

Study of constitutive models for soils under cyclic loading

Introducing a non-linear model with a spline-based backbone curve

Pedro Alonso Pinto

Instituto Superior Técnico, Technical University of Lisbon

1 Introduction

The study of problems that deal with soil-structure interaction under seismic loading conditions requires the use of a constitutive model able to reproduce the cyclic behaviour of soils from small deformations to failure. A type of models commonly used when modelling soils for certain cyclic loading scenarios is based on non-linear elasticity of the stress-strain relationship. These models have a set of rules to define first loading behaviour and to define unloading and reloading, the former being governed by so-called backbone functions. Most models of this type rely on backbone functions that have fixed formulas with a set number of parameters, which are calibrated to adapt to experimental testing data. However, the fact that these functions are dependant on fixed formulas makes them somewhat inflexible and hard to adapt to experimental data from a wide array of different soils and soil behaviours. In this context, the interest arises in investigating the consequences of replacing those fixed formulations with spline functions, which have a varying and in theory unlimited amount of parameters, and consequently the potential to be much more flexible in adapting themselves to experimental data.

This work aims to develop a new constitutive non-linear elastic soil model based on a backbone spline formulation coupled with the Masing rule. In this work some existing soil models will be analysed in their ability to replicate the essential components of cyclic loading behaviour, and then the developed model will be described and similarly tested. Finally, the analysed models and the developed model will undergo a more demanding test in which a real earthquake event will be simulated with the help of the Plaxis finite element software. The results of this test will then be compared between models, with a equivalent-linear analysis program, and with the records measured on the site of the event.

2 Soil behaviour under cyclic loading

A cyclic loading can be defined as a periodic action that when applied to a material body tends to change its stress and strain state. One of the most recognisable forms of a soil in a cyclic loading scenario happens during an earthquake. Even though seismic waves travel through rock in the majority of its journey from the source of the quake to the ground surface, the small portion of soil that is often near the surface can have a significant impact on the magnitude and nature of shaking at ground surface (Kramer, 1996). The study of the material properties of soils is therefore very important to understand the influence that the soil can have on the seismic waves (often called *site effects*), and also, conversely, the influence that the seismic waves can have on the soil's properties. Other examples of cyclic loading relevant from an engineering perspective are traffic loading in transport infrastructures and the propagation of sea waves and its effect on the seabed (Ishihara, 1996).

Most methods that deal with the characterisation of the soil under cyclic loading conditions focus on determining the evolution of two facets of the soil behaviour over the course of the loading cycles: the soil's stiffness and damping. The stiffness characteristics of the soil, in this context, can be represented by the secant shear modulus, G (or the shear modulus ratio, G/G_0 , where G_0 is the elastic shear modulus), and its damping characteristics by the damping ratio, ξ .

Of special importance is the evaluation of the stiffness and damping of soils for different strain rates. It is well known that, as the strain of a cyclic loading increases, the stiffness of soils decreases and the energy dissipation through damping increases, in most cases non-linearly (Kramer, 1996; Ishihara,

1996). This has a significant effect on the response of the soils, especially in earthquake events. Typically, soil behaviour under increasing strain in cyclic loading conditions can be thought to fall into one of these three categories, based on strain amplitude: *very small strains* ($\gamma < 10^{-5}$), where the soil exhibits linear elastic behaviour; *small to medium strains* ($\gamma = 10^{-5} - 10^{-3}$), where the soil exhibits non-linear behaviour for increasing strain, but does not exhibit significant changes in stiffness or damping properties with the application of repeating loading cycles; and *medium to large strains* ($\gamma > 10^{-3}$), where soils tend to exhibit changes in their properties not only with increasing shear strain but also with repeated application of loading cycles, and where volumetric strains start to become important.

Laboratory tests indicate that both soil stiffness and damping (and their evolution with strain) are influenced by the cyclic strain amplitude, the void ratio, the mean effective stress, the plasticity index, the overconsolidation ratio, and the number of loading cycles (Kramer, 1996; Ishihara, 1996).

Most of the damping that is thought to occur in soils for cyclic loading conditions happens through *hysteretic damping* (Verruijt, 1994), and not through *viscous damping*, which is a more common representation of damping. The hysteretic damping ratio can be expressed by (Ishihara, 1996):

$$\xi = \frac{1}{4\pi} \frac{\Delta W}{W} \quad (1)$$

where ΔW is the energy lost per loading cycle, and W the maximum elastic energy that can be stored in a unit volume of a viscoelastic body. The property ΔW is given by the area enclosed by a hysteresis loop, and W is defined as the area of the triangle bounded by a straight line defining the secant modulus at the point of maximum strain (Ishihara, 1996).

Some models, namely *non-linear, cycle-independent models* (used for soils in conditions that fall in the small to medium strain range), use the so-called *Masing rule* to model their stress-strain curve. This effectively means that, when the soil enters an unloading or reloading state, it follows a path that has the same shape as the *first loading* path, defined by a *backbone curve*, $f(\gamma)$, but scaled up by a factor of two in both plane axes. Under these conditions, equation (1) can be rewritten as (Ishihara, 1996):

$$\xi = \frac{2}{\pi} \cdot \left[\frac{2 \int_0^{\gamma_a} f(\gamma) d\gamma}{f(\gamma_a)\gamma_a} - 1 \right] \quad (2)$$

3 Overview of soil models for cyclic loading conditions

Four soil models were analysed in this work in their ability to simulate soil behaviour under cyclic loading. The choice of the models to analyse was made on the basis of their availability in the Plaxis version used (version 8). The analysed models are the following: the linear isotropic elasticity model, the Ramberg-Osgood model, the Mohr-Coulomb model, and the Hardening-Soil model. Of these models, only the Ramberg-Osgood model has been developed to deal specifically with cyclic loading conditions. However, the analysis of the other models, developed to deal primarily with static loading, is also relevant in this context, since they are used extensively in the field of geotechnical engineering.

The analysed models were tested in the finite element software Plaxis, where a simple shear test was simulated. From the shear strain–shear stress curves that were calculated by the program, the $\gamma - G/G_0$ and $\gamma - \xi$ points were determined. These were compared with each model's analytical curves to test the models' implementation in Plaxis. The results are presented in Figures 1, 2 and 3.

The models were run with the following parameters:

- Mohr-Coulomb: $E = E_{ur} = 30000 \text{ kPa}$, $\nu = 0.30$, $c = 200 \text{ kPa}$, and $\phi' = \psi' = 0$.
- Hardening-Soil: $m = 0.5$, $E_{oed}^{ref} = E_{ur}^{ref} = E_{50}^{ref} = 10000 \text{ kPa}$, $\nu_{ur} = 0.20$, $c = 50 \text{ kPa}$, $\phi' = \psi' = 0^\circ$.
- Ramberg-Osgood: $G_0 = 10000 \text{ kPa}$, $K_0 = 26000 \text{ kPa}$, $\gamma_r = 10^{-3}$, $\alpha = 50$, and $r = 2.5$.

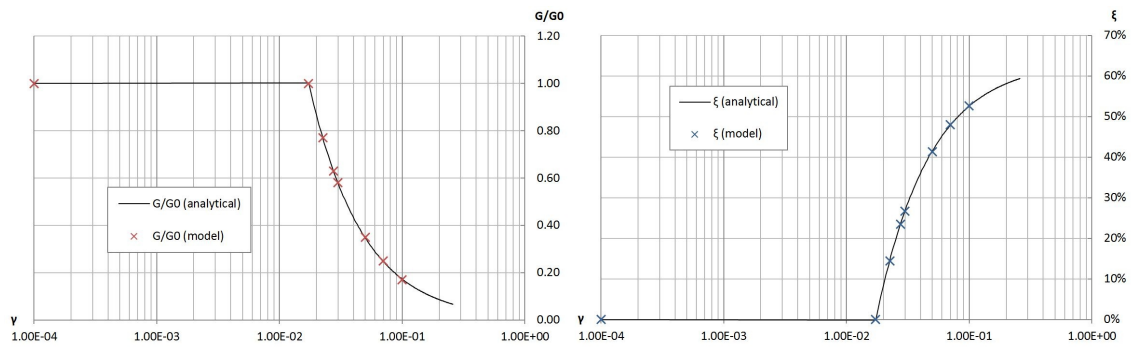


Figure 1: Analytical shear modulus and damping ratio curves and corresponding points obtained from the tests made with the Mohr-Coulomb Plaxis model.

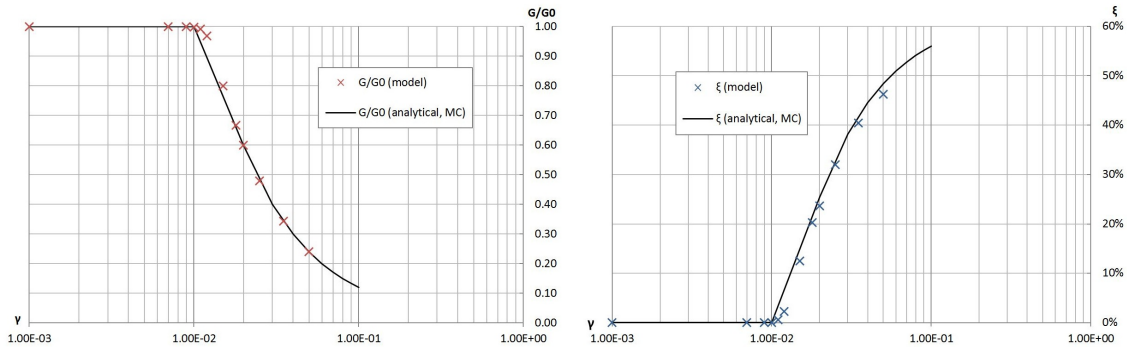


Figure 2: Analytical shear modulus and damping ratio curves (for the M-C soil model) and corresponding points obtained from the tests made with the Hardening-Soil Plaxis model.

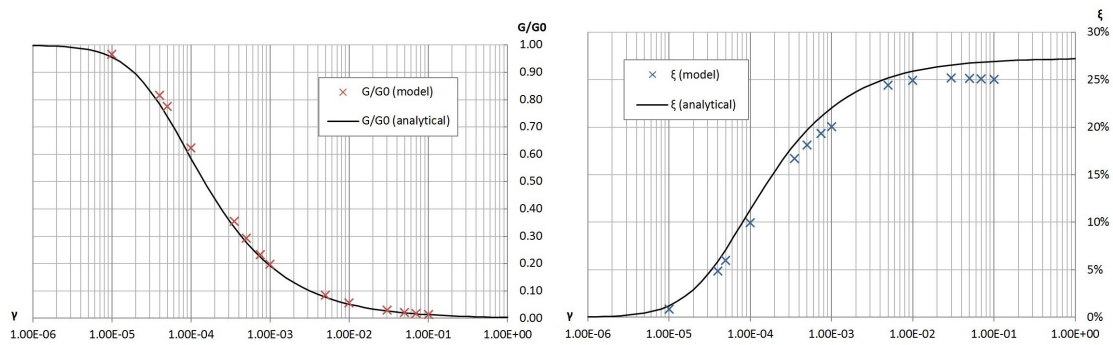


Figure 3: Analytical shear modulus and damping ratio curves and corresponding points obtained from the tests made with the Ramberg-Osgood Plaxis model.

Some conclusions were taken about each model and their ability to represent soil behaviour under cyclic loading conditions:

Mohr-Coulomb model:

- is just as limited as the linear elastic model for strains in the linear elastic range (a range which is in most cases much higher than what is typically observed in real soils);
- only simulates shear modulus reduction and damping when the soil enters a failure state, which is known not to be the case for real soil behaviour;
- does not accurately reproduce the experimentally observed G and ξ curves.

Hardening-Soil model:

- can simulate some modulus reduction and damping even before the soil enters a failure state;
- does not accurately reproduce the experimentally observed G and ξ curves (in fact, they are very similar to the ones from the Mohr-Coulomb model).

Ramberg-Osgood model:

- accounts for the reduction in shear modulus, G , with increasing strain amplitude;
- accounts for the dissipation of energy through damping, and its increase with increasing strain amplitude;
- can simulate, through elastic non-linearity, irreversible strains that occur in the soils;
- is unable to simulate soil failure;
- does not account for volumetric straining and dependence on number of loading cycles;
- even though it adequately reproduces the experimentally observed G and ξ curves' shape, its backbone curve formula may not have the flexibility to adapt itself to various different types of soil.

In conclusion, of the analysed soil models, the only one that is reasonably capable of modelling soil behaviour under cyclic loading is the Ramberg-Osgood model. However, the fact that it has a relatively stiff backbone curve definition (based on just 4 parameters), means that it may be incapable of adjusting well to the $\gamma - G/G_0$ and $\gamma - \xi$ curve shapes of a wide range of soils.

4 The Spline-based model

One of the objectives of this work was the development of a non-linear elastic soil model that, given strain-dependent modulus and damping results from laboratory tests, generates and adapts a spline-based backbone curve in stress-strain space so that the resulting strain-dependent modulus and damping curves are as close as possible to both sets of experimental results. The developed model can be classified as a non-linear cycle-independent model, similar in nature to the Ramberg-Osgood model. However, a key difference between that model and this one is that, instead of having a single formula to generate the backbone curve, this model generates a backbone curve that is a spline made up from a variable number of polynomials, with an equally variable number of parameters, which, in theory, offers more flexibility to the model.

Some of the advantages of this model are that: it has the potential to be more flexible in adapting experimental $\gamma - G/G_0$ and $\gamma - \xi$ points than other existing models; it is easy for the user to input experimental data directly, pick-and-choose experimental data points that best suit the desired behaviour, and even add in virtual data points, so as to force the model to behave in a specific manner; and, as a spline is made from piecewise defined polynomials, it is very easily analytically integrable, which speeds up its algorithmic calculations.

4.1 Methodology

Given a set of experimental data, the end result of the model is a set of polynomial coefficients that fully define a backbone curve. This backbone curve is calculated by the model to approximate the experimental data that was given as an input as well as possible. This approximation's accuracy will depend on the precision chosen for the backbone curve and the characteristics of the experimental data. The developed model follows the methodology summarised below:

1. Receive input of experimental data ($\gamma - G$ and $\gamma - \xi$ points).
2. Generate two analytical $\gamma - G$ and $\gamma - \xi$ curves that closely approximate the experimental points.
3. Based on a user-defined precision, generate a set of points for both curves using the analytical expressions (the greater the precision, the more points generated, at the cost of processing time).
4. Using the values of $\gamma - G$ at the points generated, initialise the τ_i -values of the backbone curve using $\tau = G \cdot \gamma$, obtaining a set of backbone curve points (γ_i, τ_i) .
5. Based on the backbone curve points (γ_i, τ_i) , generate the spline that defines the backbone curve.
6. Based on the backbone curve points, calculate the shear modulus error at each point defined in 3, using $\tau = (G/G_0) \cdot G_0 \cdot \gamma$ and comparing it to the analytical points obtained in 2.

7. Integrate the generated spline (see subsection 4.1.3) and calculate the damping ratio at the points defined in 3 using equation (2), and then check the error at each point by comparing the calculated values with the analytical points obtained in 2.
8. Check the overall error by adding the total errors on both shear modulus and damping values, multiplied by the corresponding error weights.
9. Minimise the error by changing the τ_i -values of the backbone curve defined at 4, and repeating steps 5 through 8.

4.1.1 Analytical approximation of shear modulus and damping ratio points

The approximation of the shear modulus and damping points using an analytical function is necessary because, while developing this model, it was observed that for it to generate well-behaved shear-dependent modulus and damping curves, it was necessary to have a sizeable amount of experimental points. If few points were supplied, the generated curves would approximate the experimental points closely, but would fluctuate wildly between them, which was undesirable. By having an analytical function approximate the experimental points, a greater number of points can be extracted from the analytical function, eliminating the problem of having few points to produce accurate results, which is often the case with experimental data. The analytical function chosen for this task is called a generalised logistic function, or Richards' function (Richards, 1959), and can be expressed by:

$$L(x) = A + \frac{K - A}{(1 + Qe^{-B(x-M)})^{1/v}} \quad (3)$$

in which: A is the lower asymptote; K is the upper asymptote; B is the function's growth rate; $v > 0$ affects where the maximum growth occurs; and M is the x of maximum growth if $Q = v$.

A significant quantity of experimental $\gamma - G/G_0$ and $\gamma - \xi$ points were tested, and the majority of them were found to be very accurately approximated by a function of this type. This means that the function, in most cases, fulfils its purpose of allowing for the generation of additional experimental points, without compromising the accuracy of the overall model.

4.1.2 Spline interpolation

A *spline curve* is a combination of piecewise polynomial curves that follow specific rules with regards to their junction points. Of the various existing types of spline curves, for the proposed model a *natural cubic spline* was chosen. The characteristics of a natural cubic spline are the following:

- it is made up from cubic polynomials (i.e. third degree polynomials);
- it passes through a set of control points, which are also its polynomial junction points;
- the first and second derivatives are continuous throughout the spline (including at the junction points);
- the second derivative at the endpoints is equal to zero.

A spline made up from cubic polynomials is malleable enough to allow good data approximation and smooth curves, but its formulation is simple enough to be easily computed.

The only necessary input in order to build a natural cubic spline is a set of control points through which the spline curve will pass. This means that a natural cubic spline made up from $n + 1$ control points will have a total of n cubic polynomials. A relatively simple algorithm to compute the construction of a natural cubic spline given a set of $n + 1$ points (x_j, y_j) is the following (Buchanan, 2010):

1. INPUT $\{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$.
2. For $i = 0, 1, \dots, n - 1$ set $a_i = y_i$; set $h_i = x_{i+1} - x_i$.

3. For $i = 1, 2, \dots, n - 1$ set $\alpha_i = \frac{3}{h_i}(a_{i+1} - a_i) - \frac{3}{h_{i-1}(a_i - a_{i-1})}$.
4. Set $l_0 = 1$; set $\mu_0 = 0$; set $z_0 = 0$.
5. For $i = 1, 2, \dots, n - 1$ set $l_i = 2(x_{i+1} - x_{i-1}) - h_{i-1}\mu_{i-1}$; set $\mu_i = \frac{h_i}{l_i}$; set $z_i = \frac{\alpha_i - h_{i-1}z_{i-1}}{l_i}$.
6. Set $l_n = 1$; set $c_n = 0$; set $z_n = 0$.
7. For $j = n - 1, n - 2, \dots, 0$ set $c_j = z_j - \mu_j c_{j+1}$; set $b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{h_j(c_{j+1} + 2c_j)}{3}$; set $d_j = \frac{c_{j+1} - c_j}{3h_j}$.
8. For $j = 0, 1, \dots, n - 1$ OUTPUT a_j, b_j, c_j, d_j .

in which a_i, b_i, c_i and d_i are the polynomial coefficients of the polynomial with index i , (x_j, y_j) are the plane coordinates of the control point with index j , and the remaining variables are auxiliary parameters whose purpose is to enhance the algorithm's efficiency.

The source code of the model that deals with the generation of the backbone curve was based off of this algorithm, with the necessary changes and additions to incorporate integration, stiffness and damping calculation, and error minimisation. The source code was written in the *Visual Basic for Applications* (VBA) language, with the support of Excel 2010.

4.1.3 Spline integration

In order to calculate the damping ratio, it is necessary to integrate the backbone curve, as per equation (2). As this model generates a backbone curve using a spline made up from piecewise polynomials, its integration can be done analytically and using very few computing resources. The integration of the backbone curve is done through the generic function:

$$I(x) = \sum_{i=0}^{k-1} \left(a_i h_i + \frac{b_i}{2} h_i^2 + \frac{c_i}{3} h_i^3 + \frac{d_i}{4} h_i^4 \right) + a_k (x - x_k) + \frac{b_k}{2} (x - x_k)^2 + \frac{c_k}{3} (x - x_k)^3 + \frac{d_k}{4} (x - x_k)^4 \quad (4)$$

in which $h_i = x_{i+1} - x_i$, and k is the index for which x_k is the greatest value of x_i that is smaller than x .

4.1.4 Error minimisation

In the model, the calculations of the error of the $\gamma - G$ and $\gamma - \xi$ curves are made using the the minimum squared error method. The total calculated error, ε_T , in each iteration is the following:

$$\varepsilon_T = \frac{w_G \cdot \sum_i \left[\left(\frac{G}{G_0} \right)_i^{\text{calc}} - \left(\frac{G}{G_0} \right)_i^{\text{exp}} \right]^2 + w_\xi \cdot \sum_i (\xi_i^{\text{calc}} - \xi_i^{\text{exp}})^2}{w_G + w_\xi} \quad (5)$$

in which w_G and w_ξ are the error weights applied to each curve, which can be specified by the user.

4.1.5 Implementation into Plaxis

The model's implementation into Plaxis was made having as a base the routine developed by P. Chitas (Chitas, 2008) for the Ramberg-Osgood model, with the necessary changes to the backbone curve formulation. Specifically, as this model was developed using Excel and VBA, and due to the significant number of parameters involved in a spline backbone curve, the VBA routine was developed in such a way that it automatically creates the *Dynamic Link Library* file (.dll file) necessary to run the model in Plaxis as a User-Defined Soil Model. Naturally, each different backbone curve generated by the model will need a separate file to load in Plaxis as a User-Defined Soil Model.

4.2 Results and comparison

The model was tested using data points extracted from some well known empirical curves by Vucetic and Dobry (1991) and Darendeli (2001). These curves are normally used as a replacement for soil analyses when experimental data is not available, and as such are a good test to evaluate the performance of the model. The results are presented in Figures 4 and 5.

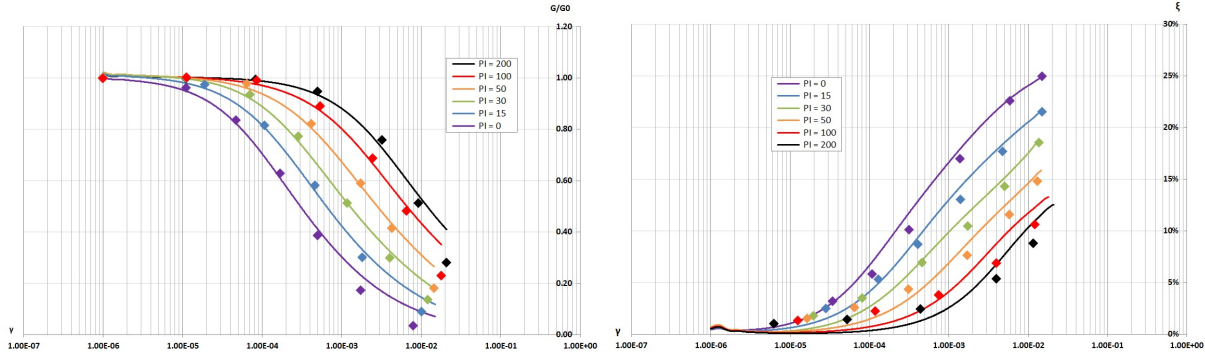


Figure 4: Stiffness and damping data points extracted from the empirical curves by Vucetic and Dobry (1991), and approximations made by running the Spline-based model (represented as lines).

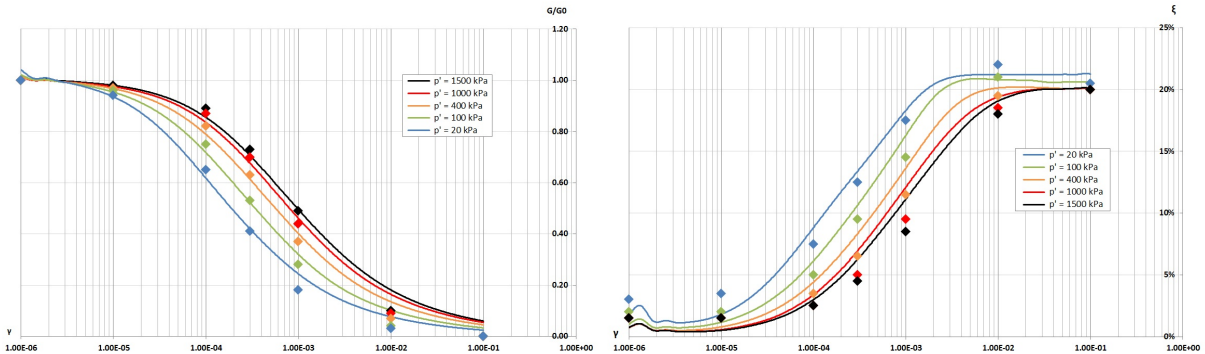


Figure 5: Stiffness and damping data points extracted from the empirical curves by Darendeli (2001), and approximations made by running the Spline-based model (represented as lines).

It can be seen that, when working with some sets of experimental points, the model has some trouble approximating the damping ratio points for low strains. The observable effect is the undervaluation of the damping ratio for lower strains. On the other hand, at higher strains the model seems to slightly overvalue the stiffness modulus ratio while significantly improving its approximation of the damping ratio points. This makes for an overall good approximation, since the higher strain values are more significant to the overall soil behaviour, and lower strain damping can be added to the model through other means.

4.3 Cyclic loading simulation with Plaxis

As was done with the analysed existing soil models in section 3, the Spline-based model was also subjected to a Plaxis cyclic loading simple shear test. The backbone curve used to test the model in Plaxis corresponds to the empirical curve by Vucetic and Dobry (1991) for PI = 15 (see Figure 4).

In Figure 6, the shear modulus and damping values calculated for each of the tests made with the Spline-based model are compared with the corresponding curves obtained with the Excel routine and with the experimental points that were at the basis of the model. The results show that the model's implementation into Plaxis is a successful one.

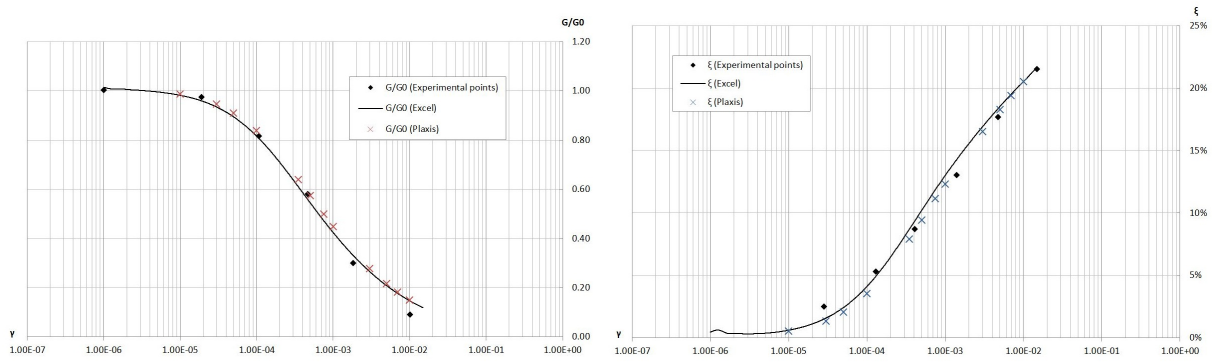


Figure 6: Empirical shear modulus and damping ratio data points from Vucetic and Dobry (1991), curves obtained by running the Excel routine, and corresponding points obtained from the tests made with the Plaxis model.

5 Site response simulation

Having developed the spline-based model algorithm in Excel and its implementation into Plaxis, the interest arose in testing both the developed model and the analysed existing models in a more complex scenario, to gauge and compare their behaviour in a more realistic geotechnical context. To this effect, the idea came about to use existent data from a known earthquake location and compare it with simulations made with Plaxis using the constitutive models for a one-dimensional ground response analysis. The event that was chosen was the Loma Prieta earthquake of 1989, specifically its recorded ground motion data records Gilroy #1 and #2 (respectively in a rock outcropping and in a soil site).

To make the earthquake simulation in Plaxis, the geology of the site was replicated using the analysed models, based on the elastic parameters and on the stiffness and damping curves proposed by Andrade and Borja (2006) for the Gilroy #2 site (see Figure 7). Each model's parameters were calibrated to conform in the best way possible to the experimental results. The site's elastic parameters used, based on Andrade and Borja (2006), were the following (the higher the index, the deeper the layer): $G_0^1 = 65$ MPa, $G_0^2 = 430$ MPa, $G_0^3 = 220$ MPa, $G_0^4 = 860$ MPa, $G_0^5 = 530$ MPa and $G_0^6 = 1050$ MPa.

Regarding the existing models, the calibrated parameters were input into Plaxis for each model. As for the Spline-based model, the backbone curves functions were compiled into User-Defined Soil Models to be used with Plaxis, using the developed algorithm. The finite element software was also carefully calibrated with regards to mesh refinement, Rayleigh and Newmark damping, etc. Each model was then run with the Gilroy #1 E-W ground motion record. The resulting sets of acceleration time series obtained at the top of the soil layers are presented in Figure 8, together with the results of running the model in the equivalent-linear tool EERA (*Equivalent-linear Earthquake site Response Analysis*), and also with the measured Gilroy #2 E-W ground motion record at ground surface. In Figure 9 the corresponding Fourier amplitude spectra and amplification functions can also be compared.

Analysing the results, it can be seen that the Ramberg-Osgood model and the Spline-based model produce the results that most closely approximate the measured records, both in the time and in the frequency domains. The Mohr-Coulomb model over-estimates the acceleration peaks in the acceleration time series, does not accurately predict the major frequency peaks, and over-estimates the amplification felt throughout the entire frequency range, especially at the higher frequencies. The results from the EERA model show an over-estimation of the acceleration peaks in the time domain, a correct identification of the major amplitude peak at the frequency $f \approx 2.5$ Hz (albeit with a wider range than what is observed in the measured records), and a shape of the amplification function that is much different from the other models and from the measured records.

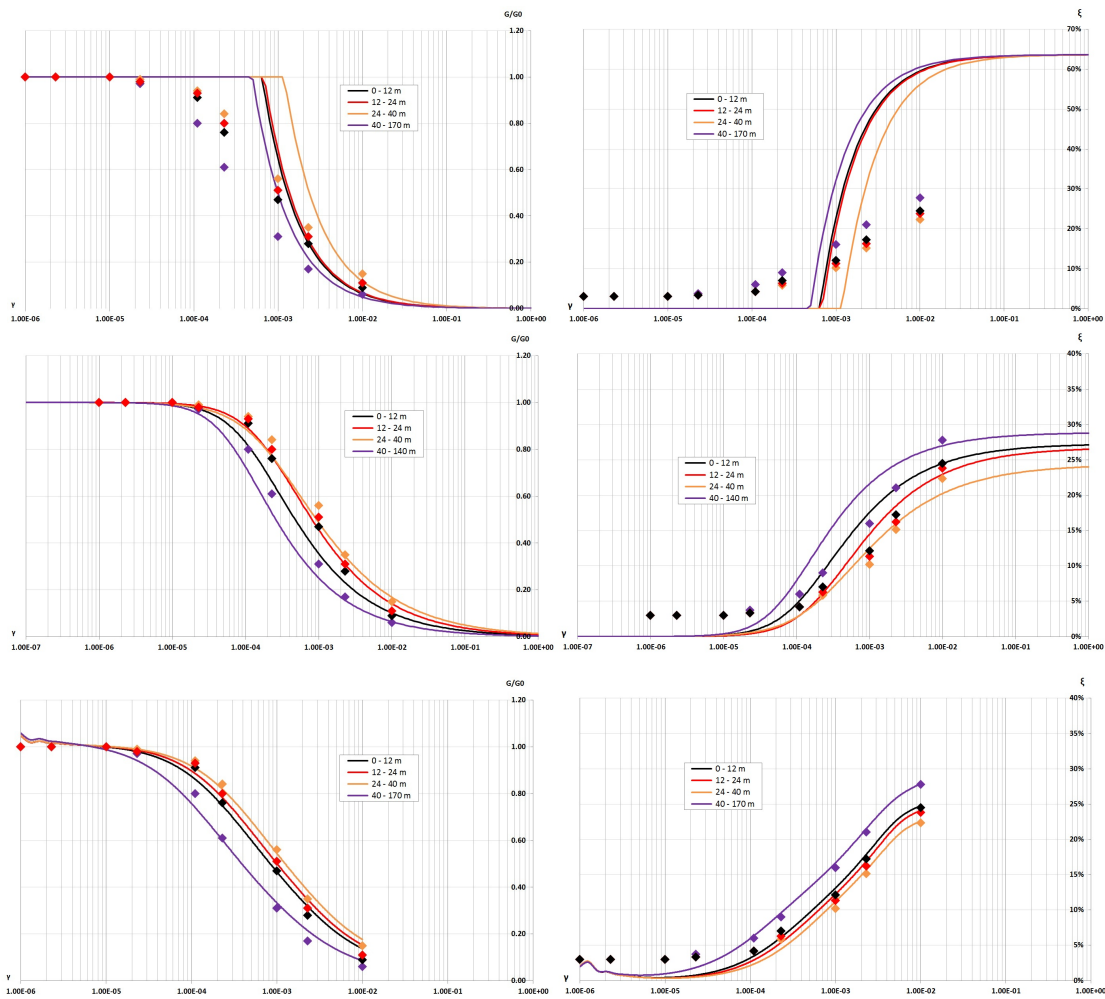


Figure 7: Stiffness and damping data points extracted from the experimental curves by Andrade and Borja (2006) for the Gilroy #2 site, and respective approximations made by running the M-C model (above), the R-O model (middle), and the Spline-based model (below).

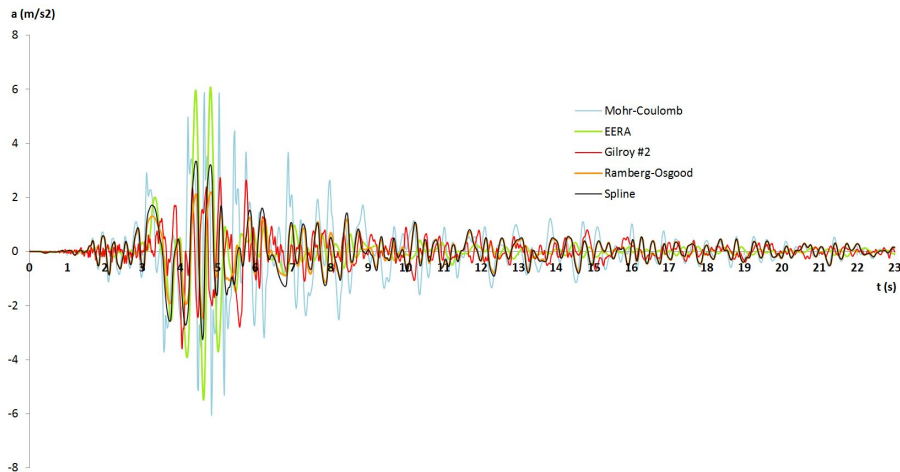


Figure 8: Comparison between the acceleration time series obtained from the Plaxis simulation for each of the models, from the EERA simulation, and from the measured Gilroy #2 E-W ground motion record.

With the obtained results, it can be said with reasonable confidence that the developed Spline-based model is well prepared to simulate soil behaviour in a cyclic loading scenario. However, the observed similarities with the R-O model may mean that the two models are essentially equivalent, and that the spline formulation does not provide any measurable advantage to the definition of the backbone curve, when compared with the R-O formulation.

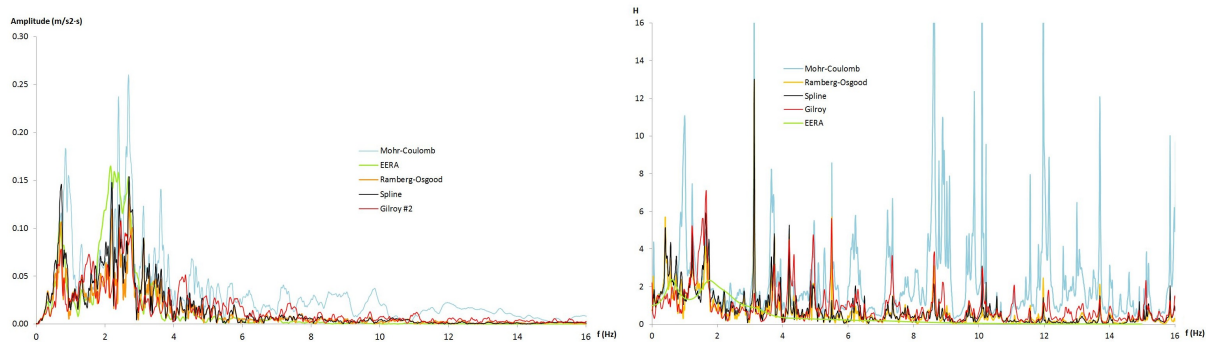


Figure 9: Comparison between the Fourier amplitude spectra (left) and amplification function (right) for each of the models, from the EERA simulation, and from the measured Gilroy #2 ground motion record.

6 Concluding remarks

The main objective of this work, the development of a constitutive non-linear elastic cyclic soil model based on a spline backbone formulation and on the Masing rule, was successfully achieved. Based on the tests made, it can be said that the model manages to adequately approximate most sets of experimental data points. The model seems to have some trouble approximating damping ratio values for very small strains. However, very small strain damping values are not typically significant to the overall soil behaviour, which means that even with this discrepancy the model is still capable of adequately simulating soil behaviour. Additionally, small-strain damping can be added in alternative ways to compensate for this, namely through Rayleigh damping. For higher strain damping values, where the model accuracy becomes more important, the model generally seems to behave in a satisfactory fashion. When the model was tested in a more realistic scenario, and compared with other existing soil models, it performed well, having adequately predicted the the measured earthquake motion records while having as an input only the ground motion record at an outcropping and the characteristics of the soil at the earthquake site. Based on the tests made, the developed model seems to behave very similarly to the R-O model, which, since all that distinguishes the models is the backbone curve definition, may mean that the spline formulation does not provide the desired increase in accuracy when compared with the R-O formulation.

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