The effects of a bounce on the spectrum of the Gravitational Waves in a metric $f(R)$-gravity

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Resumo

Este trabalho estuda as modificações no espectro de energia das ondas gravitacionais de origem cosmológica devido à presença de um “bounce” no universo primordial. O modelo do “bounce” é construído no contexto de teorias de gravidade modificada, especificamente em gravidade $f(R)$ métrica ($R$ é o escalar de curvatura), e considera um “bounce” suave, precedido por uma contracção do tipo de Sitter e sucedido por uma inflação do tipo de Sitter. Uma vez que a acção $f(R)$ obtida para descrever o “bounce” converge para a acção de Hilbert-Einstein durante a inflação, a teoria de gravidade modificada só é utilizada para descrever a evolução do universo primordial. O espectro de energia das ondas gravitacionais é analisado por via do método dos coeficientes de Bogoliubov e através de dois procedimentos: fazendo o tratamento as perturbações gravitacionais tanto no “setup” de teorias de gravidade modificadas como em Relatividade Geral. Para certas regiões do espaço de parâmetros, o “bounce” induz o aparecimento de impressões distintas na região de baixas frequências do espectro.

**Palavras-chave:** Cosmologia sem singularidade, inflação primordial, teorias modificadas de gravidade, ondas gravitacionais
Abstract

This work studies the imprints on the energy spectrum of the cosmological gravitational waves of the presence of a bounce in the early universe. The model of the bounce is constructed in a metric $f(R)$ theory of gravity, where $R$ is the scalar curvature, and considers a smooth bounce preceded by a de Sitter-like contraction phase and followed by a de Sitter-like inflation. An $f(R)$ action that converges to GR well during inflation is obtained, therefore the modified theory of gravity is only applied during the early stages of the universe. The energy spectrum of the gravitational waves is analysed through the method of the Bogoliubov coefficients by two means: taking into account the gravitational perturbations due to the modified gravitational action in the $f(R)$ setup and by simply considering those perturbations inherent to the standard Einstein-Hilbert action. For certain regions of the parameter space, distinct (oscillatory) signals appear on the spectrum for very low frequencies.

Keywords: Bouncing cosmologies, inflation, modified theories of gravity, gravitational waves
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Chapter 1

Introduction

Nowadays, the cosmological model that provides the best fit to the observational data available (from Cosmic Microwave Background (CMB) radiation \[1\], supernovae surveys \[2, 3\], baryon acoustic oscillations, gamma rays bursts), regarding the evolution and structure of the universe is the ΛCDM model complemented with an early inflationary phase. However, the rather empirical nature of this model lacks proper theoretical background for some of its features, specially with regards to the nature of the so called dark matter and dark energy. These comprise roughly \(\gtrsim 95\%\) of the matter content of the universe \[1\] and are responsible for its current expansion. In fact, the search for a theoretical model for the dark sector has led to a renewed attention to modified theories of gravity, with an attempt being made to explain the current expansion through the effects of higher order correction terms in the action. This approach is corroborated by the already known result that General Relativity (GR) is not re-normalizable \[4\] and thus, the presence of higher order terms could be required for a quantum description of gravity to be found.

The inflationary paradigm was first introduced by A. Guth \[5\] and A. Starobinsky \[6\] as a theoretical tool to explain the flatness, homogeneity and horizon problems of the Hot Big Bang model (e.g. Refs. \[7, 8\] for reviews on inflation). However, as the paradigm does not specify how inflation begins, or even how it ends, various mechanisms have been presented that can provide for an inflationary phase in the early universe. Furthermore, the existence of the Big bang singularity at the beginning of the universe is seen by many as artificial and due to a break down of GR, thus many models have been suggested, both within General Relativity \[9\] and Extended Theories of Gravity \[10, 12, 13\] (e.g. Loop Quantum Cosmology \[14\], d-Brane Cosmology \[15\], Bouncing cosmologies in \(f(R)\)-gravity \[16, 17, 18\]), removing this singularity. The wide array of models is directly connected to a lack of observational data linked to that period and shows how little we still know about the early stages of the universe.

Currently, the most promising way of probing the dynamics of the early universe is to look at the imprints of the cosmological perturbations \[19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\], originated from the quantum vacuum fluctuations during the early inflation. Due to the fact that their energy density is very small when compared to the limits of the current (and planned) GW detectors \[34, 35\], the direct detection of the GW of cosmological origin might prove difficult in the near future. The measurement of the B-mode polarization of the CMB radiation \[31, 32, 33\], might prove itself to be the best alternative to obtain information about the early stages of the universe.

In this work, one looks at the energy spectrum of the cosmological Gravitational Waves (GW) while considering a bounce in the early universe described in the setup of modified theories of gravity, specifically a metric \(f(R)\) theory of gravity \[36, 37\]. The energy spectrum of the cosmological GW is determined at the present time using the method of the continuous Bogoliubov coefficients \[38, 19, 21\], complemented by a convenient change of variables \[24, 25, 26, 27, 28, 30\].

The \(f(R)\) theories of gravity \[36, 37, 39\] have risen in popularity in this past decade due to the possibility of obtaining an effective cosmological constant. Thus, explaining the current expansion of the universe without the need for including dark energy \[40\]. As a result, an abundance of papers in \(f(R)\) gravity and cosmology has emerged, \[10, 16, 17, 37, 33\], with a focus on: \(f(R)\) late-time cosmology, \[10, 40\]; the evolution of cosmological perturbations, \[18, 14, 15, 46, 47, 48, 49, 50, 51, 52, 53, 54\]; and theoretical and observational constraints on viable \(f(R)\) actions, \[10, 59, 54, 55, 56, 57, 58\].

The use of a modified theory of gravity in this work is motivated by the fact that a bounce in a spatially flat a Friedmann-Lemaître-Robertson-Walker (FLRW) universe implies a violation of the Null
Energy Condition [59]. In addition, the inflationary phase after the bounce also involves a violation of the Strong Energy Condition [6]. By employing an $f(R)$ theory of gravity, instead of GR, one can avoid the need for inserting some kind of matter or scalar fields that violate these conditions [16]. The $f(R)$ setup is only applied during the early universe, with the bounce being preceded by a de Sitter-like contraction and followed by de Sitter-like inflation. At some point, during inflation, one switches to GR; this way, the ΛCDM model can be used to describe the late-time evolution of the universe.

The structure of this work is as follows: Chapter 2 presents a brief review on $f(R)$ gravity and FLRW cosmology in $f(R)$ gravity. The features of the model discussed in this work are introduced in Chapter 3. A general method of treating cosmological tensorial perturbations in GR and $f(R)$ and calculating the energy spectrum of the GW is presented in Chapter 4 while the numerical application and results of this method are presented in Chapter 5. In Chapter 6, a discussion of the results obtained and some final remarks are given.
Chapter 2

The $f(R)$ Theory

2.1 Equations of motion of a metric $f(R)$ theory

In Einstein’s General Relativity (GR) one derives the equations of motions of a gravitational system using the linear Hilbert-Einstein Lagrangian density:

$$\mathcal{L}^{HE} = \frac{1}{2\kappa^2} \sqrt{-g} R,$$

(2.1)

where $\kappa^2 \equiv 8\pi G$, $G$ is the gravitational constant, $g$ is the determinant of the metric and $R$ is the scalar curvature (one assumes $c = \hbar = 1$). In $f(R)$-gravity this Lagrangian density is generalized by replacing $R$ by a general function $f(R)$ of the scalar curvature $R$, hence, $\mathcal{L}^{HE}$ is substituted by:

$$\mathcal{L} = \frac{1}{2\kappa^2} \sqrt{-g} f(R),$$

(2.2)

and the action for a metric $f(R)$-gravity theory becomes:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S^{(m)}(g_{\mu\nu}, \psi).$$

(2.3)

Here $S^{(m)}$ is a matter component, dependent solely on the metric $g_{\mu\nu}$ and the matter fields $\psi$, and is related to the stress energy tensor as:

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{\mu\nu}}.$$  

(2.4)

Minimizing the total action, Eq. (2.3), everywhere and isolating the matter part on the right hand side of the equation, one obtains the following field equations [37]:

$$f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box] f_R = \kappa^2 T^{(m)}_{\mu\nu}. $$

(2.5)

These are fourth order partial differential equations in the metric and as such, admit a wider array of solutions than the Einstein equations. Notice that in the case where $f(R)$ is linear with respect to $R$, the fourth order terms vanish and Eq. (2.5) reduces to the result of General Relativity. Computing the trace of Eq. (2.5) gives:

$$f_R R - 2f + 3\Box f_R = \kappa^2 T^{(m)} ,$$

(2.6)

where $T^{(m)} = g^{\mu\nu} T^{(m)}_{\mu\nu}$ and which relates $T^{(m)}$ with $R$ differentially instead of algebraically. One immediate consequence of this is the fact that, in contrast with GR, in $f(R)$ gravity $T^{(m)} = 0$ no longer implies that $R = 0$.

One of the main interests of $f(R)$-gravity is that it serves as a guiding theory in the search for higher order corrections to GR. Following this interpretation, it is reasonable to write Eq. (2.5) in the form of the so called Generalized Einstein Equations (GEE). If one defines the usual Einstein’s tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu},$$

(2.7)
then it is possible to rewrite Eq. (2.5) as:

\[ G_{\mu\nu} = \frac{\kappa^2}{f_R} \left( T_{\mu\nu}^{(m)} + g_{\mu\nu} \frac{f - f_R R}{2\kappa^2} + \frac{\nabla_\mu \nabla_\nu + g_{\mu\nu} \Box}{\kappa^2} f_R \right), \tag{2.8} \]

This result, which contains the ten GEE of metric \( f(R) \) gravity, can be expressed in a shorter form by introducing the quantity:

\[ T^{(\text{eff})}_{\mu\nu} \equiv g_{\mu\nu} \frac{f - f_R R}{2\kappa^2} f_R + \frac{\nabla_\mu \nabla_\nu + g_{\mu\nu} \Box}{\kappa^2} f_R, \tag{2.9} \]

with Eq. (2.8) now reading:

\[ G_{\mu\nu} = \kappa^2 \left( T_{\mu\nu}^{(m)} + T^{(\text{eff})}_{\mu\nu} \right). \tag{2.10} \]

In writing the GEE in the form of Eq. (2.10), one is explicitly stating that the extra terms on Eq. (2.8), which are due to the use of an \( f(R) \) theory of gravity, can be treated as an effective stress-energy tensor, \( T^{(\text{eff})}_{\mu\nu} \). Furthermore, the fact that \( T^{(\text{eff})}_{\mu\nu} \) can be put in the form of a perfect fluid energy tensor:

\[ T^{(\text{eff})}_{\mu\nu} = \left( \rho^{(\text{eff})} + p^{(\text{eff})} \right) u_\mu u_\nu + p^{(\text{eff})} g_{\mu\nu}, \tag{2.11} \]

with energy density \( \rho^{(\text{eff})} \) and pressure \( p^{(\text{eff})} \), makes these theories of special interest in models that try to explain the dark sector of the universe. Here \( u^\mu \) is the direction of a time-like observer. In Eq. (2.10) one can also verify that an effective gravitational coupling strength, \( G^{(\text{eff})} = G / f_R \), appears in the theory. This introduces a restriction on the class of viable \( f(R) \) actions, as, in order for gravity to be attractive, one needs to guarantee the positivity of \( G^{(\text{eff})} \), which in turn implies that \( f_R > 0 \).

In \( f(R) \) gravity, matter is minimally coupled to the metric, therefore, one expects \( T_{\mu\nu} \) to be divergence-free and indeed this is the case, as one can demonstrate, using the Bianchi identities, that:

\[ \kappa^2 \nabla_\mu T^{(\text{m})}_{\mu\nu} = f_R \nabla_\mu G^{\nu}_{\mu} = 0. \tag{2.12} \]

It is noteworthy to point out that although \( T^{(\text{eff})}_{\mu\nu} \) is not a generally conserved quantity:

\[ \nabla_\nu T^{(\text{eff})}_{\mu\nu} = \frac{T^{(\text{eff})}_{\mu\nu}}{f_R} \nabla_\nu R, \tag{2.13} \]

one can always define a total energy-momentum stress tensor, Ref. [60]:

\[ T^{(T)}_{\mu\nu} \equiv \frac{T^{(\text{m})}_{\mu\nu}}{f_R} + T^{(\text{eff})}_{\mu\nu}, \tag{2.14} \]

which is divergence free. Expanding on this formulation of a total energy-momentum stress tensor, one can define the total energy density and pressure:

\[ \rho^{(T)} \equiv \frac{\rho^{(\text{m})}}{f_R} + \rho^{(\text{eff})}, \quad p^{(T)} \equiv \frac{p^{(\text{m})}}{f_R} + p^{(\text{eff})}, \tag{2.15} \]

where the quantities related to the \textit{curvature fluid} are obtained from Eq. (2.11):

\[ \rho^{(\text{eff})} = T^{(\text{eff})}_{\mu\nu} u^\mu u^\nu, \quad p^{(\text{eff})} = \frac{1}{3} h^{\mu\nu\nu} T^{(\text{eff})}_{\mu\nu}. \tag{2.16} \]

Here, \( h^{\mu\nu} = g^{\mu\nu} + g_{\mu\nu} u^\nu \) is the projected metric on the 3-space perpendicular to \( u^\mu \). If one compares the definitions in Eq. (2.16) with the definition of the effective stress-energy tensor, Eq. (2.9), then the expressions for \( \rho^{(\text{eff})} \) and \( p^{(\text{eff})} \) read:

\[ \rho^{(\text{eff})} = \frac{R f_R - f}{2\kappa^2 f_R} + \frac{\nabla^2 - \Box}{\kappa^2 f_R} f_R, \tag{2.17} \]

\[ p^{(\text{eff})} = \frac{f - R f_R}{2\kappa^2 f_R} + \frac{\nabla^2 + 4 \Box}{\kappa^2 f_R} f_R. \tag{2.18} \]
2.2 \( f(R) \) gravity in FLRW Cosmology

As in the case of GR, the generalized Friedmann equations of a metric \( f(R) \)-gravity theory can be derived by inserting the Friedmann-Lemaître-Robertson-Walker (FLRW) metric in Eq. (2.8), while assuming a perfect fluid description for the content of the universe. Separating the temporal and spatial parts, one obtains both the Friedmann equation, \[37\]:

\[
H^2 + \frac{K}{a^2} = \frac{\kappa^2}{3f_R} \left( \rho^{(m)} - \frac{f - f_{RR}R}{2\kappa^2} - 3H\frac{f_{RRR}\dot{R}}{\kappa^2} \right),
\]  

(2.19)

and the Raychaudhury equation, \[37\]:

\[
2\dot{H} + 3H^2 - \frac{K}{a^2} = -\frac{\kappa^2}{f_R} \left( \rho^{(m)} + \frac{f - f_{RR}R}{2\kappa^2} + \frac{f_{RRR}\dot{R}^2 + f_{RRR}\dot{R} + 2Hf_{RRR}\dot{R}}{\kappa^2} \right),
\]

(2.20)

where \( H \) is the Hubble parameter and a dot represents a derivative with respect to the cosmic time.

Within a FLRW space-time, the effective energy density, (2.17), and pressure, (2.18), are given by:

\[
\rho^{(eff)} = -\frac{f - f_{RR}R}{2f_R} - \frac{3Hf_{RRR}\dot{R}}{f_R},
\]

(2.21)

\[
p^{(eff)} = \frac{f - f_{RR}R}{2f_R} + \frac{f_{RRR}\dot{R}^2 + f_{RRR}\dot{R} + 2Hf_{RRR}\dot{R}}{f_{RRR}},
\]

(2.22)

which allows the modified Friedmann and Raychaudhury equations, Eqs. (2.19) and (2.20), to be written as:

\[
H^2 + \frac{K}{a^2} = \frac{\kappa^2}{3} \left( \rho^{(m)} + \rho^{(eff)} \right) = \frac{\kappa^2}{3} \rho^{(T)},
\]

(2.23)

\[
2\dot{H} + 3H^2 - \frac{K}{a^2} = -\kappa^2 \left( \frac{\rho^{(m)}}{f_R} + p^{(eff)} \right) = -\kappa^2 p^{(T)}.
\]

(2.24)

If one takes Eq. (2.23) in the limit of \( \rho^{(m)} \to 0 \) and imposes a positive gravitational coupling, i.e., \( f_R > 0 \), one finds that in a spatially flat FLRW spacetime \( \rho^{(eff)} \) has to be non-negative, as the left hand side of the equation is always positive. As in GR, the acceleration equation can be obtained from Eqs. (2.23) and (2.24) as:

\[
\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6} \left( \frac{\rho^{(m)}}{f_R} + \rho^{(eff)} + 3 \left( \frac{\rho^{(m)}}{f_R} + p^{(eff)} \right) \right) = -\frac{\kappa^2}{6} \left( \rho^{(T)} + 3p^{(T)} \right).
\]

(2.25)

The form of Eqs. (2.23) and (2.24) reinforces the interpretation of the \( f(R) \) terms in the generalized equations as the result of an effective perfect fluid. Notice that the effects of such terms are of special importance when the matter content of the universe is negligible, i.e., \( \rho^{(m)} \ll \rho^{(eff)} \) and \( |p^{(m)}| \ll |p^{(eff)}| \), as in this case the evolution of the universe is governed by the curvature fluid. This allows, for example, for an effective cosmological constant to appear in the theory and thus provides a means to obtain an inflationary de Sitter regime in the early universe, without the need of adding new scalar fields. In vacuum, the effective equation of state parameter of \( f(R) \) gravity is defined as \[37\]:

\[
w^{(eff)} = \frac{p^{(eff)}}{\rho^{(eff)}} = \frac{\frac{1}{2} (f - f_{RR}R) + f_{RRR}\dot{R}^2 + f_{RRR}\dot{R} + 2Hf_{RRR}\dot{R}}{\frac{1}{2} (f - f_{RR}R) - 3Hf_{RRR}\dot{R}}.
\]

(2.26)

Thus, the condition for a metric \( f(R) \) model to mimic a de Sitter universe, i.e., \( w^{(eff)} = -1 \), can be expressed as:

\[
\frac{f_{RRR}}{f_{RR}} = \frac{H\dot{R} - \dot{R}}{\dot{R}^2}.
\]

(2.27)
2.3 $f(R)$ gravity as a Scalar-Tensor theory with $\omega_0 = 0$

All $f(R)$ theories of gravity can be recast as a Scalar-Tensor theory \[37\], in particular a Brans-Dicke theory with a Brans-Dicke parameter $\omega_0 = 0$, thus, an extra scalar degree of freedom appears in respect to GR. In this section, one presents a brief demonstration of the equivalence between the two representations and the characteristics of this new scalar degree of freedom. In order to recast $f(R)$ gravity in a dynamically equivalent Scalar-Tensor theory, one follows, for example, Ref. \[37\] and introduces the auxiliary scalar field $\chi$:

$$
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ f(\chi) + f_\chi (\chi) (R - \chi) \right] + S^{(m)}(g_{\mu\nu}, \psi). \tag{2.28}
$$

Varying Eq. (2.28) with respect to $\chi$ reads:

$$
f_\chi + f_\chi (R - \chi) - f_\chi = 0 \Leftrightarrow f_\chi (R - \chi) = 0. \tag{2.29}
$$

Hence, if the condition $f_\chi \neq 0$ is assumed, Eq. (2.29) imposes the equivalence $\chi = R$ and one recovers the $f(R)$ action of Eq. (2.3). One now proceeds to introduce a new field $\phi \equiv f_\chi$ and define the potential:

$$
V(\phi) = \chi(\phi) - f(\chi(\phi)), \tag{2.30}
$$

which, when inserted in Eq. (2.28), result in the $f(R)$ action being recast in the Jordan frame representation of a scalar tensor theory with $\omega_0 = 0$:

$$
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \phi R - V(\phi) \right] + S^{(m)}(g_{\mu\nu}, \psi). \tag{2.31}
$$

Through this method, one adopts the dynamical field $\phi$ as the representation of the new degree of freedom of the theory. Notice, however, that the condition $f_\chi \neq 0$, see Eq. (2.29), is a sufficient but not a necessary condition for the scalar-tensor action, given in Eq. (2.28), to be equivalent to the original $f(R)$-metric action. In fact, one needs only to guarantee that the mapping $R \rightarrow \phi = f_R$ is one to one and invertible. If $f_\chi = \frac{\partial \phi}{\partial R} \neq 0$, then these requirements are automatically met, but if $f_\chi$ vanishes or is not well defined, then a general condition for the equivalence of the representations can not be found.

The new field equations, which are obtained by varying Eq. (2.31) with respect to the metric $g_{\mu\nu}$ and the field $\phi$, are just Eqs. (2.8) and (2.29) rewritten in terms of $\phi$:

$$
G_{\mu\nu} = \frac{\kappa^2}{\phi} \left( T_{\mu\nu} - \frac{1}{2\kappa^2} g_{\mu\nu} V(\phi) + \frac{(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \phi}{\kappa^2} \right), \tag{2.32}
$$

$$
R = \frac{\partial V}{\partial \phi}. \tag{2.33}
$$

For a given matter source, the evolution of the field $\phi$ is governed by the trace equation:

$$
3\Box \phi + 2V(\phi) - \phi V_\phi = \kappa^2 T. \tag{2.34}
$$

Introducing the effective potential $V^{(eff)}$, which is defined as:

$$
V^{(eff)} = \frac{\phi V_\phi - 2V(\phi)}{3}, \tag{2.35}
$$

one can express Eq. (2.34) in the form of a Klein-Gordon equation:

$$
\Box \phi - V^{(eff)} = \frac{\kappa^2}{3} T. \tag{2.36}
$$

The effective mass of the scalar field $\phi$ in the Jordan frame, is now obtained as \[13, 39\]:

$$
m^2 = V^{(eff)} = \frac{1}{3} (\phi V_\phi - V_\phi) = \frac{1}{3} \left( \frac{f_R}{f_{RR}} - R \right). \tag{2.37}
$$

From this definition, one can conclude that $\phi$ is not a tachyon as long as the condition $\frac{f_R}{f_{RR}} > R$ is met.
Chapter 3

The Model

In this chapter one introduces a model to describe the evolution of the early universe. One chooses to work in a spatially flat FLRW space-time, due to the apparent flatness of the universe today and to the fact that the treatment of cosmological perturbations is made easier if $K = 0$. An $f(R)$ framework, as described in Chap. 2, is used to describe the bounce. One also adopts specifically the metric $f(R)$ formalism, as in this case the equations for the evolution of the tensorial perturbations are more extensively studied than in the Palatini formalism. After the bounce, as the universe enters a de Sitter-like inflationary regime, one switches continuously to a GR setup. This allows a smooth connection to the radiation dominated period and the subsequent ΛCDM era that describes the late-time evolution of the universe. A modified Generalized Chaplygin Gas (mGCG) is used to interpolate the inflationary era induced by $f(R)$-gravity and the radiation dominated universe.

3.1 A bounce in the Early Universe

The existence of a bounce in the evolution of the universe is related to a minimum of the scale factor at some time $t = t_b$, [59]. While the most general condition for this minimum to occur is:

$$\exists n, \forall m < 2n : \frac{d^m a}{dt^m}(t_b) = 0 \quad \text{and} \quad \frac{d^{2n} a}{dt^{2n}}(t_b) > 0,$$

for the purposes of this work one will just require that the first derivative of the scale factor vanishes, $\dot{a}(t_b) = 0$, and that the second derivative be positive, $\ddot{a}(t_b) > 0$. Furthermore, one wants the scale factor to scale as a de Sitter-like solution after the bounce, i.e.:

$$a(t \gg t_b) \sim a^* e^{t/t^*},$$

where $a^*$ and $t^*$ are positive constants whose values are for now irrelevant.

In Ref. [59] the energy conditions were studied around a bounce in a FLRW universe within a GR setup and it was found that, for a spatially flat universe ($K = 0$), the Null Energy Condition (NEC) is always violated when the conditions $\dot{a}(t_b) = 0$ and $\ddot{a}(t_b) > 0$ are met. This means that, in GR, for a bounce to be considered one must introduce some kind of matter field in the theory that violates the NEC. To avoid the introduction of phantom matter, one uses an $f(R)$ setup to describe the evolution of the universe around the bounce; this approach has the double advantage of both allowing a bounce to occur without the presence of matter and to provide a mechanism that can lead to inflation.

To obtain an $f(R)$ model that can originate a bounce during the early universe, one uses the designer $f(R)$ methodology [57]. In this approach, instead of working with a specific family of $f(R)$ functions defined apriori and then probing the evolution of the universe, one fixes the desired behaviour for the scale factor and then solves the Friedmann equation for the function $f(R)$. However, this procedure does not uniquely determine $f(R)$, as the substitution of the scale factor in the Friedmann equation produces a second order differential equation. Therefore, some additional criteria are needed to obtain the physically viable form of the $f(R)$ function. When deriving the appropriate solution for $f(R)$ in this section, one will show that the following criteria are enough to obtain an action that is physically viable and allows for the GR setup to be recovered during the inflationary phase of the early universe:
3.1.1 Description in terms of the cosmic time

One begins by defining the scale factor during the very early universe as:

$$a(t) \equiv a_b \cosh (H_{\text{inf}} t),$$  \hspace{1cm} (3.3)

Here $a_b \equiv a(t_b)$ and $H_{\text{inf}}$ is the limiting value of the Hubble parameter during inflation, as will be shown further ahead, and thus is related to the energy scale in this period. It can easily be shown that this model describes a bounce at $t_b = 0$: $\dot{a}(0) = 0$ and $\ddot{a}(0) > 0$.

Substituting $a$, defined in Eq. (3.3), in the definition of the Hubble parameter and the scalar curvature, one obtains:

$$H(t) \equiv \frac{\dot{a}}{a} = H_{\text{inf}} \tanh(H_{\text{inf}} t),$$  \hspace{1cm} (3.4)

$$R(t) \equiv 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right] = 6H_{\text{inf}}^2 \left[ 1 + \tanh^2 (H_{\text{inf}} t) \right].$$  \hspace{1cm} (3.5)

In Fig. 3.1 the scale factor, $a$, the Hubble parameter, $H$, and the scalar curvature, $R$, are plotted near the bounce as functions of the cosmic time.

3.1.2 Description in terms of the conformal time

Using the relation between the cosmic time, $t$, and the conformal time, $\tau$, ($dt = ad\tau$) and the fact that the scale factor is explicitly defined as a function of the cosmic time, it is possible to write $\tau$ as:

$$\tau - \tau_b = \int_{0}^{t} \frac{dt'}{a(t')} = \int_{0}^{t} \frac{dt'}{a_b \cosh (H_{\text{inf}} t')}$$  \hspace{1cm} (3.6)

which, after integration, leads to:

$$\tau - \tau_b = \frac{2}{a_b H_{\text{inf}}} \arctan \tanh \left( \frac{H_{\text{inf}} t}{2} \right).$$  \hspace{1cm} (3.7)
Figure 3.2: From left to right: the scale factor, $a$; the Hubble parameter, $H$; and the scalar curvature, $R$, as functions of the conformal time.

If $\tau_b$ is fixed as $\tau_b = 0$, then Eq. (3.7) reads:

$$\tau = \frac{2}{abH_{inf}} \arctan \tanh \left( \frac{H_{inf} \tau}{2} \right). \tag{3.8}$$

This is a monotonically increasing and odd function of the cosmic time with limits:

$$\lim_{t \to \pm \infty} \tau = \pm \frac{\pi}{2abH_{inf}}. \tag{3.9}$$

Inverting Eq. (3.8) gives:

$$\tanh \left( \frac{H_{inf} \tau}{2} \right) = \tan \left( \frac{abH_{inf} \tau}{2} \right). \tag{3.10}$$

Finally, making use of Eq. (3.10) in conjunction with the relations:

$$\cosh(x) = \frac{1 + \tanh^2(x/2)}{1 - \tanh^2(x/2)} \quad \text{and} \quad \tanh(x) = \frac{2 \tanh(x/2)}{1 + \tanh^2(x/2)}, \tag{3.11}$$

$$\sec(y) = \frac{1 + \tan^2(y/2)}{1 - \tan^2(y/2)} \quad \text{and} \quad \sin(y) = \frac{2 \tan(y/2)}{1 + \tan^2(y/2)}, \tag{3.12}$$

permits a simple formulation for the dynamical variables $a(\tau)$, $H(\tau)$ and $R(\tau)$:

$$a(\tau) = ab \sec \left( abH_{inf} \tau \right), \tag{3.13}$$

$$H(\tau) = H_{inf} \sin \left( abH_{inf} \tau \right), \tag{3.14}$$

$$R(\tau) = 6H_{inf}^2 \left[ 1 + \sin^2 \left( abH_{inf} \tau \right) \right]. \tag{3.15}$$

### 3.1.3 Solution for $f(R)$

Having fixed the evolution of the scale factor, it remains to be found an $f(R)$ function that is compatible with the bounce described in Eq. (3.3). This can be done by using Eqs. (3.3), (3.4) and (3.5) to express explicitly all the temporal dependences in the Friedmann equation (2.23). The result is a second order differential equation that dictates the evolution of the function $f(R)$. Solving this equation gives a family of $f(R)$ functions that allow the desired behaviour of scale factor. The physical constraints listed above can then be used to obtain the true physical solution for $f(R)$ (see the end of the introduction to Sec. 3.1 just before the Subsec. 3.1.2).
One begins by writing the Friedmann equation in vacuum:

$$3f_R H^2 + \frac{1}{2} (f - f_R R) + 3H f_{RR} \dot{R} = 0.$$  \hspace{1cm} (3.16)

The $R$-derivatives in Eq. (3.16) can be transformed into temporal derivatives using the relations:

$$\frac{d}{dR} = \frac{1}{R} \frac{d}{dt} \quad \text{and} \quad \frac{d^2}{dR^2} = \frac{1}{R^2} \frac{d^2}{dt^2} - \frac{\dot{R}}{R^3} \frac{d}{dt},$$  \hspace{1cm} (3.17)

and Eq. (3.16) then reads:

$$3H \ddot{f} + \left[ 3H \left( H - \frac{\dot{R}}{R} \right) - \frac{1}{2} R \right] \dot{f} + \frac{1}{2} R \ddot{f} = 0.$$  \hspace{1cm} (3.18)

In a spatially flat FLRW space-time the scalar curvature and its temporal derivatives can be expressed as functions of the Hubble parameter as:

$$R = 6 \left( \dot{H} + 2\dot{H}^2 \right), \quad \ddot{R} = 6 \left( H + 4H \dot{H} \right) \quad \text{and} \quad \dddot{R} = 6 \left( \dot{H}^2 + 4H \dddot{H} + 4\dot{H}^2 \right).$$  \hspace{1cm} (3.19)

Substituting these relations in Eq. (3.18) gives:

$$H \ddot{f} - \left[ \dot{H} + H \left( H + \dddot{H} + 4H \dot{H} + 4\dot{H}^2 \right) \right] \dot{f} + \left( \dot{H} + 4H \dot{H} \right) f = 0.$$  \hspace{1cm} (3.20)

As no assumption has yet been made for the behaviour of $a(t)$ in the derivation of Eq. (3.20), this is a general result valid for any metric $f(R)$-theory in a spatially flat FLRW universe.

In the model considered in this work, where $H$ is given by Eq. (3.4), the higher order derivatives of the Hubble parameter can be expressed in terms of $H$ and $\dot{H}$ as:

$$\dddot{H} = -2H \dot{H} \quad \text{and} \quad \dddot{H} = 4H^2 \dddot{H} - 2\dot{H}^2.$$  \hspace{1cm} (3.21)

This greatly simplifies the above differential equation, as inserting Eqs. (3.21) in Eq. (3.20) reads:

$$H \dddot{f} + \left( H^2 - 2\dot{H} \right) \dot{f} + 2H \dddot{H} f = 0.$$  \hspace{1cm} (3.22)

One can emphasize the temporal dependency in the previous equation and write:

$$f(t) \equiv H_{int} \left[ 3 \tanh(H_{int} t) - 2 \coth(H_{int} t) \right] \dot{f} + 2H_{int}^2 \text{scc}^2 \left( H_{int} t \right) f = 0.$$  \hspace{1cm} (3.23)

Equation (3.23) is a second order differential equation for $f$, that can be solved analytically using the software Mapple 14. Two linearly independent solutions, $f_1$ and $f_2$, were obtained by this method:

$$f_1(t) \equiv f_1(t) + f_2(t) \equiv \frac{3 \coth(\sqrt{3} H_{int} t)}{i + \sinh(H_{int} t)} \left[ 3 \tanh(H_{int} t) + i \sqrt{3} \text{scc} \left( H_{int} t \right) \right],$$  \hspace{1cm} (3.24)

and

$$f_2(t) \equiv \frac{3 \coth(\sqrt{3} H_{int} t)}{i + \sinh(H_{int} t)} \left[ 3 \tanh(H_{int} t) + i \sqrt{3} \text{scc} \left( H_{int} t \right) \right].$$  \hspace{1cm} (3.24)

It is easy to verify that $f_1$ and $f_2$ are complex conjugates. Therefore, using a simple linear transformation:

$$f_3(t) \equiv \frac{f_1(t) + f_2(t)}{2} \quad \text{and} \quad f_4(t) \equiv \frac{f_1(t) - f_2(t)}{2i},$$  \hspace{1cm} (3.25)

one can obtain two linearly independent real-valued solutions for Eq. (3.23). However, before ensuing with this transformation, it is better to write the functions $f_1$ and $f_2$ in the polar notation:

$$f_1(t) = \sqrt{6 \tanh^2(H_{int} t) + 3 \exp \left[ (-1)^{j-1} \left( \theta_1(t) + \theta_2(t) \right) \right]},$$  \hspace{1cm} (3.26)
where $j = 1, 2$. The phases $\theta_1$ and $\theta_2$ are defined as:

$$
\theta_1(t) \equiv \arg \left[ \frac{i - \sinh(H_{\text{inf}}t)}{i + \sinh(H_{\text{inf}}t)} \right]^\frac{2\pi}{\pi} = \text{sign}(t) \frac{\sqrt{3}}{2} \arccos[1 - 2 \tanh^2(H_{\text{inf}}t)], 
$$

(3.27)

$$
\theta_2(t) \equiv \arg \left[ 3 \tanh(H_{\text{inf}}t) - i\sqrt{3} \text{sech}(H_{\text{inf}}t) \right] = \arccos \left( \frac{3 \tanh(H_{\text{inf}}t)}{\sqrt{6 \tanh^2(H_{\text{inf}}t) + 3}} \right).
$$

(3.28)

In Fig. 3.3 the phases $\theta_1$ and $\theta_2$ and the total phase $\theta_T \equiv \theta_1 + \theta_2$ are plotted with respect to the cosmic time $t$. All these functions are asymptotically constant for large $t$ and respect a specific symmetry condition: $\theta_1(t \to +\infty) = \pi$; $\theta_2(t \to +\infty) = 0$ and $\theta_2(t) = \pi - \theta_2(-t)$; $\theta_T(t \to +\infty) = \sqrt{3} \frac{\pi}{2}$ and $\theta_T(t) = \pi - \theta_T(-t)$.

From the exponential notation in Eq. (3.26) one can easily infer the sinusoidal expressions for the real-valued functions $f_3$ and $f_4$, as defined in Eq. (3.25):

$$
f_3(t) = \sqrt{6 \tanh^2(H_{\text{inf}}t) + 3 \cos (\theta_1(t) + \theta_2(t))},
$$

(3.29)

$$
f_4(t) = \sqrt{6 \tanh^2(H_{\text{inf}}t) + 3 \sin (\theta_1(t) + \theta_2(t))}.
$$

(3.30)

Given the symmetry conditions of $\theta_T$ and the cosine and sine functions, $f_3$ and $f_4$ are, respectively, odd and even functions. Both functions are plotted in Fig. 3.4.

The Friedmann equation alone does not provide any information on the physically viable values of the linear coefficients for the functions $f_3$ and $f_4$. In order to determine these coefficients and obtain the correct solution for $f(R)$, one needs to make use of the physical constraints to be imposed on the
function \( f(R) \). This analysis is simplified by changing variables and express \( f_3 \) and \( f_4 \) in terms of the reduced scalar curvature \( r \), defined as:

\[
r = \frac{R}{H_\text{inf}^2},
\]

as will be shown in the following analysis.

One proceeds by inverting Eq. (3.5):

\[
\tanh(H_\text{inf} t) = \pm \sqrt{\frac{R - 6H_\text{inf}^2}{6H_\text{inf}^2}} = \pm \sqrt{\frac{r - 6}{6}}.
\]

Here the upper sign indicates values after the bounce while the lower sign indicates values before the bounce. This notation will be maintained in the rest of this section.

Inserting Eq. (3.32) in the definition of \( f_3 \) and \( f_4 \), Eqs. (3.29) and (3.30), reads respectively:

\[
f_3(r) = \sqrt{r - 3} \cos [\theta_1(r) + \theta_2(r)],
\]

\[
f_4(r) = \sqrt{r - 3} \sin [\theta_1(r) + \theta_2(r)],
\]

while the phases \( \theta_1 \) and \( \theta_2 \) are now defined as:

\[
\theta_1(r) = \pm \frac{\sqrt{3}}{2} \arccos \left( \frac{9 - r}{3} \right),
\]

\[
\theta_2(r) = \arccos \left( \pm \sqrt{\frac{3r - 6}{2r - 3}} \right) = \frac{\pi}{2} \pm \left[ \arcsin \left( \sqrt{\frac{3r - 6}{2r - 3}} \right) \right].
\]

The functions \( \theta_1 \), \( \theta_2 \) and \( \theta_T \) are plotted in Fig. 3.5. The solutions \( f_3 \) and \( f_4 \) are plotted in Fig. 3.6, where a blue curve indicates the behaviour before the bounce and a red curve indicates the behaviour after the bounce. Notice that in each of its symmetrical branches, \( f_3 \) is a monotonic function of \( r \). On the other hand, \( f_4 \) does not change its sign at the bounce and has a local maximum at \( r = r_{\text{max}} \).

One proceeds by differentiating the functions \( f_3 \) and \( f_4 \) with respect to \( r \), thus obtaining:

\[
f_{3r} = \frac{1}{2\sqrt{r - 3}} \left\{ \cos [\theta_1(r) + \theta_2(r)] \pm \sqrt{3} \frac{\sqrt{r - 6}}{\sqrt{12 - r}} \sin [\theta_1(r) + \theta_2(r)] \right\},
\]

\[
f_{4r} = \frac{1}{2\sqrt{r - 3}} \left\{ \sin [\theta_1(r) + \theta_2(r)] \pm \sqrt{3} \frac{\sqrt{r - 6}}{\sqrt{12 - r}} \cos [\theta_1(r) + \theta_2(r)] \right\}.
\]

Here, an \( r \) subscript indicates a derivative with regards to the reduced scalar curvature. By calculating the root of Eq. (3.38) one obtains the following condition for the maximum point, \( r_{\text{max}} \), of \( f_4 \):

\[
\tan [\theta_1(r_{\text{max}}) + \theta_2(r_{\text{max}})] = \sqrt{3} \frac{\sqrt{r_{\text{max}} - 6}}{\sqrt{12 - r_{\text{max}}}}.
\]

Figure 3.5: From left to right: the phase \( \theta_1 \); the phase \( \theta_2 \) and the total phase \( \theta_T = \theta_1 + \theta_2 \) as functions of the reduced scalar curvature. The blue curves indicate the behaviour before the bounce and the red curves indicate the behaviour after.
Solving Eq. (3.39) numerically for $r_{\text{max}}$ gives the value $r_{\text{max}} \approx 9.722$.

The derivatives $f_3r$ and $f_4r$ are plotted in Fig. 3.7, where the blue curves indicate the behaviour before the bounce and the red curves indicate the behaviour after. It is possible to observe in Fig. 3.7 that both $f_3r$ and $f_4r$ diverge in the limit $r \to 12$. This is due to the factor $(\sqrt{12 - r})^{-1}$ on the second term of Eqs. (3.37) and (3.38). Therefore, in order to be possible to obtain a finite value for $f_r = C_3f_3r + C_4f_4r$ in the limit $r \to 12$ (and in particular $f_R(r \to 12) = 1$, i.e., the standard gravitational constant is recovered at the end of the $f(R)$ era), some coefficients $C_3$ and $C_4$ need to be arranged such that they eliminate the undesired terms. This gives the condition:

$$
\lim_{r \to 12} \pm C_3 \frac{1}{\sqrt{12 - r}} \sin [\theta_1(r) + \theta_2(r)] + C_4 \frac{1}{\sqrt{12 - r}} \cos [\theta_1(r) + \theta_2(r)] = 0,
$$

which implies:

$$
\frac{C_4}{C_3} = \pm \tan \left( \frac{\sqrt{3}}{2 \pi} \right). \tag{3.41}
$$

Computing the value of Eqs. (3.37) and (3.38) at $r = 6$ gives $f_3r(r = 6) = 0$ and $f_4r(r = 6) = (2\sqrt{3})^{-1}$. As the positivity of $f_r$ needs to be guaranteed for all values of $r$, one obtains the condition $C_4 > 0$. This in conjunction with Eq. (3.40) suggests the following choice for $C_3$ and $C_4$:

$$
C_3 = \pm C \cos \left( \frac{\sqrt{3}}{2 \pi} \right) \quad \text{and} \quad C_4 = C \sin \left( \frac{\sqrt{3}}{2 \pi} \right). \tag{3.42}
$$
Figure 3.8: This Fig. shows: (top left) the solution $f(r)$ in a blue curve and its asymptotic behaviour, $r \to 6$, for large $r$ in a red dashed curve; (top right) the first derivative $f'_r$; (bottom left) the second derivative $f''_r$; (bottom right) the third derivative $f'''_r$.

where $C$ is a positive constant to be determined. This choice for the coefficients has the advantage of conferring a simple form to the function $f$:

$$f(r) = C \sqrt{r - 3} \cos \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right). \quad (3.43)$$

Differentiating Eq. (3.43) repeatedly with respect to $r$ gives:

$$f'_r(r) = \frac{C}{2 \sqrt{r - 3}} \left( \cos \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right) \right. \right.$$

$$\left. + \frac{\sqrt{3}}{2} \frac{\sqrt{r - 6}}{\sqrt{12 - r}} \sin \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right) \right), \quad (3.44)$$

$$f''_r(r) = -\frac{3C}{2 \sqrt{r - 3} (12 - r)} \left( \cos \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right) \right. \right.$$

$$\left. - \frac{3 \sqrt{3}}{\sqrt{12 - r} \sqrt{r - 6}} \sin \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right) \right), \quad (3.45)$$

$$f'''_r(r) = -\frac{3C}{4 \sqrt{r - 3} (12 - r)^2} \left( \cos \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right) \right. \right.$$

$$\left. - \frac{3}{\sqrt{3} \sqrt{12 - r} (r - 6)^{3/2}} \sin \left( \frac{\sqrt{3}}{2} \left[ \pi - \arccos \left( \frac{9 - r}{3} \right) \right] + \arcsin \left( \frac{\sqrt{3} \, r - 6}{2 \, r - 3} \right) \right) \right). \quad (3.46)$$

Taking the limit of Eq. (3.44) when $r \to 12$ reads:

$$\lim_{r \to 12} f'_r = \frac{C}{2}. \quad (3.47)$$
Therefore, in order to make the theory converge to GR during inflation, one must impose \( f_R(r \to 12) = 1 \), which fixes the value for the constant \( C \) as \( C = 2H_{\text{inf}}^2 \).

In Fig. 3.8 the functions \( f \), \( f_r \), \( f_{rr} \), and \( f_{rrr} \) are plotted with \( H_{\text{inf}}^2 \) normalized to 1. One can also see in Fig. 3.8 that the second and third derivative of \( f \), and consequently all higher order derivatives, diverge at the moment of the bounce. This is due to the existence of terms containing \( (r - 6)^{-1/2} \) and \( (r - 6)^{-3/2} \) on the expression for \( f_{rr} \) and \( f_{rrr} \), Eqs. (3.45) and (3.46) respectively. Even though the right hand sides of the Friedmann, Eq. (2.19), and Raychaudhury, Eq. (2.20), equations contain terms in \( f_{rr} \) and \( f_{rrr} \), these appear in a combination that removes the divergence at the bounce. Notice that this would always need to be so, as the left hand sides of the equations are finite, and so the right hand sides need to be finite as well.

While deriving this solutions, only two of the restrictions on \( f(R) \) were used: \( f_R > 0 \), (II), and \( f_R(r \to 12) = 1 \), (III). However, it is possible to observe from Fig. 3.8 that the condition for the theory to be ghost free, (IV), is automatically met with this choice of coefficients, as \( f_{RR} > 0 \) for every value of \( R \). Hence, the only constraint that has not yet been verified is the requirement that the theory converges to GR well inside the inflation.

The Taylor expansion of \( f(R) \) around the limit value \( r = 12 \) gives:

\[
f(r) = 6 + (r - 12) + \frac{(r - 12)^2}{36} - \frac{(r - 12)^3}{1080} + O(r^4).
\]

As the coefficients of the higher order terms are small when compared to the coefficients of the constant and linear terms, one can consider simply the linear approximation of \( f(R) \) near \( r = 12 \) given by:

\[
f(r) = r - 6 + O(r^2).
\]

Hence the linearity of the action is regained during the inflationary phase, even if only approximately. To obtain some insight on the validity of this approximation during inflation, both \( f(R) \), see Eq. (3.43), and Eq. (3.49) are plotted in Fig. 3.9 and compared to the Hubble parameter. Notice that when the de Sitter-like expansion is reached, i.e. \( H \approx \text{constant} \), the solution for \( f(R) \) already has overlapped the linear approximation for some time.

### 3.1.4 Other considerations during the very early universe

Throughout this section, one has derived an action, given in Eq. (3.43), within the metric \( f(R) \) formalism that allows the scale factor to scale as in Eq. (3.3) and respects the four physical constraints imposed on \( f(R) \) and its derivatives. One will now show how the effective quantities resulting from the \( f(R) \) theory, e.g. \( \rho^{(eff)} \) and \( p^{(eff)} \), behave during the very early universe.

---

1Notice that the factor \( H_{\text{inf}}^2 \) needs to be inserted for \( f(R) \) to have the same units as the scalar curvature \( R \).
Within the assumption of vacuum, i.e., in the absence of an *inflaton* or scalar field, the effective energy density and pressure can be determined more easily from the Friedmann and Raychaudhury equations, see Eqs. (2.23) and (2.24):

$$\rho_{\text{eff}} = \frac{3}{\kappa^2} H^2 = \rho_{\text{inf}} \tanh^2 (H_{\text{inf}} t), \quad (3.50)$$

$$p_{\text{eff}} = -\frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) = -\rho_{\text{inf}} \frac{2 + \tanh^2 (H_{\text{inf}} t)}{3}. \quad (3.51)$$

Here one defines the limiting value of the energy density during inflation, $\rho_{\text{inf}}$ as:

$$\rho_{\text{inf}} \equiv E_{\text{inf}}^4 = \frac{3}{\kappa^2} H_{\text{inf}}^2. \quad (3.52)$$

From Eqs. (3.50) and (3.51), the equation of state (EoS) of the very early universe can be derived as:

$$p(\rho) = -\frac{2\rho_{\text{inf}} + \rho}{3}, \quad (3.53)$$

where the superscript was dropped for simplicity. The effective state parameter can also be derived from these results:

$$w \equiv \frac{p}{\rho} = -\frac{1}{3} - \frac{2}{3} \cotanh^2 (H_{\text{inf}} t). \quad (3.54)$$

The results of Eqs. (3.50), (3.51) and (3.54) are presented in Fig. 3.10. In order to have some sensibility on the time scale during which the bounce occurs, it was determined how long after the bounce do the density, Eq. (3.50), and pressure, Eq. (3.51), reach 99% of their limiting values, i.e.:

$$\tanh^2 (H_{\text{inf}} t_{99\%}) = 0.99 \quad \text{and} \quad \frac{2 + \tanh^2 (H_{\text{inf}} t_{99\%})}{3} = 0.99 \quad (3.55)$$

Solving Eqs. (3.55) for $t_{99\%}$ gives:

$$t_{99\%} \approx 2.993/H_{\text{inf}} \quad \text{and} \quad t_{99\%} \approx 2.439/H_{\text{inf}} \quad (3.56)$$

With the energy scales considered, which range in the interval $10^{14} \sim 10^{16}$ GeV, this corresponds to time intervals of the order of $10^{-38} \sim 10^{-42}$ seconds.

If the linear approximation of the function $f(R)$ during inflation (see Eq. (3.49)) is compared to the Lagrangian density of GR with a cosmological constant $\Lambda$:

$$S^\Lambda = \frac{1}{2\kappa^2} (R - 2\Lambda), \quad (3.57)$$

the identity $\Lambda = 3H_{\text{inf}}^2$ ensues immediately. Notice that the energy density and pressure of a cosmological constant are given by $\rho_\Lambda = \Lambda/\kappa$ and $p_\Lambda = -\rho_\Lambda$, which correspond precisely to the limiting values of Eqs. (3.50) and (3.51). It is not overstating to say that, for $r \to 12$, the universe described in a metric $f(R)$ gravity with $f(R)$ given in Eq. (3.43) perfectly mimics a vacuum state with a cosmological constant.
Λ in a GR setup. The origin of the expansion of the universe is precisely in the fact that the \(f(R)\) corrections of the theory mimic a cosmological constant after the bounce.

Before moving on to describe the end of inflation and the transition to the \(\Lambda\)CDM model, one makes a reference to the scalar sector of the theory, namely to the new scalar field \(\phi\) that appears from the use of \(f(R)\) gravity. The squared mass of the scalar degree of freedom, \(m_\phi^2\), is defined in the \(f(R)\) framework by:

\[
m_\phi^2 = \frac{1}{3} \left( \frac{f_R}{f_{RR}} - R \right) = \frac{H_{\text{inf}}^2}{3} \left( \frac{f_r}{f_{rr}} - r \right).
\]

(3.58)

Given the solution found for \(f(R)\), see Eq. (3.43), and the behaviour of its derivatives, see Fig. (3.8), one can foresee some problems regarding the field \(\phi\). As one approaches the bounce, \(r \to 6\), the second derivative \(f_{rr}\) grows to infinity; therefore, the first term of Eq. (3.58) goes to zero and \(m_\phi^2 < 0\) becomes negative, i.e. \(\phi\) becomes a tachyon. Inserting the expressions of \(f_r\) and \(f_{rr}\), given in Eqs. (3.44) and (3.45), respectively, in Eq. (3.58) gives the result:

\[
m_\phi^2 = -H_{\text{inf}}^2 \left[ 4 + \frac{(12 - r)(r - 3)}{\sqrt{3(12 - r)(r - 6)} \cot \left( \frac{\sqrt{3}}{2} \pi - \theta_T \right) - 9} \right].
\]

(3.59)

In Fig. 3.11, \(m_\phi^2\) is plotted as a function of both the reduced scalar curvature, \(r\), and the cosmic time, \(t\). The value of the reduced scalar curvature at which the mass becomes zero, \(r_{\phi}\), is defined by the condition:

\[
f_r(r_{\phi}) = f_{rr}(r_{\phi}) r_{\phi}.
\]

(3.60)

Solving Eq. (3.60) numerically gives \(r_{\phi} \approx 8.72\).

![Figure 3.11: The squared mass of the scalar field \(\phi\). From the left to the right: \(m_\phi^2/H_{\text{inf}}^2\) as a function of the reduced scalar curvature; \(m_\phi^2/H_{\text{inf}}^2\) as a function of the cosmic time.](image)

### 3.2 The Universe After The Bounce

Throughout the last section, an \(f(R)\) action was obtained capable of, after the bounce, originate an inflationary phase in the early universe. At some point, during inflation, it can be approximated by a Hilbert-Einstein action with a cosmological constant. This way, one can include the inflationary paradigm in the model of the early universe [8], without the need for any inflaton field, and can change to a GR setup so as to accommodate the \(\Lambda\)CDM model with a radiation dominated phase in the description of the late-time evolution of the universe. To describe the transition from the inflation to the radiation phase, one uses a mGCG model, as presented in [28].

#### 3.2.1 \(\Lambda\)CDM model

In GR, the Friedmann equation and Raychaudhury equation read, respectively:

\[
H^2 + \frac{K}{a^2} = \frac{\kappa^2}{3} \rho,
\]

(3.61)
and

\[ 2\dot{H} + 3H^2 - \frac{K}{a^2} = -\kappa^2 p. \]  

(3.62)

From these equations one can obtain the continuity equation:

\[ \dot{\rho} + 3H(\rho + p) = 0, \]  

(3.63)

If one defines the density parameter \( \Omega \) and the critical density at present time \( \rho_c \):

\[ \Omega \equiv \frac{\rho}{\rho_c}, \quad \text{and} \quad \rho_c \equiv \frac{3H^2}{\kappa^2}, \]  

(3.64)

then Eq. (3.61) can be recast as:

\[ \Omega - 1 = \frac{K}{a^2H^2}. \]  

(3.65)

Even though the astronomical data points for \( \Omega \approx 1 \) [1], thus most works in cosmology assume a spatially flat space-time, since the density parameter is sensible to the degree of expansion of the universe (\( \Omega a^{-2} \)), one can obtain an \( \Omega \) very close to 1 with \( K \neq 0 \) if the value of scale factor at present time is much larger than during the early stages of the universe. This is precisely how the inflationary paradigm explains the current flatness of the universe without fine-tuning the parameter \( K \).

In describing the late-time evolution of the universe, one uses the \( \Lambda \)CDM model with a radiation phase. This model assumes a spatially flat space-time with three successive phases:

- an initial phase dominated by radiation;
- an intermediate phase dominated by non-relativistic matter (cold matter and, residually, baryonic matter);
- a final phase dominated by a cosmological constant, which is the phase where the universe is currently in.

If one takes the EoS of the different matter contents of the universe to be of the form:

\[ p_j = w_j \rho_j, \]  

(3.66)

where \( w_r = 1/3 \) for radiation; \( w_c = 0 \) for non relativistic matter; \( w_r = -1 \) for the cosmological constant; then the continuity equation (3.63) gives:

\[ \rho_j = \rho_{j0} \left( \frac{a}{a_0} \right)^{-3(1+w_j)}. \]  

(3.67)

where a 0 subscript indicates evaluation at present time.

The Friedmann equation for the \( \Lambda \)CDM model becomes:

\[ \sum_j \rho_j = \rho_c, \]  

(3.68)

or, dividing by \( \rho_{c0} \) and introducing the relative density parameters \( \Omega_{j0} = \rho_{j0}/\rho_{c0} \):

\[ \Omega_{r0} \left( \frac{a_0}{a} \right)^4 + \Omega_{m0} \left( \frac{a_0}{a} \right)^3 + \Omega_{\Lambda0} = \left( \frac{H}{H_0} \right)^2. \]  

(3.69)

### 3.2.2 Modified Generalized Chaplygin Gas model (mGCG)

The Chaplygin Gas (CG) [62] is a theoretical model of a perfect fluid with an exotic EoS:

\[ p = -\frac{A}{\rho}, \]  

(3.70)

with \( A \) a positive constant. This model was first studied in the frame of d-brane cosmology, as it can be derived from the Nambu-Goto action in a \((d+1, 1)\) space-time, with a light-cone parametrization. To this day, it remains the only perfect fluid that admits a supersymmetric generalization. After the discovery
of the current expansion of the universe \cite{1, 2, 3}, the CG model gained renewed interest, specially in GR cosmology works, as it provides a simple and smooth transition from a universe dominated by dusk-like matter to an inflationary one with a cosmological constant.

After introduction of this model in FLRW cosmology Ref. \cite{62}, many generalizations of the CG model have appeared and in various contexts: as a means of unifying the dark sector of the universe \cite{63, 64, 65}; in the study of the future evolution of the universe \cite{68, 41}; as a mechanism to obtain the early inflation \cite{28, 30}.

In this work, it is required that the Chaplygin Gas interpolates the inflationary era, dictated by the $f(R)$ gravity, and the radiation dominated universe, at the beginning of the $\Lambda$CDM model. For that purpose one uses the model used in \cite{28}, where the energy density is given by:

$$\rho_{m\text{CG}} = \left( A + \frac{B}{a^{4(1+\alpha)}} \right)^{\frac{1}{1+\alpha}}. \quad (3.71)$$

Here $A$ and $B$ are positive constants and $\alpha$ is a free parameter which obeys $1 + \alpha < 0$. Notice that with this choice for the parameter $\alpha$, the Chaplygin gas behaves as a cosmological constant for small scale factors and as a radiation fluid for large scale factors:

$$\rho \simeq A^{\frac{1}{1+\alpha}}, \quad \text{for} \quad a \ll \left( \frac{B}{A} \right)^{\frac{1}{4(1+\alpha)}}, \quad (3.72)$$

$$\rho \simeq B^{\frac{1}{1+\alpha}} \frac{1}{a^4}, \quad \text{for} \quad a \gg \left( \frac{B}{A} \right)^{\frac{1}{4(1+\alpha)}}, \quad (3.73)$$

with a smooth transition in the intermediate region.

Inserting Eq. (3.71) in the continuity equation (3.63) gives the EoS of this mCGC model:

$$p_{m\text{CG}} = \frac{1}{3} \rho - \frac{4}{3} \rho^{\alpha}. \quad (3.74)$$

The value of the constants $A$ and $B$ can be fixed by comparing the asymptotic behaviour of the Chaplygin gas, Eqs. (3.72) and (3.73), with the density limit during the inflation in the $f(R)$ era and the energy density of the radiation era in the $\Lambda$CDM model:

$$A^{\frac{1}{1+\alpha}} \simeq \rho_{\text{inf}}, \quad \text{and} \quad \frac{B^{\frac{1}{1+\alpha}}}{a^4} \simeq \rho_{\text{r0}} \left( \frac{a_0}{a} \right)^4, \quad (3.75)$$

which gives $A = \rho_{\text{inf}}^{1+\alpha}$ and $B = (\rho_{\text{r0}} a_0^4)^{1+\alpha}$. 

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Chapter 4

Gravitational Waves

Cosmological (tensorial) gravitational waves originate during inflation from quantum fluctuations of the metric $g_{\mu\nu}$. The evolution and quantization of the cosmological scalar and tensorial perturbations in generalized theories of gravity [12, 15, 44], specially in metric $f(R)$ gravity, are well documented in the literature, where an extension of the canonical method for GR-based inflation is usually applied [20]. In this section, the approach of Refs. [12, 13] is followed in order to obtain the quantification of the tensorial perturbations. A Bogoliubov transformation, first considered by L. Parker, [38], and Starobinsky, [19], is then employed to describe the evolution of the quantum operators in terms of time-fixed annihilation and creation operators and the linear coefficients $\alpha$ and $\beta$, after which a final variable change ensues as described in Refs. [25, 24]. This method has the double advantage of permitting the calculation of the power and energy spectra in terms of the graviton content of the universe and providing an easy way to determine the initial conditions for the necessary numerical integrations.

4.1 Tensor Perturbations and Energy Spectrum

4.1.1 Canonical Quantization

In the conformal coordinates $(\eta, x, y, z)$ the tensorial part of the linear perturbations of a flat FLRW metric can be written as:

$$ds^2 = a^2 \left\{ -d\eta^2 + \left[ g^{(3)}_{\alpha\beta} + 2H_T(\eta)Y_{ij}(x) \right] dx^i dx^j \right\},$$

(4.1)

where $g^{(3)}_{\alpha\beta}$ is the comoving background three-space metric, $H_T(\eta)$ is the gauge-invariant mode function and $Y_{ij}(x)$ is a symmetric, tracefree and transverse harmonic function that satisfies the Helmholtz equation [71, 72]:

$$\nabla^2 Y_{ij}(x) = -k^2 Y_{ij}(x).$$

(4.2)

The temporal and spatial dependencies of the perturbations can be pieced together by defining the also traceless and transverse quantity $c_{ij}$:

$$c_{ij}(\eta, x) = H(\eta)Y_{ij}(x).$$

(4.3)

The second order perturbed action of $f(R)$, obtained from Eq. (4.1), is [12, 13]:

$$S^{(gw)} = \int d\eta \int d^3x \frac{a^2 f_R}{2} (c_{ij}' c_{ij}' - \partial_k c_{ij} \nabla^k c_{ij}),$$

(4.4)

and variation of Eq. (4.4) gives the classical equation of evolution for the tensor perturbations:

$$2aa' f_R c_{ij}' + a^2 f_R c_{ij}'' + a^2 f_R c_{ij}'' - a^2 f_R \nabla^2 c_{ij} = 0,$$

(4.5)

or equivalently:

$$c_{ij}'' + \left( 2 \frac{a'}{a} + \frac{f'_R}{f_R} \right) c_{ij}' - \nabla^2 c_{ij} = 0.$$

(4.6)
Using a change of variables \[12\, 47\, 49\] suggested by the equivalence of \(f(R)\) gravity with scalar-tensor theories \[13\, 24\]:

\[z = a\sqrt{f(R)},\]  

and:

\[\nu_g = z c_{ij} = a\sqrt{f(R) c_{ij}},\]  

one can rewrite Eq. \((4.5)\) as:

\[\nu_g'' - \left(\frac{z''}{z} + \nabla^2\right) \nu_g = 0.\]  

Since by definition the spatial part of \(c_{ij}\) satisfies the Helmholtz equation, the Laplacian operator in the last equation can be replaced, giving:

\[\nu_g'' - \left(\frac{z''}{z} - k^2\right) \nu_g = 0.\]  

It is noteworthy to point out that in an action linear in \(R\) one obtains \(z \propto a\) and the result of GR is readily obtained as \(z''/z\) reduces to \(a''/a\). \[23\, 24\, 25\, 26\, 27\]. \(^1\)

The classical perturbations \(c_{ij}\) can be decomposed into the two possible polarizations for the GW + and ×:

\[c_{ij}(\eta, x) = \sqrt{8\pi G} \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{l=1}^2 h_{l\kappa}(\eta) c_{ij}^{(l)}(k) e^{ik \cdot x},\]  

where \(l\) runs over the two polarizations (+, ×), \(c_{ij}^{(l)}(k)\) is the polarization tensor, which satisfies \(c_{ij}^{(l)}(k)e^{(l')ij}(k) = 2\delta_{ll'}\) and \(\delta_{ll'}\) is the Kronecker delta function. The contribution of each polarization can now be represented by:

\[h_{l}(\eta, x) = \sqrt{8\pi G} \int \frac{d^3k}{(2\pi)^{3/2}} h_{l\kappa}(\eta) e^{ik \cdot x},\]  

and it can be shown that \(h_l(\eta)\) also satisfies Eq. \((4.5)\).

The quantization of \(c_{ij}\) is made by defining a quantum operator, \(\hat{c}_{ij}\), associated to the classical variable which can be expressed in a Fourier expansion as:

\[\hat{c}_{ij}(\eta, x) = \sqrt{8\pi G} \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{l=1}^2 \left(\hat{h}_{l\kappa}(\eta)\hat{a}_{l\kappa}(\eta) c_{ij}^{(l)}(k) e^{ik \cdot x} + H.c.\right).\]  

Here \(H.c.\) stands for the Hermitian conjugate, the mode functions \(\hat{h}_{l\kappa}(\eta)\) obey the Wronskian condition:

\[\left[\hat{a}_{l\kappa}, \hat{a}^\dagger_{l'\kappa'}\right] = \delta_{ll'}\delta^{(3)}(k - k'), \quad \left[\hat{a}_{l\kappa}, \hat{a}_{l'\kappa'}\right] = \left[\hat{a}^\dagger_{l\kappa}, \hat{a}^\dagger_{l'\kappa'}\right] = 0.\]  

A quantum operator for the polarized perturbation can be obtained in a similar fashion:

\[\hat{h}_l(\eta, x) = \sqrt{8\pi G} \int \frac{d^3k}{(2\pi)^{3/2}} \left(\hat{h}_{l\kappa}(\eta)\hat{a}_{l\kappa}(\eta) e^{ik \cdot x} + H.c.\right),\]  

and both the total and polarized perturbation operators satisfy the same equation of motion of its classical counterparts (see Eq. \((4.5)\)).

\(^1\)It will be apparent further ahead that the transition from GR to extended theories of gravity can be achieved in a direct and simple way by replacing the quotient \(a''/a\) by \(z''/z\) in all equations.
4.1.2 Bogoliubov Transformation

The time dependent annihilation, $\hat{a}_k(\eta)$, and production, $\hat{a}_k^\dagger(\eta)$, operators can be related to the time-fixed operators $\hat{A}_k \equiv \hat{a}_k(\eta)$ and $\hat{A}^\dagger_k \equiv \hat{a}_k^\dagger(\eta)$ by means of a Bogoliubov transformation \[^{38}\] \[^{19}\]:

$$\hat{a}_k(\eta) = \alpha_{ik}(\eta)\hat{A}_k + \beta_{ik}(\eta)\hat{A}_k^\dagger,$$

(4.17)

where the coefficients $\alpha$ and $\beta$ are c-functions satisfying $\alpha(\eta) = 1$ and $\beta(\eta) = 0$, and the time-fixed operators satisfy Eq. (4.15) at time $\eta$. In Ref. \[^{38}\] it was shown that the $\beta$ coefficient is intimately related to the graviton content of the universe. In fact if $|0\rangle$ is the state containing no particles at time $\eta$ then $|\beta_k|^2$ is the average density of gravitons with polarization $l$ and wave-number $k$ in the state $|0\rangle$ at a posterior time $\eta$:

$$\langle N_k(\eta) \rangle_0 = \langle 0|\hat{a}_k^\dagger(\eta)\hat{a}_k(\eta)|0\rangle = |\beta_k(\eta)|^2.$$

(4.18)

One begins by choosing the ansatz:

$$\hat{h}_k = \frac{1}{a\sqrt{f_R}} \frac{e^{-ik(\eta-n)}}{\sqrt{k}} = \frac{e^{-ik(\eta-n)}}{z\sqrt{k}},$$

(4.19)

which can be easily verified to satisfy the Wronskian condition in Eq. (4.14) for $\hat{h}_k$. Plugging Eq. (4.19) and Eq. (4.17) in the Fourier expansion of the polarized perturbation operator, Eq. (4.16), one obtains:

$$\hat{h}_k(\eta,x) = \sqrt{8\pi G} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\hat{A}_k}{a^{3/2}\sqrt{f_R}} \psi^* e^{ikx} + H.c.$$

(4.20)

Following Ref. \[^{23}\], a new function $\psi$ is defined as:

$$\psi(\eta) = \frac{\sqrt{a}}{\sqrt{k}} \left( \alpha e^{-ik(\eta-n)} + \beta e^{ik(\eta-n)} \right),$$

(4.21)

and Eq. (4.20) can be written as:

$$\hat{h}_k(\eta,x) = \sqrt{8\pi G} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\hat{A}_k}{a^{3/2}\sqrt{f_R}} \psi^* e^{ikx} + H.c.$$

(4.22)

Inserting Eq. (4.22) in the equation of motion of $\hat{h}_k(\eta,x)$ and solving for $\psi$ gives:

$$\psi'' - \psi' \frac{a'}{a} + \psi \left[ k^2 + 3 \left( \frac{a'}{a} \right)^2 - \frac{a'}{a} \frac{f_R}{f_R} + \frac{1}{4} \left( \frac{f_R}{f_R} \right)^2 - 3 \frac{a''}{2} + \frac{1}{2} \frac{f''}{f_R} \right] = 0.$$

(4.23)

Comparing Eq. (4.24) with Eq. [9b] of Ref. \[^{23}\] one can obtain:

$$\psi'' - \psi' \frac{a'}{a} + \psi \left[ k^2 + 3 \left( \frac{a'}{a} \right)^2 - \frac{1}{2} \frac{a''}{a} \right] = \psi'.'$$

(4.24)

The solution of the inhomogeneous linear differential equation, Eq. (4.24), is given by \[^{23}\]:

$$\psi(\eta) = \frac{\sqrt{a}}{\sqrt{k}} e^{-ik(\eta-n)} + \int_{\eta}^{\eta'} G(k,\eta,\eta') \frac{z''}{z} \psi(\eta')d\eta',$$

(4.25)

where the Green function $G(k,\eta,\eta')$ is defined as:

$$G(k,\eta,\eta') = \frac{i}{2k} \sqrt{\frac{a(\eta)}{a(\eta')}} \left( e^{-ik(\eta-\eta')} - e^{ik(\eta-\eta')} \right),$$

(4.26)

and satisfies the border conditions $G(k,\eta,\eta) = 0$ and $\partial G(k,\eta,\eta')/\partial \eta = 1$ at $\eta' = \eta$. Inserting Eq. (4.21) in both sides of Eq. (4.25) leads to:

$$\alpha(\eta) + \beta(\eta)e^{2ik(\eta-n)} = 1 + \frac{i}{2k} \int_{\eta}^{\eta'} \frac{z''}{z} \left( \alpha(\eta') + \beta(\eta')e^{2ik(\eta'-n)} \right) d\eta' - \frac{i}{2k} e^{2ik(\eta-n)} \int_{\eta}^{\eta'} \frac{z''}{z} \left( \beta(\eta') + \alpha(\eta')e^{-2ik(\eta'-n)} \right) d\eta'.$$

(4.27)
This result suggests the decomposition into the system of integral equations \([23, 24]\):

\[
\alpha(\eta) = 1 + \frac{i}{2k} \int_{\eta_i}^{\eta} \frac{z''}{z} \left[ \alpha(\eta') + \beta(\eta') e^{2ik(\eta'-\eta_i)} \right] d\eta',
\]

(4.28)

\[
\beta(\eta) = -\frac{i}{2k} \int_{\eta_i}^{\eta} \frac{z''}{z} \left[ \beta(\eta') + \alpha(\eta') e^{-2ik(\eta'-\eta_i)} \right] d\eta',
\]

(4.29)

with which the conditions \(\alpha(\eta_i) = 1\) and \(\beta(\eta_i) = 0\) are automatically satisfied.

### 4.1.3 The Power and Energy Spectra

The equations (4.28) and (4.29) for the Bogoliubov coefficients encode the evolution of the tensorial perturbations, with the potential \(z''/z\) being sensitive to both the theory of gravity considered and the energy content of the universe. The integration of Eqs. (4.28) and (4.29) can be simplified by considering the additional change of variables \([24]\):

\[
X = \alpha e^{-ik(\eta-\eta_i)} + \beta e^{ik(\eta-\eta_i)},
\]

(4.30)

\[
Y = \alpha e^{-ik(\eta-\eta_i)} - \beta e^{ik(\eta-\eta_i)},
\]

(4.31)

where the constraint \(|\alpha|^2 - |\beta|^2 = 1\) imposes \(\Re(X\bar{Y}) = 1\). Substituting Eqs. (4.30) and (4.31) in Eqs. (4.28) and (4.29) and differentiating gives a system of first order differential equations for the variables \(X\) and \(Y\):

\[
X' = -ikY,
\]

(4.32)

\[
Y' = i \left( \frac{z''}{z} - k^2 \right) X.
\]

(4.33)

If one considers that the energy per graviton is \(\epsilon_\omega = \hbar \omega\), the density of graviton states is \(\langle N_\mathbf{k}(\eta) \rangle_0 = \omega^2 d\omega/(2\pi^2 c^3)\) and takes into account that there are two possible polarizations for each graviton, then, for each frequency \(\omega\), the power spectrum, \(P(\omega, \eta)\), of the GW is defined at a time \(\eta\) as \([26, 27]\):

\[
P(\omega, \eta) = \frac{2}{\omega} \sum_{l=1}^2 \frac{dE_l}{d\omega} = \frac{\epsilon_\omega \langle N_\mathbf{k}(\eta) \rangle_0 d\omega}{\pi^2 c^3} = \frac{\hbar \omega^3}{\pi^2 c^3} |\beta_\omega(\eta)|^2.
\]

(4.34)

Notice that in Eq. (4.34), and throughout the rest of this section one no longer assumes that \(c = \hbar = 1\). With the power spectrum described by Eq. (4.34), the relative logarithmic energy spectrum of gravitational waves is defined as \([21, 26, 27]\):

\[
\Omega_{GW} \equiv \frac{\rho_{GW}}{\rho_c} \partial \log \omega,
\]

(4.35)

where \(\rho_c \equiv 3H^2c^2/(8\pi G)\) is the critical energy density at present time and the gravitational wave energy density, \(\rho_{GW}\) is related to the power spectrum by \([25, 27]\):

\[
\rho_{GW} = \int P(\omega) d\omega.
\]

(4.36)

Combining Eqs. (4.34), (4.35) and (4.36), one obtains finally for the logarithmic energy spectrum \([26, 27]\):

\[
\Omega_{GW}(\omega, \eta) = \frac{8hG}{3\pi c^3 H^2} \omega^4 |\beta_\omega(\eta)|^2.
\]

(4.37)

The available observational constraints on the energy the spectrum are \([34, 29, 28]\):

- From the CMB radiation: \(h_0^2 \Omega_{GW}(\omega_{\text{hor}}, \eta_0) \lesssim 7 \times 10^{-11}\) for \(h_0 = H_0/(100\ \text{km/s/Mpc})\) and \(\omega_{\text{hor}} = 2 \times 10^{-17} h_0 \text{ rad/s}\);
- from observation of milliseconds pulsar: \(h_0^2 \Omega_{GW}(\omega_{\text{ps}}, \eta_0) < 2 \times 10^{-8}\) for \(\omega_{\text{ps}} = 2.5 \times 10^{-8} h_0 \text{ rad/s}\);
- From the Cassini spacecraft: \(h_0^2 \Omega_{GW}(\omega_{\text{Cas}}, \eta_0) < 0.014\) for \(\omega_{\text{Cas}} = 7.5 \times 10^{-6} h_0 \text{ rad/s}\);
- From the LIGO experiment: \(h_0^2 \Omega_{GW}(\omega, \eta_0) < 3.4 \times 10^{-5}\) for frequencies on the order of a few hundred rad/s;
- From BBN: \(h_0^2 \Omega(\omega, \tau_0) d\omega/\omega < 5.6 \times 10^{-6}\) for \(\omega_\eta \approx 10^{-9} \text{rad/s}\).
4.2 Cosmological Evolution

As stated above, the evolution of the tensor perturbations is described by the pair of first order linear differential equations Eqs. (4.32) and (4.33). These two equations may be merged together to give a single second order differential equation for $X$:

$$X'' + \left(k^2 - \frac{z''}{z}\right) X = 0. \tag{4.38}$$

The fact that Eq. (4.38) depends on the factor $z''/z$, where $z = a\sqrt{f(R)}$, explicitly shows the dependence of the evolution of the cosmological gravitational waves on both (i) the content of the universe throughout time and (ii) the theory that describes the gravitational interaction. In fact the cosmological GW, if detected, represent one of the best tools for probing the early universe and obtaining clues regarding the validation of the inflationary paradigm and its dynamics. In addition, the high curvature regime during the early universe makes the study of cosmological perturbations one of the best candidates to observe the effects of higher order corrections to GR, which could lead to the formulation of a consistent quantum theory of gravitation.

4.2.1 The potential $z''/z$

Equation (4.38) can be interpreted as describing an harmonic oscillator with a source term $z''/z$. In the large wave-number limit ($k^2 \gg z''/z$), the constant term dominates and the solutions are essentially sinusoidal, therefore, in this regime the graviton density is approximately constant. When the source term dominates ($k^2 \ll z''/z$), the amplitude of $X$ grows and so does the density of gravitons. In this way, $z''/z$ acts as a potential for the creation of graviton whose effects come into play when the mode passes under the potential.

Due to the specific form of $z$ one can expand the potential $z''/z$ into its GR form, $a''/a$, plus a correction term, $\Xi$:

$$\frac{z''}{z} = \frac{a''}{a} + 2\frac{a'}{a} \frac{\sqrt{f(R)}}{\sqrt{f(R)}} + \frac{(\sqrt{f(R)})''}{\sqrt{f(R)}}$$

$$= \frac{a''}{a} + \Xi. \tag{4.39}$$

From the Friedmann equations the first term in Eq. (4.39) can be written as:

$$\frac{a''}{a} = \frac{\kappa}{6} a^2 (\rho_T - 3p_T). \tag{4.40}$$

Using the definition of the density energy and pressure during the three different eras (see Eqs. (3.50), (3.51), (3.71), (3.74), (3.66) and (3.67) for the $f(R)$, mGCG and ΛCDM eras, respectively) the potential

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{figure4.png}
\end{center}
\caption{This Fig. shows the evolution of the potential $a''/a$ (blue curve) and the squared comoving wave-number, $k_H^2 \equiv 4\pi^2 a^2 H^2$, (red curve) during the three eras of the universe: (left) the $f(R)$ era; (middle) the mGCG era; (right) the ΛCDM era.}
\end{figure}
Figure 4.2: This Fig. shows (left) the $f(R)$ correction term and (right) the function $Z$ as functions of the reduced scalar curvature, $r$.

$a''/a$ can be written as:

$$a''(a) = \left\{ \begin{array}{l}
\frac{H_{\text{inf}}^2}{2} \left( 2a^2 - a_0^2 \right) \\
\frac{2}{3} \kappa^2 a^2 A \left( A + \frac{B}{\sqrt{1 + \kappa^2 A}} \right) - \frac{1}{\sqrt{1 - r}} \\
\frac{a}{6} a^2 \left( \rho_m (\frac{a_0}{a})^3 + 4 \rho_\Lambda \right).
\end{array} \right.$$  \hfill (4.41)

The second term in Eq. (4.39) represents the correction introduced by the use of the metric $f(R)$ theory instead of GR and its behaviour depends heavily on the choice of the function $f$ considered. Proceeding to develop the $\eta$-derivatives in $\sqrt{f R}$, one obtains:

$$\Xi = \frac{a'}{a} \frac{(f_R')'}{f_R} + \frac{1}{2} \frac{(f_{RR})''}{f_R} - \frac{1}{4} \left( \frac{f_R'}{f_R} \right)^2.$$  \hfill (4.42)

Expressing Eq. (4.42) in terms of $t$- and $R$-derivatives gives:

$$\Xi = \frac{1}{2} \left( a^2 \ddot{R} + 3 a \dot{a} \ddot{a} + \frac{f_{RR}}{f_R} \right) + \frac{1}{2} a^2 \ddot{R}^2 \frac{f_{RRR}}{f_R} - \frac{1}{4} a^2 \ddot{R}^2 \left( \frac{f_{RR}}{f_R} \right)^2.$$  \hfill (4.43)

This is a general result that does not depend on the form of $f$ or any of the dynamical variables of the theory. In this work, the function $f$ is defined during the early universe by Eq. (3.43), and thus, equations (3.44), (3.45) and (3.46) can be used to expand the quotients on the $R$-derivatives of $f$ in Eq. (4.43)

$$\frac{f_{RR}}{f_R} = - H_{\text{inf}}^{-2} \frac{12}{12 - r} \left[ 1 - \frac{r - 3}{r - 6} Z(r) \right],$$  \hfill (4.44)

$$\frac{f_{RRR}}{f_R} = - \frac{H_{\text{inf}}^{-4}}{(12 - r)^2} \left[ \frac{3}{2} - \frac{(r - 3)(2r - 15)}{(r - 6)^2} Z(r) \right],$$  \hfill (4.45)

where $Z(r)$ is defined as:

$$Z(r) \equiv \frac{1}{\sqrt{\frac{3}{2} \frac{\sqrt{12 - r}}{r} \cot \left( \frac{\sqrt{3}}{2} (\pi - \theta_1(r)) - \theta_2(r) \right) + 1}}.$$  \hfill (4.46)

Once the mGCG phase begins a GR description is adopted and the correction term is set to zero.

Inverting the expression for the curvature scalar, Eq. (3.3), one can express the variables $a$, $\dot{a}$, $\ddot{R}$ and $\dddot{R}$ in Eq. (4.43) as functions of $r$:

$$a(r) = a_b \sqrt{\frac{6}{12 - r}},$$  \hfill (4.47)

$$\dot{a}(r) = \pm a_b H_{\text{inf}} \sqrt{\frac{r}{12 - r}},$$  \hfill (4.48)
In Eqs. (4.48) and (4.49), the plus sign relates to values after the bounce and the minus sign to values prior to the bounce. This expressions can now be inserted in Eqs. (4.44), (4.45) and (4.46) to write \( \Xi \) as a function of the reduced curvature scalar \( r \):

\[
\Xi(r) = a_b^2 H_{\text{inf}}^2 \left( -2 \frac{2r - 9}{12 - r} + 6 \frac{r - 3}{12 - r} Z(r) - \frac{(r - 3)^2}{(12 - r)(r - 6)} Z^2(r) \right),
\]

(4.51)

the scale factor \( a \):

\[
\Xi(a) = a_b^2 H_{\text{inf}}^2 \left[ 4 - 5 \frac{a^2}{a_b^2} - \left( 6 - 9 \frac{a^2}{a_b^2} \right) Z(r(a)) - \frac{(2a_b^2 - 3a^2)^2}{4a_b^2 (a^2 - a_b^2)} Z^2(r(a)) \right],
\]

(4.52)

or the cosmological time \( t \):

\[
\Xi(t) = a_b^2 H_{\text{inf}}^2 \left\{ 4 - 5 \cosh^2 (H_{\text{inf}} t) - \left[ 6 - 9 \cosh^2 (H_{\text{inf}} t) \right] Z(r(t)) - \frac{\left[ 2 - 3 \cosh^2 (H_{\text{inf}} t) \right]^2}{4 \sinh^2 (H_{\text{inf}} t)} Z^2(r(t)) \right\}.
\]

(4.53)

Both functions \( Z \) and \( \Xi \) are depicted in Fig. 4.2. The function \( Z \) is a monotonic increasing function of \( r \) that grows from 0 at the bounce \( (r = 6) \), to \( 2/3 \) well inside the inflationary phase \( (r = 12) \). Similarly, the correction term \( \Xi \) is also a monotonic increasing function of \( r \) with:

\[
\Xi(r = 6) = - \left[ 1 + \frac{3}{4} \cot \left( \frac{\sqrt{3}}{2} \pi \right) \right] \approx -4.74278a_b^2 H_{\text{inf}}^2,
\]

(4.54)

\[
\Xi(r = 12) = \frac{1}{3} a_b^2 H_{\text{inf}}^2.
\]

(4.55)

The condition \( \Xi = 0 \) is met for \( r \approx 7.87665 \) which corresponds to \( t \approx 0.63176 H_{\text{inf}}^{-1} \). Adding the expressions for the GR potential (see Eq. (4.41)) and for the \( f(R) \) correction term (see Eq. (4.53)) gives the potential \( z''/z \) during the early universe:

\[
\frac{z''}{z}(t) = a_b^2 H_{\text{inf}}^2 \left\{ 3 - 3 \cosh^2 (H_{\text{inf}} t) - \left[ 6 - 9 \cosh^2 (H_{\text{inf}} t) \right] Z(r(t)) - \frac{\left[ 2 - 3 \cosh^2 (H_{\text{inf}} t) \right]^2}{4 \sinh^2 (H_{\text{inf}} t)} Z^2(r(t)) \right\}.
\]

(4.56)

One can see in Fig. 4.2 that during the de Sitter-like contraction and inflation, the correction term is very close to the limiting value \( \Xi(r = 12) = a_b^2 H_{\text{inf}}^2/3 \), therefore one can define the approximate potential during the de Sitter-like phases, \( z''/z(dS) \), as:

\[
\frac{z''}{z}(t \gg H_{\text{inf}}^{-1}) \approx \frac{z''(dS)}{z} \equiv a_b^2 H_{\text{inf}}^2 \left[ 2 \cosh^2 (H_{\text{inf}} t) - \frac{2}{3} \right].
\]

(4.57)

The potential \( z''/z \), given in Eq. (4.56), as well as both the GR term \( a''/a \), defined in Eq. (4.41), and the de Sitter approximation, shown in Eq. (4.57), are depicted in Fig. 4.3. One can see that \( z''/z \) is negative for \( r \lesssim 6.28663 \), which is equivalent to \( |t| \lesssim 0.222153 H_{\text{inf}}^{-1} \) and has an absolute minimum of \(-3.74278a_b^2 H_{\text{inf}}^2 \) at the bounce. Notice that, due to the definition (4.41), in GR one can only obtain a negative potential for matter that verifies \( w > 1/3 \) (e.g. stiff matter).

### 4.2.2 Solutions during the \( f(R) \) era

The complex expression obtained for \( z''/z \) gives little hope of finding an analytical solution for the differential equation Eq. (4.38). However when one considers only the simpler potentials \( a''/a \) and \( z''/z(dS) \), analytical solutions for Eq. (4.38) are indeed found. Further ahead they will prove invaluable to save
One can easily write Eqs. (4.64) and (4.64) in the polar notation as:

\[ \frac{\gamma}{a} \text{ and } \frac{\gamma'}{a} \]

Due to the fact that the factors inside the square brackets are of the form \( \frac{c}{\bar{c}} \), where the bar indicates the complex conjugate, one can easily write Eqs. (4.64) and (4.64) in the polar notation as:

\[ X_j^0(t) = \sqrt{\sinh^2(H_{\text{inf}}t) + \frac{q^2}{a^2}} \exp \left[ i \frac{(\gamma - 1)q^2}{2} \Gamma \left( \frac{1}{3} \frac{q^2}{\gamma^2} \phi_1(t) + \phi_2(t) \right) \right]. \]
where \( j = 1, 2 \) and the phases \( \phi_1 \) and \( \phi_2 \) defined by:

\[
\phi_1(t) = \pi \left[1 + \text{sign}(t)\right] - \text{sign}(t) \arccos \left[1 - 2 \text{sech}^2(H_{int})\right], \quad (4.65)
\]

and

\[
\phi_2(t) = -\pi \left[1 + \text{sign}(t)\right] + \text{sign}(t) \arccos \left[1 - 2 \frac{\gamma + q^2}{\sinh^2(H_{int}) + \gamma + q^2}\right]. \quad (4.66)
\]

From Eq. (4.64) one can immediately deduce that \( X_1^\gamma \) and \( X_2^\gamma \) are complex conjugates, and so, a
simple linear transformation can be used to obtain two independent real valued solutions of Eq. (4.66):

\[
X_3^\gamma(t) = \frac{X_1^\gamma + X_2^\gamma}{2} = \sqrt{\sinh^2(H_{int}) + \gamma + q^2} \cos \left[\text{sign}(\gamma - 1 + q^2) \frac{\sqrt{\gamma + q^2} \phi_1(t) + \phi_2(t)}{2}\right], \quad (4.67)
\]

and

\[
X_4^\gamma(t) = \frac{X_1^\gamma - X_2^\gamma}{2i} = \sqrt{\sinh^2(H_{int}) + \gamma + q^2} \sin \left[\text{sign}(\gamma - 1 + q^2) \frac{\sqrt{\gamma + q^2} \phi_1(t) + \phi_2(t)}{2}\right]. \quad (4.68)
\]

The corresponding \( Y \) real valued functions can be obtained by first differentiating Eqs. (4.67) and (4.68) and then multiplying the results by \( ia(t)/k \):

\[
iY_3^\gamma(t) = \frac{i}{q} \left[\frac{\sinh(H_{int}) \cosh^2(H_{int})}{\sinh^2(H_{int}) + \gamma + q^2} X_3^\gamma(t) - \frac{\gamma - 1 + q^2 \sqrt{\gamma + q^2}}{\sinh^2(H_{int}) + \gamma + q^2} X_4^\gamma(t)\right], \quad (4.69)
\]

and

\[
iY_4^\gamma(t) = \frac{i}{q} \left[\frac{\gamma - 1 + q^2 \sqrt{\gamma + q^2}}{\sinh^2(H_{int}) + \gamma + q^2} X_3^\gamma(t) + \frac{\sinh(H_{int}) \cosh^2(H_{int})}{\sinh^2(H_{int}) + \gamma + q^2} X_4^\gamma(t)\right]. \quad (4.70)
\]

A simple relation between the functions \( X_3^\gamma \), \( X_4^\gamma \), \( Y_3^\gamma \) and \( Y_4^\gamma \) can be obtain from Eqs. (4.67), (4.68), (4.69) and (4.70) multiplying \( X_3^\gamma \) by \( Y_4^\gamma \) and \( X_4^\gamma \) by \( Y_3^\gamma \) and then subtracting the two gives a constant which depends only on \( \gamma \) and the reduced wave-number \( q \):

\[
X_3^\gamma Y_4^\gamma - X_4^\gamma Y_3^\gamma = \frac{\gamma - 1 + q^2 \sqrt{\gamma + q^2}}{q}. \quad (4.71)
\]

**Figure 4.4:** This Fig. shows the evolution of \( X_3^\gamma \) (left) and \( X_4^\gamma \) (right) for different values of the
reduced wave-number \( q \). Solutions for \( \gamma = 2/3(\gamma = 1) \) are plotted in a solid (dashed) line.

The evolution of the functions \( X_3^\gamma \), \( X_4^\gamma \), \( Y_3^\gamma \) and \( Y_4^\gamma \) during the bounce is plotted in Figs. 4.4 and 4.5 for both potentials and for different values of the reduced wave-number \( q \). Notice that if \( q \) is high enough, all functions present an oscillatory behaviour near the bounce. This corresponds to the regime \( g(\gamma) \ll k^2 \), in which Eq. (4.56) resembles the equation of motion of an harmonic oscillator. Naturally the region of oscillatory behaviour increases with \( q \). Outside of this region the solutions grow exponentially to infinity or decay to zero, depending on the limit of the total phase:

\[
\phi_T \equiv \frac{1}{2} \text{sign}(\gamma - 1 + q^2) \left[\sqrt{\gamma + q^2} \phi_1(t) + \phi_2(t)\right], \quad (4.72)
\]

29
when $t$ goes to infinity.

For negative times, $\phi_T(t \to -\infty) = 0$ for every $q$ and so $X_3$ and $Y_3$ grow exponentially with $-t$, while $X_4$ and $Y_4$ decay rapidly to zero. This can be understood by analysing the behaviour of Eqs. (4.67), (4.68), (4.69) and (4.70) when $\phi_T$ is fixed at zero:

$$X_4^\gamma(t \ll 0) \approx \sqrt{\sinh^2(H_{\text{inf}}^t) + \gamma + q^2}, \quad X_4^\gamma(t \ll 0) \approx 0,$$

and

$$Y_4^\gamma(t \ll 0) \approx \frac{\sinh(H_{\text{inf}}^t) \cosh^2(H_{\text{inf}}^t)}{q \sqrt{\sinh^2(H_{\text{inf}}^t) + \gamma + q^2}}, \quad Y_4^\gamma(t \ll 0) \approx \frac{|\gamma - 1 + q^2|\sqrt{\gamma + q^2}}{q \sqrt{\sinh^2(H_{\text{inf}}^t) + \gamma + q^2}}.$$

Notice that $Y_4^\gamma$ decays exponentially with $t$.

Having obtained the real valued solutions of Eqs. (4.67), (4.68), (4.69) and (4.70) for the system of differential equations of Eqs. (4.32) and (4.33) one writes the desired $X$ and $Y$ functions as:

$$X = D_3^\gamma X_3^\gamma + D_4^\gamma Y_3^\gamma, \quad Y = D_3^\gamma iY_3^\gamma + D_4^\gamma iY_4^\gamma,$$

where the linear coefficients $D_3^\gamma$ and $D_4^\gamma$ can be determined by the initial conditions inherited from the Bogoliubov coefficients $|X(t_i)| = |Y(t_i)| = 1$ [25] and $X(t_i) = Y(t_i)$.

From the condition $X(t_i) = Y(t_i)$ one obtains:

$$D_4^\gamma = -\frac{X_3^\gamma(t_i) - iY_3^\gamma(t_i)}{X_4^\gamma(t_i) - iY_4^\gamma(t_i)} D_3^\gamma,$$

or, equivalently:

$$D_4^\gamma = -\left(\frac{X_3^\gamma(t_i)X_3^\gamma(t_i) + Y_3^\gamma(t_i)Y_3^\gamma(t_i)}{\sqrt{[X_3^\gamma(t_i)]^2 + [Y_3^\gamma(t_i)]^2}} + i\frac{|\gamma - 1 + q^2|\sqrt{\gamma + q^2}}{q \sqrt{[X_3^\gamma(t_i)]^2 + [Y_3^\gamma(t_i)]^2}}\right) D_3^\gamma.$$

In addition, the condition $|X(t_i)| = 1$ translates into:

$$|\gamma - 1 + q^2|^2 + \gamma + q^2 \frac{|D_3^\gamma|^2}{\sqrt{[X_3^\gamma(t_i)]^2 + [Y_3^\gamma(t_i)]^2}} = 1,$$

which gives for $|D_3^\gamma|$:

$$|D_3^\gamma| = \frac{q \sqrt{[X_3^\gamma(t_i)]^2 + [Y_3^\gamma(t_i)]^2}}{|\gamma - 1 + q^2|\sqrt{\gamma + q^2}}.$$

Therefore we can write the coefficients $D_3^\gamma$ and $D_4^\gamma$ as functions of the phase of $D_3^\gamma$, $\phi_3$:

$$D_3^\gamma = \frac{q \sqrt{[X_3^\gamma(t_i)]^2 + [Y_3^\gamma(t_i)]^2}}{|\gamma - 1 + q^2|\sqrt{\gamma + q^2}} e^{i\phi_3}.$$

Figure 4.5: This Fig. shows the evolution of $Y_3^\gamma$ (left) and $Y_4^\gamma$ (right) for different values of the reduced wave-number $q$. Solutions for $\gamma = 2/3 (\gamma = 1)$ are plotted in a solid (dashed) line.
Finally, for large wave-numbers ($q$ reduces to:

one obtains the following relation between the phase of the de Sitter solution, $\phi$

Making use of the following results, valid in the large $a$ in order to analyse the phases of the sinusoidal functions some additional calculations are required.

Expressing Eq. (4.82) in the polar notation gives for $X^{ds}$:

Furthermore, if one writes $X = D_3^d X_3^d + D_4^\gamma X_4^\gamma$ as a function of the scale factor, one obtains during the contraction phase:

While one can immediately recognize that the factors inside the square root in Eqs. (4.82) and (4.84) have the same behaviour for large $a$ and $q^2$:

in order to analyse the phases of the sinusoidal functions some additional calculations are required. Making use of the following results, valid in the large $a$ limit:

one obtains the following relation between the phase of the de Sitter solution, $\phi^{ds}$, and $\phi_T^\gamma$:

Finally, for large wave-numbers ($q^2 \gg \gamma$) one has $\sqrt{1 + 1 + \frac{\gamma - 1}{q^2}} \approx 1$ and the previous equation reduces to:

To determine the phase of $D_3^d$ one compares the solutions obtained with the de Sitter solution that corresponds to the Bunch-Davies vacuum with negative Hubble parameter $H = -H_{\text{inf}}$.

$$X^{ds} = \left(1 - i \frac{aH_{\text{inf}}}{k}\right) e^{-i \frac{2\pi}{T}}.$$  (4.82)

$$X = D_3^d \sqrt{\frac{a^2}{a_b^2} + \gamma - 1 + q^2} \times \cos \left\{ \frac{\text{sign} (\gamma - 1 + q^2)}{2} \sqrt{\gamma + q^2 \arccos \left(1 - 2 \frac{a_b^2}{a^2}\right) - \arccos \left(1 - 2 \gamma + q^2 - \frac{q^2}{a^2/a_b^2 + \gamma - 1 + q^2}\right)} \right\}.$$  (4.84)

$$\sqrt{a^2 + \frac{k^2}{H_{\text{inf}}^2}} \approx a_b \sqrt{\frac{a^2}{a_b^2} + \gamma - 1 + q^2},$$  (4.85)

$$\arctan \frac{aH_{\text{inf}}}{k} \approx \frac{\pi}{2} - \frac{k}{aH_{\text{inf}}} + \frac{k^3}{3 H_{\text{inf}}^3 a^3} + O \left(\frac{1}{a^4}\right),$$  (4.86)

$$\arccos \left(1 - 2 \frac{a_b^2}{a^2}\right) \approx 2 q \frac{a_b}{a} + \frac{q^3}{3 a^2 + a_b^3} + O \left(\frac{1}{a^4}\right),$$  (4.87)

$$\arccos \left(1 - 2 \frac{\gamma + q^2}{a^2/a_b^2 + \gamma - 1 + q^2}\right) \approx 2 \sqrt{\gamma + q^2} \frac{a_b}{a} + \frac{\gamma + q^2}{3 \gamma^2} \frac{a_b^3}{a^3} + O \left(\frac{1}{a^4}\right),$$  (4.88)

$$\phi^{ds} \approx \frac{\pi}{2} + \frac{\gamma - 1}{q^2} \sqrt{1 + \frac{\gamma}{q^2} \phi_T^\gamma}.$$  (4.89)

$$\phi^{ds} \approx \frac{\pi}{2} + \phi_T^\gamma.$$  (4.90)
Making use of the relations in Eqs. (4.85) and (4.90) one finds that, in order for $X = D_3^\gamma X_3^\gamma + D_4^\gamma X_4^\gamma$ to converge to a Bunch-Davies like solution in the limit of large $a$ and $q$, one must have:

$$e^{i\phi_3}(q^2 \gg \gamma, a \gg a_b) = -i, \quad \text{and} \quad e^{i\phi_4}(q^2 \gg \gamma, a \gg a_b) = -1.$$  \hspace{1cm} (4.91)

Combining these results with Eqs. (4.80) and (4.81) results in:

$$D_3^\gamma = -i q \sqrt{[X_3^\gamma(t_i)]^2 + [Y_3^\gamma(t_i)]^2} / (\gamma - 1 + q^2) \sqrt{\gamma + q^2},$$  \hspace{1cm} (4.92)

and

$$D_4^\gamma = \frac{1}{\sqrt{[X_4^\gamma(t_i)]^2 + [Y_4^\gamma(t_i)]^2}} \left( -1 + i q \left[ X_3^\gamma(t_i)X_4^\gamma(t_i) + Y_3^\gamma(t_i)Y_4^\gamma(t_i) \right] / (\gamma - 1 + q^2) \sqrt{\gamma + q^2} \right),$$  \hspace{1cm} (4.93)

where it can be demonstrated that in the limit of large wave-numbers the imaginary term in Eq. (4.93) disappears, and so $e^{i\phi_4}(q^2 \gg \gamma, a \gg a_b) \approx -1$. 


Chapter 5

Results

In order to obtain the energy spectrum of the GW, the set of differential Eqs. \(1.32\) and Eq. \(1.33\) needs to be solved from an initial time before the bounce, throughout the different eras of the universe considered and until the present time. The integration of these equations requires a numerical approach as for the most part, the complicated expression for the potential \(z''/z\) prevents an analytical solution to be found. In this chapter one outlines the parameters of the model and the methods used for the numerical integrations necessary. The energy spectrum of the GW computed for different values of the parameters and using two methods for treating the cosmological perturbations. The results are presented in the last section.

5.1 Parameters of the model

Due to the fact that three different models for the evolution of the universe are used in this work, one for each of the three eras, one is left with a great number of parameters:

- From the model for the bounce during the \(f(R)\) era:
  - the initial time of the \(f(R)\) era - \(t_{\text{ini}}\);
  - the value of the scale factor at the bounce - \(a_b\);
  - the energy scale during inflation - \(E_{\text{inf}}\);

- From de mGCG model:
  - the positive constant related to the inflationary phase - \(A\);
  - the positive constant related to the radiation dominated phase - \(B\);
  - the exponent parameter \(\alpha\);

- From the \(\Lambda\)CDM model with a radiation phase:
  - the relative energy density of radiation - \(\Omega_r\);
  - the relative energy density of baryonic matter - \(\Omega_m\);
  - the relative energy density of the cosmological constant - \(\Omega_{\Lambda}\);
  - the value of the Hubble parameter at present time - \(H_0\);
  - the value of the scale factor at present time - \(a_0\).

However, astronomical observations and the requirement of a smooth connection between the various eras, greatly reduces the number of independent parameters available.

As was seen before, the values of the positive constants \(A\) and \(B\) of the mGCG model are fixed by other parameters:

\[
A = E_{\text{inf}}^{4(1+\alpha)}, \quad \text{and} \quad B = \left(\frac{3}{\kappa^2 \Omega_r H_0^2 a_0^2}\right)^{1+\alpha}.
\]  

(5.1)
In addition, from the seven-year WMAP release \([\Pi]\), one obtains:

\[
\Omega_m = 0.266, \quad \Omega_\Lambda = 0.734, \quad \text{and} \quad H_0 = 70.4 \text{km/s/Mpc}.
\] (5.2)

Finally, following previous works, see Refs. \([28]\), one defines the residual energy density of radiation and the value of the scale factor at present time as:

\[
\Omega_r = 8 \times 10^{-5}, \quad \text{and} \quad a_0 = 10^{58},
\] (5.3)

leaving only four independent parameters in the model: \(t_{\text{ini}}, a_0, E_{\inf} \) and \(\alpha\). The role of each of these parameters is discussed further ahead in this section.

Besides these parameters, it is important to define range of wave-numbers scanned when calculating the energy spectrum of GW. The minimum wave-number corresponds to the mode that is leaving the horizon at present time, i.e.:

\[
k_{\text{min}} = k_{\text{hor}} \equiv a_0 H_0,
\] (5.4)

which corresponds to an angular frequency \(\omega_{\text{hor}} \approx 1.43 \times 10^{-17} \text{rad/s}\). The maximum wave-number scanned is defined by the maximum of the potential \(a'/'a\) during the mGCG era, which occurs at some \(a = a_{\text{max}}\) defined by:

\[
\frac{\partial}{\partial a} a^2 \left( A + \frac{B}{a^{4(1+\alpha)}} \right) \bigg|_{a_{\text{max}}} = 0.
\] (5.5)

Solving for \(a_{\text{max}}\) gives:

\[
a_{\text{max}} = \left[ - (1 + 2\alpha) \frac{B}{A} \right]^{-\frac{1}{4(1+\alpha)}}.
\] (5.6)

If the expression found for \(a_{\text{max}}\) is now inserted in the definition of the potential \(a''/a\) during the mGCG era, Eq. (4.41), one obtains:

\[
k_{\text{max}} = 2 \frac{\kappa^2}{3} \left[ A B (1 + 2\alpha) \left( \frac{1 + 2\alpha}{2\alpha} \right)^2 \right]^{-\frac{1}{4(1+\alpha)}},
\] (5.7)

or, substituting \(A\) and \(B\) by eq. (5.1):

\[
k_{\text{max}} = 2 \sqrt{\frac{\kappa^2}{3} \Omega_r a_0^2 E_{\inf}^2 \left( 1 + 2\alpha \left( \frac{1 + 2\alpha}{2\alpha} \right)^2 \right) ^{-\frac{1}{4(1+\alpha)}}},
\] (5.8)

5.1.1 Initial time

The behaviour of the \(X\) and \(Y\) solutions for large negative times, Eq. (4.73), indicates that an eternal de Sitter like state cannot happen before the bounce, as no set of linear coefficients \(D_3^{(\gamma)}\) and \(D_4^{(\gamma)}\), defined in Eqs. (4.92) and (4.93), fixes \(|X(t_{\text{ini}})| = |Y(t_{\text{ini}})| = 1\) when \(t_{\text{ini}} \to -\infty\). Thus in order to use the method of Bogoliubov coefficients, as presented in Ref. [33], a vacuum state with zero density of gravitons must be set at a finite time before the bounce. This effectively defines the amount of the de Sitter-like contraction that the universe goes through before the bounce and suggests the existence of a different state for the universe before this contraction.

At the beginning of the integration in the \(f(R)\) era, one can define a range of wave-numbers that verify the condition \(k^2 < z''/z\), i.e. the corresponding modes are in the regime of graviton creation. Hence, if one defines the wave-number \(k_I\):

\[
k_I \equiv \sqrt{\frac{z''(t_{\text{ini}})}{z}} \approx a_b H_{\inf} \sqrt{2 \cosh^2(H_{\inf} t_{\text{ini}}) - \gamma}
\] (5.9)

and if \(k_I > k_{\text{hor}}\), one expects to see an increase on the energy spectrum of the GW for wave-numbers \(k < k_I\). Since the potential \(z''/z\) grows exponential with \(-t\), one expects that the increase on the spectrum is more intense and affects more wave-numbers with increasing \(t_{\text{ini}}\).
Figure 5.1: Regions creation of gravitons before the bounce for (left) the GR potential $a''/a$ and (right) the $f(R)$ potential $z''/z$. The approximated potential $z''/z^{dS}$ is plotted in a dashed blue curve in the graphic on the right. The range of frequencies affected by a rise in the graviton density before the bounce, and the duration of the interval of time during which this rise takes place, increase with higher $|t_{ini}|$. The value of $k_{\text{hor}}^2$ is plotted (dashed grey curve) for $a_b = 2000$ and $E_{\text{inf}} = 1.5 \times 10^{16}$GeV.

5.1.2 Scale factor at the bounce and the Energy density during inflation

Either on the expressions of the potentials $a''/a$ and $z''/z$, Eqs. (4.41) and (4.56), respectively, or in the definition of the reduced wave-number, $q$, the parameter $a_b$ never appears isolated; instead it is always coupled to the variable $H_{\text{inf}}$, which is related to the energy scale during inflation:

$$H_{\text{inf}}^2 = \frac{k^2}{3} E_{\text{inf}}^4. \tag{5.10}$$

This suggests that one should analyse the effects of changing the product $a_b H_{\text{inf}}$, instead of just $a_b$. In particular, the dependence of the solutions $X$ and $Y$ on $q$ suggests that a change in $a_b H_{\text{inf}}$, which induces a rescaling of the correspondence $q(k)$, should result in a horizontal shift of the imprints on the energy spectrum of the GW.

In Fig. 5.2 one shows the evolution of the potentials $a''/a$ and $z''/z$ near the bounce for different values of $a_b H_{\text{inf}}$. For comparison, the value of $k_{\text{hor}}^2$ is also plotted for $a_b = 2000$ and $E_{\text{inf}} = 1.5 \times 10^{16}$GeV. As the value of $a_b H_{\text{inf}}$ increases, the potentials rise in comparison with $k_{\text{hor}}^2$, and so more modes should be significantly affected by the bounce in the early universe. Notice also that $k_1 \propto a_b H_{\text{inf}}$. Nevertheless, if the value $a_b H_{\text{inf}}$ becomes too low, the modes affected may enter the region of frequencies inaccessible today ($k < k_{\text{hor}}$). Thus, in this case, the energy spectrum of gravitational waves may not have any particular imprints of the bounce.

Comparing the expressions in Eqs. (4.67) and (4.68) with $\gamma = 1$ and $\gamma = 2/3$ reveals that:

$$X_j^1(k^2) = X_j^{2/3} \left( k^2 + \frac{a_b^2 H_{\text{inf}}^2}{3} \right) \tag{5.11}$$

where $j = 3, 4$. This relation between the $X$ solutions suggests the relation:

$$|\beta^{(1)}|^2(k) \sim |\beta^{(2/3)}|^2 \left( \sqrt{k^2 + k_{II}^2} \right) \tag{5.12}$$

where $k_{II}$ is defined as:

$$k_{II} = \frac{a_b H_{\text{inf}}}{\sqrt{3}}, \tag{5.13}$$

This suggests that the differences in the spectra should be particularly noticeable for $k \lesssim k_{II}$. For modes with large wave-number, the constant term inside the square root becomes negligible ($k \gg k_{II}$), therefore, the imprints of the bounce on the energy spectra of the GW should be independent of the theory used to describe the cosmological perturbations. It is noteworthy to point out that the correspondence in Eq. (5.11) is not carried on to the $Y$ functions, since by definition $Y = iX/k$. Therefore, the relation (5.12) cannot be considered exact.
The range of frequencies that “see” the bounce rises with \( a_0 H_{\text{inf}} \). The negative values of \( z''/z \) are plotted in dashed curves. The value of \( k_{\text{II}}^2 \) is plotted (dashed grey curve) for \( a_0 = 2000 \) and \( E_{\text{inf}} = 1.5 \times 10^{16} \text{GeV} \). The value of \( a_0 \) varies from the blue curve (top) to the red curve (bottom) as: \( a_0 = 2 \times 10^2; a_0 = 2 \times 10^3; a_0 = 2 \times 10^4; a_0 = 2 \times 10^5; a_0 = 2 \times 10^6 \). The value of \( E_{\text{inf}} \) is fixed at \( 1.5 \times 10^{16} \text{GeV} \).

The relation (5.12) was obtained using the approximate solutions for the potential \( z''/z \). Although these are only valid during the de Sitter-like phases, one can argue that for large \( k \) the modes should be insensible to the bounce. Taking the maximum value of the correction term \( \Xi \) as a gauge for the modes that “see” the true form of the potential \( z''/z \), one defines the wave-number \( k_{\text{III}} \):

\[
k_{\text{III}} \equiv \sqrt{\max |\Xi|} = \sqrt{|\Xi_0|} \approx 1.935 a_0 H_{\text{inf}}.
\]  

This way the approximate solutions are used during the entire \( f(R) \) era if \( k \gg k_{\text{III}} \), and only during the de Sitter-like phases if otherwise. As \( k_{\text{II}} < k_{\text{III}} \), the relation (5.12) is not expected to hold. Nevertheless, for high wave-numbers, the spectra obtained with \( \gamma = 1 \) and \( \gamma = 2/3 \) should be indistinguishable.

As an isolated parameter, the energy scale \( E_{\text{inf}} \) only appears in the definition of the constant \( A \) of the mGCG model, Eqs. (3.52) and (3.75), and only affects the form of the potential \( a''/a \) and the comoving wave number \( k_H = 2\pi a H \) during this era (see Fig. 5.3). From the results obtained in Refs. [28, 30], it is expected that an increase of the energy scale imposes a vertical positive shift of the whole energy spectrum of the GW. Also, the maximum frequency that passes under \( a''/a \) scales quadratically with \( E_{\text{inf}} \), as by Eq. (5.17).

Figure 5.2: The dependence on \( a_0 H_{\text{inf}} \) of (left) the GR potential \( a''/a \) and (right) the \( f(R) \) potential \( z''/z \). The value of \( k_{\text{III}}^2 \) is plotted as a function of \( t(H_{\text{inf}}^{-1}) \). The value of \( k_{\text{III}}^2 \) is plotted (dashed grey curve) for \( a_0 = 2000 \) and \( E_{\text{inf}} = 1.5 \times 10^{16} \text{GeV} \). The value of \( a_0 \) varies from the blue curve (top) to the red curve (bottom) as: \( a_0 = 2 \times 10^2; a_0 = 2 \times 10^3; a_0 = 2 \times 10^4; a_0 = 2 \times 10^5; a_0 = 2 \times 10^6 \). The value of \( E_{\text{inf}} \) is fixed at \( 1.5 \times 10^{16} \text{GeV} \).

Figure 5.3: The dependence on \( E_{\text{inf}} \) of (continuous curve) the potential \( a''/a \) and (dashed curve) the squared comoving wave-number \( k_H^2 \), during the mGCG era. The maximum value of the potential rises with \( E_{\text{inf}} \). The value of \( E_{\text{inf}} \) varies from the blue curve (top) to the red curve (bottom) as: \( E_{\text{inf}} = 1.5 \times 10^{16} \text{GeV}; E_{\text{inf}} = 0.5 \times 10^{16} \text{GeV}; E_{\text{inf}} = 1.5 \times 10^{15} \text{GeV}; E_{\text{inf}} = 0.5 \times 10^{15} \text{GeV}; E_{\text{inf}} = 1.5 \times 10^{14} \text{GeV} \). The value of the parameter \( \alpha \) is fixed at \(-1.04 \), which is obtained in Ref. [28] as the best fit to the WMAP5 results.
5.1.3 The mGCG parameter $\alpha$

The primary effect of the parameter $\alpha$ of the mCGC model is to define how *abrupt* is the transition between the inflation and the radiation dominated phases. As the value of $\alpha$ steps away from the value $-1$, the maximum of the potential becomes higher, see Eq. (5.7), and the decrease of the potential during the radiation phase becomes steeper, thus lowering the value of $a_2$. In Ref. [28], it was shown that, besides an increase of the maximum frequency, the increase of the parameter $\alpha$ results in a softer decay of the energy spectrum in the intermediate frequency range and and abrupt decay in the very high frequencies. No particular imprints on the lower frequency range are expected from this parameter.

![Figure 5.4](image)

**Figure 5.4:** The dependence on the parameter $\alpha$ of (continuous curve) the potential $a''/a$ and (dashed curve) the squared comoving wave-number $k_H^2$. The maximum value of the potential rises and the potential decay more rapidly during the radiation phase with increasing $E_{\text{inf}}$. The value of $\alpha$ varies from the blue curve (top) to the red curve (bottom) as: $\alpha = -1.03; \alpha = -1.04; \alpha = -1.05; \alpha = -1.10; \alpha = -1.50$. The energy scale is fixed at $1.5 \times 10^{16}$GeV.

5.2 Numerical Integrations

The numerical integrations necessary to obtain the energy spectrum were performed using the software Wolfram Mathematica 8. The BDF method of the routine NDSolve was chosen, following the suggestion of Ref. [28]. For each wave-number $k$ the integration was divided in two parts:

(i) a first integration during the $f(R)$-gravity era performed in terms of the cosmic time;  
(ii) a final integration during the mCGC and $\Lambda$CDM eras performed in terms of the scale factor.

Furthermore, two different methods were used to determine the evolution of the cosmological perturbations:

(a) the background is described using an $f(R)$ setup while the perturbations are treated in GR;  
(b) both the background and the perturbations are described in an $f(R)$ setup.

A transition between eras is considered to occur in such a way that the potential $z''/z$ is always continuous, i.e., the moment of transition from the first to the second era, $a_1$, is implicitly defined by:

$$
\frac{z''}{z} (a_1) = \frac{a''_{\text{mCGC}}}{a} (a_1), \tag{5.15}
$$

while the moment of transition from the second to the third era, $a_2$, is defined by:

$$
a''_{\text{mCGC}} (a_2) = \frac{a''_{\Lambda\text{CDM}}}{a} (a_2). \tag{5.16}
$$

Using the definition of the scale factor during around the bounce, see Eq. (3.3), one can also express the transition from the first to the second era in terms of the cosmic time, at $t = t_1$:

$$
t_1 \equiv \arccos \left( \frac{a_1}{a_b} \right). \tag{5.17}
$$
5.2.1 Integration during the $f(R)$ era

During the first era, the integration is done in terms of the cosmic time so as to make sure that the effects of the bounce are taken into account when calculating the energy spectrum of the GW. The integration is started at an initial time $t_{ini}$, fixed well into the contraction phase, and continued until the end of the $f(R)$ era, at $t = t_1$, Eq. (5.17). By guaranteeing that the initial conditions are set sufficiently far away from the bounce, one is able to use the analytical solutions found in Chapter 4 to set the initial values of the variables $X$ and $Y$.

One begins by rewriting the differential system of Eqs. (4.32) and (4.33) as:

$$\dot{X}(t) = -i \frac{k}{a(t)} Y(t), \quad (5.18)$$

$$\dot{Y}(t) = -i \frac{k^2 - z''/z}{ka(t)} X(t), \quad (5.19)$$

where the potential $z''/z$ is given in Eq. (4.56). Separating the real and imaginary parts of Eqs. (5.18) and (5.19), gives two pairs of coupled first order differential equations in terms of the real valued variables $X^{re}$, $X^{im}$, $Y^{re}$ and $Y^{im}$:

$$\dot{X}^{re}(t) = \frac{k}{a(t)} Y^{im}(t), \quad \text{and} \quad \dot{Y}^{im}(t) = \frac{k^2 - z''/z}{ka(t)} X^{re}(t), \quad (5.20)$$

$$\dot{X}^{im}(t) = -\frac{k}{a(t)} Y^{re}(t), \quad \text{and} \quad \dot{Y}^{re}(t) = -\frac{k^2 - z''/z}{ka(t)} X^{im}(t). \quad (5.21)$$

In addition, one can use the solutions in Eqs. (4.67), (4.68), (4.69) and (4.68), with linear coefficients given in Eqs. (4.92) and (4.93), to write $X^{re}$, $X^{im}$, $Y^{re}$ and $Y^{im}$ at the initial time $t_{ini}$:

$$X^{re}(t_{ini}) = Y^{re}(t_{ini}) = \frac{X^1(t_{ini})}{\sqrt{|X^1(t_{ini})|^2 + |Y^1(t_{ini})|^2}}, \quad (5.22)$$

$$X^{im}(t_{ini}) = Y^{im}(t_{ini}) = -\frac{Y^1(t_{ini})}{\sqrt{|X^1(t_{ini})|^2 + |Y^1(t_{ini})|^2}}. \quad (5.23)$$

For $\gamma = 1$, i.e., within a GR treatment of the gravitational perturbations, the Eqs. (4.67), (4.68), (4.69) and (4.68) give exact solutions for the differential equations (5.18) and (5.19), with the potential $a''/a$ instead of $z''/z$. This allows to obtain the values of $X$ and $Y$ at the end of the $f(R)$ era, while skipping the numerical integration altogether during this period. Defining the linear coefficients as in Eqs. (4.92) and (4.93) gives:

$$X^{re}(t) = \frac{X^1(t)}{\sqrt{|X^1(t_{ini})|^2 + |Y^1(t_{ini})|^2}}, \quad (5.24)$$

$$X^{im}(t) = \frac{-1}{q \sqrt{1 + q^2}} \left( \sqrt{|X^1(t_{ini})|^2 + |Y^1(t_{ini})|^2} X^1_1(t) - \frac{X^1_3(t_{ini}) X^1_4(t_{ini}) + Y^1_3(t_{ini}) Y^1_4(t_{ini})}{\sqrt{|X^1_1(t_{ini})|^2 + |Y^1_1(t_{ini})|^2}} X^1_4(t) \right), \quad (5.25)$$

$$Y^{re}(t) = \frac{1}{q \sqrt{1 + q^2}} \left( \sqrt{|X^1(t_{ini})|^2 + |Y^1(t_{ini})|^2} Y^1_3(t) - \frac{X^1_3(t_{ini}) X^1_4(t_{ini}) + Y^1_3(t_{ini}) Y^1_4(t_{ini})}{\sqrt{|X^1_1(t_{ini})|^2 + |Y^1_1(t_{ini})|^2}} Y^1_4(t) \right), \quad (5.26)$$

$$Y^{im}(t) = \frac{Y^1(t)}{\sqrt{|X^1(t_{ini})|^2 + |Y^1(t_{ini})|^2}}. \quad (5.27)$$
However, in the case of $\gamma = 2/3$ the Eqs. 4.67, 4.68, 4.69 and 4.68 represent solutions for the approximated potential $z''/z^{\alpha}$, Eq. 4.57, which is valid only during the de Sitter-like phases. This means, the initial conditions are set using Eqs. (5.22) and (5.23) and a numerical integration is necessary to obtain the values of the variables $X$ and $Y$ at the end of the $f(R)$ era. Nevertheless, for large $k > k_{III}$ it is considered that the modes are insensitive to the exact form of the potential $z''/z$ around the bounce and the following solutions are used during the entire $f(R)$ era:

$$X^{re}(t) = \frac{X^{2/3}(t)}{\sqrt{[X^{2/3}(t_{ini})]^2 + [Y^{2/3}(t_{ini})]^2}}.$$  (5.28)

$$X^{im}(t) = \frac{-q}{|q| - 1/3}\left(\sqrt{[X^{2/3}(t_{ini})]^2 + [Y^{2/3}(t_{ini})]^2}X^{2/3}(t)
- \frac{X^{2/3}(t_{ini})X^{2/3}(t)}{\sqrt{[X^{2/3}(t_{ini})]^2 + [Y^{2/3}(t_{ini})]^2}}\right),$$  (5.29)

$$Y^{re}(t) = \frac{-q}{|q| - 1/3}\left(\sqrt{[X^{2/3}(t_{ini})]^2 + [Y^{2/3}(t_{ini})]^2}Y^{2/3}(t)
- \frac{X^{2/3}(t_{ini})X^{2/3}(t)}{\sqrt{[X^{2/3}(t_{ini})]^2 + [Y^{2/3}(t_{ini})]^2}}\right),$$  (5.30)

$$Y^{im}(t) = \frac{Y^{2/3}(t)}{\sqrt{[X^{2/3}(t_{ini})]^2 + [Y^{2/3}(t_{ini})]^2}}.$$  (5.31)

### 5.2.2 Integration during the mGCG and $\Lambda$CDM eras

At the beginning of the mGCG era, one switches variables and performs the numerical integrations in terms of the scale factor. For that, the set of differential Eqs. 4.32 and Eq. 4.33 are recast as:

$$\frac{\partial X}{\partial a}(a) = -\frac{k}{a}Y(a),$$  (5.32)

$$\frac{\partial Y}{\partial a}(a) = -\frac{k^2 - a''/a}{ka'}X(a),$$  (5.33)

where the potential $a''/a$ is now defined as:

$$\frac{a''}{a}(a) = \begin{cases} \frac{2}{3}ka^2 A \left( A + \frac{B}{a^{2(n+1)}} \right)^{-\frac{n}{n+1}}, & a_1 < a < a_2 \\ \frac{c}{6}a^2 \left( \rho_m \left( \frac{a_0}{a} \right)^3 + 4\rho_\Lambda \right), & a > a_2 \end{cases}$$  (5.34)

and the derivative $a'$ can be determined by:

$$a'(a) = a^2 \sqrt{\frac{k^2}{3} - \rho} = \begin{cases} \sqrt{\frac{k^2}{3}}a^2 \left( A + \frac{B}{a^{2(n+1)}} \right)^{-\frac{1}{n+1}}, & a_1 < a < a_2 \\ \sqrt{\frac{k^2}{3}}a^2 \left( \rho_0 \frac{a_0^3}{a} + \rho_m \frac{a_0}{a} + \rho_\Lambda \right)^{1/2}, & a > a_2 \end{cases}$$  (5.35)

Once more, the two differential equations are decomposed in a pair of coupled real valued differential equations:

$$\frac{\partial X^{re}}{\partial a}(a) = \frac{k}{a}Y^{im}(a), \quad \text{and} \quad \frac{\partial Y^{im}}{\partial a}(a) = \frac{k^2 - a''/a}{ka'}X^{re}(a),$$  (5.36)

$$\frac{\partial X^{im}}{\partial a}(a) = -\frac{k}{a}Y^{re}(a), \quad \text{and} \quad \frac{\partial Y^{re}}{\partial a}(a) = -\frac{k^2 - a''/a}{ka'}X^{im}(a),$$  (5.37)
In principle, the integration of Eq. (5.36) is started at \( a = a_1 \), Eq. (5.15), with the initial values of the variables \( X^{re}, X^{im}, Y^{re} \) and \( Y^{im} \) being defined by the results of the integration during the \( f(R) \) era, and until the present time, at \( a = a_0 \). However, in the regime \( k^2 \gg a''/a \), the \( X \) and \( Y \) functions have a sinusoidal behaviour, hence \( |\beta|^2 \) is approximately constant. Since the integration of these oscillatory regimes is very time consuming and not very rewarding (the changes in \( |\beta|^2 \) are not appreciable), one stops the integration when the modes are well inside the Hubble horizon.

Let \( a_{\text{exit}}^{(k)} \) and \( a_{\text{entry}}^{(k)} \) be the values of the scale factor when, for a given wave-number \( k \), the mode leaves and re-enters the horizon, respectively. One defines \( a_{\text{exit}}^{(k)} \) and \( a_{\text{entry}}^{(k)} \) implicitly as:

\[
 k = a_{\text{exit}}^{(k)} H \left( a_{\text{exit}}^{(k)} \right), \quad \text{with} \quad a_{\text{exit}}^{(k)} < \left( \frac{B}{A} \right)^{\frac{1}{1 + \gamma}}, \tag{5.38}
\]

\[
 k = a_{\text{entry}}^{(k)} H \left( a_{\text{entry}}^{(k)} \right), \quad \text{with} \quad a_{\text{entry}}^{(k)} > \left( \frac{B}{A} \right)^{\frac{1}{1 + \gamma}}, \tag{5.39}
\]

and considers that the mode is well inside the horizon if:

\[
 a < 10^{-2} a_{\text{exit}}^{(k)}, \quad \text{or} \quad a > 10^{2} a_{\text{entry}}^{(k)}, \tag{5.40}
\]

This way, for each wave-number \( k \) the integration in only performed inside the interval \([a_{\text{min}}, a_{\text{max}}]\), where:

\[
 a_{\text{min}}^{(k)} = \max \left\{ a_1; 10^{-2} a_{\text{exit}}^{(k)} \right\}, \tag{5.41}
\]

and

\[
 a_{\text{max}}^{(k)} = \min \left\{ a_0; 10^{2} a_{\text{entry}}^{(k)} \right\}. \tag{5.42}
\]

For the modes which verify \( a_{\text{min}}^{(k)} > a_1 \), the analytical solutions of the first era are prolonged through the mGCG era to obtain the initial values for the integration at \( a_{\text{exit}}^{(k)} \). An exception is made if \( k < k_{\text{lll}} \) and \( \gamma = 2/3 \) as the mode “sees” the true form of the potential \( z''/z \) and the analytical solutions are not valid during and after the bounce. In this case the numerical integration is started at \( a_1 \) even if the condition (5.41) is met.

### 5.3 Numerical Results

In this section one presents the energy spectra of the GW obtained for different values of the parameters \( \{t_{\text{ini}}, a_0, E_{\text{inf}}, \alpha\} \). As expected, the existence of a bounce in the early universe affects the spectrum only in the low frequency range, where a highly oscillatory regime is present in contrast with the smooth plateau in the intermediate frequencies and the rapid decay in the high frequency range. This oscillatory regime appears both in a GR treatment and in an \( f(R) \) treatment of the perturbations, which indicates that this feature of the spectrum is due to the effects of the bounce and not to the theory used to analyse the tensorial perturbations. Similar oscillations have been obtained in works of loop quantum cosmology, as first pointed out by Afonso et al [29, 70].

The features of the spectra obtained in a GR and in an \( f(R) \) treatment are presented in Figs. 5.5. As discussed before, the two spectra overlap for high frequencies \( (k > k_{\text{lll}}) \) and are distinguishable only in the low frequency range. In particular, an extra local minimum occurs for the \( f(R) \) treatment at:

\[
 k = k_{\text{lll}}. \tag{5.43}
\]

In Fig. 5.6 the dependence of the imprints of the bounce can be observed for different values of \( t_{\text{ini}} \), where the peaks on the low frequency range of the spectrum are enhanced. The results corroborate the analysis made in Subsec. 5.1.1, as \( t_{\text{ini}} \) is increased the height of the peaks grows and the range of frequencies affected is wider.

The effects of the constant \( a_0 H_{\text{inf}} \) can be observed in the Figs. 5.7, 5.8 and 5.9. When either parameter \( a_0 \) or \( H_{\text{inf}} \) is changed and the other is kept constant (Figs. 5.7 and 5.8), the imprints of the bounce are shifted horizontally on the spectrum. As the value of \( a_0 H_{\text{inf}} \) increases the peaks are moved to the right on the spectrum (higher frequencies), and when its value decreases the peaks are moved to the left (lower frequencies). However, when the two parameter are changed but their product is kept constant (Fig. 5.9).
the position of the peaks on the spectrum is not altered. Notice that in some of the spectra the peaks do not appear at all. This corresponds to the situation when the value of $a_b H_{inf}$ is low enough to move the peaks to frequencies lower than $\omega_{hor}$.

Finally, Fig. 5.10 shows the effects on the energy spectra of varying the value of the parameter $\alpha$. As referred in the Subsec. 5.1.3, there is no change on the spectrum in the low frequency range. The effects are only noticeable for the intermediate to high frequencies. As the value of $|\alpha|$ increases, the spectrum decays more slowly at the intermediate plateau and drops abruptly as the frequencies approach $\omega_{max}$. The value of $\omega_{max}$ also increases with $|\alpha|$.
Figure 5.5: Comparison of the energy spectrum of the GW obtained using a GR (red curve) and a $f(R)$ (blue curve) treatment for the tensorial perturbations.

<table>
<thead>
<tr>
<th>$t_{ini}(H_{inf}^{-1})$</th>
<th>$a_0$</th>
<th>$E_{inf}$ (GeV)</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\omega_{max}$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^2$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$1.865 \times 10^5$</td>
<td>$3.221 \times 10^5$</td>
<td>$2.242 \times 10^5$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$1$</td>
<td>$2.246 \times 10^5$</td>
<td>$3.221 \times 10^5$</td>
<td>$2.242 \times 10^5$</td>
</tr>
</tbody>
</table>
Figure 5.6: Energy spectra of the GW for different values of $t_{\text{ini}}$. The parameters $a_0$, $E_{\text{inf}}$ and $\alpha$ are fixed.

<table>
<thead>
<tr>
<th>$t_{\text{ini}}/(H_{\text{inf}}^{-1})$</th>
<th>$a_0$</th>
<th>$E_{\text{inf}}$ (GeV)</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\omega_{\text{max}}$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.99322</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.04</td>
<td>2/3</td>
<td>$2.2097 \times 10^4$</td>
<td>$3.22127 \times 10^{51}$</td>
<td>$2.2415 \times 10^5$</td>
</tr>
<tr>
<td>-3.68825</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.04</td>
<td>2/3</td>
<td>$2.2097 \times 10^4$</td>
<td>$3.22127 \times 10^{51}$</td>
<td>$2.2415 \times 10^5$</td>
</tr>
<tr>
<td>-4.60507</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.04</td>
<td>2/3</td>
<td>$2.2097 \times 10^4$</td>
<td>$3.22127 \times 10^{51}$</td>
<td>$2.2415 \times 10^5$</td>
</tr>
<tr>
<td>-5.29829</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.04</td>
<td>2/3</td>
<td>$2.2097 \times 10^4$</td>
<td>$3.22127 \times 10^{51}$</td>
<td>$2.2415 \times 10^5$</td>
</tr>
</tbody>
</table>
Figure 5.7: Energy spectra of the GW for different values of $a_b$. 

<table>
<thead>
<tr>
<th>$t_{\text{inf}}(H_{\text{inf}}^{-1})$</th>
<th>$a_b$</th>
<th>$E_{\text{inf}}$ (GeV)</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\omega_{\text{max}}$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^2$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$2.619 \times 10^3$</td>
<td>$3.221 \times 10^{51}$</td>
<td>$2.242 \times 10^5$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$2.210 \times 10^4$</td>
<td>$3.221 \times 10^{51}$</td>
<td>$2.242 \times 10^5$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^4$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$1.865 \times 10^5$</td>
<td>$3.221 \times 10^{51}$</td>
<td>$2.242 \times 10^5$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^5$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$1.574 \times 10^6$</td>
<td>$3.221 \times 10^{51}$</td>
<td>$2.242 \times 10^5$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^6$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$1.329 \times 10^7$</td>
<td>$3.221 \times 10^{51}$</td>
<td>$2.242 \times 10^5$</td>
</tr>
</tbody>
</table>
Figure 5.8: Gravitational wave spectra for different values of \(a_b\) and \(E_{\text{inf}}\) with \(a_bE_{\text{inf}}\) constant.

<table>
<thead>
<tr>
<th>(t_{\text{ini}}(H_{\text{inf}}^{-1}))</th>
<th>(a_b)</th>
<th>(E_{\text{inf}}) (GeV)</th>
<th>(\alpha)</th>
<th>(\gamma)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(\omega_{\text{max}}) (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2.993)</td>
<td>(2 \times 10^7)</td>
<td>(1.5 \times 10^{14})</td>
<td>-1.04</td>
<td>2/3</td>
<td>(1.574 \times 10^8)</td>
<td>(6.047 \times 10^{51})</td>
<td>(2.242 \times 10^3)</td>
</tr>
<tr>
<td>(-2.993)</td>
<td>(1.8 \times 10^6)</td>
<td>(0.5 \times 10^{15})</td>
<td>-1.04</td>
<td>2/3</td>
<td>(1.548 \times 10^7)</td>
<td>(5.131 \times 10^{51})</td>
<td>(7.472 \times 10^3)</td>
</tr>
<tr>
<td>(-2.993)</td>
<td>(2 \times 10^5)</td>
<td>(1.5 \times 10^{15})</td>
<td>-1.04</td>
<td>2/3</td>
<td>(1.865 \times 10^6)</td>
<td>(4.415 \times 10^{51})</td>
<td>(2.242 \times 10^4)</td>
</tr>
<tr>
<td>(-2.993)</td>
<td>(1.8 \times 10^4)</td>
<td>(0.5 \times 10^{16})</td>
<td>-1.04</td>
<td>2/3</td>
<td>(1.834 \times 10^5)</td>
<td>(3.745 \times 10^{51})</td>
<td>(7.472 \times 10^4)</td>
</tr>
<tr>
<td>(-2.993)</td>
<td>(2 \times 10^3)</td>
<td>(1.5 \times 10^{16})</td>
<td>-1.04</td>
<td>2/3</td>
<td>(2.210 \times 10^4)</td>
<td>(3.221 \times 10^{51})</td>
<td>(2.242 \times 10^5)</td>
</tr>
</tbody>
</table>
Figure 5.9: Energy spectra of the GW for different values of $E_{\text{inf}}$. 

<table>
<thead>
<tr>
<th>$t_{\text{ini}}(H_{\text{inf}}^{-1})$</th>
<th>$a_0$</th>
<th>$E_{\text{inf}}$ (GeV)</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\omega_{\text{max}}$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{14}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$3.105 \times 10^4$</td>
<td>$6.047 \times 10^{31}$</td>
<td>$2.2415 \times 10^3$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$0.5 \times 10^{15}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$2.841 \times 10^4$</td>
<td>$5.131 \times 10^{31}$</td>
<td>$7.4717 \times 10^3$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{15}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$2.619 \times 10^4$</td>
<td>$4.415 \times 10^{31}$</td>
<td>$2.2415 \times 10^4$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$0.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$2.396 \times 10^4$</td>
<td>$3.745 \times 10^{31}$</td>
<td>$7.4717 \times 10^4$</td>
</tr>
<tr>
<td>$-2.993$</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>$-1.04$</td>
<td>$2/3$</td>
<td>$2.210 \times 10^4$</td>
<td>$3.221 \times 10^{31}$</td>
<td>$2.2415 \times 10^5$</td>
</tr>
</tbody>
</table>
**Figure 5.10:** Energy spectra of the GW for different values of the mGCG parameter $\alpha$.

<table>
<thead>
<tr>
<th>$t_{\text{rad}}(H^{-1}_{\text{inf}})$</th>
<th>$a_0$</th>
<th>$E_{\text{inf}}$ (GeV)</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\omega_{\text{max}}$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.993</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.03</td>
<td>2/3</td>
<td>$6.709 \times 10^3$</td>
<td>$1.927 \times 10^{12}$</td>
<td>$1.245 \times 10^4$</td>
</tr>
<tr>
<td>-2.993</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.04</td>
<td>2/3</td>
<td>$2.210 \times 10^4$</td>
<td>$3.221 \times 10^{11}$</td>
<td>$2.242 \times 10^5$</td>
</tr>
<tr>
<td>-2.993</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.05</td>
<td>2/3</td>
<td>$6.882 \times 10^4$</td>
<td>$5.843 \times 10^{10}$</td>
<td>$1.271 \times 10^6$</td>
</tr>
<tr>
<td>-2.993</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.10</td>
<td>2/3</td>
<td>$9.841 \times 10^6$</td>
<td>$4.854 \times 10^{10}$</td>
<td>$4.111 \times 10^7$</td>
</tr>
<tr>
<td>-2.993</td>
<td>$2 \times 10^3$</td>
<td>$1.5 \times 10^{16}$</td>
<td>-1.50</td>
<td>2/3</td>
<td>$2.694 \times 10^7$</td>
<td>$6.716 \times 10^7$</td>
<td>$7.000 \times 10^7$</td>
</tr>
</tbody>
</table>
Chapter 6

Conclusions

In this work it was investigated the imprints left on the energy spectrum of the cosmological gravitational waves, at present time, by the presence of a particular bounce in the early universe. Since the existence of a bounce in a GR setup implies violation of the null energy condition, \[ f(R) \] gravity theory [10, 36, 37] was used to describe the dynamics of the early universe and the “designer” \( f(R) \) methodology was employed to obtain an \( f(R) \) action compatible with the desired behaviour for the scale factor. The late time evolution of the universe was described using the \( \Lambda \)CDM model and a mGCG model. [31], was used to connect the inflationary era, that begins after the bounce, with the radiation epoch. This way, a smooth transition between the two phases (inflation and radiation) can be obtained without affecting the low frequencies of the spectrum, where the main effects of the bounce should appear.

The energy spectrum of the GW was computed using the method of the Bogoliubov coefficients, [38, 23, 24, 25, 26, 27], to determine the evolution of the graviton density. The tensorial perturbations were treated during the early universe both in a GR approach (with an \( f(R) \) background) and in a full \( f(R) \) treatment, [10, 58, 45, 46, 47]. A set of four independent parameters was obtain for the model of universe considered within this work: the value of the scale factor at the time of the bounce, \( a_b \); the initial time \( t_{\text{ini}} \) of the integration that defines the amount of contraction before the bounce; the energy scale during the inflationary era, \( E_{\text{inf}} \), and the mGCG parameter \( \alpha \). The energy spectrum of the gravitational waves was determined for different values of each of this parameters.

After computing the energy spectrum, it was confirmed that the presence of the bounce only affects the low frequency region of the spectrum, while leaving the intermediate and high frequency regions unaltered. In the low frequencies, various peaks appear on the spectrum, whose position and intensity depend mainly on \( a_b H_{\text{inf}} \) and \( t_{\text{ini}} \) where \( H_{\text{inf}} \) is related to the energy density during the early inflation. The horizontal position of the peaks is defined by \( a_b H_{\text{inf}} \) as increasing \( a_b H_{\text{inf}} \) displaces the peaks to higher frequencies. On the other hand, raising \( t_{\text{ini}} \) enhances the peaks of the spectrum and increases the range of modes affected by the bounce. As expected, changing the mGCG parameter \( \alpha \) does not affect the low frequency range of the spectrum.

As the intensity of the peaks rise rapidly with \( t_{\text{ini}} \), the CMB radiation limits on the energy density of the cosmological Gravitational Waves [26, 34] can be used to constraint the amount of contraction allowed before the bounce. The fact that the graviton density constraints the amount of the de Sitter-like contraction and does not allow for an eternal contraction in the past suggests the existence of a different initial state for the universe that precedes the contraction. All the other observational constraints on the energy density of the cosmological gravitational waves are satisfied.

One concludes by noting that the detection of the results obtained in this work are unlikely to be observed in the near future [34]. Nevertheless, the future measurement of the B-mode polarization of the CMB radiation might start to shed some light on the dynamics of the early stages of the universe. Thus, a natural continuation of this work would be the imprints of the bounce on the polarization of the CMB radiation, particularly the B-modes [31, 32, 33]. Additionally, other types of bounces might be considered in order to avoid the problems that arise in the scalar sector.
Appendix A

Appendix

A.1 Derivation of the functions $Y_3^γ$ and $Y_4^γ$

The functions $Y_j$, where $j = 3, 4$, can be obtained from the respective functions $X_j^γ$ by the relation:

$$Y_j^γ = \frac{a(t)}{k}X_j^γ(t). \quad (A.1)$$

One reminds here that the scale factor is defined as:

$$a(t) = a_b \cosh(H_{\text{inf}}t), \quad (A.2)$$

and the functions $X_j^γ$ were derived in Chapter 4 as:

$$X_3^γ(t) = \sqrt{\sinh^2(H_{\text{inf}}t) + \gamma + q^2} \left[ \text{sign}(\gamma - 1 + q^2) \sqrt{\gamma + q^2 \phi_1(t) + \phi_2(t)} \right], \quad (A.3)$$

and

$$X_4^γ(t) = \sqrt{\sinh^2(H_{\text{inf}}t) + \gamma + q^2} \left[ \text{sign}(\gamma - 1 + q^2) \sqrt{\gamma + q^2 \phi_1(t) + \phi_2(t)} \right]. \quad (A.4)$$

The phases $\phi_1$ and $\phi_2$ in Eqs. (A.3) and (A.4) are defined as:

$$\phi_1^γ(t) = \pi [1 + \text{sign}(t)] - \text{sign}(t) \arccos \left[ 1 - 2\text{sech}^2(H_{\text{inf}}t) \right], \quad (A.5)$$

and

$$\phi_2^γ(t) = -\pi [1 + \text{sign}(t)] + \text{sign}(t) \arccos \left[ 1 - 2 \frac{\gamma + q^2}{\sinh^2(H_{\text{inf}}t) + \gamma + q^2} \right]. \quad (A.6)$$

As they will be of use right ahead, one starts by differentiating $\phi_1^γ$ and $\phi_2^γ$:

$$\dot{\phi}_1^γ(t) = \frac{\text{sign}(t)}{\sqrt{1 - [1 - 2\text{sech}^2(H_{\text{inf}}t)]^2}} \frac{\partial}{\partial t} \left[ 1 - 2\text{sech}^2(H_{\text{inf}}t) \right]$$

$$= \frac{H_{\text{inf}} \text{sign}(t)}{\sqrt{4\text{sech}^2(H_{\text{inf}}t) - 4\text{sech}^4(H_{\text{inf}}t)}} \frac{4\text{sech}^2(H_{\text{inf}}t) \tanh(H_{\text{inf}}t)}{4\text{sech}^2(H_{\text{inf}}t) \tanh(H_{\text{inf}}t) - 4\text{sech}^4(H_{\text{inf}}t)}$$

$$= 2H_{\text{inf}} \text{sign}(t) \text{sech}(H_{\text{inf}}t) \tanh(H_{\text{inf}}t)$$

$$= 2H_{\text{inf}} \text{sech}(H_{\text{inf}}t) \tanh(H_{\text{inf}}t)$$

$$= 2H_{\text{inf}} \text{sech}(H_{\text{inf}}t) \tanh(H_{\text{inf}}t)$$

$$= 2H_{\text{inf}} \text{sech}(H_{\text{inf}}t) \tanh(H_{\text{inf}}t)$$

51
\[
\dot{\phi}_3^2(t) = -\frac{\text{sign}(t)}{\sqrt{1 - \left[1 - 2\frac{\gamma + q^2}{\sinh^2(H_{int}t) + \gamma + q^2}\right]^2} \frac{\partial}{\partial t} \left[1 - 2\frac{\gamma + q^2}{\sinh^2(H_{int}t) + \gamma + q^2}\right]}
\]

\[
= -\frac{H_{int}\text{sign}(t)}{\sqrt{4\sinh^2(H_{int}t) + \gamma + q^2} - 4\frac{\gamma + q^2}{\sinh^2(H_{int}t) + \gamma + q^2}} 4\frac{\gamma + q^2}{\sinh^2(H_{int}t) + \gamma + q^2} \left(\cosh(H_{int}t) \sinh(H_{int}t)\right)
\]

\[
= -2H_{int}\frac{\gamma + q^2}{\sinh^2(H_{int}t) + \gamma + q^2} \left(\cosh(H_{int}t) \sinh(H_{int}t)\right)
\]

\[
= -2H_{int}\frac{\gamma + q^2}{\sinh^2(H_{int}t) + \gamma + q^2} \left(\cosh(H_{int}t) \sinh(H_{int}t)\right)
\]

Thus, the derivative of the total phase \(\phi_3^2\) is:

\[
\dot{\phi}_3^2(t) = \text{sign}(\gamma - 1 + q^2) \sqrt{\gamma + q^2} \frac{\dot{\phi}_1^2(t) + \dot{\phi}_2^2(t)}{2}
\]

\[
= H_{int}\text{sign}(\gamma - 1 + q^2) \left[\frac{\text{sech}(H_{int})}{\sinh^2(H_{int}t) + \gamma + q^2} \cosh(H_{int})\right]
\]

\[
= H_{int}|\gamma - 1 + q^2| \frac{\text{sech}(H_{int})}{\sinh^2(H_{int}t) + \gamma + q^2}
\]

One can now differentiate \(X_3^2\) and \(X_4^2\) and obtain:

\[
X_3^2(t) = \frac{\partial}{\partial t} \sqrt{\sinh^2(H_{int}t) + \gamma + q^2} \cos[\phi_3^2(t)] - \sqrt{\sinh^2(H_{int}t) + \gamma + q^2} \phi_3^2(t) \sin[\phi_3^2(t)]
\]

\[
= H_{int} \left[\text{cosh}(H_{int}t) \sinh(H_{int}t) \cos[\phi_3^2(t)] - \frac{|\gamma - 1 + q^2|}{\sqrt{\sinh^2(H_{int}t) + \gamma + q^2}} \text{sech}(H_{int}t) \sin[\phi_3^2(t)]\right]
\]

\[
= H_{int} \left[\frac{\text{cosh}(H_{int}t) \sinh(H_{int}t)}{\sinh^2(H_{int}t) + \gamma + q^2} \phi_3^2(t) - \frac{|\gamma - 1 + q^2|}{\sqrt{\sinh^2(H_{int}t) + \gamma + q^2}} \text{sech}(H_{int}t) \sin[\phi_3^2(t)]\right]
\]

and

\[
X_4^2(t) = \frac{\partial}{\partial t} \sqrt{\sinh^2(H_{int}t) + \gamma + q^2} \sin[\phi_3^2(t)] + \sqrt{\sinh^2(H_{int}t) + \gamma + q^2} \phi_3^2(t) \cos[\phi_3^2(t)]
\]

\[
= H_{int} \left[\text{cosh}(H_{int}t) \sinh(H_{int}t) \sin[\phi_3^2(t)] + \frac{|\gamma - 1 + q^2|}{\sqrt{\sinh^2(H_{int}t) + \gamma + q^2}} \text{sech}(H_{int}t) \cos[\phi_3^2(t)]\right]
\]

\[
= H_{int} \left[\frac{\text{cosh}(H_{int}t) \sinh(H_{int}t)}{\sinh^2(H_{int}t) + \gamma + q^2} \phi_3^2(t) + \frac{|\gamma - 1 + q^2|}{\sqrt{\sinh^2(H_{int}t) + \gamma + q^2}} \text{sech}(H_{int}t) \sin[\phi_3^2(t)]\right]
\]

Finally, inserting these results in Eq. (A.1) reads:

\[
Y_3^2(t) = \frac{1}{q} \left[\frac{\cos^2(H_{int}t) \sinh(H_{int}t)}{\sinh^2(H_{int}t) + \gamma + q^2} X_3^2(t) - \frac{|\gamma - 1 + q^2|}{\sqrt{\sinh^2(H_{int}t) + \gamma + q^2}} X_3^2(t)\right]
\]

and

\[
Y_4^2(t) = \frac{1}{q} \left[\frac{\cos^2(H_{int}t) \sinh(H_{int}t)}{\sinh^2(H_{int}t) + \gamma + q^2} X_4^2(t) + \frac{|\gamma - 1 + q^2|}{\sqrt{\sinh^2(H_{int}t) + \gamma + q^2}} X_4^2(t)\right]
\]
Bibliography


