Fock space and Fock-Toeplitz operators

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1 Fock space

The work on Fock space has a long list of contributions, see e.g. [1, 8, 14, 36].

1.1 Properties of the Fock space

To begin we define the Fock space and we prove some elementary properties. Consider the measure $d\mu(z) := e^{-|z|^2} dA(z)$, where dA is the Lebesgue area measure in $\mathbb{C}$. Let $1 \leq p < +\infty$. A measurable function $f : \mathbb{C} \to \mathbb{C}$ belong to the Fock space $\mathbb{F}^p(\mathbb{C})$ if f is entire and

$$\|f\|_p := \left(\int_{\mathbb{C}} |f(z)|^p \, d\mu(z) \right)^{1/p} < +\infty.$$

The conjugate exponent of p is denoted by p'. If $p = 1$, let $p' := +\infty$. If $1 < p < +\infty$, let p' be the unique complex number such that $1/p + 1/p' = 1$.

It follows from the Hölder inequality that if $1 \leq p \leq q < +\infty$ and if $f \in \mathbb{F}^q(\mathbb{C})$, then $\|f\|_p \leq \pi^{1/q'} \|f\|_q$. Therefore, we have that

$$\mathbb{F}^q(\mathbb{C}) \subseteq \mathbb{F}^p(\mathbb{C}) \subseteq \mathbb{F}^1(\mathbb{C}), \quad 1 \leq p \leq q < +\infty.$$

In the following theorem we show that the point-evaluation functionals are continuous on the Fock space. As a consequence, the Fock space is a *reproducing kernel Hilbert space*.

Theorem 1.1.2. Let $1 \leq p < +\infty$ and $z \in \mathbb{C}$. If $f \in \mathbb{F}^p(\mathbb{C})$, then

$$|f(z)| \leq C_{z,p} \|f\|_p, \tag{1}$$

where $C_{z,p} > 0$ is a constant which only depends on z and p. If $K \subseteq \mathbb{C}$ is a nonempty compact, then exist a constant $C_{K,p} > 0$, which only depends on K and p, such that

$$\sup_{z \in K} \{|f(z)|\} \leq C_{K,p} \|f\|_p. \tag{2}$$

We show that the Fock space is identified with a closed subspace of $L^p(\mathbb{C}, d\mu)$. Hence, we have the following proposition.

Theorem 1.1.4. If $1 \leq p < +\infty$, then $\mathbb{F}^p(\mathbb{C})$ is a Banach space.

The next proposition is helpful, for example to prove that the Taylor series of f converge weakly to f

Proposition 1.1.5. Let $1 < p < +\infty$ and f_n be a sequence in $\mathbb{F}^p(\mathbb{C})$. Then f_n weakly converge to $f \in \mathbb{F}^p(\mathbb{C})$ if and only if there exist $C \geq 0$ such that $\|f_n\|_p \leq C$ and $f_n \to f$, uniformly on the compacts subsets of $\mathbb{C}$.

The set of all polynomials in the complex variable z is denoted by $\mathbb{P}[z]$. It is clear that $\mathbb{P}[z]$ is contained in the Fock space. In the next section we will prove that $\mathbb{P}[z]$ is dense in $\mathbb{F}^2(\mathbb{C})$.

Proposition 1.1.7. If $1 \leq p < +\infty$, then $\mathbb{P}[z] \subset \mathbb{F}^p(\mathbb{C})$.

1.2 Fock Kernel and Projection

In this section we define a *reproducing kernel Hilbert space*. First, we check that the Fock space $\mathbb{F}^2(\mathbb{C})$ is a *reproducing kernel Hilbert space* and we show some elementary properties. Such properties reamin valid in any *reproducing kernel Hilbert space*, see e.g. [3, 34]. We explicitly compute the reproducing kernel in the Fock space, which is said to be the Fock kernel.

A space of functions H over a domain (open, connected and nonempty set) $U \subset \mathbb{C}$ with complex output is said to be a *reproducing kernel Hilbert space* if H is a Hilbert space and if the point-evaluations functionals are continuous for each $z \in U$. It follows from the Riesz Theorem that exist a unique $k : U \times U \to \mathbb{C}$ such that if $y \in U$, then

$$f(y) = \langle f(.), k(., y)\rangle \quad \text{and} \quad k(., y) \in H.$$

From Theorem 1.1.4, we know that the Fock space $\mathbb{F}^2(\mathbb{C})$ is a closed subspace of a Hilbert space. Acoording to the Theorem 1.1.2, the point-evaluations functionals are continuous on the Fock spaces. Therefore $\mathbb{F}^2(\mathbb{C})$ is a *reproducing kernel Hilbert space*. Let $z \in \mathbb{C}$. From the Riesz Theorem, we conclude that exist a unique κ_z such that

$$f(z) = \langle f, \kappa_z\rangle = \int_{\mathbb{C}} f(w)\overline{\kappa_z(w)}d\mu(w), \quad f \in \mathbb{F}^2(\mathbb{C}).$$

It's clear that the orthogonal of $\{\kappa_z\}_{z\in\mathbb{C}}$ is trivial. Therefore, $\{\kappa_z\}_{z\in\mathbb{C}}$ is dense in $\mathbb{F}^2(\mathbb{C})$. The Fock kernel is given by

$$K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad K(z, w) := \overline{\kappa_z(w)}.$$

The following properties hold in the Fock space.

Proposition 1.2.2. If $z, w \in \mathbb{C}$, then the Fock kernel respect the following properties:
(a) $\overline{K(z, .)} \in \mathbb{F}^2(\mathbb{C})$;
(b) $\overline{K(z, w)} = K(w, z)$;
(c) $K(z, z) = \|K(., z)\|_2^2 = \|\overline{K(z, .)}\|_2^2$;
(d) $K(z, w)$ is the unique reproduction kernel of the Fock space.

The orthogonal projection from $L^2(\mathbb{C}, d\mu)$ to $\mathbb{F}^2(\mathbb{C})$ is denoted by P and it is said to be the Fock projection. The Fock projection is a linear integral operator and its kernel is the Fock kernel.

Proposition 1.2.3. Let $f \in L^2(\mathbb{C}, d\mu)$. Then

$$Pf(z) = \int_{\mathbb{C}} K(z, w)f(w)d\mu(w), \quad z \in \mathbb{C}.$$

Using the Green formulas (see [14, page 77]), we calculate the inner product of a function $f \in \mathbb{F}^2(\mathbb{C})$ with z^n, for every $n \in \mathbb{N}_0$.

Proposition 1.2.4. Let $n \in \mathbb{N}_0$. If $f \in \mathbb{F}^2(\mathbb{C})$, then $\langle f, z^n\rangle = \pi f^{(n)}(0)$.

It follows from the previous Proposition and from the Taylor development that $\mathbb{P}[z]$ is dense in $\mathbb{F}^2(\mathbb{C})$. Moreover, the set of all monomials $z^n/\sqrt{\pi n!}$ ($n \in \mathbb{N}_0$) is an orthonormal base for $\mathbb{F}^2(\mathbb{C})$.

Proposition 1.2.6. Let e_n be a orthonormal base of $\mathbb{F}^2(\mathbb{C})$, with $n \in \mathbb{N}_0$. The Fock kernel is given by

$$K(z, w) = \sum_{j=0}^{\infty} \overline{e_j(w)}e_j(z), \quad z, w \in \mathbb{C}.$$

Corollary 1.2.7. The Fock kernel is explicitly given by $K(z, w) = e^{\overline{w}z}/\pi$, where $z, w \in \mathbb{C}$.

1.3 Minimization problems

The properties of a *reproducing kernel Hilbert space* allow us to study the following minimization problems.

Let $w, \alpha \in \mathbb{C}$. We will consider the following subset of $\mathbb{F}^2(\mathbb{C})$

$$M_{w,\alpha} := \left\{ f \in \mathbb{F}^2(\mathbb{C}) : f(w) = \alpha \right\}.$$

The set $M_{w,\alpha}$ is nonempty, since it contains the constant functions.

Proposition 1.3.1. Let $w, \alpha \in \mathbb{C}$. Then there exist a unique $f_0 \in M_{w,\alpha}$ such that $\inf_{f \in M_{w,\alpha}} \|f\|_2 = \|f_0\|_2$. Furthermore

$$f_0(z) = \alpha \frac{K(w,z)}{K(w,w)} = \alpha e^{\overline{w}z - |w|^2} \quad \text{and} \quad \|f_0\|_2 = |\alpha|\sqrt{\pi} e^{-\frac{|w|^2}{2}}.$$

Let $w_j, \alpha_j \in \mathbb{C}$, with $j = 0, 1, .., n$. Define

$$M := \left\{ f \in \mathbb{F}^2(\mathbb{C}) : f(w_j) = \alpha_j,\ j = 0, 1, .., n \right\}.$$

Note that M is nonempty, because there is a interpolation polynomial that respect the interpolation conditions and it belongs to the Fock space.

Proposition 1.3.2. Let $w_j, \alpha_j \in \mathbb{C}$, with $j = 0, 1, .., n$ and M defined above. Then exist a unique $F \in M$ such that $\inf_{f \in M} \|f\|_2 = \|F\|_2$. Moreover exist constants $c_j \in \mathbb{C}$ such that

$$F(z) = \sum_{j=0}^{n} c_j K(z, w_j). \tag{3}$$

2 Fock-Toeplitz operators

The Fock-Toeplitz operators have been studied in the last decades, e.g. [6, 8, 11, 23, 25].

2.1 Properties of the Fock-Toeplitz Operator

In this section we define the Fock-Toeplitz, Fock-Hankel and Weyl operators and we show some of its properties. Let (X, Ω, ν) be a measurable space with the $\sigma-$finite measure ν and let $g : X \to X$ be a measurable function. The multiplication operator M_g in $L^2(X, d\nu)$ is given by $M_g f := fg$, where $f \in L^2(X, d\nu)$.

The Fock-Toeplitz operator T_g with symbol g is the composition of the projection P and the multiplication operator M_g defined for each $f \in \mathbb{F}^2(\mathbb{C})$ such that $gf \in L^2(\mathbb{C}, d\mu)$, i.e.

$$T_g : \mathcal{D} \subset \mathbb{F}^2(\mathbb{C}) \to \mathbb{F}^2(\mathbb{C}), \quad T_g := PM_g,$$

where $\mathcal{D}$ is the set of function $f \in \mathbb{F}^2(\mathbb{C})$ such that $gf \in L^2(\mathbb{C}, d\mu)$. First, we study the continuity of the Fock-Toeplitz operator with symbols z and $\overline{z}$.

Proposition 2.1.2. If $f \in \mathbb{P}[z]$, then

$$T_{\overline{z}} f = \frac{1}{\pi} \frac{\partial}{\partial z} f.$$

It follows from the Proposition 1.2.4 that $\|T_{\overline{z}} z^k\|_2 = k\|z^{k-1}\|_2/\pi = \|z^k\|_2/\pi$, for every $k \in \mathbb{N}$. Then $T_{\overline{z}}$ is a bounded linear operator. On the other hand, we have that $\|T_z z^k\|_2 = (k+1)\|z^k\|_2$, for $k \in \mathbb{N}_0$. So T_z is not continuous. From now on we will only consider Fock-Toeplitz operators with essentially bounded symbols.

Proposition 2.1.3. Let g and g_k be an essentially bounded functions and let λ_k be a complex numbers, $k = 1, 2$. Then

(a) $T_{\lambda_1 g_1 + \lambda_2 g_2} = \lambda_1 T_{g_1} + \lambda_2 T_{g_2}$;
(b) $\|T_g\| \leq \|g\|_\infty$;
(c) $T_g^* = T_{\overline{g}}$.

Let $a \in \mathbb{C}$. The normalized Fock kernel is denoted by k_a and it is given by

$$k_a(z) := \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{e^{\overline{a}z - \frac{|a|^2}{2}}}{\pi}, \quad z \in \mathbb{C}. \tag{4}$$

Let φ_a be given by $\varphi_a := e^{\frac{|a|^2}{2} + 2iIm(\overline{a}z)}$. It is clear that φ_a an essentially bounded function. The Fock-Toeplitz operator with symbol φ_a is called the Weyl operator and it is denoted by W_a, i.e.

$$W_a : \mathbb{F}^2(\mathbb{C}) \to \mathbb{F}^2(\mathbb{C}), \quad W_a := T_{\varphi_a}. \tag{5}$$

The Weyl operator is explicitly given by $W_a f(z) = k_a(z) f(z-a)$, where $z \in \mathbb{C}$. We extend the Weyl operator to $L^2(\mathbb{C}, d\mu)$. The Weyl operator is unitary in $L^2(\mathbb{C}, d\mu)$ and its adjoint is given by $W_a^* f(z) = \sqrt{\pi} k_{-a}(z) f(z+a)$, where $z \in \mathbb{C}$.

Let g be an essentially bounded function. The Fock-Hankel operator with symbol g is defined by

$$H_g : \mathbb{F}^2(\mathbb{C}) \to L^2(\mathbb{C}, d\mu), \quad H_g := (I-P)M_g.$$

Note that, if $f \in \mathbb{F}^2(\mathbb{C})$, then $H_g f \in \mathbb{F}^2(\mathbb{C})^\perp$ and $M_g f = T_g f + H_g f$.

Proposition 2.1.4. Let g and g_k be an essentially bounded functions and let λ_k be a complex numbers, $k = 1, 2$. Then,

(a) $H_{\lambda_1 g_1 + \lambda_2 g_2} = \lambda_1 H_{g_1} + \lambda_2 H_{g_2}$;
(b) $\|H_g\| \leq \|g\|_\infty$;
(c) $H_g^* : L^2(\mathbb{C}, d\mu) \to \mathbb{F}^2(\mathbb{C}), \quad H_g^* = PM_{\overline{g}}(I-P)$;
(d) $T_{g_1 g_2} - T_{g_1} T_{g_2} = (H_{\overline{g}_1})^* H_{g_2}$.

The Fock-Toeplitz operator is uniquely defined by its symbol.

Proposition 2.1.6. If $g \in L^\infty(\mathbb{C}, d\mu)$, then $T_g = 0$ if and only if $g = 0$ almost everywhere.

Let H be a Hilbert space. The set of the linear continuous operators acting on H is denoted by $\mathcal{L}(H)$, and the set of linear compact operators in $\mathcal{L}(H)$ is denoted by $\mathcal{K}(H)$. To simplify notation, we write $\mathcal{K}$ instead of $\mathcal{K}(\mathbb{F}^2(\mathbb{C}))$. Let V_∞ denote the set of essentially bounded function having limit zero at infinity point, i.e.

$$V_\infty := \left\{ f \in L^\infty(\mathbb{C}, d\mu) : \lim_{|z| \to +\infty} f(z) = 0 \right\}.$$

The set of essentially bounded symbol such that the Fock-Toeplitz operator is compact is denoted by $\mathfrak{B}$, i.e.

$$\mathfrak{B} := \{ f \in L^\infty(\mathbb{C}, d\mu) : T_f \in \mathcal{K} \}.$$

Furthermore, we define

$$\Gamma := \{ f \in L^\infty(\mathbb{C}, d\mu) : H_f \in \mathcal{K} \}.$$

The sets V_∞, $\mathfrak{B}$ and Γ are closed in $L^\infty(\mathbb{C}, d\mu)$. V_∞ and $\mathfrak{B}$ are invariant for conjugation.

Since the set of functions with compact support is contained in $\mathfrak{B}$, then $V_\infty \subset \mathfrak{B}$.

2.2 Berezin Transform

In this section we define the Berezin transform and prove some of its properties. The Berezin transform is also study in others *reproducing kernel Hilbert spaces*, for example in the Bergman space, see e.g. [24], and the Hardy space, see e.g. [35].

The Berezin transform of a linear continuous operator A in $\mathbb{F}^2(\mathbb{C})$ is given by

$$\tilde{A} : \mathbb{C} \to \mathbb{C}, \quad \tilde{A}(a) := \langle A k_a, k_a \rangle. \tag{6}$$

Proposition 2.2.1. Let $A, A_k \in \mathcal{L}(\mathbb{F}^2(\mathbb{C}))$ and let λ_k be a complex numbers, with $k = 1, 2$. Then

(a) $\tilde{A} \in L^\infty(\mathbb{C}, d\mu)$;
(b) $\|\tilde{A}\|_\infty \leq \|A\|$;
(c) If $B = \lambda_1 A_1 + \lambda_2 A_2$, then $\tilde{B} = \lambda_1 \tilde{A}_1 + \lambda_2 \tilde{A}_2$.

The Berezin transform is injective.

Proposition 2.2.3. Let $A \in \mathcal{L}(\mathbb{F}^2(\mathbb{C}))$. Then $A = 0$ if and only if $\tilde{A} = 0$.

The Berezin transform of a function $f \in L^\infty(\mathbb{C}, d\mu)$ is defined by the Berezin transform of the Fock-Toeplitz operator with symbol f, i.e. $\tilde{f}(a) := \tilde{T}_f(a)$. The Berezin transform of a function f is explicitly given by

$$\tilde{f}(a) = \frac{1}{\pi} \int_{\mathbb{C}} f(z) e^{-|z-a|^2} dA(z).$$

Let $m \in \mathbb{N}$. The m-th iteration of the Berezin transform of the function f is denoted by $\tilde{f}^{(m)}$.

BC denote the set of bounded continuous functions, and C_0 denote the set of continuous functions that go to zero in the infinity, i.e. $C_0 := BC \cap V_\infty$. The sets BC and C_0 are closed and conjugate invariant. The normalized Fock kernel k_a is weakly convergent to 0, as $|a| \to \infty$.

Proposition 2.2.6. If $f \in \mathfrak{B}$, then $\tilde{f} \in C_0$.

If $A \in \mathcal{L}(L^2(\mathbb{C}, d\mu))$ or $A \in \mathcal{L}(\mathbb{F}^2(\mathbb{C}))$, then we weakly define respectively the operator

$$\hat{A} : L^2(\mathbb{C}, d\mu) \to L^2(\mathbb{C}, d\mu), \quad \hat{A} := \frac{1}{\pi} \int_{\mathbb{C}} W_a^* A W_a d\mu(a)$$

or

$$\hat{A} : \mathbb{F}^2(\mathbb{C}) \to \mathbb{F}^2(\mathbb{C}), \quad \hat{A} := \frac{1}{\pi} \int_{\mathbb{C}} W_a^* A W_a d\mu(a).$$

This new operator is related with the Berezin transform and the Fock-Toeplitz, Fock-Hankel or multiplication operator, as is indicated in the following propositions.

Proposition 2.2.7. If $A \in \mathcal{L}(\mathbb{F}^2(\mathbb{C}))$, then $\hat{A} = T_{\tilde{A}}$.

Corollary 2.2.8. If $f \in L^\infty(\mathbb{C}, d\mu)$, then $\hat{T}_f = T_{\tilde{f}}$.

Proposition 2.2.9. If $f \in L^\infty(\mathbb{C}, d\mu)$, then $\hat{M}_f = M_{\tilde{f}}$.

Corollary 2.2.10. If $f \in L^\infty(\mathbb{C}, d\mu)$, then $\widehat{H_f P} = H_{\tilde{f}} P$.

In the next proposition is given an estimative for the norm of a Fock-Toeplitz operator.

Proposition 2.2.12. If $f \in L^\infty(\mathbb{C}, d\mu)$, then

$$\|\tilde{f}\|_\infty \leq \|T_f\| \leq 2\|\widetilde{|f|}\|_\infty.$$

2.3 Compact semi commutator

In this section we characterize the largest set Q such that if $f, g \in Q$, then the semi commutator $T_f T_g - T_{gf}$ is compact, following [7]. For this characterization the set of functions such that its oscillation inside a ball of radius 1 goes to zero, as the center of the ball goes to the infinity plays a important role. The characterization of Q will be useful to study the character of Fredholm of a Fock-Toeplitz operator, see next section. Let Q be the set of essentially bounded function f such that H_f and $H_{\overline{f}}$ are compacts operators, i.e.

$$Q := \left\{ f \in L^\infty(\mathbb{C}, d\mu) : H_f, H_{\overline{f}} \in \mathcal{K} \right\} = \Gamma \cap \Gamma^*,$$

where $\Gamma^* := \left\{ \overline{f} : f \in \Gamma \right\}$. Acoording to *(d)* of the Proposition 2.1.4, we have the next proposition.

Proposition 2.3.1. If $f \in Q$, then $T_f T_g - T_{fg}$ and $T_g T_f - T_{gf}$ are compacts operators, for all $g \in L^\infty(\mathbb{C}, d\mu)$. Q is the largest closed subset of $L^\infty(\mathbb{C}, d\mu)$ such that if $f, g \in Q$, then $T_f T_g - T_{gf} \in \mathcal{K}$.

The commutator of two linear operator A and B is the operator $[A, B] := AB - BA$.

Proposition 2.3.2. If $f \in L^\infty(\mathbb{C}, d\mu)$, then $f \in Q$ if and only if $[M_f, P] \in \mathcal{K}(L^2(\mathbb{C}, d\mu))$.

We define ESV to be the set of essentially bounded function such that the oscillation inside a ball of radius 1 goes to zero, as the center of the ball goes to the infinity, i.e.

$$ESV := \left\{ f \in L^\infty(\mathbb{C}, d\mu) : \lim_{r\to+\infty} \sup_{\substack{|z-w|\leq 1\\ |z|\geq r}} |f(z) - f(w)| = 0 \right\}.$$

The set ESV is closed in $L^\infty(\mathbb{C}, d\mu)$ and invariant for conjugation. Next we give some examples of functions in ESV.

A function $g : \mathbb{C} \to \mathbb{C}$ is called a homogeneous function if there exist $g_\theta : S^1 \to \mathbb{C}$ (S^1 is the unit circle) such that $g(z) = g_\theta(z/|z|)$, with $z \in \mathbb{C}\setminus\{0\}$. The function g_θ is called the homogeneous part of g.

Proposition 2.3.3. Let $g \in L^\infty(\mathbb{C}, d\mu)$ a homogeneous function, with the homogeneous part g_θ. Then $g \in ESV$ if and only if g_θ is continuous.

We write that the function $f : \mathbb{C} \to \mathbb{C}$ has a limit in the infinity if there exist $c \in \mathbb{C}$ such that $f(\infty) := \lim_{|z|\to+\infty} f(z) = c$.

Proposition 2.3.4. Let $f \in L^\infty(\mathbb{C}, d\mu)$. If f has a limit in the infinity, then $f \in ESV$.

Example 2.3.5. $\{V_\infty + \lambda : \lambda \in \mathbb{C}\} \subset ESV$.

A function $f : \mathbb{C} \to \mathbb{C}$ is called radial if there exist $f_r : \mathbb{R}_0^+ \to \mathbb{C}$ ($\mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$) such that $f(z) = f_r(|z|)$, with $z \in \mathbb{C}$. The function f_r is called the radial part of f.

Proposition 2.3.6. Let $f \in L^\infty(\mathbb{C}, d\mu)$ a radial function, with the radial part f_r continuous differentiable. Then $f \in ESV$ if and only if $f_r'(\infty) = 0$.

Let $\Lambda(\epsilon) := \{f \in BC : |f(a) - f(b)| \leq \epsilon|a - b|\}$, where $\epsilon > 0$.

Theorem 2.3.10. The following conditions are equivalent:
(a) $f \in ESV$,
(b) $f - \tilde{f} \in V_\infty$,
(c) $f - \tilde{f}^{(m)} \in V_\infty$, for all $m = 1, 2, \ldots$,
(d) $f \in \bigcap_{\epsilon>0}(\Lambda(\epsilon)) + V_\infty$.

The next equality of sets follows from the Proposition 2.2.6 and the Theorem 2.3.10.

Proposition 2.3.11. $ESV \cap \mathfrak{B} = V_\infty$.

The set $Q \cap \mathfrak{B}$ is characterized, by the Berezin transform.

Proposition 2.3.12. $\Gamma \cap \mathfrak{B} = Q \cap \mathfrak{B} = \{f : |f|^2 \in \mathfrak{B}\}$.

Proposition 2.3.13. $V_\infty \subset Q \cap \mathfrak{B}$.

We show that ESV is inside Q, by the Proposition 2.3.2 and *(d)* of the Theorem 2.3.10.

Theorem 2.3.14. $ESV \subset Q$.

The next two results give us a characterization of Q.

Theorem 2.3.17.

$$\Gamma = ESV + Q \cap \mathfrak{B} = Q.$$

Proposition 2.3.18.

$$Q \cap \mathfrak{B} = \left\{ f \in L^\infty(\mathbb{C}, d\mu) : \widetilde{|f|^2} \in C_0 \right\}.$$

2.4 Character of Fredholm of the Fock-Toeplitz operators

Our in this section we prove a criterion for the Fredholm character of Fock-Toeplitz operator. We state some of the standard facts on C*-algebra theory, see e.g. [4, 15, 30], and on Fredholm Theory, see e.g. [10, 13].

The sets Q, ESV, $\mathcal{L}(H)$ and $BCESV := BC \cap ESV$ are C*-algebra, where H denote a Hilbert space. d have the respectively ideal $Q \cap \mathfrak{B}$, V_∞, $\mathcal{K}(H)$ and C_0 are ideals respectively of Q, ESV, $\mathcal{L}(H)$ and $BCESV := BC \cap ESV$.

For $A \subset L^\infty(\mathbb{C}, d\mu)$, the algebra generated by $\{T_f : f \in A\}$ is denoted $\tau(A)$. We prove that $\tau(V_\infty)$ and $\tau(Q)$ are irreducible C*-algebras.

Proposition 2.4.7.

$$\tau(V_\infty) = \mathcal{K} \subset \tau(Q)$$

Theorem 2.3.10 and 2.3.17 leads to the next proposition.

Proposition 2.4.8. The following C*-algebras are isomorphic $\tau(Q)/\mathcal{K}$, $Q/(Q \cap \mathfrak{B})$, ESV/V_∞ and $BCESV/C_0$.

For each Fredholm operator we define its Fredholm index by

$$ind\, T = dim\, Ker\, T - codim\, Im\, T.$$

Let $f \in BCESV$, $R > 0$ and $m > 0$ such that if $|z| \geq R$, then $|f(z)| \geq m$. The winding number of f is the number of loops of $[0,1] \ni t \mapsto f(re^{2\pi it})$ for any $r > R$ around the origin, and it is denoted by $index\, f$. The next theorem give us a necessarily and sufficient condition to a Fock-Toeplitz operator with symbol in $BCESV$ be a Fredholm operator.

Theorem 2.4.15. Let $f \in BCESV$. T_f is a Fredholm operator if and only if exist $m > 0$ and $R > 0$ such that $|f(z)| \geq m$, for all $z \in \mathbb{C}$ with $|z| \geq R$. Moreover $Ind\, T_f = -index\, f$.

3 Bargmann transform and localization operators

In this chapter, following [18, 28] we introduce the short time Fourier transform and give some of its properties. After, we introduce and study the Bargmann transform. The results presented can be found in [20, 22]. Finally, we introduce the Localization operators and give a relation with the Fock-Toeplitz operators.

3.1 Short time Fourier transform

Let $f, g \in L^2(\mathbb{R})$. We define the short time Fourier transform by

$$V_g f(x,y) := \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2i\pi ty}dt, \quad x, y \in \mathbb{R}.$$

The translation of a real function g by $x \in \mathbb{R}$ is denoted by τ_x, and is given by $\tau_x g(t) := g(t-x)$, with $t \in \mathbb{R}$. Let $y \in \mathbb{R}$. The modulation operator M_y of a real function is defined by the multiplication operator by the following function $e_y(t) := e^{2\pi iyt}$, where $t \in \mathbb{R}$.

Note if $f, g \in L^2(\mathbb{R})$, then

$$V_g f(x,y) = \langle f, M_y \tau_x g\rangle, \quad x, y \in \mathbb{R}. \tag{7}$$

It follows from the previous equality and the uniform continuity of τ_x and M_y with respect to x and y, respectively, that the short time Fourier transform is uniformly continuous.

Proposition 3.1.1. If $f, g \in L^2(\mathbb{R})$, then $V_g f$ is uniformly continuous in $\mathbb{R}^2$.

Let $\mathcal{S}$ denote the Schwartz space and let $\mathcal{F}$ denote the Fourier transform in $\mathcal{S}$. For every $f, g \in \mathcal{S}$, the short time Fourier transform respect that

$$V_g f(x,y) = \mathcal{F}(f\tau_x\overline{g})(y), \quad x, y \in \mathbb{R}. \tag{8}$$

Considering the equality above and properties of the Fourier transform we prove the next proposition. This will allow us to prove that a short time Fourier transform is bounded and to calculate its inverse.

Proposition 3.1.2. If $f_k, g_k \in L^2(\mathbb{R})$, with $k = 1, 2$, then $\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle\langle g_2, g_1\rangle$.

The short time Fourier transform is a continuous operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$.

Theorem 3.1.3. If $g \in L^2(\mathbb{R})$, then the short time Fourier transform with window g is a continuous operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$ and

$$\|V_g\| = \|g\|_2.$$

If $\|g\|_2 = 1$, then the short time Fourier transform is isometric between $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$.

It follows from the Theorem 3.1.3 and (7) that $M_y\tau_x g$ with $x, y \in \mathbb{R}$ is dense in $L^2(\mathbb{R})$, for every $g \in L^2(\mathbb{R})$ not null. We prove the inverse formula of the short time Fourier transform, by the Proposition 3.1.2 .

Proposition 3.1.5. Let $g, h \in L^2(\mathbb{R})$ be such that $< g, h > \neq 0$. Then, for every $f \in L^2(\mathbb{R})$ we have that

$$f(t) = \frac{1}{< h, g >}\int_{\mathbb{R}^2} V_g f(x, y) M_y\tau_x h(t)dxdy, \quad t \in \mathbb{R}. \tag{9}$$

3.2 Bargmann transform

In this section we show that the Bargamnn transform is a unitary operator from $L^2(\mathbb{R})$ to $\mathbb{F}^2(\mathbb{C})$. We argue by means of a relation between the Bargmann transform and the short time Fourier transform. The Bargmann transform of a function $f \in L^2(\mathbb{R})$ is given by

$$\mathcal{B}f(z) := \frac{2^{1/4}}{\pi^{1/2}}\int_{\mathbb{R}} f(t)e^{2\pi tz-(\pi t)^2-\frac{z^2}{2}}dt, \quad z \in \mathbb{C}. \tag{10}$$

Let $\phi(x) := 2^{1/4}e^{-(\pi x)^2}$. The relation between the short time Fourier transform and the Bargmann transform is the following.

Proposition 3.2.1. If $x, y \in \mathbb{R}$ and $z = x + iy$, then

$$\frac{1}{\sqrt{\pi}}V_\phi f(\frac{x}{\pi}, -y) = e^{ixy}\mathcal{B}f(z)e^{-\frac{|z|^2}{2}}, \quad f \in L^2(\mathbb{R}).$$

The following proposition is proved, by checking the Cauchy-Riemann equation.

Proposition 3.2.2. If $f \in L^2(\mathbb{R})$, then $\mathcal{B}f \in \mathbb{H}(\mathbb{C})$.

The next theorem follows from the Theorem 3.13, the Propositions 3.2.1 and 3.2.2.

Theorem 3.2.3. The Bargmann transform is a isometry from $L^2(\mathbb{R})$ to $\mathbb{F}^2(\mathbb{C})$.

We Prove that the Bargamnn transform is subjective. Hence we have the following theorem.

Theorem 3.2.5. The Bargmann transform is a unitary operator from $L^2(\mathbb{R})$ to $\mathbb{F}^2(\mathbb{C})$.

Note $\mathcal{B}^* = B^{-1}$. The inverse of the Bargmann transform is given by

$$\mathcal{B}^{-1}F(t) = \frac{2^{1/4}}{\pi^{1/2}}\int_{\mathbb{C}} F(z)e^{2\pi t\bar{z}-(\pi t)^2-\frac{\bar{z}^2}{2}-|z|^2}dA(z), \quad F \in \mathbb{F}^2(\mathbb{C}) \quad \text{and} \quad t \in \mathbb{R}.$$

3.3 Localization operators

Finally we relate the localization operators, which are presented below, and the Fock-Toeplitz operators. Let $h \in \mathbb{F}^2(\mathbb{C})$ and $f \in L^\infty(\mathbb{C}, d\mu)$. The localization operators with window h is the operator in the Fock space defined by

$$< L_f^{(h)} g_1, g_2 >= \frac{1}{\pi}\int_{\mathbb{C}} f(a) < g_1, W_a h >< W_a h, g_2 > dA(a), \quad g_1, g_2 \in \mathbb{F}^2(\mathbb{C}).$$

First, we prove that the localization operator is a linear bounded operator in the Fock space.

Proposition 3.3.1. If $h \in \mathbb{F}^2(\mathbb{C})$ and if $f \in L^\infty(\mathbb{C}, d\mu)$, then $\|L_f^{(h)}\| \leq \pi^2\|f\|_\infty\|h\|_2^2$.

We show a particular relation between the localization operators with constant window and the Fock-Toeplitz operators.

Proposition 3.3.2. If $f \in L^{\infty}(\mathbb{C}, d\mu)$, then $L_f^{(1)} = \pi^2 T_f$.

Let $BC^{\infty}(\mathbb{C})$ denote the set of complex functions that have bounded continuous partial derivation of any order. The partial derivation in order to z and $\overline{z}$ of the order $k \in \mathbb{N}_0$ is denoted by ∂^k and $\overline{\partial}^k$, respectivelly.

Theorem 3.3.3. If $h \in \mathbb{P}[z] \subset \mathbb{F}^2(\mathbb{C})$, then exist a differential operator $D = D^{(h)} \in \mathbb{P}[\partial, \overline{\partial}]$, such that

$$L_f^{(h)} = \pi^2 T_{Df},$$

for all $f \in BC^{\infty}(\mathbb{C})$.

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