j \in \overline{\Lambda}\}$ is a basis for I/Q. In particular, $c_i + I = 0$.

As we chose an arbitrary $i \leqslant n$, this shows that $c_1 + I = \ldots = c_n + I = 0$, and thus

$$\{x_1 + I^2, \ldots, x_n + I^2\}$$

is a basis for I/I^2 over R/I. Therefore, I/I^2 is a free module over R/I.

Now all that remains is to show that $IQ = I^2$. We always have $IQ \subseteq I^2$, so all we need to show is that $I^2 \subseteq IQ$. For simplicity, let us assume, without loss of generality, that $Q = (x_1, \ldots, x_d)$. Let $y \in I^2 \subseteq Q$. Pick $c_1, \ldots, c_d \in R$ such that $y = c_1x_1 + \cdots + c_dx_d$. As $y \in I^2$, we have

$$(c_1 + I)(x_1 + I^2) + \ldots + (c_d + I)(x_d + I^2) = (c_1x_1 + \ldots + c_dx_d) + I^2 = y + I^2 = 0$$

and therefore $c_1 + I = \ldots = c_d + I = 0$, that is, $c_1, \ldots, c_d \in I$. But then for every $1 \leqslant i \leqslant d$, $c_ix_i \in IQ$, and thus $y \in IQ$, as desired. ☐

Definition 3.2.7 (Cohen-Macaulay type)**.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. For each Cohen-Macaulay R-module M we set

$$r(M) := \lambda\left(\operatorname{Ext}_R^s(R/\mathfrak{m}, M)\right)$$

where $s := \dim(M)$, and we call $r(M)$ the *Cohen-Macaulay type* of M.

Lemma 3.2.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. Then $r(R) = \operatorname{type}(R)$. In particular, R is a Gorenstein ring if and only if $r(R) = 1$.*

Proof. Let $d := \dim(R)$. By 0.5.7, we have

$$\dim_{R/\mathfrak{m}} \operatorname{Ext}_R^d(R/\mathfrak{m}, R) = \lambda\left(\operatorname{Ext}_R^d(R/\mathfrak{m}, R)\right)$$

and thus $r(R) = \operatorname{type}(R)$. By 1.4.16,

$$r(R) = \operatorname{type}(R) = 1$$

if and only if R is Gorenstein. ☐

Definition 3.2.9 (Good ideal)**.** Let $(R, \mathfrak{m})$ be a Gorenstein local ring and I an $\mathfrak{m}$-primary ideal. We say that I is a *good ideal* if $I^2 = QI$ and $(Q :_R I) = I$ for any minimal reduction Q of I.

Corollary 3.2.10. *[6, Corollary 2.6] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field, $\dim(R) = d$ and I an $\mathfrak{m}$-primary ideal that is not a parameter ideal. Suppose I is an Ulrich ideal and let $n = \mu(I) > d$. Then, for any minimal reduction Q of I,*

(1) $(Q :_R I) = I$.

(2) $(n - d)r(R/I) = r(I/Q) \leqslant r(R)$.

Proof.

(1) As we have seen in the proof of 3.2.5, $I/Q \cong (R/I)^{n-d}$. So let $f\colon I/Q \longrightarrow (R/I)^{n-d}$ be an isomorphism. We want to show that $(Q :_R I) = I$. As I is an Ulrich ideal we have $I^2 = IQ \subseteq Q$. So let us show that $(Q :_R I) \subseteq I$.

Let $x \in I$ be such that $f(x+Q) = (1+I, 0, \ldots, 0) \in (R/I)^{n-d}$. As $bx \in Q$, then $b(x+Q) = 0$, and thus

$$(b+I, 0, \ldots, 0) = b\,(1+I, 0, \ldots, 0) = b\,f(x+Q) = f(b(x+Q)) = f(0) = 0,$$

which means that $b \in I$.

(2) By (1),

$$\mathrm{Ann}_R\left(\frac{I}{Q}\right) = (Q :_R I) = I,$$

and thus

$$\dim(I/Q) = \dim\left(\frac{R}{\mathrm{Ann}_R(I/Q)}\right) = \dim\left(\frac{R}{I}\right) = 0.$$

Therefore, using 0.1.12,

$$r\,(I/Q) = \lambda\left(\mathrm{Hom}_R\left(R/\mathfrak{m}, (R/I)^{n-d}\right)\right)$$
$$= \lambda\left((\mathrm{Hom}_R(R/\mathfrak{m}, R/I))^{n-d}\right) = (n-d)\lambda\,(\mathrm{Hom}_R(R/\mathfrak{m}, R/I)) = (n-d)\,r\,(R/I)\,.$$

Also, as Q is $\mathfrak{m}$-primary, $\dim(R/Q) = 0$, and thus

$$r\,(R/Q) = \lambda\left(\mathrm{Ext}^0_R\,(R/\mathfrak{m}, R/Q)\right) = \lambda\,(\mathrm{Hom}_R\,(R/\mathfrak{m}, R/Q))\,.$$

Since $Q\,(R/\mathfrak{m}) = 0$, similarly to what we did on 1.4.15, we get

$$\lambda\,(\mathrm{Hom}_R\,(R/\mathfrak{m}, R/Q)) = \lambda\left(\mathrm{Hom}_{R/Q}\,(R/\mathfrak{m}, R/Q)\right)$$

and using 1.4.6, as Q is a parameter ideal, then

$$\mathrm{Hom}_{R/Q}\,(R/\mathfrak{m}, R/Q) \cong \mathrm{Ext}^d_R\,(R/\mathfrak{m}, R)\,,$$

so that

$$r\,(R/Q) = r\,(R)\,.$$

Consider the canonical inclusion

$$0 \longrightarrow I/Q \longrightarrow R/Q\ .$$

Applying $\mathrm{Hom}_R(R/\mathfrak{m}, -)$ to this exact sequence, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/\mathfrak{m}, I/Q) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{m}, R/Q)\ ,$$

so that

$$\lambda\left(\operatorname{Hom}_R(R/\mathfrak{m}, I/Q)\right) \leqslant \lambda\left(\operatorname{Hom}_R(R/\mathfrak{m}, R/Q)\right),$$

and therefore,

$$r(I/Q) \leqslant r(R/Q).$$

Using our previous calculations, we get

$$(n-d)\, r\,(R/I) = r(I/Q) \leqslant r(R/Q) = r(R)$$

as desired.

$\square$

Proposition 3.2.11. *Let $(R, \mathfrak{m})$ be an Artinian Gorenstein local ring. If $M \neq 0$ is a finitely generated R-module with $\operatorname{Ann}_R(M) = 0$, then M is free.*

Corollary 3.2.12. *[6, Corollary 2.6] Let $(R, \mathfrak{m})$ be a Gorenstein local ring with infinite residue field, $\dim(R) = d$ and I an $\mathfrak{m}$-primary ideal that is not a parameter ideal. The following conditions are equivalent:*

(1) *I is an Ulrich ideal.*

(2) *I is good and $\mu(I) = \dim(R) + 1$.*

(3) *I is good and R/I is a Gorenstein ring.*

Proof. Every Ulrich ideal is good. Indeed, for any minimal reduction Q of I, $I^2 = IQ$ from the definition, and $(Q :_R I) = I$ by 3.2.10.

Let us show that $(1) \Rightarrow (2), (3)$. By 3.2.10 and 3.2.8,

$$(\mu(I) - \dim(R))\, r(R/I) \leqslant r(R) = 1.$$

As $\mu(I) - \dim(R) \geqslant 1$, considering I is not a parameter ideal and must then be generated by more than $d = \dim(R)$ elements, then

$$r(R/I) \leqslant (\mu(I) - \dim(R))\, r(R/I) \leqslant 1.$$

Thus, $r(R/I)$ is either 0 or 1. Suppose that $r(R/I) = 0$. As $\dim(R/I) = 0$, then

$$0 = r(R/I) = \lambda\left(\operatorname{Ext}^0_R(R/\mathfrak{m}, R/I)\right) = \lambda\left(\operatorname{Hom}_R(R/\mathfrak{m}, R/I)\right),$$

and so $\operatorname{Hom}_R(R/\mathfrak{m}, R/I) = 0$.

However, we will construct a non-zero $f \in \operatorname{Hom}_R(R/\mathfrak{m}, R/I)$. Since I is an $\mathfrak{m}$-primary ideal, $\sqrt{I} = \mathfrak{m}$. As R is Noetherian, then $\mathfrak{m}$ is a finitely generated ideal. By 0.4.4, for big enough $s \geqslant 1$ we have $\mathfrak{m}^s \subseteq I$. Let s be the minimal such s, meaning that $\mathfrak{m}^s \subseteq I$ and $\mathfrak{m}^{s-1} \nsubseteq I$. Pick $a \in \mathfrak{m}^{s-1}$, $a \notin I$. Then $\mathfrak{m}a \subseteq I$.

Consider f: $R/\mathfrak{m} \longrightarrow R/I$ given by

$$f(r+\mathfrak{m}) = ra + I.$$

Such f well-defined and clearly a homomorphism of R-modules. It is indeed well-defined, because

$$r+\mathfrak{m} = 0 \Rightarrow r \in \mathfrak{m} \Rightarrow ra \in I \Rightarrow ra + I = 0 \Rightarrow f(r+\mathfrak{m}) = 0.$$

Also, since $a \notin I$, then $f(1+\mathfrak{m}) = a + I \neq 0$, and thus $f \neq 0$. Therefore, we cannot have $r(R/I) = 0$, and thus $r(R/I) = 1$. The previous inequaly becomes

$$1 \leqslant \mu(I) - \dim(R) \leqslant 1,$$

and thus $\mu(I) - \dim(R) = 1$, which shows that $\mu(I) = \dim(R) + 1$. Note that R/I is a Cohen-Macaulay local ring because it is Artinian. We also showed, using 3.2.8, that

$$\operatorname{type}(R/I) = r(R/I) = 1,$$

and by 1.4.16, that implies that R/I is Gorenstein. So we showed both $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$.

To show $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$, notice that in both cases we already have $I^2 = QI$ for any minimal reduction Q of I, since I is a good ideal.

$(2) \Rightarrow (1)$

Let Q be a minimal reduction of I. We know that $\mu(I) = \dim(R) + 1 = \mu(Q) + 1$. Take a minimal generating set $x_1, \ldots, x_d$ of Q. By 3.1.23, we can find $x_{d+1} \in I$ such that $I = (x_1, \ldots, x_{d+1})$. Write $x = x_{d+1}$. As I/Q is generated by $x + Q$, then I/Q is a cyclic R-module. Since I is a good ideal, we also know that $(Q :_R I)$, which can be restated as $I = \operatorname{Ann}_R(I/Q)$.

Let $M = Rm$ be a cyclic module over R with $\operatorname{Ann}_R(M) = I$. Then the canonical surjection $R \longrightarrow M$ sending $1 \mapsto m$ has kernel I, so that $M \cong R/I$. This implies that $I/Q \cong R/I$, and thus I/Q is a free R/I-module. By 3.2.5, I is an Ulrich ideal.

$(3) \Rightarrow (1)$ Since R/I is a Noetherian ring, by 0.3.3, and $\dim(R/I) = 0$, then R/I is an Artinian ring, by 0.3.6. Then R/I is an Artinian Gorenstein local ring. Also, I/Q is a finitely generated R/I-module with

$$\operatorname{Ann}_{\frac{R}{I}}\left(\frac{I}{Q}\right) = \frac{(Q :_R I)}{I} = I/I = 0.$$

By 3.2.11, I/Q is a free I/Q-module. By 3.2.5, this implies that I is an Ulrich ideal. ☐

Example 3.2.13. We can use 3.2.12 to find Ulrich ideals of Gorenstein rings. We saw in 1.4.4 that the ring $R = \frac{K[[x,y,z]]}{(x^3-y^2,z^2-x^2y)}$ is Gorenstein with $\dim(R) = \operatorname{depth}(R) = 1$. We can show that $R \cong K[[t^4, t^6, t^7]]$. Let $I = (t^4, t^6)$ and $Q = (t^4)$. Clearly, $\mu(I) = 2 = \dim(R) + 1$. Moreover,

$$I^2 = (t^8, t^{10}, t^12) = (t^8, t^{10}, t^4 t^8) = (t^8, t^{10}) = (t^4)(t^4, t^6) = QI.$$

This implies that $I \subseteq (Q : I)$. Suppose that $I \subsetneq (Q : I)$. Then we would have $(t^7) \cap (Q : I) \neq 0$, and thus there exist $a_n \in K$, $n \geqslant 0$, such that

$$f(t) = \sum_{n=0}^{\infty} a_n t^{7n} \in (Q : I).$$

Moreover, $t^4 f(t) \in Q = (t^4)$. Let n be such that $a_n \neq 0$. Then $t^{7n+4} \in t^{4m}$ for some m. Similarly, and $t^{7n+6} = t^{4k}$. Then $4|7n+4$ and $4|7n+6$, which is clearly impossible. Therefore, $I = (Q : I)$. Then I is an Ulrich ideal with Q a minimal reduction.

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Index

List of Symbols

$(I :_R J)$	$\{r \in R : rJ \subseteq I\}$, page 11
$\mathrm{Ann}_R(M)$	Annihilator of M in R, page 11
$\mathrm{Ann}_R(m)$	Annihilator of m in R, page 11
ara(I)	There exists $\sqrt{J} = \sqrt{I}$ with J generated by ara(I) elements, page 73
$\mathrm{Ass}(M)$	Associated primes of M, page 11
Č$(\underline{x})$	Čech complex, page 65
$\deg(m)$	Degree of a homogeneous element m in a graded module, page 76
$\deg(r)$	Degree of a homogeneous element r in a graded ring, page 76
depth(M)	Depth of the module M, page 10
$\dim(R)$	Krull dimension of R, page 5
$\ell(\mathcal{F})$	Analytic spread of $\mathcal{F}$, page 85
$\ell(I)$	Analytic spread of I, page 85
$\mathrm{gr}_{\mathcal{F}}(R)$	Associated graded ring of $\mathcal{F}$, page 84
$\mathrm{gr}_I(R)$	Associated graded ring of I, page 84
grade(I, M)	Grade of M with respect to I, page 10
$\mathrm{H}^i_{\underline{x}}(M)$	H^i(Č$(\underline{x}, M))$, page 65
ht(I)	Height of I, page 5
inj dim$_R(M)$	Injective dimension M as an R-module, page 15
$\lambda(M)$	Length of the module M, page 8
$\mathcal{A}(R)$	Category of Artinian R-modules, page 31
$\mathcal{M}(R)$	Category of finitely generated R-modules, page 31
$\mathcal{N}(R)$	Category of Noetherian R-modules, page 31

$\mu(I)$	Minimal number of generators of I, page 4
$\mu_i(M, P)$	i-th Bass number of M with respect to P, page 53
proj dim(M)	Projective dimension M, page 15
reg(M)	Castelnuovo-Mumford regularity of R, page 81
$\mathrm{Spec}(R)$	Set of prime ideals in R, page 3
$\sqrt{I}$	Radical of the ideal I, page 7
$\mathrm{Supp}(M)$	Support of M, page 12
$\mathrm{type}(R)$	Type of R, page 45
$a(M)$	a-invariant of M, page 79
$a_i(M)$	a_i-invariant of M, page 79
$E(M)$	Injective envelope of M, page 26
$e_0(I)$	Multiplicity of I, page 91
$F(\mathcal{F})$	Fiber cone of $\mathcal{F}$, page 85
$F(I)$	Fiber cone of I, page 85
$H_I^i(M)$	Local Cohomology, page 61
$H_M(t)$	Hilbert function of M at t, page 90
M_P	Localization of M in $R \setminus P$, for P a prime ideal, page 3
M_U	Localization of M in the multiplicative set U, page 3
$R(\mathcal{F})$	Rees Algebra of $\mathcal{F}$, page 84
$r(\mathcal{F})$	Reduction number of $\mathcal{F}$, page 85
$r(I)$	Reduction number of I, page 85
$r_J(\mathcal{F})$	Reduction number of $\mathcal{F}$ with respect to J, page 85
$r_J(I)$	Reduction number of I with respect to J, page 85
$r_R(M)$	Cohen-Macaulay type of M, page 97
$V(I)$	Prime ideals containing the ideal I, page 12