The effects of a bounce on the spectrum of the Gravitational Waves in a metric f(R)-gravity

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(Dated: October, 2012)

This work studies the imprints on the energy spectrum of the cosmological gravitational waves of the presence of a bounce in the early universe. The model of the bounce is constructed is a metric $f(R)$ theory of gravity, where R is the scalar curvature, and considers a smooth bounce preceded by a de Sitter-like contraction phase and followed by a de Sitter-like inflation. An $f(R)$ action that converges to GR well during inflation is obtained, therefore the modified theory of gravity is only applied during the early stages of the universe. The energy spectrum of the gravitational waves through is analysed through the method of the Bogoliubov coefficients by two means: taking into account the gravitational perturbations due to the modified gravitational action in the $f(R)$ setup and by simply considering those perturbations inherent to the standard Hilbert-Einstein action. For certain regions of the parameter space, distinct (oscillatory) signals appear on the spectrum for very low frequencies.

PACS numbers: 98.80.-k, 98.80.Bp, 95.36.+x, 04.30.-w
Keywords: Bouncing cosmologies, inflation, modified theories of gravity, gravitational waves

I. INTRODUCTION

In this work, we look at the energy spectrum of the cosmological Gravitational Waves (GW), while considering a bounce in the early universe described in the setup of modified theories of gravity, specifically a metric $f(R)$ theory of gravity [1–3]. The energy spectrum of the cosmological GW is determined at the present time using the method of the continuous Bogoliubov coefficients [4–6], complemented by a convenient change of variables [7–13]. The use of a modified theory of gravity in this work is motivated by the fact that a bounce in a spatially flat a Friedmann-Lemaître-Robertson-Walker (FLRW) universe implies a violation of the Null Energy Condition [14]. In addition, the inflationary phase after the bounce also involves a violation of the Strong Energy Condition [15]. By employing an $f(R)$ theory of gravity, instead of GR, we avoid the need for inserting some kind of matter or scalar fields that violate these conditions [16–18]. The $f(R)$ setup is only applied during the early universe, with the bounce being preceded by a de Sitter-like contraction and followed by de Sitter-like inflation. At some point, during inflation, one switches to GR; this way, the ΛCDM model can be used to describe the late-time evolution of the universe.

The structure of this work is: Section II presents a brief review on $f(R)$ cosmology. The features of the model discussed in this work are introduced in Section III. A general method of treating cosmological tensorial perturbations in GR and $f(R)$ and calculating the energy spectrum of the GW is presented in Section IV, while the numerical application and results of this method are presented in Section V. In Section VI, a discussion of the results obtained and some final remarks are given.

II. f(R) GRAVITY

In metric $f(R)$ gravity, the Hilbert-Einstein action is generalized by replacing the scalar curvature R in the action by a general function $f(R)$ [2]:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S^{(m)}(g_{\mu\nu}, \psi). \tag{2.1}$$

Here, $\kappa^2 \equiv 8\pi G$, G is the gravitational constant and g is the determinant of the metric (one assumes $c = \hbar = 1$). The matter component of the action, $S^{(m)}$, depends solely on the metric $g_{\mu\nu}$ and the matter fields ψ, and is related to the stress energy tensor in the usual way.

The generalized Friedmann and Raychaudhury equations are obtained by minimizing Eq. (2.1) and inserting the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which gives:

$$H^2 = \frac{\kappa^2}{3f_R}\left(\rho - \frac{f(R) - f_R R}{2\kappa^2} - 3H\frac{f_{RR}\dot{R}}{\kappa^2}\right), \tag{2.2}$$

$$2\dot{H} + 3H^2 = \frac{\kappa^2}{f_R}\left(p + \frac{f(R) - f_R R}{2\kappa^2} + \frac{f_{RRR}\dot{R}^2 + f_{RR}\ddot{R} + 2Hf_{RR}\dot{R}}{\kappa^2}\right), \tag{2.3}$$

The extra terms that appear in these equations can be written as the effective the energy density and pressure of a "cosmological" fluid:

$$\rho^{(eff)} = -\frac{f - f_R R}{2f_R} - \frac{3Hf_{RR}\dot{R}}{f_R}, \tag{2.4}$$

$$p^{(eff)} = \frac{f - f_R R}{2f_R} + \frac{f_{RRR}\dot{R}^2 + f_{RR}\ddot{R} + 2Hf_{RR}\dot{R}}{f_{RR}}, \tag{2.5}$$

By taking Eq. (2.2) in the limit of $\rho^{(m)} \to 0$ and imposing a positive gravitational coupling, i.e., $f_R > 0$, one finds that, in a spatially flat FLRW spacetime, $\rho^{(eff)}$ has to be non-negative, as the left hand side of the equation is always positive.

Any metric $f(R)$ theory of gravity can be cast in the form a Scalar-Tensor theory, in particular a Brans-Dicke theory with a Brans-Dicke parameter $\omega_0 = 0$, and therefore, it introduces a new scalar degree of freedom in regards to General Relativity. In metric $f(R)$ gravity, a convenient representation of this degree of freedom is the scalar field $\phi \equiv f_R$whose squared mass is given by [3, 19, 20]:

$$m^2 = \frac{1}{3}\left(\frac{f_R}{f_{RR}} - R\right). \tag{2.6}$$

III. THE MODEL

In this work a phenomenological bounce is considered in the early stages of a spatially flat FLRW universe ($K = 0$) as a way to avoid the big bang singularity. The choice of working in a spatially flat FLRW space-time is due to the apparent flatness of the universe today and to the fact that the treatment of cosmological perturbations is made easier if $K = 0$. An $f(R)$ framework, specifically the metric $f(R)$ formalism, is used to describe the bounce. After the bounce, as the universe enters a de Sitter-like inflationary regime, one switches continuously to a GR setup. This way, the ΛCDM model, preceded by a radiation phase, can be used to describe the late-time evolution of the universe. A modified Generalized Chaplygin Gas (mGCG), [12], is used to interpolate the inflationary era induced by $f(R)$-gravity and the radiation dominated universe.

A. A bounce in the early universe

Inspired on the de Sitter solution for a closed FLRW space-time, we define the scale factor around as:

$$a(t) = a_b \cosh\left[H_{\text{inf}}(t)\right] \tag{3.1}$$

where $a(t)$ is the scale factor, a_b is a constant quantifying the size of the universe at the bounce. In addition, H_{inf} is related to the energy scale of inflation just after the bounce [1]:

$$H_{\text{inf}}^2 = \frac{\kappa^2}{3} E_{\text{inf}}^4. \tag{3.2}$$

[1] Notice at this regard that $\ddot{a} > 0$ at all times and in particular for $t > t_b$. In addition, $\dot{a} > 0$ for $t > t_b$.

Since the existence of such a bounce is prohibited for a flat space-time in GR unless exotic matter, which violates the null energy conditions, is invoked, [14], an $f(R)$ theory of gravity is employed to the described the evolution of the universe during the early stages. The appropriate $f(R)$ action is obtained by fixing the behaviour of the scale factor and then solving the Friedmann equation for for the function $f(R)$ ("designer" $f(R)$ gravity). However, this procedure does not uniquely determine $f(R)$, as the substitution of the scale factor in the Friedmann equation produces a second order differential equation. Therefore, some additional criteria are needed to obtain the physically viable form of the $f(R)$ function. In this work, it will be required that the $f(R)$ action verifies the following constraints:

- $f(R)$ must become linear well into the inflationary phase, so as to recover the Hilbert-Einstein action.
- f_R has to be positive, so that gravity is attractive, i.e. $G^{(eff)} > 0$;
- During inflation one must have $f_R \to 1$, so that the effective gravitational coupling converges asymptotically to the gravitational constant G;
- f_{RR} has to be positive to guarantee that the Dolgov-Kawasaki stability condition is met and the theory is ghost free.

Substituting Eq. (3.1) in the Friedmann equation with $K = 0$ and $\rho^{(m)} = 0$, gives the following differential equation for the function $f(R)$:

$$\ddot{f} + H_{\text{inf}}\left[3\tanh(H_{\text{inf}}t) - 2\text{cotanh}(H_{\text{inf}}t)\right]\dot{f} + 2H_{\text{inf}}^2 \text{sech}^2(H_{\text{inf}}t) f = 0. \tag{3.3}$$

This second order differential equation for f was solved analytically using the software *Mapple 14* and a general solution $f = C_1 f_1 + C_2 f_2$ was found as a linear combination of the real-valued functions

$$f_1(t) = \sqrt{6\tanh^2(H_{\text{inf}}t) + 3}\cos\left[\theta_1(t) + \theta_2(t)\right], \tag{3.4}$$

$$f_2(t) = \sqrt{6\tanh^2(H_{\text{inf}}t) + 3}\sin\left[\theta_1(t) + \theta_2(t)\right]. \tag{3.5}$$

where the phases θ_1 and θ_2 are defined as:

$$\theta_1(t) = \text{sign}(t)\frac{\sqrt{3}}{2}\arccos[1 - 2\tanh^2(H_{\text{inf}}t)], \tag{3.6}$$

$$\theta_2(t) = \arccos\left(\frac{3\tanh(H_{\text{inf}}t)}{\sqrt{6\tanh^2(H_{\text{inf}}t) + 3}}\right). \tag{3.7}$$

The linear coefficients C_1 and C_2 can be obtained by considering the constraints set on $f(R)$ in the beginning of the section and in fact we find that ir order for $f_R =$

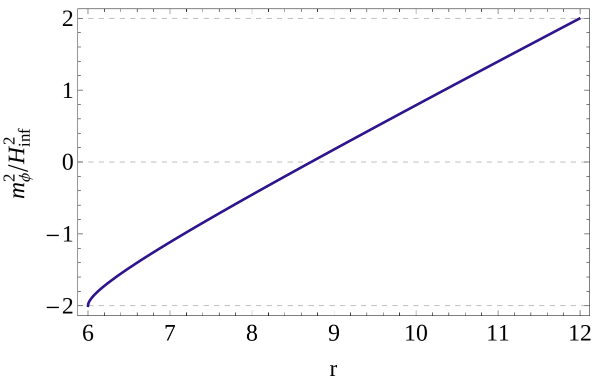


FIG. 1. The squared mass of the scalar field ϕ as a function of the reduced scalar curvature; $m_\phi^2/H_{\rm inf}^2$ as a function of the cosmic time.

$C_1 f_{1R} + C_2 f_{2R}$ to be finite and positive for all R, the coefficients C_1 and C_2 must respect:

$$C_1 = \pm C \cos\left(\frac{\sqrt{3}}{2}\pi\right) \quad \text{and} \quad C_2 = C \sin\left(\frac{\sqrt{3}}{2}\pi\right). \tag{3.8}$$

The value of the positive constant C is obtained by imposing that $f_R \to 1$ as $R \to 12H^2_{textnormalinf}$, which gives $C = 2H^2_{\rm inf}$. With this choice of coefficients, the function $f(R)$ can be written as:

$$f(r) = 2H^2_{\rm inf}\sqrt{r-3}\cos\left\{\frac{\sqrt{3}}{2}\left[\pi - \arccos\left(\frac{9-r}{3}\right)\right]\right. \\ \left. + \arcsin\left(\sqrt{\frac{3}{2}\frac{r-6}{r-3}}\right)\right\}, \tag{3.9}$$

where $r \equiv R/H^2_{textnormalinf}$. The Taylor expansion of the solution in Eq. (3.9) around the limit value $r = 12$ gives:

$$f(R) = R - 6H^2_{\rm inf} + O(R^2), \tag{3.10}$$

with the coefficients of the higher order terms being small when compared to the coefficients of the constant and linear terms. If the linear approximation of the function $f(R)$ during inflation (see Eq. (3.10)) is compared to the Lagrangian density of GR with a cosmological constant Λ:

$$S^\Lambda = \frac{1}{2\kappa^2}\left(R - 2\Lambda\right). \tag{3.11}$$

the identity $\Lambda = 3H^2_{\rm inf}$ ensues immediately. The solution Eq. (3.9) perfectly mimics a vacuum state with a cosmological constant Λ in a GR setup. The origin of the expansion of the universe is precisely in the fact that the $f(R)$ corrections of the theory mimic a cosmological constant after the bounce.

The solution $f(R)$ and its first two R- derivatives are plotted in Fig. 2. In the first graphic, the solution $f(R)$ is compared to the limiting behaviour in Eq. (3.10). It can be immediately verified that the solution found satisfies all the physical constraints imposed above. It is noteworthy to point out that f_{RR}, and consequently all higher order derivatives of $f(R)$, diverges at the bounce. However, in the Friedmann and Raychaudhury equations, these high order derivatives appear in such combinations that the divergence vanishes and the right hand sides of the equations are always finite.

Inserting the solution obtained for $f(R)$ in Eq. (2.6) gives the squared mass of the scalaron, ϕ, as a function of r:

$$m_\phi^2 = -H^2_{\rm inf}\left[4 + \frac{(12-r)(r-3)}{\sqrt{3(12-r)(r-6)}\cot\left(\frac{\sqrt{3}}{2}\pi - \theta_T\right) - 9}\right]. \tag{3.12}$$

Here, it was introduced the total phase $\theta_T \equiv \theta_1 + \theta_2$. Numerically, it can be found that $m_\phi^2 = 0$ roughly when the scalar curvature is about $r \approx 8.72$, and that $m_\phi^2 < 0$ for smaller values. Thus, near the bounce, the scalaron becomes a tachyon. The evolution of the squared mass of the scalaron with r is plotted in Fig. (1).

B. The Universe After The Bounce

The late time evolution of the universe is described using the ΛCDM model with a radiation phase in a GR setup. The Friedmann equation for the ΛCDM model is thus:

$$\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{\Lambda 0} = \left(\frac{H}{H_0}\right)^2, \tag{3.13}$$

where a 0 subscript indicates evaluation at present time and the density parameter is defined as:

$$\Omega_{j0} \equiv \frac{\kappa^2}{3H_0^2}\rho_{j0} \tag{3.14}$$

The transition from the inflationary phase originated by the $f(R)$ gravity and the latter radiation phase is made using a mGCG model, as used in Ref. [12]. For this model, the energy density is given by:

$$\rho_{mGCG} = \left(A + \frac{B}{a^{4(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}, \tag{3.15}$$

where A and B are positive constants and α is a free parameter which obeys $1 + \alpha < 0$. With this choice for the parameter α, the Chaplygin gas behaves as a cosmological constant for small scale factors and as a radiation fluid for large scale factors, with a smooth transition in the intermediate region. The value of the constants A and B can be fixed by comparing the asymptotic behaviour

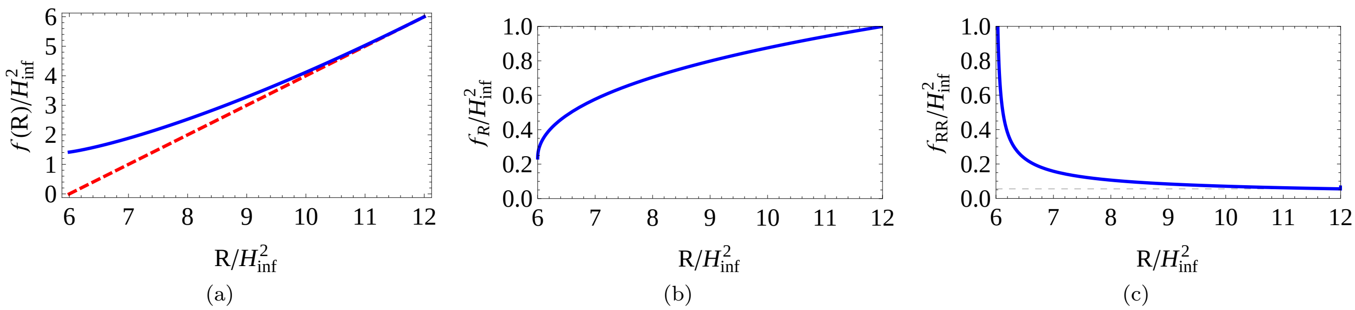


FIG. 2. These plots show from left to right: (a) the behaviour of $f(R)$ as a function of R/H_{inf}^2 (see the blue curve) and the Hilbert-Einstein action $R - 6H_{\mathrm{inf}}^2$ (see the red curve) corresponding to the limiting behaviour of $f(R)$; (b) the behaviour of f_R as a function of R/H_{inf}^2; (c) the behaviour of f_{RR} as a function of R/H_{inf}^2.

of the Chaplygin gas with the density limit during the inflation in the $f(R)$ era and the energy density of the radiation in the ΛCDM modelwhich gives $A = \rho_{\mathrm{inf}}^{1+\alpha}$ and $B = (\rho_{r0} a_0^4)^{1+\alpha}$.

IV. GRAVITATIONAL WAVES

Cosmological (tensorial) gravitational waves originate during inflation from quantum fluctuations of the metric $g_{\mu\nu}$. In this work, the approach of Refs. [19, 22] is followed in order to obtain the quantification of the tensorial perturbations. A Bogoliubov transformation, first considered By L. Parker, [6], and Starobinsky, [4], is then employed to describe the evolution of the quantum operators in terms of time-fixed annihilation and creation operators and the linear coefficients α and β, after which a final variable change ensues as described in Refs. [8, 9]. This method has the double advantage of permitting the calculation of the power and energy spectra in terms of the graviton content of the universe and providing an easy way to determine the initial conditions for the necessary numerical integrations.

A. Tensor Perturbations and Energy Spectrum

In the conformal coordinates (η, x, y, z) the tensorial part of the linear perturbations of a flat FLRW metric can be written as:

$$ds^2 = a^2 \left\{ -d\eta^2 + \left[g_{ij}^{(3)} + 2c_{ij}(\eta, \mathbf{x}) \right] dx^i dx^j \right\}, \quad (4.1)$$

where $g_{\alpha\beta}^{(3)}$ is the comoving background three-space metric and c_{ij} is a traceless and transverse quantity that describes the temporal and spatial dependencies of the perturbations.

It can be shown that the quantum operator that describes the contribution of the two possible polarizations for the GW ($+$ and $\times$) is [19, 22]:

$$\hat{h}_l(\eta, \mathbf{x}) = \sqrt{8\pi G} \int \frac{\mathbf{d^3k}}{(2\pi)^{3/2}} \left(\tilde{h}_{l\mathbf{k}}(\eta) \hat{a}_{l\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + H.c. \right). \quad (4.2)$$

Here $H.c.$ stands for the Hermitian conjugate, the mode functions $\tilde{h}_{l\mathbf{k}}(\eta)$ obey the Wronskian condition:

$$\tilde{h}_{l\mathbf{k}} (\tilde{h}_{l\mathbf{k}}^*)' - \tilde{h}_{l\mathbf{k}}^* (\tilde{h}_{l\mathbf{k}})' = \frac{i}{a^2 f_R}, \quad (4.3)$$

and $\hat{a}_{l\mathbf{k}}(\eta)$ is the annihilation operator for gravitons, which obeys the usual commutation relations.

The time dependent annihilation, $\hat{a}_{l\mathbf{k}}(\eta)$, and production, $\hat{a}_{l\mathbf{k}}^\dagger(\eta)$, operators can be related to the time-fixed operators $\hat{A}_{l\mathbf{k}} \equiv \hat{a}_{l\mathbf{k}}(\eta_i)$ and $\hat{A}_{l\text{-}\mathbf{k}}^\dagger \equiv \hat{a}_{l\text{-}\mathbf{k}}^\dagger(\eta_i)$ by means of a Bogoliubov transformation [4, 6]:

$$\hat{a}_{l\mathbf{k}}(\eta) = \alpha_{l\mathbf{k}}^*(\eta) \hat{A}_{l\mathbf{k}} + \beta_{l\mathbf{k}}(\eta) \hat{A}_{l\text{-}\mathbf{k}}^\dagger, \quad (4.4)$$

where the coefficients α and β are c-functions satisfying $\alpha(\eta_i) = 1$ and $\beta(\eta_i) = 0$. In Ref. [6] it was shown that if $|0\rangle$ is the state containing no particles at time η_i, then $|\beta_{l\mathbf{k}}|^2$ is the average density of gravitons with polarization l and wave-number $\mathbf{k}$ in the state $|0\rangle$ at a posterior time η :

$$\langle N_{l\mathbf{k}}(\eta) \rangle_0 = \langle 0 | \hat{a}_{l\mathbf{k}}^\dagger(\eta) \hat{a}_{l\mathbf{k}}(\eta) | 0 \rangle = |\beta_{l\mathbf{k}}(\eta)|^2. \quad (4.5)$$

If the following change of variables [8] is considered:

$$X = \alpha e^{-ik(\eta - \eta_i)} + \beta e^{ik(\eta - \eta_i)}, \quad (4.6)$$

$$Y = \alpha e^{-ik(\eta - \eta_i)} - \beta e^{ik(\eta - \eta_i)}, \quad (4.7)$$

where the constraint $|\alpha|^2 - |\beta|^2 = 1$ imposes $\Re(X\bar{Y}) = 1$, then the evolution of the tensorial perturbation is dictated by the set of differential equations:

$$X' = -ikY, \quad (4.8)$$

$$Y' = \frac{i}{k} \left(\frac{z''}{z} - k^2 \right) X. \quad (4.9)$$

The the potential z''/z encodes the dependence of this evolution on both the theory of gravity considered and the energy content of the universe.

If we consider that the energy per graviton is $\epsilon_\omega = \hbar\omega$, the density of graviton states is $\langle N_{\mathbf{k}}(\eta)\rangle_0 = \omega^2 d\omega/(2\pi^2c^3)$ and takes into account that there are two possible polarizations for each graviton, then, for each frequency ω, the relative logarithmic energy spectrum of gravitational waves is defined as [5, 10, 11]:

$$\Omega_{GW}(\omega,\eta) = \frac{8\hbar G}{3\pi c^5 H^2}\omega^4|\beta_\omega(\eta)|^2. \tag{4.10}$$

The available observational constraints on the energy the spectrum are [10, 12, 23] :

- From the CMB radiation: $h_0^2\Omega_{GW}(\omega_{hor},\eta_0) \lesssim 7\times 10^{-11}$ for $h_0 = H_0/(100$ km/s/Mpc) and $\omega_{hor} = 2\times 10^{-17}h_0$ rad/s;
- from observation of milliseconds pulsar: $h_0^2\Omega_{GW}(\omega_{pul},\eta_0) < 2\times 10^{-8}$ for $\omega_{pul} = 2.5\times 10^{-8}h_0$ rad/s;
- From the Cassini spacecraft: $h_0^2\Omega_{GW}(\omega_{Cas},\eta_0) < 0.014$ for $\omega_{Cas} = 7.5\times 10^{-6}h_0$ rad/s;
- From the LIGO experiment: $h_0^2\Omega_{GW}(\omega,\eta_0) < 3.4\times 10^{-5}$ for frequencies on the order of a few hundred rad7s;
- From BBN: $h_0^2\Omega(\omega,\tau_0)d\omega/\omega < 5.6\times 10^{-6}$ for $\omega_n \approx 10^{-9}$rad/s.

B. Cosmological Evolution

As stated above, the evolution of the tensor perturbations is described by the pair of first order linear differential equations Eqs. (4.8) and (4.9). These two equations may be merged together to give a single second order differential equation for X:

$$X'' + \left(k^2 - \frac{z''}{z}\right)X = 0. \tag{4.11}$$

The fact that Eq. (4.11) depends on the factor z''/z, where $z = a\sqrt{f_R}$, explicitly shows the dependence of the evolution of the cosmological gravitational waves on both (i) the content of the universe throughout time and (ii) the theory that describes the gravitational interaction.

Equation (4.11) can be interpreted as describing an harmonic oscillator with a source term $\frac{z''}{z}$. In the large wave-number limit ($k^2 \gg z''/z$), the constant term dominates and the solutions are essentially sinusoidal, therefore, in this regime the graviton density is approximately constant. When the source term dominates ($k^2 \ll z''/z$), the amplitude of X grows and so does the density of gravitons. In this way, z''/z acts as a potential for the creation of graviton whose effects come into play when the mode passes under the potential

Due to the specific form of z one can expand the potential z''/z into its GR form, a''/a, plus a correction term, Ξ:

$$\begin{aligned}\frac{z''}{z} &= \frac{a''}{a} + 2\frac{a'}{a}\frac{(\sqrt{f_R})'}{\sqrt{f_R}} + \frac{(\sqrt{f_R})''}{\sqrt{f_R}}\\ &= \frac{a''}{a} + \Xi.\end{aligned} \tag{4.12}$$

The first term in Eq. (4.12) can be defined for the three eras in terms of the scale factor as:

$$\frac{a''}{a}(a) = \begin{cases} H_{\rm inf}^2\left(2a^2 - a_b^2\right) \\ \frac{2}{3}\kappa^2a^2A\left(A + \frac{B}{a^{4(1+\alpha)}}\right)^{-\frac{\alpha}{1+\alpha}} \\ \frac{\kappa^2}{6}a^2\left(\rho_m\left(\frac{a_0}{a}\right)^3 + 4\rho_\Lambda\right). \end{cases} \tag{4.13}$$

The second term in Eq. (4.12) represents the correction introduced by the use of the metric $f(R)$ theory instead of GR and its behaviour depends heavily on the choice of the function f considered. After some lengthy calculation, we obtain for Ξ:

$$\begin{aligned}\Xi(t) = &H_{\rm inf}^2a_b^2\Bigg\{4 - 5\cosh^2(H_{\rm inf}t)\\ &+ \left[9\cosh^2(H_{\rm inf}t) - 6\right]Z(t)\\ &- \frac{\left[3\cosh^2(H_{\rm inf}t) - 2\right]^2}{4\sinh^2(H_{\rm inf}t)}Z^2(t)\Bigg\},\end{aligned} \tag{4.14}$$

where the $Z(t)$ is defined as:

$$Z(t) = \frac{\sqrt{3\sinh^2(H_{\rm inf}t)}}{\cot\left(\frac{\sqrt{3}}{2}\theta_1(t) + \theta_2(t)\right) + \sqrt{3\sinh^2(H_{\rm inf}t)}}. \tag{4.15}$$

The function $Z(t)$ is a monotonic increasing function of $|t|$ that goes from $Z(0) = 0$ to the limiting value during the de Sitter like expansion $Z(t\to\infty) = 2/3$. In this regime $\Xi \approx H_{\rm inf}^2a_b^2/3$, therefore, we can make the approximation:

$$\frac{z''}{z}(t \gg H_{\rm inf}^{-1}) \approx \frac{z''}{z}^{dS} H_{\rm inf}^2a_b^2\left[2\cosh^2(H_{\rm inf}t) - 2/3\right]. \tag{4.16}$$

Very close to the bounce, the potential z''/z becomes negative ($z''/z < 0$ approximately for $|t| < 0.22H_{\rm inf}^{-1}$), and its precisely at the bounce that the deviations from the GR term are maximum: $\max|\Xi| = |\Xi(0)| = 4.74278H_{\rm inf}^2a_b^2$. We note that, in GR, the potential a''/a only becomes negative whenever $\rho - 3p < 0$, for example for stiff matter.

The complex expression obtained for z''/z gives little hope of finding an analytical solution for the differential equation Eq. (4.11). However when one considers only the simpler potentials a''/a and $z''/z^{(dS)}$, analytical so-

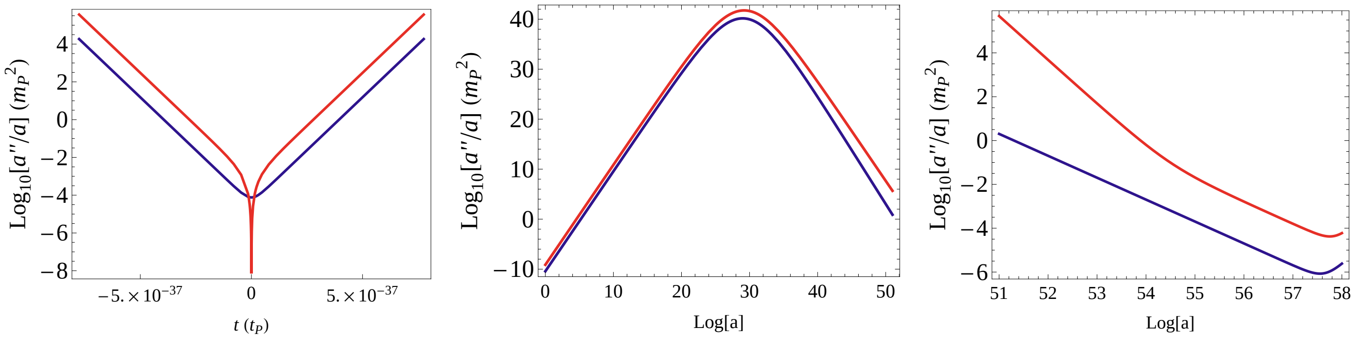


FIG. 3. This Fig. shows the evolution of the potential a''/a (blue curve) and the squared comoving wave-number, $k_H^2 \equiv 4\pi^2 a^2 H^2$, (red curve) during the three eras of the universe: (left) the $f(R)$ era; (middle) the mGCG era; (right) the ΛCDM era.

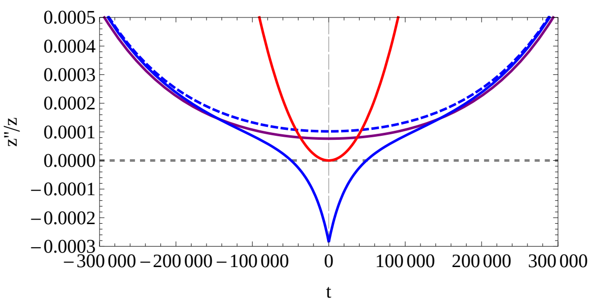


FIG. 4. This Fig. shows: (i) the potential z''/z (see the blue curve) and its asymptotic behaviour (see the blue dashed line); (ii) the potential a''/a (see the purple curve); (iii) the comoving wave-number a^2H^2(see the red curve); as functions of cosmic time and near the bounce. The variables are presented in Planck units

lutions for Eq. (4.11) can indeed be found:

$$
\begin{aligned}
X_1^\gamma(t) &= \sqrt{\sinh^2\left(H_{\rm inf}t\right)+\gamma+q^2} \\
&\times \cos\left[\mathrm{sgn}(\gamma-1+q^2)\frac{\phi_1^\gamma(t)\sqrt{\gamma+q^2}-\phi_2^\gamma(t)}{2}\right], \\
X_2^\gamma(t) &= \sqrt{\sinh^2\left(H_{\rm inf}t\right)+\gamma+q^2} \\
&\times \sin\left[\mathrm{sgn}(\gamma-1+q^2)\frac{\phi_1^\gamma(t)\sqrt{\gamma+q^2}-\phi_2^\gamma(t)}{2}\right], \quad (4.17)
\end{aligned}
$$

where the phases ϕ_1 and ϕ_2 are defined as:

$$
\begin{aligned}
\phi_1^\gamma(t) &= \pi+\mathrm{sgn}(t)\arccos\left[2\mathrm{sech}^2(H_{\rm inf}t)-1\right], \\
\phi_2^\gamma(t) &= \pi+\mathrm{sgn}(t)\arccos\left[\frac{2\left(\gamma+q^2\right)}{\sinh^2(H_{\rm inf}t)+\gamma+q^2}-1\right],
\end{aligned}
\quad (4.18)
$$

Here, we have introduced the reduced wave-number $q = k/a_bH_{\rm inf}$ and the GR results, for a''/a, and the de Sitter approximation, for $z''/z^{(dS)}$, are recovered by setting $\gamma = 1$ and $\gamma = 2/3$, respectively.

V. RESULTS

In order to obtain the energy spectrum of the GW, the set of differential Eqs. (4.8) and Eq. (4.9) needs to be solved from an initial time before the bounce, throughout the different eras of the universe considered and until the present time. The integration of these equations requires a numerical approach, as for the most part, the complicated expression for the potential z''/z prevents us to find analytical solution. In this section we outline the parameters of the model and the methods used for the numerical integrations necessary. The energy spectrum of the GW is computed for different values of the free parameters and using two methods for treating the cosmological perturbations.

A. Parameters of the model

In our model we can identify a set of four independent parameters:

1. the time when the initial conditions are set, $t_{\rm ini}$;
2. the scale factor at the time of the bounce, a_b;
3. the energy scale during the inflationary era, $E_{\rm inf}$;
4. the value of the parameter α in the mGCG model.

The positive constants A and B of the mGCG model are fixed as:

$$
A = E_{\rm inf}^{4(1+\alpha)}, \quad \text{and} \quad B = \left(\frac{3}{\kappa^2}\Omega_r H_0^2 a_0^4\right)^{1+\alpha}. \quad (5.1)
$$

In addition, the parameters of the ΛCDM model are taken from the seven-year WMAP release [24]:

$$
\Omega_m = 0.266, \quad \Omega_\Lambda = 0.734, \quad \text{and} \quad H_0 = 70.4\text{km/s/Mpc}. \quad (5.2)
$$

Finally, following previous works (see Ref. [12]), one defines the residual energy density of radiation and the value of the scale factor at present time as:

$$\Omega_r = 8 \times 10^{-5}, \quad \text{and} \quad a_0 = 10^{58}, \tag{5.3}$$

The minimum wave-number scanned when computing the energy spectrum of the gravitational waves corresponds to the mode that is leaving the horizon at present time, i.e.:

$$k_{\min} = k_{\text{hor}} \equiv a_0 H_0, \tag{5.4}$$

which is equivalent to an angular frequency of $\omega_{\text{hor}} \approx 1.43 \times 10^{-17}$rad/s. The maximum wave-number scanned is defined by the maximum of the potential a''/a during the mGCG era:

$$k_{\max} = 2\sqrt{\frac{\kappa^2}{3}\Omega_r} H_0 a_0^2 E_{\text{inf}}^2 \left[(1+2\alpha)\left(\frac{1+2\alpha}{2\alpha}\right)^{2\alpha}\right]^{\frac{1}{2(1+\alpha)}} \tag{5.5}$$

B. Numerical Integrations

The numerical integrations necessary to obtain the energy spectrum were performed using the software *Wolfram Mathematica 8*. The *BDF* method of the routine *NDSolve* was chosen, following the suggestion of Ref. [12]. For each wave-number k the integration was divided in two parts:

- a first integration during the $f(R)$-gravity era performed in terms of the cosmic time;
- a final integration during the mCGC and ΛCDM eras performed in terms of the scale factor.

Furthermore, two different methods were used to determine the evolution of the cosmological perturbations:

- the background is described using an $f(R)$ setup while the perturbations are treated in GR;
- both the background and the perturbations are described in an $f(R)$ setup.

The transition from the first to the second era, a_1, and from the second to the third era, a_2, are considered to occur in such a way that the potential z''/z is always continuous.

Integration during the $f(R)$ era

During the first era, the integration is done in terms of the cosmic time so as to make sure that the effects of the bounce are taken into account when calculating the energy spectrum of the GW. The integration is started at an initial time t_{ini}, fixed well into the contraction phase, and continued until the end of the $f(R)$ era, at $t = t_1 \equiv \arccos(a_1/a_b)$. By guaranteeing that the initial conditions are set sufficiently far away from the bounce, one is able to use the analytical solutions found in Chaper IV to set the initial values of the variables X and Y.

For $\gamma = 1$, i.e., within a GR treatment of the gravitational perturbations, the functions X and Y obtained are exact solutions of the differential equations (4.8) and (4.9), and therefore, are used to obtain the values of X and Y at the end of the $f(R)$ era, while skipping the numerical integration altogether during this period.

However, in the case of $\gamma = 2/3$ the the functions X and Y represent solutions for the approximated potential z''/z^{dS}, which is valid only during the de Sitter-like phases. This means, the initial conditions are set using the analytical solutions of Eq. (4.17) and a numerical integration is necessary to obtain the values of the variables X and Y at the end of the $f(R)$ era. Nevertheless, for large k it is considered that the modes are insensible to the exact form of the potential z''/z around the bounce and the approximate solutions are used during the entire $f(R)$ era

Integration during the mGCG and ΛCDM eras

At the beginning of the mGCG era, we switch variables and perform the numerical integrations in terms of the scale factor. In principle, the integration is started at $a = a_1$ and is stopped at the present time, at $a = a_0$. The initial values of the variables X and Y are defined by the results of the integration during the $f(R)$ era. However, in the regime $k^2 \gg a''/a$, the X and Y functions have a sinusoidal behaviour, hence $|\beta|^2$ is approximately constant, thus, to save computational time, the integration is stopped when the modes are well inside the Hubble horizon. This way, for the modes that are still well inside the horizon when the mGCG era starts, the analytical solutions of the first era are prolonged through the mGCG era to obtain the initial values for the integration at $10^{-2}a_{\text{exit}}^{(k)}$, where $a_{\text{exit}}^{(k)}$ defines the moment when the mode exits the horizon, i.e., $k = a_{\text{exit}}^{(k)} H(a_{\text{exit}}^{(k)})$. An exception is made for small k and $\gamma = 2/3$, as the mode "sees" the true form of the potential z''/z and the analytical solutions are not valid during and after the bounce. In this case the numerical integration is always started at a_1.

C. Numerical Results

The energy spectra of the GW obtained for different values of the parameters $(t_{ini}, a_b, E_{\text{inf}}, \alpha)$ are presented in Figs. 5, 6, 7, 8, 9 and 10. As expected, the existence of a bounce in the early universe affects the spectrum only in the low frequency range, where a highly oscillatory regime is present, in contrast with the smooth plateau in the intermediate frequencies and the rapid decay in the high frequency range. This oscillatory regime appears both in a GR treatment and in an $f(R)$ treatment of the perturbations, which indicates that this feature of the spectrum is due to the effects of the bounce and not to the theory used to analyse the tensorial perturbations. Similar oscillations have been obtained in works of loop quantum cosmology, as first pointed out by Afonso et al [25, 26].

The features of the spectra obtained in a GR and in an $f(R)$ treatment are presented in Figs. 5. The two spectra overlap for high frequencies and are distinguishable only in the low frequency range. In particular, an extra local minimum occurs for the $f(R)$ treatment at:

$$k = \frac{a_b H_{\rm inf}}{\sqrt{3}}. \tag{5.6}$$

In Fig. 6 the dependence of the imprints of the bounce can be observed for different values of $t_{\rm ini}$, where the peaks on the low frequency range of the spectrum are enhanced.

The effects of the constant $a_b H_{\rm inf}$ can be observed in the Figs. 7, 8 and 9. When either parameter a_b or $H_{\rm inf}$ is changed and the other is kept constant (Figs. 7 and 8), the imprints of the bounce are shifted horizontally on the spectrum. However, when the two parameter are changed but their product is kept constant (Fig. 9), the position of the peaks on the spectrum is not altered. Notice that in some of the spectra the peaks do not appear at all. This corresponds to the situation when the value of $a_b H_{\rm inf}$ is low enough to move the peaks to frequencies lower than $\omega_{\rm hor}$.

Finally, Fig. 10 shows the effects on the energy spectra of varying the value of the parameter α. There is no change on the spectrum in the low frequency range, with the effects only being noticeable for the intermediate to high frequencies. As the value of $|\alpha|$ increases, the spectrum decays more slowly at the intermediate plateau and drops abruptly as the frequencies approach $\omega_{\rm max}$. The value of $\omega_{\rm max}$ also increases with $|\alpha|$.

VI. CONCLUSIONS

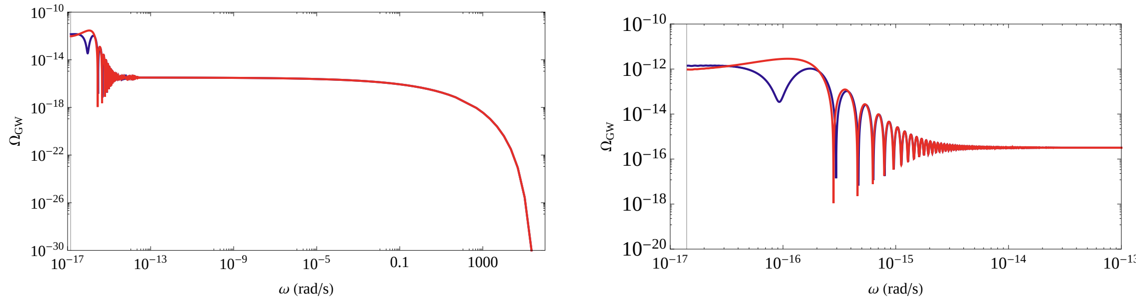
In this work it was investigated the imprints left on the energy spectrum of the cosmological gravitational waves, at present time, by the presence of a particular bounce in the early universe. Since the existence of a bounce in a GR setup implies violation of the null energy condition, [14], an $f(R)$ gravity theory [1, 2, 27] was used to describe the dynamics of the early universe and the "designer" $f(R)$ methodology was employed to obtain an $f(R)$ action compatible with the desired behaviour for the scale factor. The late time evolution of the universe was described using the ΛCDM model and a mGCG model, [28], was used to connect the inflationary era, that begins after the bounce, with the the radiation epoch. This way, a smooth transition between the two phases (inflation and radiation) can be obtained without affecting the low frequencies of the spectrum, where the main effects of the bounce should appear.

The energy spectrum of the GW was computed using the method of the Bogoliubov coefficients, [6–11], to determine the evolution of the graviton density. The tensorial perturbations were treated during the early universe both in a GR approach (with an $f(R)$ background) and in a full $f(R)$ treatment, [20, 27, 29, 30]. A set of four independent parameters was obtain for the model of universe considered within this work: the value of the scale factor at the time of the bounce, a_b; the initial time $t_{\rm ini}$ of the integration that defines the amount of contraction before the bounce; the energy scale during the inflationary era, $E_{\rm inf}$, and the mGCG parameter α. The energy spectrum of the gravitational waves was determined for different values of each of this parameters.

After computing the energy spectrum, it was confirmed that the presence of the bounce only affects the low frequency region of the spectrum, while leaving the intermediate and high frequency regions unaltered. In the low frequencies, various peaks appear on the spectrum, whose position and intensity depend mainly on $a_b H_{\rm inf}$ and $t_{\rm ini}$ where $H_{\rm inf}$ is related to the energy density during the early inflation. The horizontal position of the peaks is defined by $a_b H_{\rm inf}$ as increasing $a_b H_{\rm inf}$ displaces the peaks to higher frequencies. On the other hand, raising $t_{\rm ini}$ enhances the peaks of the spectrum and increases the range of modes affected by the bounce. As expected, changing the mGCG parameter α does not affect the low frequency range of the spectrum.

As the intensity of the peaks rise rapidly with $t_{\rm ini}$, the CMB radiation limits on the energy density of the cosmological Gravitational Waves [10, 23] can be used to constraint the amount of contraction allowed before the bounce. The fact that an eternal de Sitter-like contraction cannot be obtained in the past suggests the existence of a different initial state for the universe that precedes the contraction. All the other observational constraints on the energy density of the cosmological gravitational waves are satisfied.

One concludes by noting that the detection of the results obtained in this work are unlikely to be observed in the near future [23]. Nevertheless, the future measurement of the B-mode polarization of the CMB radiation might start to shed some light on the dynamics of the early stages of the universe. Thus, a natural continuation of this work would be the imprints of the bounce on the polarization of the CMB radiation, particularly the B-modes [35–37]. Additionally, other types of bounces might be considered in order to avoid the problems that arise in the scalar sector.



(a) (b)

FIG. 5. Comparison of the energy spectrum of the GW obtained using a GR (red curve) and a $f(R)$ (blue curve) treatment for the tensorial perturbations.

	$\mathbf{t}_{\rm ini}(H_{\rm inf}^{-1})$	$\mathbf{a_b}$	$\mathbf{E}_{\rm inf}$ (GeV)	α	γ	$\mathbf{a_1}$	$\mathbf{a_2}$	$\omega_{\rm max}$ (rad/s)
	-2.993	2×10^2	1.5×10^{16}	-1.04	$2/3$	1.865×10^5	3.221×10^{51}	2.242×10^5
	-2.993	2×10^3	1.5×10^{16}	-1.04	1	2.246×10^5	3.221×10^{51}	2.242×10^5

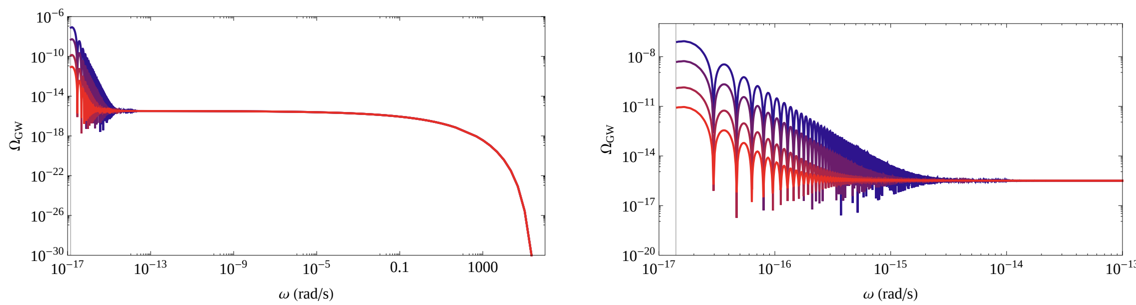


FIG. 6. Energy spectra of the GW for different values of $t_{\rm ini}$. The parameters a_b, $E_{\rm inf}$ and α are fixed.

	$\mathbf{t}_{\rm ini}(H_{\rm inf}^{-1})$	$\mathbf{a_b}$	$\mathbf{E}_{\rm inf}$ (GeV)	α	γ	$\mathbf{a_1}$	$\mathbf{a_2}$	$\omega_{\rm max}$ (rad/s)
	-2.99322	2×10^3	1.5×10^{16}	-1.04	$2/3$	2.2097×10^4	3.22127×10^{51}	2.2415×10^5
	-3.68825	2×10^3	1.5×10^{16}	-1.04	$2/3$	2.2097×10^4	3.22127×10^{51}	2.2415×10^5
	-4.60507	2×10^3	1.5×10^{16}	-1.04	$2/3$	2.2097×10^4	3.22127×10^{51}	2.2415×10^5
	-5.29829	2×10^3	1.5×10^{16}	-1.04	$2/3$	2.2097×10^4	3.22127×10^{51}	2.2415×10^5

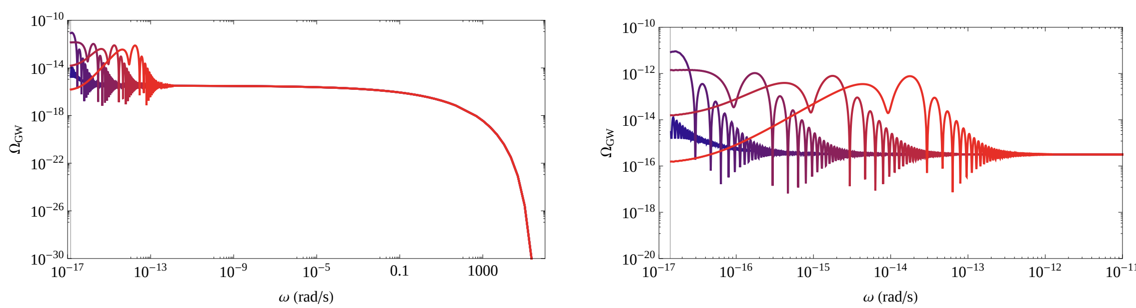


FIG. 7. Energy spectra of the GW for different values of a_b.

	$\mathbf{t}_{\rm ini}(H_{\rm inf}^{-1})$	$\mathbf{a_b}$	$\mathbf{E}_{\rm inf}$ (GeV)	α	γ	$\mathbf{a_1}$	$\mathbf{a_2}$	$\omega_{\rm max}$ (rad/s)
	-2.993	2×10^2	1.5×10^{16}	-1.04	$2/3$	2.619×10^3	3.221×10^{51}	2.242×10^5
	-2.993	2×10^3	1.5×10^{16}	-1.04	$2/3$	2.210×10^4	3.221×10^{51}	2.242×10^5
	-2.993	2×10^4	1.5×10^{16}	-1.04	$2/3$	1.865×10^5	3.221×10^{51}	2.242×10^5
	-2.993	2×10^5	1.5×10^{16}	-1.04	$2/3$	1.574×10^6	3.221×10^{51}	2.242×10^5
	-2.993	2×10^6	1.5×10^{16}	-1.04	$2/3$	1.329×10^7	3.221×10^{51}	2.242×10^5

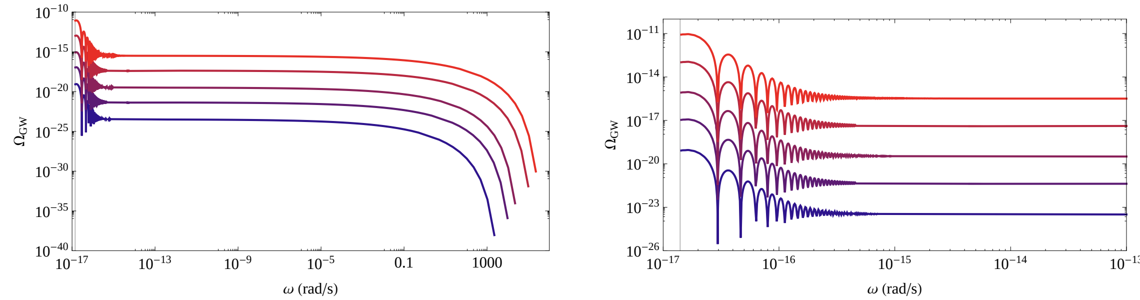


FIG. 8. Gravitational wave spectra for different values of a_b and $E_{\rm inf}$ with $a_b E_{\rm inf}$ constant.

	$\mathbf{t}_{\rm ini}(H_{\rm inf}^{-1})$	$\mathbf{a_b}$	$\mathbf{E}_{\rm inf}$ (GeV)	α	γ	$\mathbf{a_1}$	$\mathbf{a_2}$	$\omega_{\rm max}$ (rad/s)
	-2.993	2×10^{7}	1.5×10^{14}	-1.04	$2/3$	1.574×10^{8}	6.047×10^{51}	2.242×10^{3}
	-2.993	1.8×10^{6}	0.5×10^{15}	-1.04	$2/3$	1.548×10^{7}	5.131×10^{51}	7.472×10^{3}
	-2.993	2×10^{5}	1.5×10^{15}	-1.04	$2/3$	1.865×10^{6}	4.415×10^{51}	2.242×10^{4}
	-2.993	1.8×10^{4}	0.5×10^{16}	-1.04	$2/3$	1.834×10^{5}	3.745×10^{51}	7.472×10^{4}
	-2.993	2×10^{3}	1.5×10^{16}	-1.04	$2/3$	2.210×10^{4}	3.221×10^{51}	2.242×10^{5}

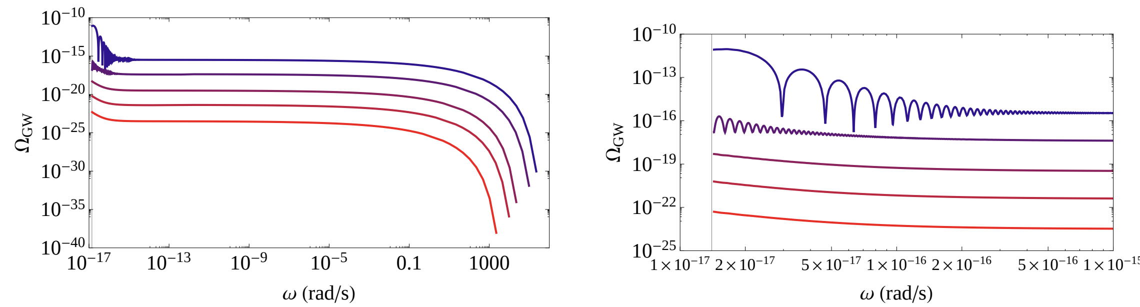


FIG. 9. Energy spectra of the GW for different values of $E_{\rm inf}$.

	$\mathbf{t}_{\rm ini}(H_{\rm inf}^{-1})$	$\mathbf{a_b}$	$\mathbf{E}_{\rm inf}$ (GeV)	α	γ	$\mathbf{a_1}$	$\mathbf{a_2}$	$\omega_{\rm max}$ (rad/s)
	-2.993	2×10^{3}	1.5×10^{14}	-1.04	$2/3$	3.105×10^{4}	6.047×10^{51}	2.2415×10^{3}
	-2.993	2×10^{3}	0.5×10^{15}	-1.04	$2/3$	2.841×10^{4}	5.131×10^{51}	7.4717×10^{3}
	-2.993	2×10^{3}	1.5×10^{15}	-1.04	$2/3$	2.619×10^{4}	4.415×10^{51}	2.2415×10^{4}
	-2.993	2×10^{3}	0.5×10^{16}	-1.04	$2/3$	2.396×10^{4}	3.745×10^{51}	7.4717×10^{4}
	-2.993	2×10^{3}	1.5×10^{16}	-1.04	$2/3$	2.210×10^{4}	3.221×10^{51}	2.2415×10^{5}

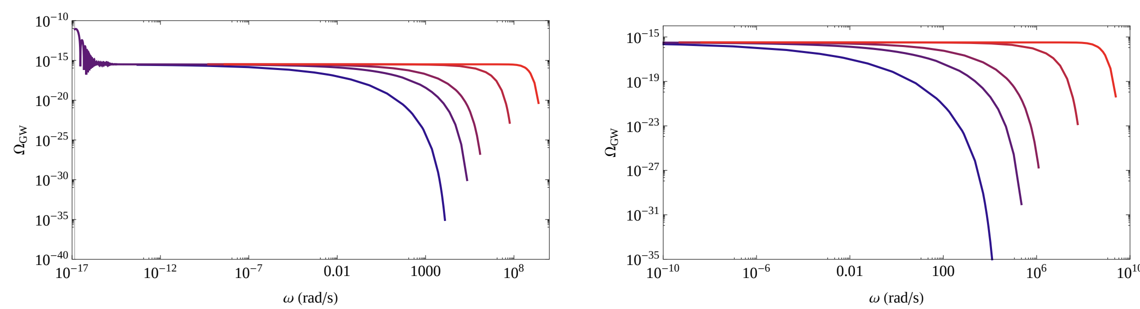


FIG. 10. Energy spectra of the GW for different values of the mGCG parameter α.

	$\mathbf{t}_{\rm ini}(H_{\rm inf}^{-1})$	$\mathbf{a_b}$	$\mathbf{E}_{\rm inf}$ (GeV)	α	γ	$\mathbf{a_1}$	$\mathbf{a_2}$	$\omega_{\rm max}$ (rad/s)
	-2.993	2×10^{3}	1.5×10^{16}	-1.03	$2/3$	6.709×10^{3}	1.927×10^{52}	1.245×10^{4}
	-2.993	2×10^{3}	1.5×10^{16}	-1.04	$2/3$	2.210×10^{4}	3.221×10^{51}	2.242×10^{5}
	-2.993	2×10^{3}	1.5×10^{16}	-1.05	$2/3$	6.882×10^{4}	5.843×10^{50}	1.271×10^{6}
	-2.993	2×10^{3}	1.5×10^{16}	-1.10	$2/3$	9.841×10^{6}	4.854×10^{47}	4.111×10^{7}
	-2.993	2×10^{3}	1.5×10^{16}	-1.50	$2/3$	2.694×10^{7}	6.716×10^{37}	7.000×10^{7}

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