Flux tubes in $U(1)$ Lattice Gauge Theory

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Resumo

Neste trabalho implementamos um código numérico com vista a estudar vários aspectos de Teoria de Gauge na Rede $U(1)$, principalmente propriedades da fase confinante.

Começamos com uma introdução geral ao tema de Teoria de Gauge na Rede.

Utilizamos correlações de loops de Polyakov para estudar tubos de fluxo em $U(1)$ compacto a $4D$ e o potencial estático electrão positrão em teoria de gauge na rede. Utilizando operadores de campo medimos directamente os campos eléctrico e magnético. Estudamos ainda a evolução do potencial com $\beta$, assim como outras quantidades já bem conhecidas. De modo a melhorar a razão sinal-ruído na fase confinante, aplicamos o algoritmo multinível de Lüscher e Weiss.

Estudamos um intervalo de temperaturas de $0.05 T_c$ a $0.40 T_c$ e os nossos resultados seguem as previsões da Teoria de Cordas Efectiva em todo o intervalo. Conseguimos observar a transição do comportamento logarítmico para o linear.

O nosso código está completamente escrito em CUDA, e corremo-lo em unidades de processamento gráfico de geração FERMI da NVIDIA, de modo a atingir o desempenho necessário aos nossos cálculos.

Palavras-chave: Teoria de Gauge na Rede, $U(1)$, tubos de fluxo, Cromodinâmica Quântica na Rede, potencial estático, Teoria de Cordas Efectiva
Abstract

In this work we implement a numerical code to study several aspects of $U(1)$ Lattice Gauge Theory, mainly properties of confining phase.

We start with a general introduction to the subject of Lattice Gauge Theory.

We utilize Polyakov loop correlations to study 4D compact $U(1)$ flux tubes and the static electron-positron potential in lattice gauge theory. By using field operators we are able to probe directly the electric and magnetic fields. Also we study the evolution of the potential with $\beta$ as well as other already well-known quantities. In order to improve the signal-to-noise ratio in the confinement phase, we apply the Lüscher-Weiss multilevel algorithm.

We study a range of temperatures from 0.05 $T_c$ to 0.40 $T_c$ and our results follow the Effective String Theory prediction for the relation between the the static charge distance and the width of the tube flux in the whole range. We are able to observe the transition from logarithmic to linear behavior.

Our code is completely written in CUDA, and we run it in NVIDIA FERMI generation GPU’s, in order to achieve the necessary performance for our computations.

Keywords: Lattice Gauge Theory, $U(1)$, flux tubes, Lattice QCD, static potential, Effective String Theory
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Chapter 1

Introduction and Motivation

Lattice Quantum Field Theory is an area of study increasingly important in Quantum Field Theory. The recent important advances, both theoretical and of computational capacity, open new horizons of study and allow this theory to reach areas it could not before.

Lattice Quantum Field Theory opens the possibility of a better understanding of confining mechanisms an area of study closed to the usage of perturbative tools.

In the context of this formalism we can study of the details of the interactions in order to compare then with other theoretical models, like Effective String Theory, and get a better understanding of the virtues and limitations of this kind of effective models. It is important to check these predictions in the context of different groups. This result is already well set in 3D for $SU(2)$ gauge group and some other groups (like $\mathbb{Z}_2$) and some studies are being conducted in this moment like with $SU(3)$ gauge group. Some studies have been done in $U(1)$ gauge group although its results are not yet completely well established. We decided to study the structure of flux tubes produced by static charges in $U(1)$ lattice gauge theory in order to try to give a contribution to settle this subject.

Although sometimes the introductory parts are written in a more general frameset, the objective of this dissertation is to study $U(1)$ lattice gauge theory. It assumes knowledge of basic quantum field theory by the reader.

We decided to write our code completely in CUDA because, although we have the drawback of being significantly more difficult to program, we can achieve much better performances due to the possibility of writing a highly parallelized code. CUDA requires knowledge about the architecture of the GPU to be used and careful planning of the memory usage to achieve better results.

We advise the reader non familiarized with the notation to take a glance at appendix A on notation and definitions frequently used throughout this text and maybe use it as reference.

In this chapter we introduce some steps on building a Lattice Quantum Field Theory as well as a small introduction of the relevant results of Effective Field Theory in the context of this work. In the second chapter we introduce the numerical methods used in our calculations and present the results in third chapter. Finally a conclusion where we make a summary of our results and present some paths that, in our opinion, can be worth exploring in the sequence of this work.
1.1 Lattice Quantum Field Theory

Lattice Quantum Field Theory is a discretized Quantum Field Theory that allows to perform non-perturbative numerical Quantum Field Theory computations, as well as some analytical calculations, including perturbative ones.

Lattice Gauge Theory (LGT) was first proposed by Wilson in 1974 [2] as an alternative ultraviolet cutoff method, allowing to perform analytical computations and obtaining continuum limits.

Soon after people started to explore the possibilities that lattice approach opens in direct numerical evaluation of gauge theories. In 1981 Hamber and Parisi publish for the first time a hadronic mass spectrum obtained from lattice QCD in quenched approximation [3], demonstrating in practice the possibility of performing this kind of calculations. With time Lattice Gauge Theory became regarded as the natural way for calculating numerical non-perturbative quantities.

Progressively people obtained a much deeper understanding of systematical errors and the 2000’s, with the development of algorithms and computational power, have seen the appearance of unquenched calculations, with the effects of quark loops being taken into account.

Although in 2001 people feared it would be impossible to reach realistic quark masses, independently of computational power available, due to an enormous rise in the calculation time needed with the algorithms available, the recent development of new algorithms solved this problem and opened the possibility of approaching limits where realist calculations can be performed.

In this moment this calculations are being done and Lattice Quantum Field theory seems to have a promising role in next years in physics.

Lattice QCD seems the best approach to get an understanding of the physics where perturbative tools fail, being an extremely important approach in the context of QCD due to the fact of this theory being non-pertubative in its low energy regime (cf. figure 1.1).

![Figure 1.1: Evolution of $\alpha_s$ with the scale as function of energy scale (figure from [1]).](image-url)
Spacetime Discretization

In Lattice Quantum Field Theory we want to build a theory that can evaluate the path integral directly in a computer. As computers cannot deal with a continuous spacetime the first obvious step is to discretize it. There are several possible approaches to this, being that the most widely used is the discretization of space time in an isotropic hypercubic lattice with a certain lattice spacing between points. This spacing can be determined \textit{a posteriori} by the appropriate matching of lattice quantities with physical measured quantities or can just be taken as a parameter which computed quantities depend on.

This isotropic hypercubic lattice is the simplest choice that can be taken and, although other approaches have been tried sporadically (e.g. \cite{4,5}), it remains the almost exclusively used discretization. This hypercube will have dimensions $N_S \times N_S \times N_S \times N_T$ and periodical boundary conditions in all directions\textsuperscript{1}.

The next step consists in introducing the fields in the lattice in a way that they preserve the proprieties we are interested in. This is done by considering the fermions as Grassman variables laying in the lattice sites and the gauge fields as group elements in the links between lattice sites. So we assign the gauge field in the links the average value of the field in between the two nearest points. We will see that this definition allows to preserve gauge invariance.

We denote the fermions by $\psi$ and the gauge field by $U_\mu$ (cf. figure 1.2).

\begin{equation}
\psi(x) \rightarrow V(x)\psi(x) \quad (1.1)
\end{equation}

\begin{equation}
\bar{\psi}(x) \rightarrow \bar{\psi}(x)V^{-1}(x) \quad (1.2)
\end{equation}

\begin{equation}
U_\mu(x) \rightarrow V(x)U_\mu(x)V^{-1}(x + \hat{\mu}) \quad (1.3)
\end{equation}

\begin{equation}
U^{-1}_\mu(x) \rightarrow (V(x)U_\mu(x)V^{-1}(x + \hat{\mu}))^{-1} = V(x + \hat{\mu})U^{-1}_\mu(x)V^{-1}(x). \quad (1.4)
\end{equation}

We will now check an example of a gauge invariant quantity on the lattice. Let us consider a closed product of links in the lattice, for example the smallest possible one $P_{\mu\nu}$, the plaquette, a product of four

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.2.png}
\caption{Fermions and gauge fields.}
\end{figure}

\textbf{Gauge invariance}

It is important to preserve the gauge symmetry of the original theory. To do so we need to understand how to build gauge invariant quantities in the lattice we just defined. First of all let us transcribe the notion of gauge freedom to the lattice. Gauge freedom is the freedom to perform a local gauge transformation of our fields, which can be expressed as the multiplication of a gauge element locally. In lattice a general gauge transformation can be parametrized as

\begin{equation}
\psi(x) \rightarrow V(x)\psi(x) \quad (1.1)
\end{equation}

\begin{equation}
\bar{\psi}(x) \rightarrow \bar{\psi}(x)V^{-1}(x) \quad (1.2)
\end{equation}

\begin{equation}
U_\mu(x) \rightarrow V(x)U_\mu(x)V^{-1}(x + \hat{\mu}) \quad (1.3)
\end{equation}

\begin{equation}
U^{-1}_\mu(x) \rightarrow (V(x)U_\mu(x)V^{-1}(x + \hat{\mu}))^{-1} = V(x + \hat{\mu})U^{-1}_\mu(x)V^{-1}(x). \quad (1.4)
\end{equation}

\textsuperscript{1}Again this choice is not unique. Recently there have been attempts to work with lattices that are not periodic in time \cite{6}, which allows to solve some topology related technical problems, although it is not yet well explored.
neighbor links (cf. figure 1.3). We have

\[ P_{\mu\nu}(x) \equiv U_\mu(x) \, U_\nu(x + \hat{\mu}) \, U^{-1}_\mu(x + \hat{\nu}) \, U^{-1}_\nu(x). \]  \hspace{1cm} (1.5)

Performing a general gauge transformation we have \( P'_{\mu\nu}(x) \)

\[ P'_{\mu\nu}(x) = V(x) \, U_\mu(x) \, V^{-1}(x + \hat{\mu}) \, V(x + \hat{\mu}) \, U_\nu(x + \hat{\mu}) \, V^{-1}(x + \hat{\mu} + \hat{\nu}) \times \]
\[ \times V(x + \hat{\mu} + \hat{\nu}) \, U^{-1}_\mu(x + \hat{\nu}) \, V(x + \hat{\nu}) \, U^{-1}_\nu(x) \, V^{-1}(x) = \]
\[ = V(x) \, U_\mu(x) \, U_\nu(x + \hat{\mu}) \, U^{-1}_\mu(x + \hat{\nu}) \, U^{-1}_\nu(x) \, V^{-1}(x) \]  \hspace{1cm} (1.6)

We can take the trace of \( P'_{\mu\nu} \) and use the cyclic property of the trace obtaining

\[ \text{Tr} \left[ P'_{\mu\nu}(x) \right] = \text{Tr} \left[ V(x) \, U_\mu(x) \, U_\nu(x + \hat{\mu}) \, U^{-1}_\mu(x + \hat{\nu}) \, U^{-1}_\nu(x) \right] = \]
\[ = \text{Tr} \left[ U_\mu(x) \, U_\nu(x + \hat{\mu}) \, U^{-1}_\mu(x + \hat{\nu}) \, U^{-1}_\nu(x) \right] = \text{Tr} \left[ P_{\mu\nu}(x) \right] \]  \hspace{1cm} (1.7)

so we can conclude that this quantity \( \text{Tr} P_{\mu\nu} \) is gauge invariant.

Now that we have an idea about how to get gauge invariant quantities we can do it in a more general way. It is easy to check that when we have an ordered product of links if we perform a gauge transformation all the transformation matrices vanish except the first and the last ones. If \( U(x, y; C) \) denotes that product for any ordered path \( C \) starting at \( x \) and ending at \( y \) it transforms as

\[ U(x, y; C) \to V(x)U(x, y; C)V^{-1}(y). \]  \hspace{1cm} (1.8)

So, to get a gauge invariant we need to eliminate those transformation matrices in the edges of the path. We have two aways to do so

- taking the beginning and the end to be the same point and take the trace of \( U(x, y; C) \) (a generalization of our plaquette example)

\[ \text{Tr} \left[ U(x, x; C) \right] \to \text{Tr} \left[ U'(x, x; C) \right] = \text{Tr} \left[ V(x)U(x, x; C)V^{-1}(x) \right] = \text{Tr} \left[ U(x, x; C) \right] \]  \hspace{1cm} (1.9)

\[ \begin{array}{c}
\text{Figure 1.3: Plaquette.} \\
\end{array} \]
• taking the beginning and the end of the path to be, respectively, an antifermion and a fermion

\[ \bar{\psi}(x)U(x, y; \mathcal{C})\psi(y) \to \bar{\psi}(x)U^{t}(x, y; \mathcal{C})\psi'(y) = \bar{\psi}(x)V^{-1}(x)V(x, y; \mathcal{C})V^{-1}(y)V(y)\psi(y) \]

\[ = \bar{\psi}(x)U(x, y; \mathcal{C})\psi(y). \quad (1.10) \]

**Action**

We have discretized the spacetime, it is now important to understand how we can build an action in our new theory. This choice is not unique, we can accept several actions since they respect gauge invariance and have the right continuum limit.

The correct limit of the gauge action in continuum should be the well-known Yang-Mills action

\[ S_G = -\frac{1}{4} \int d^4x \, \text{Tr} F_{\mu\nu}F^{\mu\nu}. \quad (1.11) \]

We can start by taking our example of the plaquette operator we defined before. We will do this in the Abelian case, the case of most interest for us in this dissertation. The non-Abelian case is not very different (although the calculations are somewhat lengthier and more complicated). We have

\[ P_{\mu\nu}(x) = U_\mu(x) \, U_\nu(x + \hat{\mu}) \, U_\mu^{-1}(x + \hat{\nu}) \, U_\nu^{-1}(x) \quad (1.12) \]

It makes sense to define

\[ U_\mu(x) \equiv e^{i a g [ A_\mu(x + \hat{x})]} \quad (1.13) \]

once the link values are defined in the middle point of the link. This way in the new variables the plaquette will be

\[ P_{\mu\nu}(x) = e^{i a g [ A_\mu(x + \hat{\mu}) + A_\nu(x + \hat{\nu}) - A_\mu(x + \hat{n} + \hat{x}) - A_\nu(x + \hat{n} + \hat{x})]} = \\
e^{i a g [ A_\mu(r - \frac{\hat{n}}{2}) + A_\nu(r + \hat{n} + \hat{x}) - A_\mu(r + \hat{n} + \hat{x}) - A_\nu(r - \frac{\hat{n}}{2})]} \quad (1.14) \]

where \( r \) is the coordinate of the center of the plaquette. Noting that \( A_\mu(r - \frac{\hat{n}}{2}) = A_\mu(r) + \partial_\nu A_\mu(-\frac{\hat{n}}{2}) + \mathcal{O}(a^2) \)

we have in the exponent

\[ i a g [ A_\mu(r - \frac{\hat{n}}{2}) - A_\mu(r + \frac{\hat{n}}{2}) + A_\nu(r + \frac{\hat{n}}{2}) - A_\nu(r - \frac{\hat{n}}{2})] = \\
i a g [ A_\mu(r) - a \partial_\nu A_\mu - A_\mu(r) - a \partial_\nu A_\mu + A_\nu(r) + a \partial_\nu A_\nu - A_\nu(r) + a \partial_\mu A_\nu + \mathcal{O}(a^2)] = \\
i a g [ \partial_\mu A_\nu - \partial_\nu A_\mu + \mathcal{O}(a^2)] = i a^2 g F_{\mu\nu} + i \mathcal{O}(a^3). \quad (1.15) \]

Expanding now the exponential we can get

\[ P_{\mu\nu}(x) = e^{i a^2 g F_{\mu\nu} + i \mathcal{O}(a^3)} = 1 + i a^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F^2_{\mu\nu} + i \mathcal{O}(a^3) + \mathcal{O}(a^5). \quad (1.16) \]

So the real part of plaquette contains the square of the components of the electromagnetic tensor\(^3\). This way

\(^3\)Although not relevant here it is easy to check that, unlike with non-Abelian theories, in \( U(1) \) it is possible to probe the electric field directly using the imaginary part of the plaquette.
we can take the action to be

\[
S_g = \frac{1}{g^2} \sum_x \sum_{\mu<\nu} \text{Re} [1 - P_{\mu\nu}(x)] = \beta \sum_x \sum_{\mu<\nu} \text{Re} [1 - P_{\mu\nu}(x)]
\] (1.17)

where we adopt the usual convention \( \beta = \frac{1}{g^2} \).

It is easy to check that this action has as continuum limit the usual Yang-Mills action

\[
S_g = \frac{1}{g^2} \sum_x \sum_{\mu<\nu} \text{Re} [1 - P_{\mu\nu}(x)] = \frac{a^4}{2} \sum_x \sum_{\mu<\nu} F_{\mu\nu}^2 =
\]

\[
= \frac{a^4}{4} \sum_{x, \mu, \nu} F_{\mu\nu}^2 \rightarrow \frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu}.
\] (1.18)

The trace is not needed since we are working with \( U(1) \) gauge theory and \( U(1) \) group elements commute. In a more general settlement of non-Abelian theories we would have \( \text{Tr} \, \text{Re} [1 - P_{\mu\nu}(x)] \).

This action is the simplest possible and is known as Wilson action. It is possible to build many other actions taking into account more loop contributions (e.g. [7]). Some of these actions can cancel terms in the expansion up to higher orders, making the action converge faster to the continuum. They have although the disadvantage of bringing more complexity and taking more time to compute.

**Strong coupling expansion**

We can calculate prediction of this theory in the limit of strong coupling \( (\beta \rightarrow 0 \iff g \rightarrow \infty) \).

We want to calculate the average density of plaquette in this limit. This quantity is given by

\[
\langle P_{\mu\nu} \rangle = Z^{-1} \int dU P_{\mu\nu} e^{-S}
\] (1.19)

where \( S \), the action, is the Wilson action we introduced before.

When \( \beta \rightarrow 0 \), the integral

\[
\int dU P_{\mu\nu} \, e^{-S} \rightarrow 0
\] (1.20)

because

\[
\int dU P_{\mu\nu} = 0.
\] (1.21)

Basically this reflects the fact that, in the strong coupling limit, the links values are independent of each other so the average should be 0.

Consequently we will have

\[
\int dU P_{\mu\nu} \, P_{\alpha\gamma}^* = \delta_{\mu\alpha} \delta_{\nu\gamma}
\] (1.22)

because in \( U(1) \) we have \( P_{\mu\nu}^* = P_{\mu\nu}^{-1} \).
The next step is to expand the $e^{-S}$ in a $\beta$ series

$$e^{-S} = \exp\left[-\beta \left(1 - \sum_{\mu<\nu} \text{Re} P_{\mu\nu}\right)\right] = e^{-\beta} \exp\left[\beta \sum_{\mu<\nu} \text{Re} P_{\mu\nu}\right] = e^{-\beta} \exp\left[\frac{\beta}{2} \sum_{\mu<\nu} (P_{\mu\nu} + P_{\mu\nu}^*)\right].$$

(1.23)

So, if we replace the action expansion in average plaquette integral, we get

$$\langle P_{\alpha\gamma} \rangle = Z^{-1} \int dU P_{\alpha\gamma} e^{-S} = Z^{-1} \int dU P_{\alpha\gamma} e^{-\beta} \left[1 + \frac{\beta}{2} \sum_{\mu<\nu} (P_{\mu\nu} + P_{\mu\nu}^*) + \mathcal{O}(\beta^2)\right] =$$

$$= Z^{-1} e^{-\beta} \left[\int dU P_{\alpha\gamma} + \frac{\beta}{2} \int dU P_{\alpha\gamma} \sum_{\mu<\nu} (P_{\mu\nu} + P_{\mu\nu}^*) + \mathcal{O}(\beta^2)\right].$$

(1.24)

For the reasons stated before the first integral evaluates to 0 and the second one to 1 so the result is

$$\langle P_{\alpha\gamma} \rangle = Z^{-1} e^{-\beta} \left[\frac{\beta}{2} + \mathcal{O}(\beta^2)\right].$$

(1.25)

It is easy to check that

$$Z = \int dU \ e^{-S} = e^{-\beta} \int dU \left[1 + \frac{\beta}{2} \sum_{\mu<\nu} (P_{\mu\nu} + P_{\mu\nu}^*) + \mathcal{O}(\beta^2)\right] = e^{-\beta} \left[1 + \mathcal{O}(\beta^2)\right]$$

(1.26)

$$Z^{-1} = e^\beta \left[1 + \mathcal{O}(\beta^2)\right]^{-1} = e^\beta \left[1 - \mathcal{O}(\beta^2)\right]$$

(1.27)

So we get

$$\langle P_{\alpha\gamma} \rangle = \frac{\beta}{2} + \mathcal{O}(\beta^2).$$

(1.28)

As in this limit the link variables become independent of each other the result does not depend on spacetime dimension.

**Monte Carlo Methods**

We discretized spacetime and introduced the fields of the theory in the lattice. Now we should find a way to compute quantities in our new theory. For that we should find a way of calculating the Feynman Path Integral. For that usually people resort on Monte Carlo methods, in practice introducing a new layer of discretization. The process consists in taking the path integral and discretizing it by taking into account only a finite number of field configurations which are representative of the system in thermal equilibrium.

We start with the partition function

$$Z = \int \mathcal{D}\phi \ e^{iS[\phi]}$$

(1.29)
and the average of an operator

\[ \langle O \rangle = \frac{\int D\phi \, O[\phi] \, e^{iS[\phi]}}{\int D\phi \, e^{iS[\phi]}}. \quad (1.30) \]

The next step is to effectuate a Wick rotation, in order to be in an Euclidean space.

\[ Z_E = \int D\phi_E \, e^{-S_E[\phi_E]} \quad (1.31) \]

and

\[ \langle O \rangle_E = \frac{\int D\phi_E \, O_E[\phi_E] \, e^{-S_E[\phi_E]}}{\int D\phi_E \, e^{-S_E[\phi_E]}}. \quad (1.32) \]

where the subscript \( E \) indicates euclidean version, which we will use almost always. From now on, for the sake of simplicity, we will drop the \( E \) subscript whenever the situation is not dubious.

This way we have

\[ \langle O \rangle = \int D\phi \, O[\phi] \, P[\phi] \quad (1.33) \]

where

\[ P[\phi] \, D\phi \propto e^{-S[\phi]} \, D\phi. \quad (1.34) \]

Denominating a configuration for \( x^{(\alpha)} \), we can approximate the integral in configurations for the following sum in \( N_{cf} \) configurations

\[ \langle O \rangle \approx \bar{O} \equiv \frac{1}{N_{cf}} \sum_{\alpha=1}^{N_{cf}} O[x^{(\alpha)}] \quad (1.35) \]

with a correspondent standard deviation

\[ \sigma_{\bar{O}} = \sqrt{\text{Var} \left[ \frac{1}{N_{cf}} \sum_{\alpha=1}^{N_{cf}} O[x^{(\alpha)}] \right]} = \sqrt{\frac{1}{N_{cf}^2} \text{Var} \left[ \sum_{\alpha=1}^{N_{cf}} O[x^{(\alpha)}] \right]} = \sqrt{\frac{1}{N_{cf}} \text{Var} [O[x^{(\alpha)}]]} = \sigma_O N_{cf} \propto \frac{1}{\sqrt{N_{cf}}}. \quad (1.36) \]

assuming that the configurations are independent Bienaymé’s formula guarantees that we can perform the following step\(^4\)

\[ = \sqrt{\frac{1}{N_{cf}^2} \sum_{\alpha=1}^{N_{cf}} \text{Var} [O[x^{(\alpha)}]]} = \sqrt{\frac{\sigma_{\bar{O}}^2}{N_{cf}}} \propto \frac{1}{\sqrt{N_{cf}}}. \quad (1.37) \]

\(^4\)Bienaymé’s formula is a theorem that states that the variance of the sum of independent random variables is equal to the sum of the variances of those variables. For a proof check for example \(^8\).
1.2 Temperature

We want to study the field at different temperatures so we need to be able to estimate it. The temperature is given by the inverse length of the lattice in temporal direction

\[ T(\beta, N_t) = \frac{1}{N_t a(\beta)}. \] (1.38)

Thus we need to introduce a scale from which we can extract the lattice spacing \( a \). There are several ways of doing this, the simplest one being probably the normalization to string tension. For this purpose we study a widely used scale introduced by Sommer in 1993 [9] known as Sommer scale (\( r_0 \)). He argues that this parameter has lower systematical and statistical errors than the traditional methods.

It consists of choosing a specific point in the force, taken as the derivative of the potential, and using it as reference distance to normalize quantities. \( r_0 \) is defined through the relation \( r_0^2 F(r_0) = 1.65 \). Details about our specific implementation are given in section 2.4.

1.3 Effective String Theory

Effective string theory is an effective bosonic string theory that is believed to model successfully many aspects of confining theories. This was proposed long ago and was initially supported by two arguments [10]

- meson spectroscopy could be explained assuming a string-like interaction between quarks,
- the discovery that in strong coupling limit the interquark potential rises linearly.

Many papers were published about this subject and many predictions of effective string theory have been studied. It predicts the formation of a flux tube between charges, giving origin to an asymptotically linear potential, responsible by the confinement property of the theory.

Its most famous result is the shape of the confining potential in leading order [11]

\[ V(r) = A + \sigma r + \frac{\gamma}{r} + O(1/r^3) \] (1.39)

where \( \gamma = \frac{(d-2)\pi}{24} \) and \( d \) is the dimension of the spacetime. This potential have some striking properties namely being independent of the gauge group we chose for the interaction. Also the numerical value of \( \gamma \), known as Lüscher term, is a dimensionless constant thought to be independent of gauge group, just depending on the dimensionality of spacetime. This fact is confirmed by many high precision studies in lattice using different groups and spacetime dimensions [10,12–18].

Recently there have been progress in this area with the calculation of the energy spectrum of the theory up to higher orders (e.g. [19]).

Also a well-known result of effective string theory is the evolution of string tension with temperature. A 2-loops expression for it is [20]

\[ \sigma(N_t) = \sigma_0 - \frac{(d-2)\pi}{6N_t^2} - \frac{(d-2)^2\pi^2}{72\sigma_0 N_t^4} \] (1.40)

where \( \sigma_0 \) is the value of string tension at zero temperature.
Other important results of this theory are the predictions about the width of the flux tube. The theory predicts that the flux tube width should increase logarithmically with the distance between the sources at zero temperature and linearly at high temperature at leading order [20]

- zero temperature \( w_{20}^2(r/2) = \frac{d-2}{2\pi\sigma} \log(\frac{r}{\rho_0}) \)
- finite temperature \( w_{20}^2(r/2) = \frac{d-2}{2\pi\sigma} \log(\frac{N_t}{4\rho_0}) + \frac{d-2}{4N_t\sigma} r + O(e^{-2\pi r/N_t}) \)

where \( N_t \) is the temporal extent of the lattice, \( d \) is the dimension of spacetime and \( \rho_0 \) is some distance scale to be fitted.

These expressions have been recently shown [21,22] to be correspondingly the low and high temperature limits of the same expression that incorporates the dependence of temperature explicitly

\[
\begin{align*}
  w_{20}^2(r/2) &= \frac{d-2}{2\pi\sigma} \log\left(\frac{r}{\rho_0}\right) + \frac{d-2}{\pi\sigma} \log\left(\frac{\eta(2iu)}{\eta^2(iu)}\right) \\
\end{align*}
\]

where \( \eta \) is Dedekind \( \eta \) function and \( u = \frac{N_t}{2\pi} \) incorporates the temperature dependence.

This width is calculated in the lattice through the computation of

\[
\begin{align*}
  w^2(r/2) &= \frac{\int dx_+ x_+^2 O(x_\perp)}{\int dx_+ x_+^2} \\
\end{align*}
\]

where \( O(x_\perp) \) is an operator that represents the flux tube (e.g. the density of energy) and the integral is evaluated in the mediator plane of the charges.
Chapter 2

Methods

We describe the numerical methods used to obtain our results.

2.1 Generation of configurations

We want to generated a set of configurations which should be decorrelated of each other (ideally independent), so we can guarantee that our estimate for the error is correct.

We define the autocorrelation of configurations as

$$C(x^{(a)}, x^{(b)}) = \frac{1}{V \times d} \left| \sum_x \sum_\mu \left[ U^{(a)}_\mu (x) \right]^{-1} U^{(b)}_\mu (x) \right|.$$  \hspace{1cm} (2.1)

Errors estimation

We use Jackknife method for error estimation. This method was introduced in 1956 by Quenouille \[23\] and two years latter by John Tukey \[24\].

It provides a method for estimating the variance of a general unknown distribution. It consists in systematically taking points out of the average and checking how much does it change, relating this change to the variance of the distribution.

If we have a dataset with \(N\) measurements of an observable \(x\) with average \(\bar{x}\) we define \(\bar{x}_j = \frac{1}{N-1} \sum_{i \neq j} x_i\) (the average extracting the measurement \(j\)). So for the variance of the average \(\bar{x}\) we have the following expression \[25\]

$$\sigma^2_x = \frac{N - 1}{N} \sum_{i=1}^{N} (\bar{x}_j - \bar{x})^2.$$  \hspace{1cm} (2.2)

It can also be generalized to a method in which the data is group in bins of a certain size, where an appropriate choice of bin sizes allows to solve the problems related to the absence of statistical independence between consecutive measurements.

Initial configuration

We implement two possible initial configurations: one with all gauge elements aligned (known as cold start) and one for which the group elements are drawn from an uniform random distribution (known as hot start).
If enough updates are applied they should produce identical results, converging to the set of states that characterizes the thermal equilibrium to the chosen $\beta$.

**Metropolis algorithm**

Metropolis algorithm is a Monte Carlo algorithm that allows to draw samples from a probability distribution. It produces a Markov chain of lattice configurations when applied sequentially to an initial configuration.

It is a simple algorithm that consists in the following procedure [26]:

- generate a new random link $\theta_{\text{new}}$ with uniform probability within the group
- if the action of the new link decreases accept it automatically, otherwise accept it with probability $e^{-S_{\text{new}}}/e^{-S_{\text{old}}}$.

An alternative to this is the Heat-bath algorithm. The difference is that instead of generating a new element with uniform probability it generates it using $e^{-S}$ as weight and always accepts the new element (since it is already distributed according to the right distribution). This method has several advantages, namely smaller correlation times due to always accepting the new element, but in $U(1)$ it is not possible to implement a pure Heat-bath algorithm\(^1\), although some variations that approximate this method are possible (e.g. [27]).

**Overrelaxation**

In some regions of $\beta$ pure Metropolis method exhibits a long autocorrelation time between configurations (cf. figure 2.2). Because of that the need for a method which decorrelates faster becomes important.

Overrelaxation is a widely used method for this propose. It was first introduced by Stephen Adler in 1981 [28, 29]. It decorrelates the system faster without changing the value of the action (so it changes a configuration to a different one which has equal probability of occurrence). There are several variations on this method, all of them having the same base idea.

The one we implement corresponds in practice to a reflection of the value of the links in relation to the minimum action (cf. figure 2.1) so

\[
e^{i\theta} \to e^{i(\theta_{\text{min}} + (\theta_{\text{min}} - \theta))} = e^{i(2\theta_{\text{min}} - \theta)}
\]

where $\theta_{\text{min}}$ is to the phase that would minimize the action given fixed links around it. As Wilson action is symmetric for reflections about its minimum this change keeps the action constant. We implement this by applying overrelaxation to a direction at a time keeping the others constant.

The minimum link that minimizes the action can be easily calculated.

\[
S_{\mu} = \beta \sum_{\nu \neq \mu} (1 - \text{Re}[e^{i\theta_{\mu\nu}^*}]) = 6\beta - \beta \text{ Re} \left[ \sum_{\nu \neq \mu} e^{i\theta_{\mu\nu}} \right] = 6\beta - \beta \text{ Re} \left[ e^{i\theta_{\mu}} \sum_{\nu \neq \mu} e^{i(\theta_{\mu\nu} - \theta_{\mu})} \right] = (2.4)
\]

\(^1\)For constructing a heat-bath algorithm we should integrate and invert $dp(U) = dUe^{-S}$. Developing $dp(U) = dUe^{-S}$ we get to $dp(\theta) \propto e^{\beta |W_{\theta}| \cos(\theta)}d\theta$ which integral can not be written as a closed form. The alternative is to use numerical approximations of this method, which can get results close to a pure heat-bath method.
factorizing the staple

\[ = 6\beta - \beta \text{Re} \left[ e^{i\theta} W_\mu \right] \equiv 6\beta - \beta |W_\mu| \text{Re} \left[ e^{i(\theta_\mu - \theta)} \right] = 6\beta - \beta |W_\mu| \cos(\theta_\mu + \theta_\perp) \]  \hspace{1cm} (2.5)

Now we can find the $\theta_\mu$ that minimizes the action.

\[ \frac{dS_\mu}{d\theta_\mu} \bigg|_{\theta_{\mu\text{min}}} = 0 \Rightarrow \sin(\theta_{\mu\text{min}} + \theta_\perp) = 0 \leftrightarrow \theta_{\mu\text{min}} = -\theta_\perp + k\pi \]  \hspace{1cm} (2.6)

Replacing in the previous expression for overrelaxation we obtain

\[ e^{i\theta} \rightarrow e^{i(-2\theta_\perp - \theta + 2k\pi)} \leftrightarrow \theta \rightarrow -2\theta_\perp - \theta + 2k\pi. \]  \hspace{1cm} (2.7)

The freedom of choosing $k$ value reflects the periodicity of the action and ensures we can project the result in our $[-\pi, \pi]$ working interval.

This method decorrelates configurations much faster than using metropolis method alone (cf. figure 2.2 and figure 2.3).
2.2 Multihit

A technique frequently applied to reduce the statistical error is multihit. It consists in a replacement of the links in time direction with effective links calculated through an average process against a constant background.

This new variables have the same average of the old links but a smaller variance so they have smaller statistical errors than the traditional links. Most of the times this average is evaluated stochastically, but in $U(1)$, as we will see, this substitution can be easily done analytically. This analytical process is frequently referred to as link integration.

The procedure corresponds to the following substitution \[ \text{[30,31]} \]

\[ U_{\mu}(x) \to \frac{\int dU \exp(-\beta S)}{\int dU \exp(-\beta S)} = \frac{I_1(\beta |W_{\mu}(x)|)}{I_0(\beta |W_{\mu}(x)|)} W_{\mu}(x) \]  

where $I_0$ and $I_1$ are the modified Bessel functions of order 0 and 1 and $W_{\mu}(x)$ is the sum of the staples of the $U_{\mu}(x)$ link. If calculating correlations between links in the same temporal layer, they should be spaced of at least two lattice spacings to ensure the result is correct.

The general formula for modified Bessel functions of type 1 is

\[ I_\alpha(x) = \frac{1}{\pi} \int_0^\pi \exp[x \cos(\theta)] \cos(\alpha \theta) d\theta - \frac{\sin(\alpha \pi)}{\pi} \int_0^\infty \exp(-x \cosh t - \alpha t) dt. \]  

So, in particular,

\[ I_0(x) = \frac{1}{\pi} \int_0^\pi \exp[x \cos(\theta)] d\theta \]  

and

\[ I_1(x) = \frac{1}{\pi} \int_0^\pi \exp[x \cos(\theta)] \cos(\theta) d\theta. \]
Now we should rewrite the action in a good way for our purpose

$$S = \beta \sum_{\nu \neq \mu} (1 - \text{Re} \left[ e^{i\theta_{\mu}} \right]) = 6\beta - \beta \text{Re} \left[ \sum_{\nu \neq \mu} e^{i\theta_{\nu}} \right] = 6\beta - \beta \text{Re} \left[ e^{-i\theta_{\mu}} \sum_{\nu \neq \mu} e^{i(\theta_{\mu} + \theta_{\nu})} \right] =$$

we can factorize the staple

$$= 6\beta - \beta \text{Re} \left[ e^{-i\theta_{\mu}} W_{\mu} \right] = 6\beta - \beta |W_{\mu}| \text{Re} \left[ e^{i(\theta_{\mu} + \theta_{\nu})} \right] = 6\beta - \beta |W_{\mu}| \cos(\theta_{\mu} - \theta_{\nu})$$

Then we should rewrite the integrals in multihit expression in terms of the modified Bessel function

$$\int dU e^{-S} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta} e^{-6\beta |W_{\mu}| \cos(\theta_{\nu} - \theta_{\mu})} = \frac{e^{-6\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{i\theta' + i\theta} |W_{\mu}| \cos \theta'$$

where we performed a change of integration variable $\theta \to \theta' + \theta_{\nu}$. We can change the interval $[-\pi - \theta_{\nu}, \pi - \theta_{\nu}]$ into $[-\pi, \pi]$ again since the integrand has period $2\pi$

$$= \frac{e^{-6\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{i\theta' + i\theta} |W_{\mu}| \cos \theta' = \frac{e^{-6\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} (\cos \theta' + i \sin \theta') e^{i\theta} |W_{\mu}| \cos \theta' =$$

the part of the integrand proportional to $\cos \theta'$ is even and the part proportional to $\sin \theta'$ is odd and they are integrated in a symmetric interval around the origin so

$$= \frac{e^{-6\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} \cos \theta' e^{i\theta} |W_{\mu}| \cos \theta' = e^{-6\beta} \frac{W_{\mu}}{|W_{\mu}|} I_1(\beta |W_{\mu}|)$$

where we took into account that $\cos \theta_{\nu} = \frac{W_{\mu}}{|W_{\mu}|}$ and $\frac{1}{\pi} \int_{0}^{\pi} d\theta' \cos \theta' e^{i\theta} |W_{\mu}| \cos \theta' = I_1(\beta |W_{\mu}|)$.

The second integral can be obtained with equivalent calculations to be

$$\int dU e^{-S} = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-6\beta |W_{\mu}| \cos(\theta_{\nu} - \theta_{\mu})} = \frac{e^{-6\beta}}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} e^{i\theta' + i\theta} |W_{\mu}| \cos \theta'$$

$$= e^{-6\beta} \frac{1}{\pi} \int_{0}^{\pi} d\theta' e^{i\theta} |W_{\mu}| \cos \theta' = e^{-6\beta} I_0(\beta |W_{\mu}|). \tag{2.13}$$

So we get the final result

$$U_{\mu}(x) \to \frac{\int dU \exp(-S)}{\int dU \exp(-S)} = \frac{e^{-6\beta} I_1(\beta |W_{\mu}(x)|) W_{\mu}(x)}{e^{-6\beta} I_0(\beta |W_{\mu}(x)|) |W_{\mu}(x)|} = \frac{I_1(\beta |W_{\mu}(x)|) W_{\mu}(x)}{I_0(\beta |W_{\mu}(x)|) |W_{\mu}(x)|}. \tag{2.14}$$

### 2.3 Lüscher-Weisz Multilevel

Multilevel algorithm was purposed by Martin Lüscher and Peter Weisz in 2001 [32] as a method for achieving exponential error reduction in some lattice gauge theory calculations. It is useful when the lattice is in confining phase and a local action is used, especially for evaluating average values of Polyakov or Wilson
loops.

Method

This method is inspired on the multihit method and explores the locality of the action by factorizing the path integral into smaller integrals calculated over sublattices. We will follow Lüscher and Weisz’s paper \[32\] in this section, focusing on Polyakov loop, since it is the structure we are most interested in our work, and applying it to $U(1)$ group.

We will start by introducing two-link operators since they play an essential in the formulation of the theory. Two-link operators are structures defined from the tensor product of two links in the same time layer at a certain distance $r$ (cf. figure 2.4). This structure represents the propagation in time of a pair of fermions from time $t$ to time $t + 1$ and they can be regarded as the basic constituents of the Polyakov loop correlation or of the temporal part of Wilson loop. They are defined as the following

$$T(x, t, r, \mu) = U_0^*(x + t\hat{0}) U_0(x + t\hat{0} + r\hat{\mu}). \quad (2.15)$$

We can note that this structure would be much more complicated in non-Abelian theories, since although in $U(1)$ the tensor product reduces to the usual product of two complex numbers this is not the case in general.

Now we can rewrite the Polyakov loop correlation in terms of these newly defined variables

$$P(x) P(x + r\hat{\mu}) = \left(U_0(x) \ldots U_0(x + T\hat{0})\right)^* \left(U_0(x + r\hat{\mu}) \ldots U_0(x + T\hat{0} + r\hat{\mu})\right) = \mathbb{T}(x, 0, r, \mu) \ldots \mathbb{T}(x, t, r, \mu) \ldots \mathbb{T}(x, T, r, \mu). \quad (2.16)$$

Next we will see how we can break the Polyakov loop correlation written like this in a product of averages over sublattices.

To do so we need to find a way of isolating sublattices from within our lattice. These sublattices are time-slices of our original lattice, contained between two hyperplanes of constant time $x_0$ and $y_0$. If we take two of those hyperplanes and hold their spatial links constant we can isolate the dynamics of the sublattice from the rest. This can be done thanks to the locality of the action (because the action only depends on plaquettes adjacent to it). This way we can calculate subaverages of quantities inside the smaller lattice. We follow the usual convention and denote the sublattice expectation values with square brackets $[\ldots]$ and expectation values over the whole lattice as $\langle \ldots \rangle$.

![Figure 2.4: Two-link operator, as part of a Polyakov loop.](image-url)
It is now possible to separate the integral in a hierarchical integration process with several intermediate levels. This integrals satisfy identities like \( T(x,t)T(x,t+1) = [T(x,t)]T(x,t+1) \), so for example we can calculate the average of the Polyakov loop correlation like this

\[
\langle P^*P \rangle = \langle [T(x,t)]T(x,t+1)\cdots[T(x,2r-2)]T(x,2r-1) \rangle. \tag{2.17}
\]

It is easy to check that the innermost average corresponds to a multihit process because it fixes the spatial links everywhere and averages the link. This way the multilevel algorithm can be seen as a generalization of multihit.

### Polyakov loop correlation

For calculating Polyakov loop correlation is useful to introduce some auxiliary quantities that will be averaged in the sublattices.

Here we follow approximately the notation defined in [31]. We start by defining the operator

\[
T^{(2)}(x,t,r,\mu) = T(x,t-r,\mu)T(x,t+1,r,\mu). \tag{2.18}
\]

If we define the first average as an average in sublattices of thickness 2 we have Polyakov loop correlation

\[
\langle P^*P \rangle = \langle [T^{(2)}(x,0,r,\mu)]T^{(2)}(x,2,r,\mu)\cdots[T^{(2)}(x,2r-2,\mu)]T^{(2)}(x,2r-1,\mu) \rangle. \tag{2.19}
\]

In our code we implemented one more level in order to achieve a further reduction on errors, calculating

\[
\langle P^*P \rangle = \langle [[[T^{(2)}(x,0,r,\mu)]T^{(2)}(x,2,r,\mu)]\cdots[T^{(2)}(x,2r-4,\mu)]T^{(2)}(x,2r-3,\mu)] \rangle. \tag{2.20}
\]

In practice the algorithm proceeds in a nested scheme as follows
Electromagnetic Field

In section 1.1 we saw how to expand the plaquette

\[ P_{\mu\nu} = e^{ia^2 g F_{\mu\nu} + i\mathcal{O}(a^3)} = 1 + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu}^2 + i\mathcal{O}(a^3) + \mathcal{O}(a^5). \]  

(2.21)

It follows that we can measure electromagnetic field tensor components through the study of the imaginary part of the plaquette operator

\[ \text{Im} P_{\mu\nu} = a^2 g F_{\mu\nu} + \mathcal{O}(a^3). \]  

(2.22)

Since \( g = 1/\sqrt{\beta} \) we have

\[ a^2 F_{\mu\nu} = \sqrt{\beta} \text{Im} P_{\mu\nu} + \mathcal{O}(a^3). \]  

(2.23)

Similarly if we want to study the squared electromagnetic field we can take the real part of the plaquette

\[ a^4 F_{\mu\nu}^2 = 2\beta (1 - \text{Re} P_{\mu\nu}) + \mathcal{O}(a^6). \]  

(2.24)

We want to study the field produced by static charges at a certain distance of each other so we should correlate our operator with a Polyakov loop correlation representing those charges, getting

\[ \langle O \rangle_{P^* P} = \frac{\langle P^* P O \rangle}{\langle P^* P \rangle} - \langle O \rangle \]  

(2.25)

where \( O \) stands for any operator we want to measure and \( \langle O \rangle_{P^* P} \) stands for the expectation value of \( O \)
produced by the charges represented by $P^*P$.

Figure 2.6: Multilevel method for electromagnetic field.

To implement a multilevel algorithm for this calculation it is useful to introduce some more quantities, besides $T$ and $T^{(2)}$,

$$
T^{(0)}(x_0, x, t, r, \mu) = T(x, t, r, \mu)O(x_0) + T(x, t, r, \mu)O(x_0)
$$

$$
T^{(2)}(x, t, r, \mu) = T^{(0)}(x, t + 1, r, \mu)O(x_0) + T^{(0)}(x, t + 1, r, \mu)O(x_0)
$$

$$
T^{(4)}(x, t, r, \mu) = T^{(2)}(x, t + 1, r, \mu)O(x_0) + T^{(2)}(x, t + 1, r, \mu)O(x_0)
$$

With this notation we can write

$$
\langle P^*(0)P(r\bar{\mu})O(x_0) \rangle = \frac{1}{N^d S d} \sum_x \sum_{\mu} \left[ T^{(4)}(x_0, x, 0, r, \mu)T^{(4)}(x, 4, r, \mu) + T^{(4)}(x_0, x, 4, r, \mu)T^{(4)}(x, 0, r, \mu) + \ldots \right]
$$

which can be calculated in a way analogous to the Polyakov loop correlation.

2.4 Sommer scale

As stated before, we use Sommer $r_0$ to establish a scale in the lattice.

To determine it we interpolate a function $F(r) = f_1 + \frac{f_2}{r}$ which constitutes a good local approximation to the force $\bar{F}$. To do so we compute the force $F(r) = V(r) - V(r - 1)$, as a discretized derivative of the
potential and numerically solve the system

\begin{align}
f_1 + \frac{f_2}{r^2} &= V(r) - V(r - 1) \quad (2.30) \\
f_1 + \frac{f_2}{(r+1)^2} &= V(r + 1) - V(r) \quad (2.31) \\
r_0^2 \left( f_1 + \frac{f_2}{r_0^2} \right) &= 1.65 \quad (2.32)
\end{align}

to obtain the $r_0$ value. The value of $\bar{r}_r$ is chosen to be a tree-level improved variable as to eliminate the $O(a^2)$ term from $r_0$. This is done choosing the value of $[9]$

\begin{align}
\bar{r}_r &= \left[ 4\pi \frac{G(r) - G(r - d)}{a} \right]^{-1/2} \quad (2.33) \\
G(r) &= \frac{1}{4a} \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{\cos(k_1 r/a)}{4\sum_{j=1}^{3} \sin^2(k_j/2)}. \quad (2.34)
\end{align}

This definition yields a considerable improvement for small $r_0$ over choice of $\bar{r}_r = r + \frac{a}{2}$, but becomes negligible at bigger values since the force becomes constant.

Having determined $r_0(\beta) = r_0/a(\beta)$ scale we can use it to estimate the temperature at which we are working. Since the temperature is

\[ T = \frac{1}{N_t a(\beta)}. \quad (2.35) \]

From here we can easily get

\begin{align}
\frac{T(\beta, N_t)}{T_c} &= \frac{N_t c a(\beta_c)}{N_t a(\beta)} = \frac{N_t c r_0(\beta)}{N_t r_0(\beta_c)} \quad (2.36)
\end{align}

where $T_c$ is the reference critical temperature for confining phase transition.

### 2.5 CUDA

CUDA (Computer Unified Device Architecture) is a computer architecture developed by NVIDIA that allows to explore the graphics processing units (GPU’s) for general purpose calculations. We use CUDA extensions of C++ programming language, which consist a set of extensions for this language that allow to use many features of C++, like classes and templates in GPU calculations.

Due to the challenges faced by graphics processing, GPU architecture have evolved into devices extremely well suited for highly parallelized code execution, with many cores and high memory bandwidth. This is ideal for lattice QCD calculations, because this calculations typically involve the analysis of all points of lattice independently which results in highly parallelizable calculations.

We use NVIDIA Fermi generation GPU’s that support double precision operations and allow the necessary performance for our calculations.

For our calculations we resorted mainly to four GPU’s, two NVIDIA GeForce GTX 580 and two NVIDIA Tesla C2075 (specifications can be found in table [2.1]).
<table>
<thead>
<tr>
<th></th>
<th>GeForce GTX 580</th>
<th>Tesla C2075</th>
</tr>
</thead>
<tbody>
<tr>
<td>CUDA capability</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Multiprocessors (MP)</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>Cores per MP</td>
<td>32</td>
<td>32</td>
</tr>
<tr>
<td>Total number of cores</td>
<td>512</td>
<td>448</td>
</tr>
<tr>
<td>Global memory</td>
<td>3072 MB GDDR5</td>
<td>6144 MB GDDR5</td>
</tr>
<tr>
<td>Shared memory (per SM)</td>
<td>48 KB or 16 KB</td>
<td>48 KB or 16 KB</td>
</tr>
<tr>
<td>L1 cache (per SM)</td>
<td>16 KB or 48 KB</td>
<td>16 KB or 48 KB</td>
</tr>
<tr>
<td>L2 cache (chip wide)</td>
<td>768 KB</td>
<td>768 KB</td>
</tr>
<tr>
<td>Clock rate</td>
<td>1.57 GHz</td>
<td>1.15 GHz</td>
</tr>
<tr>
<td>Memory Bandwidth</td>
<td>192.4 GB/s</td>
<td>144 Gb/s</td>
</tr>
<tr>
<td>Device with ECC support</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table 2.1: Specifications of the GPU’s used in this work.

**Code implementation**

For the configurations we store each lattice link in the form of a phase $\theta$, so we store one double precision floating point number per lattice link. This way of storing has several advantages, namely being always projected in $U(1)$ circle and mapping multiplications of group elements to sums of the phases ($e^{i\theta_1} \times e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$). Whenever we need to have sums or operations that fall out of the $U(1)$ group in general (e.g. when we apply multihit), we store the numbers in the complex form $a + ib$, thus using 2 double precision floating point numbers.

CUDA organizes the parallel threads into calls of kernels, organized in a grid of blocks, which are executed in parallel in arbitrary order. To each thread CUDA assigns a threadId and a blockId that uniquely identify the thread. With these ID’s we compute the position of the thread as a set of four lattice indices, corresponding to a position in the four directions of spacetime. The CUDA version we use, CUDA 4.1, allows the use of 3D indices in both grids and blocks, so in 4D kernels we use the first index to encode x and y directions, the second to z and the third to t.

As the threads are not guaranteed to run in any specific order we should ensure that the different calls do not interfere with each other. For that purpose, whenever we run update operations on a lattice that depend only on neighbor sites (e.g. Metropolis or overrelaxation updates), we separate the calls in odd and even points (when the sum $x + y + z + t$ is odd or even). This way we can update the odd points, running all the directions, synchronize, and update the even points, running all the directions again, making sure they keep independent.

Whenever the operations depend only on reading the lattice, thus not interfering with each other, we calculate all the lattice points at the same time (e.g. calculating plaquette or Polyakov loop, that only perform reading operations in the lattice).

To calculate averages we resort to a reduction algorithm that sums all the points in a lattice. We use an implementation based on the one provided in NVIDIA SDK (for documentation check [35]).

CUDA has access to several kinds of memory, which should be well managed to optimize the program. The NVIDIA FERMI memory architecture is summarized in figure 2.7.

For optimizing the memory accesses whenever we need to effectuate read-only operations on a configuration, we store it in texture memory, which is global memory but has a special memory cache that optimizes this kind of accesses.

Multilevel algorithm requires a large amount of memory because we should store the correlations of the
\( T^{(2)} \) and \( T^{(4)} \) operator with the plaquette for all the space points. This way for each point we want to test the field we should store \( N_s^3 N_t \) (lattice volume) \( \times 3 \) (spacial directions) \( \times n \) (time-like links are grouped into sets of n) \( \times 2 \times \) size of double. For a \( 24^4 \) with two levels (2 and 4) lattice it yields approximately 11.4 Mb per point where we wish to calculate the field.

### 2.6 Other computational resources

Although our lattice code is completely written in CUDA, we make use of Mathematica and qtiplot for fits, numerical integrations, and some other general purpose calculations, as well as, of Mathematica and gnuplot for plotting and creating the figures presented in this work.
Chapter 3

Results

In this chapter we state and analyze the main results obtained in our studies.

3.1 Average plaquette density

Average plaquette density is known to have a phase transition, known as bulk transition, which corresponds to a transition in the lattice spacing $a$.

This phase transition can be clearly observed as a peak in plaquette susceptibility ($\langle P_{\mu}^2 \rangle - \langle P_{\mu} \rangle^2$) (cf. figure 3.1).

![Figure 3.1: Plaquette susceptibility with $\beta$ in a 4D $8^4$ lattice.](image)

We study $1 - \langle P_{\mu\nu} \rangle$ which is the quantity most often studied in literature instead of $\langle P_{\mu\nu} \rangle$. We present our results for several spacetime dimensions (cf. figure 3.2). We can verify that they follow the expected limits in strong and weak coupling regimes. We find that the phase transition details depend significantly on the dimension of spacetime, maybe even changing the order of phase transition.

Our results for plaquette reproduce the results that can be found in many publications.

3.2 Confining phase transition

Confining/non-confining transition can be studied through the analysis of different parameters. The best parameter to study this are Polyakov loop expectation value or plaquette susceptibility. Polyakov loop
Figure 3.2: Average plaquette density in function of $\beta$ for 2D (a), 3D (b), 4D (c) and 5D (d). The error bars are much smaller than the size of the points in the graph. The functions plotted are the strong and the weak coupling limits calculated for the theory. The vertical line corresponds to $\beta_c$ extracted from Polyakov loop expectation value.
expectation value is 0 in confining phase and rises fast after the phase transition, making it possible to use it as an order parameter for this phase transition.

We calculated Polyakov loop expectation value to identify with precision the location of the phase transition in the lattice sizes we use (typically $24^4$). Our results are presented in the following tables and graphs.

This transition is more obvious on smaller lattices so we chose a $8^4$ lattice to illustrate it (cf. figure 3.3).

![Figure 3.3: Polyakov loop expectation value with $\beta$ in a 4D $8^4$ lattice (the error bars are smaller than the size of the points).]

3.3 Sommer scale

We use Sommer $r_0$ to establish a scale in the lattice so we can determine the temperature.

We obtain the error in $r_0$ by averaging the results of the difference of the result with the interpolation including an extra term of order $1/r^4$.

We study the temperature normalized to critical temperature at each the system becomes unconfined. In order to determine it we first had to estimate the critical temperature. We determined the plaquette susceptibility (cf. figure 3.4) and fitted the peak of phase transition in a system with $N_t = 4$. We obtained the approximate value of $\beta_c = 1.003$ and $r_0/a(\beta_c) = 9.10 \pm 0.97$, which yields for critical temperature $T_c \approx 2.3r_0$.

![Figure 3.4: Plaquette susceptibility in function of $\beta$ in a lattice $24^3 \times 4$.]
Then we calculated the temperatures for $\beta = 1$ and $N_t$ in the range of 4 to 24. To estimate the error in temperature we extrapolated the error induced by the error in critical temperature, since this was calculated close to the phase transition having a significantly bigger error than the other temperatures.

Our results for both $r_0$ and temperature are stated in table 3.1.

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$r_0/a$</th>
<th>$T/T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>04</td>
<td>3.60 ± 0.02</td>
<td>0.40 ± 0.04</td>
</tr>
<tr>
<td>08</td>
<td>2.898 ± 0.003</td>
<td>0.16 ± 0.02</td>
</tr>
<tr>
<td>12</td>
<td>2.882 ± 0.004</td>
<td>0.11 ± 0.01</td>
</tr>
<tr>
<td>16</td>
<td>2.87 ± 0.02</td>
<td>0.079 ± 0.008</td>
</tr>
<tr>
<td>20</td>
<td>2.86 ± 0.04</td>
<td>0.063 ± 0.007</td>
</tr>
</tbody>
</table>

Table 3.1: Dependence of $r_0/a$ and temperature with $N_t$.

We can see that the $r_0/a$ grows with $\beta$ which is in agreement with our expectations of lattice spacing getting smaller with $\beta$.

### 3.4 Multilevel convergence

The multilevel parameters should be tuned to obtain better results. The number of necessary iterates grows exponentially with the interquark distance [32], in order to compensate the exponential growing of the relative error in Polyakov correlations (due to exponential decay of Polyakov correlations). To illustrate this we show the behavior of Polyakov loop correlation with the number of multilevel iterates for different distances (figure 3.5). It is clear from the figure that the Polyakov loop correlations fall exponential with the distance as well as that the number of multilevel iterates to obtain a stable results grow in an approximately exponential way too.

![Figure 3.5: Polyakov loop correlation convergence with multilevel iterate.](image)

### 3.5 Potential

Static potential can be obtained through the study of Polyakov loop since we have

$$\langle P^*(0)P(r) \rangle = e^{-N_t V(r)}.$$  (3.1)
This way we have

$$V(r) = -\frac{1}{N_t} \ln \langle |P^*(0)P(r)| \rangle.$$  \hfill (3.2)

**Zero temperature**

We computed the static charge potential with several values of $\beta$ with and without multilevel. For the confining phase we obtain values compatible with the potential described by effective string theory, for the non-confining phase we obtain an $1/r$ dipole potential.

For $\beta = 1.00$, in the confining phase, we calculate the potential from 100 multilevel configurations, each one with 100 level 4 and 1000 level 2 multilevel iterates, with multihit method for further error reduction (except for $r/a = 1$). We fit the results (figure 3.6) and extract a value for string tension of $\sigma = 0.16719 \pm 0.00030$, if we force the Lüscher term to be constant ($\chi^2/dof = 0.140$). With Lüscher term as a fit parameter we obtain $\sigma = 0.1666 \pm 0.0022$ and Lüscher term 0.274 ± 0.061 ($\chi^2/dof = 0.147$), in a good agreement with the expected value of $\pi/12 = 0.2618$. We do not include the first two points in the fit.

For $\beta = 3.00$, now in the non-confining phase, we can observe the string tension going to zero. Multilevel is not needed in this region since the results are numerically much more stable. We use 1000 configurations, 10 iterates away from each other, and obtain the results in figure 3.7. From the fit we extract the values of $\sigma = -0.000194 \pm 0.000019$ and for the coefficient of the term in $1/r 0.03468 \pm 0.00062$ ($\chi^2/dof = 0.526$).

From the potential fits we can extract the value of the string tension $\sigma$ in function of $\beta$ (cf. figure 3.8). We can observe the string tension going to zero at deconfining phase transition. We extract also the value of Lüscher term which is compatible with the expected value in the confining phase of $\pi/12$ (cf. figure 3.9).
Figure 3.7: Potential of two static charges at $\beta = 3$. The error bars are much smaller than the size of the points in the graph.

Table 3.2: Fit parameters of potential of two static charges at different $\beta$. The fit function is $V(r) = A + b/r + \sigma r$. 

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$b$</th>
<th>$\sigma$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.960</td>
<td>0.46 ± 0.24</td>
<td>0.374 ± 0.012</td>
<td>0.201</td>
</tr>
<tr>
<td>0.970</td>
<td>0.37 ± 0.12</td>
<td>0.3336 ± 0.0059</td>
<td>0.0625</td>
</tr>
<tr>
<td>0.980</td>
<td>0.32 ± 0.10</td>
<td>0.2878 ± 0.0046</td>
<td>0.113</td>
</tr>
<tr>
<td>0.990</td>
<td>0.319 ± 0.071</td>
<td>0.2313 ± 0.0030</td>
<td>0.0471</td>
</tr>
<tr>
<td>0.995</td>
<td>0.25 ± 0.15</td>
<td>0.2046 ± 0.0057</td>
<td>0.335</td>
</tr>
<tr>
<td>1.000</td>
<td>0.274 ± 0.061</td>
<td>0.1666 ± 0.0022</td>
<td>0.147</td>
</tr>
<tr>
<td>1.005</td>
<td>0.260 ± 0.063</td>
<td>0.1273 ± 0.0026</td>
<td>0.0315</td>
</tr>
<tr>
<td>1.010</td>
<td>0.265 ± 0.032</td>
<td>0.0557 ± 0.0011</td>
<td>0.000293</td>
</tr>
<tr>
<td>1.0105</td>
<td>0.2344 ± 0.0029</td>
<td>−0.00095 ± 0.00013</td>
<td>0.00799</td>
</tr>
<tr>
<td>1.011</td>
<td>0.2291 ± 0.0047</td>
<td>−0.00122 ± 0.00021</td>
<td>0.0245</td>
</tr>
<tr>
<td>1.012</td>
<td>0.2194 ± 0.0070</td>
<td>−0.00138 ± 0.00031</td>
<td>0.113</td>
</tr>
<tr>
<td>1.013</td>
<td>0.2139 ± 0.0074</td>
<td>−0.00140 ± 0.00032</td>
<td>0.160</td>
</tr>
<tr>
<td>1.015</td>
<td>0.1947 ± 0.0049</td>
<td>−0.00106 ± 0.00016</td>
<td>0.0349</td>
</tr>
<tr>
<td>1.020</td>
<td>0.1896 ± 0.0089</td>
<td>−0.00133 ± 0.00038</td>
<td>0.822</td>
</tr>
<tr>
<td>1.030</td>
<td>0.1641 ± 0.0043</td>
<td>−0.00084 ± 0.00014</td>
<td>0.238</td>
</tr>
<tr>
<td>1.050</td>
<td>0.1478 ± 0.0035</td>
<td>−0.00073 ± 0.00011</td>
<td>0.320</td>
</tr>
<tr>
<td>1.100</td>
<td>0.1276 ± 0.0025</td>
<td>−0.000617 ± 0.000082</td>
<td>0.616</td>
</tr>
<tr>
<td>1.500</td>
<td>0.0794 ± 0.0021</td>
<td>−0.000422 ± 0.000061</td>
<td>0.401</td>
</tr>
<tr>
<td>2.000</td>
<td>0.0543 ± 0.0017</td>
<td>−0.000280 ± 0.000052</td>
<td>0.683</td>
</tr>
<tr>
<td>2.500</td>
<td>0.0420 ± 0.0013</td>
<td>−0.000222 ± 0.000039</td>
<td>0.633</td>
</tr>
<tr>
<td>3.000</td>
<td>0.03468 ± 0.00062</td>
<td>−0.000194 ± 0.000019</td>
<td>0.526</td>
</tr>
<tr>
<td>5.000</td>
<td>0.02001 ± 0.00025</td>
<td>−0.0001120 ± 0.0000081</td>
<td>0.254</td>
</tr>
</tbody>
</table>
Figure 3.8: Dependence of string tension with $\beta$ in 4D.

Figure 3.9: Dependence of $1/r$ term with $\beta$ in 4D (normalized to $\pi/12$).
Finite temperature

Also we calculated the potential with smaller temporal lattice extents. These graphs are analogous to the presented for $N_t = 24$. Our results are in table 3.3. It is interesting to verify that these results follow closely the theoretical prediction of effective string theory for the evolution of string tension with $N_t$ (cf. figure 3.10).

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$0.126 \pm 0.004$</td>
</tr>
<tr>
<td>8</td>
<td>$0.1893 \pm 0.0001$</td>
</tr>
<tr>
<td>12</td>
<td>$0.198 \pm 0.003$</td>
</tr>
<tr>
<td>16</td>
<td>$0.204 \pm 0.007$</td>
</tr>
<tr>
<td>20</td>
<td>$0.203 \pm 0.006$</td>
</tr>
<tr>
<td>24</td>
<td>$0.205 \pm 0.006$</td>
</tr>
</tbody>
</table>

Table 3.3: $\sigma$ in function of $N_t$ for $\beta = 1.00$.

Figure 3.10: $\sigma$ extracted from potential fit at $\beta = 1.00$ for different $N_t$ values. The curve is the theoretical expectation from Effective String Theory at 2-loops ($\chi^2 = 0.006$). Notice that the expression is not fitted, since the only parameter ($\sigma_0$) is fixed by zero temperature result.

3.6 Flux tube profile

We calculated the $E_x$ tube profile in the middle plan between the charges using 100 multilevel configurations\(^1\).

To the result we fit the ansatz suggested in [20]

$$ \frac{\langle P^* P F_{\mu\nu} \rangle}{\langle P^* P \rangle} = A \exp(-x_\perp^2/s) \frac{1 + B \exp(-x_\perp^2/s)}{1 + D \exp(-x_\perp^2/s)} $$ (3.3)

As expected we can notice that the flux tube (figures 3.11 and 3.12) gets broader at bigger charge separations. In the next section (3.7), we quantify this result through the calculation of the flux tube width.

\(^1\)These results were presented at Excited QCD 2012 International Meeting. For the corresponding proceeding check [36]. About the width of the flux tube the results we present are an improvement over the ones since now we went to $r/a = 8$, instead of just 6, and we calculated the flux tube width using $\langle E^2 + B^2 \rangle$, instead of the $\langle E_x^2 \rangle$ presented at the conference.
Figure 3.11: Flux tube profile for several charge distances at $\beta = 1.00$ and $N_t = 24$ ($T = 0.053T_c$).

Figure 3.12: Flux tube profile for several charge distances at $\beta = 1.00$ and $N_t = 4$ ($T = 0.40T_c$).
3.7 Flux tube width

We integrate \( (E^2 + B^2) \) ansatz fits to calculate the flux tube width. The errors are estimated using a jackknife algorithm.

Although the data confirms the theoretical expression for flux tube width from Effective String Theory \( \sigma \) extracted from these is not compatible with the value of \( \sigma \) obtained from the potential.

We study field width for several temperatures, corresponding to \( N_t = 4, 8, 12, 24 \) (cf. results table 3.4).

<table>
<thead>
<tr>
<th>( N_t )</th>
<th>( T/T_c )</th>
<th>( a^2 \sigma )</th>
<th>( \rho_0/a )</th>
<th>( \chi^2/dof )</th>
<th>( R )</th>
<th>( a^2 \omega^2(r/a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>0.053</td>
<td>0.268 ± 0.044</td>
<td>0.50 ± 0.16</td>
<td>0.27</td>
<td>2</td>
<td>1.61 ± 0.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>2.54 ± 0.15</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>2.98 ± 0.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>3.22 ± 0.16</td>
</tr>
<tr>
<td>12</td>
<td>0.11</td>
<td>0.18367 ± 0.00092</td>
<td>0.7590 ± 0.0068</td>
<td>0.000026</td>
<td>2</td>
<td>1.68 ± 0.16</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>2.88 ± 0.15</td>
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<td></td>
<td></td>
<td></td>
<td>6</td>
<td>3.60 ± 0.38</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>4.16 ± 0.82</td>
</tr>
<tr>
<td>8</td>
<td>0.16</td>
<td>0.2008 ± 0.0069</td>
<td>0.693 ± 0.030</td>
<td>0.16</td>
<td>2</td>
<td>1.68 ± 0.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>2.80 ± 0.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>3.42 ± 0.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>4.26 ± 0.29</td>
</tr>
<tr>
<td>4</td>
<td>0.40</td>
<td>0.153 ± 0.019</td>
<td>0.52 ± 0.17</td>
<td>0.30</td>
<td>2</td>
<td>2.67 ± 0.78</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>4.63 ± 0.15</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>6.66 ± 0.52</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
<td>7.900 ± 0.066</td>
</tr>
</tbody>
</table>

Table 3.4: Widths and fit parameters of the width expression.

We can realize that the flux tube width consistently grows with the charge separation. We studied several temperatures in order to observe the transition between the zero temperature limit (pure logarithmic dependence) and the high temperature regime (pure linear behavior). To make the transition more obvious we show repeated plots in logarithmic and linear scale (cf. figure 3.13).
Figure 3.13: Squared width in function of charge separation for 24 (a), 12 (b), 8 (c) and 4 (d). The figures on the left are plots of $a^2w^2(r/a)$ vs log$(r/a)$ and on the right $a^2w^2(r/a)$ vs $r/a$. 
3.8 Field in finite temperature

Field in finite temperature is numerically more stable than in zero temperature allowing us to calculate some field results without having to resort on multilevel algorithm.

We can see that the charges’ influence is confined to their neighborhood, being the far away regions dominated by vacuum fluctuations only (cf. figures 3.14, 3.16 and 3.17).

We used 20000 independent configurations, separated by 10 steps of 1 Montecarlo iterate and 3 Overrelaxation iterates separating each configuration.

![Electric field in charges’ plan for charge separations of 2, 4, 6 and 8 lattice units in a 24^3 × 4 4D lattice at \( \beta = 1 \).](image)

Figure 3.14: Electric field in charges’ plan for charge separations of 2, 4, 6 and 8 lattice units in a 24^3 × 4 4D lattice at \( \beta = 1 \).

3.9 Field in non-confining phase

We can calculate the field in non-confining phase also. In this phase the results are much more stable numerically and there is no need for using multilevel algorithm. We expect to obtain the Coulomb field result in the low coupling limit of the theory. We show some examples of our results in a 4D \( \beta = 3 \) lattice with a charge separations in the range 2-8 lattice units, compared with Coulomb law with periodic boundary
conditions (cf. figures 3.15, 3.18 and 3.19). We can notice that, unlike the field in confining phase, the charges’ influence extends much further away from them (compare with figure 3.17).

For these results we used 20000 configurations, with 20 steps of 1 Monte Carlo iterate and 3 Overrelaxation iterates separating each configuration.

Figure 3.15: Electric field in charges’ plan for charge separations of 2, 4, 6 and 8 lattices units in a 24^4 4D lattice at $\beta = 3$. 

Figure 3.16: Absolute value of electric field in charges’ plan for charge separations of 2, 4, 6 and 8 lattices units in a $24^3 \times 8$ 4D lattice at $\beta = 1$. 
Figure 3.17: Field line plot of electric field in charges' plan for charge separations of 2, 4, 6 and 8 lattices units in a $24^3 \times 4$ 4D lattice at $\beta = 1$. The colors correspond to a threshold to make it easier to visually separate the stronger field caused by the charges from the small random fluctuations of the background (where red corresponds to the stronger field).
Figure 3.18: Absolute value of electric field in charges’ plan for charge separations of 2, 4, 6 and 8 lattices units in a $24^3 \times 8$ 4D lattice at $\beta = 3$. 
Figure 3.19: Field line plot of electric field in charges' plan for charge separations of 2, 4, 6 and 8 lattices units in a $24^4$ 4D lattice at $\beta = 3$. In red we have our result and in blue the Coulomb result.
Chapter 4

Conclusion

In this work we successfully implemented a gpu code to generate and study $U(1)$ field configurations. We applied it to the study of $U(1)$ flux tubes in confining phase, as well as some other already well-known quantities.

Our results for potential are in a good agreement with the predictions of the effective string model in the confining phase, not only showing an asymptotically linear potential, but also being compatible with a Lüscher term of $\pi/12$.

In the non-confining phase the results follow the expected Coulomb $1/r$ potential, with the string tension going fast to zero.

For flux tube profile we find a broadening of the flux tube according to effective string theory predictions. However our result for $\sigma$ with potential and with flux tube broadening are not compatible, it would be interesting to verify if we can account for this discrepancy using a higher order result for the theoretical prediction. In a study with SU(2) group in 3D (cf. [20]) going further than leading order is important to account for the correct result at small distances. We calculated the flux tubes up to $r/a = 8$ that to our knowledge is the longest distance obtained up to now without duality transformations in $U(1)$. Also the lattice size used in this work ($24^4$) is bigger than most of the lattice sizes used in literature in $U(1)$ studies (typically $\leq 16^4$). Further research can be done, to increase the precision of the result with more configurations and to obtain more points to the fit, possibly by calculating the $r/a$ odd points and including temperatures higher than $0.4 T_c$.

We report a widening of the flux tube compatible with the Effective String Theory predictions, where [31] report an almost constant width of the flux tube, although their analysis and operators used to probe field are different. Using dually transformed lattices [37] also finds a logarithmic growth of the flux tube width at zero temperature. It might be interesting to deepen this study in order to understand better the relation between these results.

The usage of up-to-date numerical methods as multilevel as well as advanced computing technology as making usage of the GPU’s processing capabilities through CUDA allows the possibility of doing this kind of computations without having to resort on big clusters of CPU’s.
Appendix A

Notation and Definitions

This appendix lists some usual notation and definitions used throughout the text. It is intended to be used as reference.

- Greek indexes (e.g. $\mu$)
  refer to directions in lattice. Direction 0 is the temporal direction, the others are space directions.

- lattice point $x$
  refers to a specific point in lattice with coordinates given by $x$. It is a shortcut to a vector of 4 components $x^\mu$.

- lattice point $x + \hat{\mu}$
  refers to the next point in the lattice in $\mu$ direction.

- lattice link $U_\mu(x)$
  refers to a link that connects lattice points $x$ and $x + \hat{\mu}$. Its value is defined in the middle point of $x$ and $x + \hat{\mu}$ (sometimes denoted by $x + \frac{\hat{\mu}}{2}$) and is an element of the gauge group. When specifying where is the exact position of the link defined the following notation is used $U_\mu(x) \equiv e^{i\alpha[A_\mu(x + \frac{\hat{\mu}}{2})]}$.

- $U(1)$ phase $\theta_\mu(x)$
  used in the context of $U(1)$ lattice gauge theory. Refers to the phase of the link $U_\mu(x)$ so that $U_\mu(x) \equiv e^{i\theta_\mu(x)}$.

- plaquette $P_{\mu\nu}(x)$
  smallest possible loop, built from the closed product of four links along the directions $\mu$ and $\nu$,
  $P_{\mu\nu}(x) \equiv e^{i\theta_{\mu\nu}(x)} \equiv U_\mu(x)U_\nu(x + \hat{\mu})U_\nu^*(x + \hat{\nu})U_\mu^*(x)$.

- staple $W_\mu(x)$
  sum of all the products of three links around a certain link $U_\mu(x)$
  $W_\mu(x) \equiv \sum_{\nu \neq \mu} [U_\nu(x)U_\mu(x + \hat{\nu})U_\mu^*(x + \hat{\mu}) + U_\nu^*(x - \hat{\nu})U_\mu(x - \hat{\nu})U_\nu(x - \hat{\nu} + \hat{\mu}) \equiv |W_\mu(x)| \ e^{i\theta_\mu(x)}$

- Polyakov loop $P(x)$
  product of all the links in time direction in one spatial point $P(x) \equiv \prod_{t=0}^{N_t} U_0(x + t\hat{0})$.

- configuration $x^{(\alpha)}$
  the notation $x^{(\alpha)}$ is used to denote a specific configuration. A sum on alpha $\sum_{\alpha=1}^{N_c} x^{(\alpha)}$ is used to refer
to a summation over all the $N_{c_f}$ configurations available. When several configurations are available $U_\mu^{(\alpha)}(x)$ may be used to specify to which configuration does the link $U_\mu(x)$ belong.
Bibliography


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[33] “Nvidia specifications for nvidia geforce gtx 580 card.”

[34] “Nvidia specifications for nvidia tesla c2075 card.”

