Castelnuovo-Mumford regularity and Ulrich ideals

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Local cohomology was created by Grothendieck to prove Lefschetz-type theorems in Algebraic Geometry, but this theory soon proved to be useful in more algebraic contexts as well. Local cohomology plays an important role in modern Commutative Algebra, mainly because of the importance of the a_i-invariants and the Castelnuovo-Mumford regularity and their connection with recent research.

The study of local cohomology is directly linked to the study of injective envelopes, injective dimension and Gorenstein rings. The theory of injective envelopes was mainly developed by Eben Matlis ([4]) and relates to the study of Gorenstein rings. These rings were first introduced by Alexander Grothendieck ([3]), and they relate to a duality property of singular plane curves studied by Gorenstein ([2]).

Among the many applications of local cohomology, the recent topic of Ulrich ideals arises from decades of study that started with Northcott and Ress' reductions in the 1950s ([5]) and Sally's study of Abyankar's inequality in the 1970s and its connection to blow-up algebras ([7]). Blow-up algebras form a class of graded rings which represent fibrations of a variety with fibers which are affine spaces. This class, that appears in many constructions in Commutative Algebra and Algebraic Geometry, includes polynomial rings, the Rees Algebra and the associated graded ring.

In chapter 1 we study injective modules. Every module M can be embedded in an injective module. Fixing an injective module I containing M, we can find an injective module E such that $M \subseteq E \subseteq I$ and that is minimal with respect to this property. This injective module E is unique up to isomorphism and independent of the choice of I. We denote it by $E(M)$ and call it the injective envelope of M. As an example, we can see that $E(\mathbb{Z}) = \mathbb{Q}$ as modules over $\mathbb{Z}$.

Using injective envelopes, we can characterize injective modules over Noetherian rings:

Theorem. *Let R be a Noetherian ring and E an injective R-module Then E is of the form*

$$E = \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} (E(R/\mathfrak{p}))^{\mu_{\mathfrak{p}}}$$

where $\mu_{\mathfrak{p}}$ are cardinals.

The fact that any module can be embedded in an injective module allows us to build injective resolutions. If M is a module over R, an injective resolution of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

where I_i is an injective module for each $i \geqslant 0$. If there exists an injective resolution of M such that for some n we have $I_i = 0$ for $i \geqslant n$, we say that I has finite injective dimension. The minimal such n is the injective dimension of M. Otherwise, M is said to be of infinite injective dimension.

A Noetherian local ring of finite injective dimension is said to be a Gorenstein ring. Gorenstein rings are Cohen-Macaulay. Moreover, the injective dimension of a Gorenstein ring coincides with its Krull dimension.

In chapter 2 we study local cohomology. We fix a ring R and consider an ideal I in R. For each R-module M, consider

$$\Gamma_I(M) = \{e \in M \mid \exists\, s \geqslant 1 : I^s e = 0\}.$$

In section 2.1 we show that Γ_I is a left exact functor. Therefore, we can consider its right derived functors $\mathrm{H}^i_I(-)$. The modules $\mathrm{H}^i_I(M)$ are the local cohomology modules of I with respect to M.

The most important case is when we consider a Noetherian local ring and its maximal ideal. In that case, local cohomology modules measure depth and dimension of finitely generated modules:

Theorem. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finitely generated R-module. Then*

$$\dim(M) = \sup\left\{i \colon \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\right\}$$

and

$$\mathit{depth}(M) = \inf\left\{i : \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\right\}.$$

If we consider a graded ring $R = \oplus_{n \geqslant 0} R_n$, a homogeneous ideal I and a graded R-module $M = \oplus_{n \geqslant 0} M_n$, the local cohomology modules $\mathrm{H}^i_I(M)$ are also graded R-modules. We write $R_+ = \oplus_{n \geqslant 1} R_n$. If R is a Noetherian ring and $(R_0, \mathfrak{m}_0)$ is a Noetherian local ring, then $N = (\mathfrak{m}_0, R_+)$ is the unique maximal homogeneous ideal. The behavior of R with respect to N mimics the behavior of a Noetherian local ring with respect to its unique maximal ideal. Moreover, we have the following result:

Proposition 0.1. *Let $R = \bigoplus_{n \geqslant 0} R_n$ be a graded ring for which $(R_0, \mathfrak{m}_0)$ is a local ring and consider the maximal ideal $N := (\mathfrak{m}_0, R_+)$. Let M be a graded R-module. Then, for every $i \geqslant 0$, there exists $N \geqslant 0$ such that for all $n \geqslant N$,*

$$\left(\mathrm{H}^i_N(M)\right)_n = 0.$$

This allows us to define the a_i-invariants and the Castelnuovo-Mumford regularity for a graded R-module M. For each i, the a_i-invariant of M is defined as

$$a_i(M) := \sup\left\{j \colon \mathrm{H}^i_N(M)_j = 0\right\}.$$

If M is Cohen-Macaulay, only $a_d(M)$ contains non-trivial information. In that case, we denote it by $a(M)$.

Under the same conditions, the Castelnuovo-Mumford regularity of M is defined as

$$\mathrm{reg}(M) := \max\{a_i(M) + i \mid i \geqslant 0\}.$$

In chapter 3 we study Ulrich ideals, which relate to reductions of ideals and to the theory of blow-up algebras. For two ideals $J \subseteq I$, J is a reduction of I if $I^{n+1} = JI^n$ for some integer $n \geqslant 1$. We say that J is a minimal reduction of I if it is minimal with respect to containment. Blow-up algebras include the associated graded ring, which corresponds to an important geometric construction. If R is the localization at the origin of the coordinate ring of an affine variety passing through 0, then the associated graded ring of R is the coordinate ring of the tangent cone of the variety considered, which is the cone composed of all the lines that are limiting positions of secant lines to the variety at 0. For an ideal I in the ring R, the associated graded ring of I is defined as

$$\mathrm{gr}_I(R) := R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots = \bigoplus_{n \geqslant 0} I^n/I^{n+1}.$$

For Cohen-Macaulay local rings $(R, \mathfrak{m})$, Abhyankar proved an inequality ([1]) that relates the minimal number of generators of $\mathfrak{m}$, $\mu(\mathfrak{m})$, the dimension of R and the e_0-multiplicity of $\mathfrak{m}$, $e_0(\mathfrak{m})$, which measures the length of R/J for J a minimal reduction of $\mathfrak{m}$:

$$e_0(\mathfrak{m}) \geqslant \mu(\mathfrak{m}) - \dim(R) + 1.$$

Sally showed ([6]) that in the case of equality, the associated graded ring of $\mathfrak{m}$ is Cohen-Macaulay, and that for any minimal reduction J of $\mathfrak{m}$, $\mathfrak{m}^2 = J\mathfrak{m}$. Ulrich ideals appear as a result of the attempts to obtain similar results for $\mathfrak{m}$-primary ideals.

We can define Ulrich ideals via reductions or via conditions on the associated graded ring. Consider a Cohen-Macaulay local ring $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal I. We say that I is an Ulrich ideal if the following conditions are satisfied:

(1) For any minimal reduction J of I, $I^2 = JI$.

(2) The R/I-module I/I^2 is free.

Equivalently, I is an Ulrich ideal if

(1) $\mathrm{gr}_I(R)$ is Cohen-Macaulay and $a(\mathrm{gr}_I(R)) \leqslant 1 - \dim(R)$.

(2) The R/I-module I/I^2 is free.

If $d = \dim(R)$, a parameter ideal is an $\mathfrak{m}$-primary ideal generated by d elements. Every parameter ideal is an Ulrich ideal.

Over Gorenstein rings, Ulrich ideals are always good ideals. An $\mathfrak{m}$-primary ideal I of a local Gorenstein ring is a good ideal if for some minimal reduction J of I, we have $I^2 = JI$ and $(J :_R I) = I$. These ideals are good because the associated graded ring of a good ideal is always Gorenstein.

We finish with a characterization of Ulrich ideals over Gorenstein rings:

Theorem. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $R/\mathfrak{m}$ infinite,* $\dim(R) = d$ *and I an $\mathfrak{m}$-primary ideal that is not a parameter ideal. The following conditions are equivalent:*

(1) *I is an Ulrich ideal*

(2) *I is good and $\mu(I) = \dim(R) + 1$*

(3) *I is good and R/I is a Gorenstein ring.*

References

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