Classification of Fiber Bundles over the Riemann Sphere

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Abstract

In this work we study mainly Riemann surfaces and vector Bundles. We also introduce many other structures that will be useful. On the second chapter we will study principal bundles, that are a more general case of fiber bundle. In the end, we focus on vector bundles that come from principal bundles with orthogonal or symplectic structure groups.

1 Vector Bundles

We start this exposition with the very basic concepts. The first one is a Riemann surface, which is a 2-dimensional real manifold where we can define holomorphic functions coherently.

For every point of a Riemann surface, we can find a neighborhood of it that is homeomorphic to an open set of $\mathbb{C}$. Furthermore, when two such neighborhoods overlap, we know that their homeomorphisms are related by an holomorphic function.

Example 1. The projective line $\mathbb{P}^1$, also known as the Riemann sphere, is defined as the quotient $(\mathbb{C}^2 \setminus \{(0,0)\})/\sim$, with $(x,y) \sim \lambda(x,y) = (\lambda x, \lambda y)$ for any $\lambda \in \mathbb{C}^*$. A point in $\mathbb{P}^1$ is denoted by $(x : y)$, when (x,y) is a representative of the class it defines through $\sim$.

We use two coordinate charts to cover $\mathbb{P}^1$. They are:

$$\begin{array}{cc} U_0 = \{(1:y) : y \in \mathbb{C}\} & U_1 = \{(x:1) : x \in \mathbb{C}\} \\ \phi_0 : U_0 \to \mathbb{C} & \phi_1 : U_1 \to \mathbb{C} \\ (1:y) \mapsto y & (x:1) \mapsto x \end{array}$$

We may also define n-dimensional complex manifold for greater dimensions (a Riemann surface corresponds to the case $n = 1$) in a similar way to a Riemann surface. The main difference is that n-dimensional complex manifold will be locally homeomorphic to an open set of $\mathbb{C}^n$.

Our main example of a complex manifold with dimension greater than 1 is a vector bundle. In a sense, this is a Riemann surface with a copy of $\mathbb{C}^n$ attached to each one of its points.

A vector bundle E has a smooth structure of a $(n+1)$-dimensional complex manifolds and a projection π onto a Riemann surface S. Moreover, for any point of S, there is a neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{C}^n$, and this homeomorphism preserves the projection onto S. When two such neighborhoods overlap, they differ by a linear automorphism on $\mathbb{C}^n$, for every point of S. In this context we call the vector spaces $\mathbb{C}^n$, fibers.

Example 2. When working on $\mathbb{P}^1$, using the standard cover $\{U_0, U_1\}$ of example 1, we may set a vector bundle with transition function $g_{01} = z^m$, where $z = \phi_0$ are the coordinates on U_0. This way, we obtain a line bundle that we denote by $\mathcal{O}(m)$.

We can obtain more examples of vector bundles through certain operations. We shall consider: the direct sum, the dual and the tensor product of vector bundles.

Example 3. Consider E and F vector bundles with rank m and n, respectively, and transition functions A_{ij} and B_{ij}, respectively, for a covering $\{U_i\}$ of S.

(1) The direct sum $E \oplus F$ is a vector bundle with rank $m+n$ and whose transition functions are given by:

$$C_{ij} := \begin{bmatrix} A_{ij} & \mathbf{0} \\ \mathbf{0} & B_{ij} \end{bmatrix}.$$

(2) The dual of E, is the vector bundle E^* with rank m, whose transition functions are given by:

$$A'_{ij} := (A_{ij}^{-1})^T.$$

(The superscript T is used for the transpose of a matrix.)

(3) The tensor product $E \otimes F$ is a vector bundle with rank mn, whose transition functions are given by:

$$C_{ij}(u_j \otimes v_j) := u_i \otimes v_i = (A_{ij} u_j) \otimes (B_{ij} v_j)$$

This defines $C_{ij}(x)$ for a basis of $E_x \otimes F_x$, so that we may define it everywhere by linearity.

1.1 Sheaves

To describe vector bundles we use another structures, which are the sheaves over S. As we aren't interested in discussing sheaves in all its generality, we shall regard a sheaf $\mathcal{F}$ as an assignment from every open set U of S to an abelian group $\mathcal{F}(U)$, which is commonly the functions with domain U. For the inclusion of sets $U \subset V$ we also consider morphisms between $\mathcal{F}(V)$ and $\mathcal{F}(U)$, which are commonly given by the restriction of a function to a smaller domain.

For the purpose of this work, the most important sheaf will be the sheaf of holomorphic sections of a vector bundle E over S, which is denoted $\mathcal{O}(E)$. A holomorphic section on $U \subset S$ is a holomorphic map from U to E which preserves the projection onto S.

Another useful tool to describe vector bundles and sheaves is cohomology. To construct the cohomology groups of a given sheaf $\mathcal{F}$ over S

we will use a general covering $\mathcal{U} = \{U_\alpha\}$. The most basic concept is a p-cochain, which is an assignment of every intersection of $p+1$ open sets $U_{\alpha_0}, \dots, U_{\alpha_p}$ to an element $g_{\alpha_0 \dots \alpha_p} \in \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_p})$. When the order of the sets is changed, we also change $g_{\alpha_0 \dots \alpha_p} = \mathrm{sgn}(\sigma) \cdot g_{\sigma(\alpha_0 \dots \alpha_p)}$ according to the sign of the permutation. The set of all p-cochains on the covering $\mathcal{U}$ is denoted by $C^p(\mathcal{U}, \mathcal{F})$.

We define the coboundary map $\partial : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ for each $g \in C^p(\mathcal{U}, \mathcal{F})$, by:

$$(\partial g)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{k=0}^{p+1} (-1)^k g_{\alpha_0 \dots \widehat{\alpha_k} \dots \alpha_{p+1}}$$

A particularity of the coboundary map is that $\partial^2 = 0$. Moreover, we define the sets $Z^p(\mathcal{U}, \mathcal{F})$, $B^p(\mathcal{U}, \mathcal{F}) \subset C^p(\mathcal{U}, \mathcal{F})$, $p \in \mathbb{N}$, as the kernel and image of ∂. As $\partial^2 = 0$, it follows immediately that $B^p(\mathcal{U}, \mathcal{F}) \subset Z^p(\mathcal{U}, \mathcal{F})$. Thus, the p-th cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ is well-defined:

$$\mathrm{H}^p(\mathcal{U}, \mathcal{F}) := \frac{Z^p(\mathcal{U}, \mathcal{F})}{B^p(\mathcal{U}, \mathcal{F})}$$

Furthermore, we can define a cohomology group independently of the covering, using successively finer coverings and taking the direct limit:

$$\mathrm{H}^p(S, \mathcal{F}) := \varinjlim_{\mathcal{U}} \mathrm{H}^p(\mathcal{U}, \mathcal{F})$$

1.2 Fundamental Results

Before stating some of the most important results for vector bundles, we introduce two integer valued invariants. They are the degree of a vector bundle E and the genus of a Riemann suface S. Intuitively, the degree concerns the sum of the multiplicities of the zeros of a section of E, when there exists any, and the genus concerns the number of tori on the connected sum of S, or simply the number of holes of S.

We now present the results.

Theorem 1 (Riemann-Roch)**.** *Let S be a compact connected Riemann surface with genus g and E a vector bundle over S. We have:*

$$\dim_{\mathbb{C}} \mathrm{H}^0(S, \mathcal{O}(E)) - \dim_{\mathbb{C}} \mathrm{H}^1(S, \mathcal{O}(E)) = \deg E + (1-g) \cdot \mathrm{rk}\, E$$

Theorem 2 (Serre's Duality)**.** *Let E be a vector bundle over S. We have:*

$$\mathrm{H}^1(S, E) \cong \mathrm{H}^0(S, E^* \otimes K)^*$$

The following result provides us with a decomposition of every vector bundle over $\mathbb{P}^1$, as line bundles, which are vector bundles with rank 1.

Theorem 3 (Grothendieck)**.** *For any vector bundle E with rank n over $\mathbb{P}^1$, there are integers $a_1, \dots, a_n$, unique up to permutation, such that E decomposes as*

$$E \cong \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_n).$$

2 Principal Bundles

We will now study a more general version of Grothendieck's Theorem that applies to principal bundles. A principal bundle E with structure group G over S is similar to a vector bundle, with the difference that the set of fibers may be other than $\mathbb{C}^n$, and there is an action of E that preserves the projection onto S.

We may also describe a principal bundle up to isomorphism using the elements of a cohomology group. However, to allow a more general structure group we extend definition the first cohomology group for non-Abelian groups, which is not an immediate observation.

When we consider the local trivializations of a principal bundle, we may associate it to the class of cohomology group $\mathrm{H}^1(S, \mathcal{O}_S(G))$ defined by its transition functions. As an example, we may see that, up to isomorphism, a vector bundle of rank n over a Riemann surface S is a class in $\mathrm{H}^1(S, \mathcal{O}_S(GL(n, \mathbb{C})))$.

2.1 Grothendieck's Theorem for Principal Bundles

Now, we would like to state Grothendieck's Theorem for principal bundles. For this, we shall consider principal bundles with structure G a complex reductive Lie group. Moreover, we shall consider H a Cartan group of G, N the normalizer of H inside G and W the Weyl group of G, which is the discrete group given by $G = N/H$.

Observe that conjugation by an element of N stabilizes $H \subset G$, so it defines a map in the classes $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(H))$. However, conjugation by an element inside H acts trivially in the classes of $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(H))$, so we can say that the Weyl group $W = N/H$ acts on $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(H))$ through conjugation.

The inclusion $H \hookrightarrow G$ induces a map $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(H)) \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(G))$. Furthermore, as $N \subset G$, conjugation with an element of N will act trivially in the classes of $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(G))$. So, the following map is well defined:

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(H))/W \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(G)) \tag{1}$$

Our main result is the following:

Theorem 4 (Grothendieck)**.** *The map defined in (1), induced by the inclusion of groups, is bijective.*

2.2 Particular Cases

In this section we consider the cases of principal bundles with structure groups $O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. Of course, through inclusion, these will be vector bundles. What differentiates these principal bundles from the general vector bundles is that these possess an holomorphic symmetric or antisymmetric (respectively) non-degenerate bilinear form on each fiber. We may call this type of forms quadratic or symplectic, respectively.

For both this structure groups we see that:

Theorem 5. *The maps induced by the inclusions* $O(n,\mathbb{C}) \hookrightarrow GL(n,\mathbb{C})$ *and* $Sp(2n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{C})$

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(O(n,\mathbb{C}))) \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(GL(n,\mathbb{C})))$$

and

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(Sp(2n,\mathbb{C}))) \to \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(GL(2n,\mathbb{C})))$$

are injective.

Furthermore, we can see that a vector bundle E, comes from a principal bundles with orthogonal structure group, if and only $E \cong E^*$. The same applies for vector bundles with even rank and the symplectic structure group.

As a quite unexpected corollary, one may see that a vector bundle with even rank, comes from a principal bundle with orthogonal structure group, if and only it also comes from one with symplectic structure group.

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