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Dynamics of perfect fluids in nonminimally coupled gravity

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To the memory of Carlos Duarte.

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Resumo

Este trabalho estuda uma teoria alternativa de gravidade que apresenta um acoplamento não-mínimo entre curvatura e matéria. Para obter este efeito, o sector da matéria da acção das teorias $f(R)$ é modificado para $f_2(R)\mathcal{L}_m$, onde f_2 é uma função genérica do escalar da curvatura, R. Mostra-se que esta adição leva a características novas e potencialmente interessantes, nomeadamente, uma modificação da lei de conservação do tensor energia-momento. Explora-se esta alteração no contexto dos fluidos perfeitos, e argumenta-se que as suas consequências podem ser relevantes tanto a nível estelar como galáctico. Relativamente ao último, introduz-se um mecanismo que simula os efeitos da matéria escura.

Alguns dos resultados apresentados nesta tese seguem o trabalho desenvolvido em [35].

Palavras-chave: Teorias modificadas da gravidade, Fluidos perfeitos, Matéria escura, Dinâmica Newtoniana modificada.

Abstract

This work studies an alternative theory of gravity in which a nonminimal coupling between curvature and matter is present. To achieve that, the matter sector of the action of $f(R)$ theories is modified to read $f_2(R)\mathcal{L}_m$, where f_2 is a generic function of the curvature scalar, R. It is shown that this addition leads to some new and potentially interesting features, most notably, a modification of the conservation law for the stress-energy tensor. This alteration is explored in the context of perfect fluid matter, and it is argued that its consequences may have relevance both at stellar and galactic level. Regarding the latter, a dark matter mimicking mechanism is introduced.

Some of the results presented in this thesis follow the work developed in [35].

Keywords: Modified theories of gravity, Perfect fluids, Dark matter, Modified Newtonian dynamics.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Nearly a century has passed since Einstein proposed his theory of General Relativity (GR) in 1915, and, so far, no experiment was performed that could clearly claim to have deprived it from the status of prime model of gravitational interaction [1, 2]. However, as more astrophysical and cosmological data is gathered everyday, it is starting to become apparent that the theory has some shortcomings. Indeed, the first puzzling observations date back to 1933, when Zwicky posed the "missing mass problem" for galaxy clusters [3, 4], although only 40 years later, through the work of Rubin and Ford [5, 6], would it receive proper attention from the scientific community. A satisfactory answer has been pending ever since. More recently, datasets from Cosmic Microwave Background Radiation (CMBR) and supernovae surveys seem to indicate that there is something amiss in the energy budget of the Universe, the so-called *dark energy*, which is also held responsible for the observed cosmological accelerated expansion [7, 8, 9]. To complicate things further, the theoretical grounds of General Relativity were shaken when no way was found to renormalize it [10, 11].

Facing this surprising picture of the Universe, one has to admit that our current theoretical framework seems somewhat limited and requiring further refinement. The simplest alternative that adequately fits the experimental observations (recently, this has been disputed, see [12] for details) is the concordance model or Λ-CDM (Λ-Cold Dark Matter), which is the combination of GR with a cosmological constant Λ, dark matter and inflation, the latter usually based on some scalar field called inflaton. Besides not explaining the origin of inflation or the nature of dark matter by itself, this approach is burdened with the well-known cosmological constant problems [13, 14], which make it more of an empirical fit to the data whose theoretical motivation can be regarded as quite poor.

Another way to look at the issues described above is to regard them as the signature of a modified theory of gravity. It is worth pointing out that this idea, which may seem rather radical at first, is not new: indeed, back in 1919, Weyl was the first to consider including higher order invariants to the theory's action functional [15]. Nowadays, the literature on the subject is quite vast, with TeVeS (Tensor-Vector-Scalar) and $f(R)$ theories establishing the paradigm [16, 17]. As to whether these alternatives should be taken seriously or not, Sotiriou and Faraoni [16] put it in the following way: "It is rather pointless to argue whether such a perspective would be better or worse than any of the other solutions already proposed. It is definitely a different way to address the same problems and, as long as these problems do not find a plausible, well accepted and simple solution, it is worth pursuing all alternatives.".

In this work, a different model is considered, one in which the matter sector of the action is modified by introducing a nonminimal coupling between the matter Lagrangian density and the curvature [23]. This proposal can be regarded as an extension of the typical $f(R)$ theories (where the linear scalar curvature term of the Einstein-Hilbert action is replaced by a more general function of R), though in general a joint treatment of both effects is too involved, and one resorts to the assumption that the gravitational sector of the action remains unaltered, focusing full attention on the modified matter term. This thesis deals with the dynamics of perfect fluids in the context of the abovementioned theory.

The study of perfect fluids is important because they play a fundamental role in almost every theory of gravitation, including GR. It is thought that a perfect fluid description of matter adequately fits a large number of astrophysical and cosmological scenarios, such as stars, galaxies, and the Universe itself, when considered at sufficiently large scales. It is by far the most used way to describe matter, and one of the two most used forms of the stress-energy tensor (the other being vacuum). For these reasons, any candidate theory of gravity must take into account this type of matter, and nonminimally coupled gravity in particular, because it reduces to $f(R)$ theory in vacuum, rendering the coupling irrelevant.

The structure of this thesis is as follows: chapter 2 introduces the general features of the model mentioned above, particularly those that have direct relevance to the rest of the work. The main results are presented in chapter 3, where the dynamics of perfect fluids in this model are discussed, following the ideas introduced in [35]. In the first place, the consequences for static, spherically symmetric distributions of matter are analyzed, the generalized hydrostatic equilibrium and Tolman-Oppenheimer-Volkoff equations are derived for the weak coupling limit, and a brief introduction to the strong coupling regime is given. Secondly, the case of nonstatic fluids is considered, it is shown that the Newtonian potential may be generalized in a rather straightforward way to include the effects of the nonminimal coupling, and a discussion about the applicability of this result to the simulation of galactic dark matter halos follows. Chapter 4 concludes this thesis.

Chapter 2

Nonminimal coupling between geometry and matter

2.1 General Relativity and $f(R)$ theories as motivation

As it is well-known, the field equations of GR can be derived, through the variational principle, from the Einstein-Hilbert action:

$$S_{EH} = \int [\kappa R + \mathcal{L}_m] \sqrt{-g} d^4x, \tag{2.1}$$

where $\mathcal{L}_m$ is the Lagrangian density of matter, R is the Ricci scalar, g is the metric determinant, and $\kappa = \frac{c^4}{16\pi G}$. From now on, one shall work with a system of units where $c = 1$. Deriving Einstein's field equations (EFE) from (2.1) can actually be accomplished using two different variational formalisms, *metric* and *Palatini*. In the first one, the connection is taken to be Levi-Civita's, and, as such, entirely defined by the metric. Variation of the action is then performed with respect to the metric tensor. In the latter, it is assumed that the connection defines a set of independent variables, and the action is varied with respect to both the metric tensor and this independent connection, yielding two sets of equations. This formalism further assumes that the matter part of the action does not depend on the independent connection. This assumption plays a crucial role in the theory: it implies that both the parallel transport and the covariant derivative must be defined using the Levi-Civita connection of the metric (as discussed in [16]), and enables the derivation of the EFE from an action linear in the gravitational sector[1]. Performing the variation, one ends up with the well-known set of equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor and the matter energy-momentum is defined, as usual, by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}. \tag{2.3}$$

The commonly called "$f(R)$ theories" generalize action (2.1) by replacing the linear term in R with a general function of the same quantity, that is, by considering

$$S = \int \left[\frac{1}{2} f(R) + \mathcal{L}_m\right] \sqrt{-g} d^4x. \tag{2.4}$$

[1] One can generalize this procedure by assuming that the Lagrangian density of matter depends on both the metric and the independent connection. This leads to the so-called *metric-affine* formalism. Like most generalizations, it includes enriched phenomenology at the cost of more challenging mathematical complexity. We shall not go into detail here, and, instead, refer the interested reader to [16], where the case of *metric-affine* $f(R)$ theory is discussed.

In this case, the two variational formalisms mentioned above yield different sets of field equations (although it is possible to show that the independent connection can be expressed solely in terms of the metric tensor, and, therefore, acts as some sort of auxiliary field). Consequently, when dealing with $f(R)$ gravity, one has to specify both the action and the variational formalism being used. In this work, only the metric formalism is considered[2].

Varying action (2.4) with respect to the metric gives

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \triangle_{\mu\nu}f'(R) = T_{\mu\nu}, \tag{2.5}$$

where $\triangle_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$ and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. One immediately realizes that the field equations are now fourth order in the metric, in general. The trace of eq. (2.5) is

$$f'(R)R - 2f(R) + 3\Box f'(R) = T. \tag{2.6}$$

These theories have received a significant deal of attention lately, mainly due to the possibility of explaining the origin of dark energy, and, consequently, the origin of the current accelerated expansion of the Universe. Although certainly an interesting alternative, we shall not go into further detail in this work. For comprehensive reviews on $f(R)$ theories see [16, 17, 18].

More recently, an extension of action (2.4) exhibiting not only a non-linear $f(R)$ term in the gravitational sector of the Lagrangian density, but also a nonminimal coupling (NMC) between $\mathcal{L}_m$ and the scalar curvature, R, has been proposed [23, 24]. The remaining of this chapter will be used to provide an overall view of this model. Naturally, and despite the fact that the theory is quite recent, there is already a significant amount of literature on the subject, and, therefore, it will not be possible to go into detail in every aspect. The main features and results of the model will be presented, and, whenever necessary, references to detailed papers will be given.

2.2 Action and Field Equations

Following the discussion above, one postulates an action of the form

$$S = \int \left[\frac{1}{2}f_1(R) + f_2(R)\mathcal{L}_m\right]\sqrt{-g}d^4x, \tag{2.7}$$

where $f_i(R)$ are arbitrary functions of the scalar curvature. Setting $f_2(R) = 1$ one recovers the usual $f(R)$ theories, while this choice with $f_1(R) = 2\kappa\,(R - 2\Lambda)$ originates the standard Einstein-Hilbert action (with cosmological constant Λ).

Varying action (2.7) with respect to the metric coefficients yields the field equations

$$\left(F_1 + 2F_2\mathcal{L}_m\right)R_{\mu\nu} - \frac{1}{2}f_1 g_{\mu\nu} = \triangle_{\mu\nu}\left(F_1 + 2F_2\mathcal{L}_m\right) + f_2 T_{\mu\nu}, \tag{2.8}$$

where one defines $F_i(R) = f_i'(R)$ for convenience, and the argument of the functions is omitted. Taking the trace of eq. (2.8), one obtains

$$\left(F_1 + 2F_2\mathcal{L}_m\right)R - 2f_1 = -3\Box\left(F_1 + 2F_2\mathcal{L}_m\right) + f_2 T. \tag{2.9}$$

One may write the field equations in a more "natural" way by making explicit the Einstein tensor,

$$G_{\mu\nu} = \frac{1}{2\kappa^{eff}}\left(T_{\mu\nu} + T_{\mu\nu}^{eff}\right), \tag{2.10}$$

where $\kappa^{eff} = \frac{H}{2f_2}$, $H \equiv F_1 + 2F_2\mathcal{L}_m$ and

$$T_{\mu\nu}^{eff} = \frac{1}{2}g_{\mu\nu}\left(\frac{f_1}{f_2} - \frac{R}{f_2}H\right) + \frac{\triangle_{\mu\nu}H}{f_2}. \tag{2.11}$$

[2] It has been shown that $f(R)$ theories using Palatini formalism exhibit several undesirable features, and their viability has been seriously called into question. For a review of the problems, see [16].

Notice that the general relativistic choice $f_1(R) = 2\kappa(R - 2\Lambda)$, $f_2(R) = 1$ yields $H = \kappa$, $\kappa^{eff} = \kappa$ and $T^{eff}_{\mu\nu} = -2\kappa\Lambda g_{\mu\nu}$. Although this way of writing the field equations is rather suggestive because of its resemblance to the EFE with an effective matter source term and an effective coupling constant, one should be careful in interpreting the physical meaning of these: note, for example, that while in common $f(R)$ theories $T^{eff}_{\mu\nu}$ can be interpreted as a curvature or geometric contribution to the source term (in the sense that it has no direct contribution from matter fields), this is not the case anymore when a NMC is present, since there is also a dependence on $\mathcal{L}_m$. Moreover, even though the definition of a κ^{eff} is useful in $f(R)$ theories because it helps explain the Dolgov-Kawasaki instability (see [19, 20] for a derivation of the instability condition, and [21] for a physical interpretation), it is still unclear whether its definition is physically justified in NMC gravity. Nevertheless, it will be shown in chapter 3 that a connection with a possible violation of the Strong Equivalence Principle may exist.

The introduction of a second arbitrary function of R in the action adds greater freedom to the theory. Most of the results derived in this work are valid for the general form of the model, mainly because one of its most striking features, the non-conservation of $T_{\mu\nu}$ (discussed in the next section), depends only on the function $f_2(R)$. However, when necessary, the assumption $f_1(R) = 2\kappa R$ will be made. This is because a joint study of both $f_1(R)$ and $f_2(R)$, although desirable, seems unattainable at the moment, as the mathematical treatment would be too involved. Since this thesis intends to study the effects of a NMC, it seems reasonable to fix $f_1(R)$ and work only with $f_2(R)$. Making that assumption, the field equations (2.8) reduce to

$$\left(1 + \frac{F_2}{\kappa}\mathcal{L}_m\right) R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\kappa}\triangle_{\mu\nu}\left(F_2\mathcal{L}_m\right) + \frac{1}{2\kappa}f_2 T_{\mu\nu}, \tag{2.12}$$

and the trace reads

$$\left(1 - \frac{F_2}{\kappa}\mathcal{L}_m\right) R = \frac{3}{\kappa}\Box\left(F_2\mathcal{L}_m\right) - \frac{1}{2\kappa}f_2 T. \tag{2.13}$$

2.3 Conservation equation for $T_{\mu\nu}$, extra force, and the choice of $\mathcal{L}_m$

By now the reader has probably noticed one intriguing feature of this model: the existence of terms containing $\mathcal{L}_m$ in the field equations. Indeed, in both GR and $f(R)$ theories (as well as most other alternative gravity candidates) the matter contribution to the equations appears solely through the stress-energy tensor $T_{\mu\nu}$. This raises the question 'What form should the Lagragian density of matter, $\mathcal{L}_m$, take?'. Before answering it, two additional features of the theory will be presented that should provide a better understanding of the importance of this choice.

Taking the covariant derivative of the field equations (2.8) one obtains, after some manipulation (the details are presented in the Appendix A)

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2}\left[g^{\mu\nu}\mathcal{L}_m - T^{\mu\nu}\right]\nabla_\mu R. \tag{2.14}$$

This equation, which contrasts starkly with the usual $\nabla_\mu T^{\mu\nu} = 0$ of GR and $f(R)$ theories, is certainly one of the most interesting features of the model, and its dependence on the function f_2, but not on f_1, makes it a prime candidate to study the effects of a NMC without the need to consider the consequences of a nonlinear curvature term in the geometric part of the action. Physically, this equation has been interpreted as an exchange of energy and momentum between curvature and matter [23]. More specifically, in the context of the equivalence with a scalar-tensor theory (with two scalar fields, $\phi = R$ and $\psi = \mathcal{L}_m$, see [25]), it displays that same exchange between the matter and curvature fields [26]. The same feature is observed when one performs a conformal transformation on the metric, typically from the *Jordan* to the *Einstein* frame, where matter is coupled to the geometric conformal factor [17], although in this case it has a more fundamental status, as it is not possible, in general, to find a conformal transformation that makes the coupling between matter and curvature minimal [22].

Since the conservation law for $T_{\mu\nu}$ can be used to derive the matter equations of motion, one expects that the nonzero term on the RHS of eq. (2.14) may modify those equations. Naturally, different forms of $T_{\mu\nu}$ yield different expressions. In most applications, two choices are particularly important: vaccum, $T_{\mu\nu} = 0$, and perfect fluids. Since in the former the coupling between curvature and matter is irrelevant, only the perfect fluid case will be considered. Describing that matter type by the usual stress-energy tensor

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}, \tag{2.15}$$

where ρ and p are, respectively, the energy density and the pressure of the fluid, and u^μ is the four-velocity, obeying $u^\mu u_\mu = -1$, it can be shown that the equation of motion for a fluid element is

$$\frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta = f^\nu, \tag{2.16}$$

where τ is the affine parameter,

$$f^\nu = \frac{1}{\rho + p}\left[\frac{F_2}{f_2}\left(\mathcal{L}_m - p\right)\nabla_\mu R - \nabla_\mu p\right] h^{\mu\nu}, \tag{2.17}$$

and the projection operator, $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, was introduced. The details of the derivation are presented in the Appendix A. Regarding eq. (2.17), notice that the second term on the RHS is just the usual pressure gradient acceleration. However, the first represents an extra force per unit mass that arises from the existence of a NMC between matter and geometry, and, like eq. (2.14), it is a distinctive feature of the model. Most of the results derived in this work follow from these two equations.

The previous discussions on the conservation law of $T^{\mu\nu}$ and on the equation of motion for a perfect fluid in NMC gravity have served to emphasize the importance that the choice of $\mathcal{L}_m$ has in the model. Indeed, eq. (2.17) predicts the existence of an additional acceleration which depends on the Lagrangian density of matter. This seems rather odd, as there are several choices for $\mathcal{L}_m$ that yield the stress-energy tensor (2.15): Schutz [36] argued in 1970 that $\mathcal{L}_m = p$ is a "natural" choice. Three years later, Hawking and Ellis [37] showed that $\mathcal{L}_m = -\rho$ could also be used. Finally, Brown [38] settled the issue in 1993 by showing that both choices (and, in fact, many others) are admissible. In the case of NMC gravity, this degeneracy must be broken, or otherwise it would imply different accelerations from supposedly equivalent choices of $\mathcal{L}_m$. Indeed, it has been shown that there is a rather natural way out of this problem. A brief sketch of the reasoning will be presented. For a detailed account, see [26].

According to Ref. [38], the matter action of a perfect fluid may be written in the quite general form

$$S_m = \int \left[-\sqrt{-g}\rho + J^\mu V_\mu + B^\mu_{;\mu}\right] d^4x, \tag{2.18}$$

where J^μ is the flow vector of the particle number density, V_μ is a covariant vector (its particular form is not important here. See Ref. [38] for details), and $B^\mu_{;\mu}$ is a surface term, responsible, in the case of GR, for the equivalence between the various options of $\mathcal{L}_m$. Indeed, it can be shown that different choices of $B^\mu_{;\mu}$ ultimately yield different Lagrangian densities of matter [38]. However, there is one very important point to be made: matter actions containing Lagragian densities such as $\mathcal{L}_m = p$ or $\mathcal{L}_m = -\rho$ are, in fact, "on-shell" representations[3] of the original matter action (2.18). In GR (and in $f(R)$ theories) $\mathcal{L}_m$ appears in the equations solely through $T_{\mu\nu}$, and, as such, this distinction is not particularly important. By contrast, when one has a NMC (that is, $F_2 \neq 0$), the terms of $\mathcal{L}_m$ that are nonminimally coupled do appear in the field equations. The question is then "To which terms of the original action (2.18) should the NMC be applied?" (it can be shown that the NMC does not affect the "on-shell" representations [26]). Different choices yield different field equations, and, consequently, different forms of eq. (2.17). As an example, suppose one applies the NMC to the first two terms of (2.18), that is, $\mathcal{L}_m^{NMC} = -\rho + \frac{J^\mu V_\mu}{\sqrt{-g}}$, where $\mathcal{L}_m^{NMC}$ is the part of the lagrangian density of matter that is nonminimally coupled. Then, this is the term that appears in the field equations, independently of its "on-shell" representation.

The discussion above points out that there is, in fact, additional freedom in this model, at least when dealing with perfect fluids: besides choosing the functional form of f_2 (and, possibly, f_1), one also has to specify which terms of the original action should be coupled to the nonminimal curvature term. Arguably,

[3] An "on-shell" representation is one that is correct only when a set of equations (in this case, the fluid equations of motion derived from (2.18)) holds. Take, for example, the function $f(x, y) = x^2 + y^2$. If one imposes that the variables x and y must be related by $x^2 + y^2 = 1$, then the "on-shell" representation of f would be $f = 1$.

the "natural" choice is to couple only the first term of action (2.18), that is, take $\mathcal{L}_m^{NMC} = -\rho$ and postulate a nonminimally coupled matter action of the form

$$S'_m = \int \left[-\sqrt{-g} f_2(R) \rho + J^\mu V_\mu + B^\mu_{;\mu} \right] d^4x. \tag{2.19}$$

This is motivated by two reasons. The first is simplicity: if one couples the nonminimal curvature term to $J^\mu V_\mu$, for example, it may happen that the field equations contain terms whose physical interpretation is unclear, as is the case of some fields present in the vector V_μ. On the other hand, the term $-\rho$ is always well-defined in the context of perfect fluids. The second is related to the motivation of the model: when it was first introduced in [23], the idea was to insert a NMC between curvature and the minimally coupled terms of $\mathcal{L}_m$. From action (2.18), one sees that $\mathcal{L}_m = -\rho + \frac{J^\mu V_\mu}{\sqrt{-g}} + \frac{B^\mu_{;\mu}}{\sqrt{-g}}$, which shows that only the first term is minimally coupled. Additionally, this choice has another desirable feature: by assuming $\mathcal{L}_m^{NMC} = -\rho$, the "on-shell" representation of action (2.19) becomes rather irrelevant, as the energy density will remain unchanged by "on-shell" modifications (unlike J^μ or V_μ). Therefore, the choice $\mathcal{L}_m = -\rho$ (where the superscript NMC was dropped for convenience) will be used for the rest of this work, and eqs. (2.14;2.16;2.17) read

$$\nabla_\mu T^{\mu\nu} = -\frac{F_2}{f_2} \left[g^{\mu\nu} \rho + T^{\mu\nu} \right] \nabla_\mu R, \tag{2.20}$$

$$\frac{du^\nu}{d\tau} = f^\nu - \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta, \tag{2.21}$$

$$f^\nu = -\frac{1}{\rho + p} h^{\mu\nu} \nabla_\mu p - \frac{F_2}{f_2} h^{\mu\nu} \nabla_\mu R. \tag{2.22}$$

Notice that the extra force is now independent of the energy-matter distribution. This characteristic will be important later, when discussing the generalization of the Newtonian gravitational potential.

2.4 Applications of the model

Since its introduction a few years ago [23, 24], this theory has been applied to a wide range of scenarios, often suggesting new and intriguing phenomenology. However, as stated previously, one cannot go into detail in every aspect, since that would be time and space-consuming, and, ultimately, unnecessary to the main subject of this work, introduced in the next chapter. Instead, a list of its most interesting applications follows, with references to detailed work on the subject:

- Introducing an extra force in the equation of motion of perfect fluids [23].
- Consistency with experimental values of at least some Parametrized post-Newtonian (PPN) parameters [25].
- Clarifying the role of the Lagrangian density of matter in theories exhibiting a NMC between matter fields and curvature, as well as breaking the equivalence of different choices for $\mathcal{L}_m$ present in GR and $f(R)$ theories [26].
- New phenomenology in the dynamics of stellar equilibrium brought about by a generalized Tolman-Oppenheimer-Volkoff (TOV) equation for the linear case $f_2(R) = 1 + \lambda R$ and the choice $\mathcal{L}_m = p$ [27].
- Mimicking dark matter in galaxies and clusters of galaxies [28, 29].
- Mimicking dark energy, the cosmological constant and the accelerated expansion of the Universe [30, 31].
- Natural conditions for preheating in inflationary models [32].
- Introducing the possibility of traversable wormholes and time machines, through the violation of generalized energy conditions, without resorting to exotic forms of matter [33, 34].

Finally, it is worth pointing out that, at the moment of writing, no experimental test had definitively ruled out the existence of a NMC between matter and curvature.

Chapter 3

Dynamics of perfect fluids in nonminimally coupled gravity

The study of perfect fluids is of great importance in geometric theories of gravity. It is thought that they describe quite accurately the matter content of several cosmological and astrophysical situations, such as stars, galaxies and the Universe, when considered on a cosmological scale (~ 100 Mpc). Moreover, both the cosmological constant and quintessence scenarios may be mapped into perfect fluid descriptions [?]. As such, and since one is interested in studying the impact of a NMC between matter and geometry, which excludes the vacuum scenario, it seems quite natural to consider perfect fluid matter distribution. That shall be the case in what follows.

This chapter contains the main results of this thesis. Some of them are compiled in Ref. [35].

3.1 Static, spherically symmetric matter distributions

The case of static, spherically symmetric fluid distributions derives its importance from the study of stars, which are adequately described as self-gravitating objects. In most situations, the gravitational fields involved are so weak that GR can be ignored. However, there are notable exceptions, such as neutron or supermassive stars. The consequences that a nonminimal coupling between matter and curvature can have on stellar equilibrium are analyzed in this section.

A generalized equation of hydrostatic equilibrium

One starts with the "standard" spherically symmetric metric

$$\begin{aligned} g_{tt} &= -B(r), & (3.1)\\ g_{rr} &= A(r), & (3.2)\\ g_{\theta\theta} &= r^2, & (3.3)\\ g_{\phi\phi} &= r^2 \sin^2\theta, & (3.4)\\ g_{\mu\nu} &= 0 \quad \text{for } \mu \neq \nu, & (3.5) \end{aligned}$$

and the perfect fluid stress-energy tensor (repeated here for convenience)

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}. \tag{3.6}$$

Since the fluid is at rest, the velocity four-vector has components

$$\begin{aligned} u_r = u_\theta = u_\phi &= 0, & (3.7)\\ (u_t)^2 = \frac{1}{-g^{tt}} &= B(r). & (3.8) \end{aligned}$$

The last equality follows from $g^{\mu\nu}u_\mu u_\nu = -1$.

The generalized equation of hydrostatic equilibrium can be derived using eqs. (2.21;2.22), repeated here in a combined way

$$u^\mu \nabla_\mu u^\nu = -\frac{1}{\rho + p} h^{\mu\nu} \nabla_\mu p - \frac{F_2}{f_2} h^{\mu\nu} \nabla_\mu R. \tag{3.9}$$

Before writing the equation explicitly, it is convenient to write the extra acceleration term, $-\frac{F_2}{f_2} h^{\mu\nu}\nabla_\mu R$, in a different form. To do so, recall that R, the curvature scalar, is a function of four spacetime coordinates, here denoted as x^0, x^1, x^2, x^3. On the other hand, the scalar coupling function, f_2, receives only one argument, that is, $f_2 : \mathbb{R} \to \mathbb{R}$. However, for some purposes it is better to consider f_2 as a function of the spacetime coordinates, which can be achieved through the composition $f_2\left(x^0, x^1, x^2, x^3\right) = f_2(\cdot) \circ R\left(x^0, x^1, x^2, x^3\right)$. Using the chain rule, it is clear that $\nabla_\mu f_2 = \frac{df_2}{dR}\nabla_\mu R = F_2 \nabla_\mu R$, and the extra term can be written as $-\frac{F_2}{f_2} h^{\mu\nu}\nabla_\mu R = -h^{\mu\nu}\frac{\nabla_\mu f_2}{f_2} = -h^{\mu\nu}\nabla_\mu \log|f_2|$.

Taking $\nu = r$ (the remaining choices yield trivial $0 = 0$ relations), eq. (3.9) reads

$$\frac{B'(r)}{B(r)} = -\frac{2}{\rho + p} p'(r) - \frac{2}{f_2(r)} f_2'(r), \tag{3.10}$$

where the prime represents differentiation with respect to the radial coordinate r. This generalizes the usual hydrostatic equilibrium equation of GR [39] by adding the term $-\frac{2}{f_2(r)} f_2'(r)$ to the RHS.

It is worth noting that the extra term bears a strong resemblance to the common pressure gradient acceleration. Indeed, one is tempted to write $f_2(r) = \rho_{NMC}(r) + p_{NMC}(r)$ with $\rho'_{NMC}(r) = 0$, and interpret the coupling as a sort of "geometric fluid" with constant energy density (although this last property seems to have no physical meaning). Despite not facilitating the mathematical treatment, this view is rather suggestive, and it will be useful in several occasions.

In GR, the pressure gradient, whatever its origin may be, plays a vital role, because it prevents the gravitational collapse of the star. Indeed, even when one is dealing with Newtonian stars, for which $\rho \gg p$, it is necessary to take into account $p'(r)$. However, the introduction of a NMC brings new dynamics to the stellar equilibrium. Even though one expects tiny corrections in regions where the curvature is very faint (discussed in detail next section), there is no *a priori* reason to believe the same holds in other situations. If the RHS of eq. (3.10) is dominated by the coupling term, one should not exclude the possibility of having stars supported by a "geometric pressure". More importantly, it may prevent the formation of singularities, if this pressure is enough to avoid complete collapse, and hint at a behaviour expected in a quantum gravity theory. These hypotheses are currently under study, and no concluding results have been found so far. As such, they will not be discussed further in this work, although the mathematical structure for their treatment will be derived later on.

Finally, before proceeding to a detailed discussion of the weak coupling limit, one explores the relation between f_2 and the gravitational redshift. Integrating eq. (3.10) from a generic value of the radial coordinate r to the radius of the star, R_s, yields the equation

$$B(r) = \frac{B(R_s)\left[f_2(R_s)\right]^2}{\left[f_2(r)\right]^2} \exp\left[2\int_r^{R_s} \frac{p'(s)}{\rho(s) + p(s)} ds\right]. \tag{3.11}$$

Since one has vacuum for $r > R_s$, the boundary conditions read $B(R_s) = 1 - \frac{2GM}{R_s}$ and $f(R_s) \simeq 1$, where M is the mass of the star[1]. The gravitational redshift of radiation emitted at $r \leqslant R_s$ and observed by someone at $R_e > R_s$ is

$$1 + z_{NMC}(r) = \left[\frac{B(R_e)}{B(r)}\right]^{\frac{1}{2}} = f_2(r)\left[\frac{1 - \frac{2GM}{R_e}}{1 - \frac{2GM}{R_s}}\right]^{\frac{1}{2}} \exp\left[-\int_r^{R_s} \frac{p'(s)}{\rho(s) + p(s)} ds\right]. \tag{3.12}$$

[1] Two remarks are in order here: first, the boundary condition on $B(r)$, which follows from a comparison with GR, does not take into account the effects that the NMC may have (for example, in the weak coupling limit, discussed in the next section, M should be substituted by an effective mass, M^*). However, that does not influence the argument in a significative way. Second, admitting $f(R_s) \simeq 1$ does not mean that $f_2(R = 0) = 1$. When dealing with very small densities (and, hence, very small curvatures), one must be particularly careful because, contrary to what may be thought, choosing the correct behaviour of f_2 as R (the curvature scalar) approaches 0 is a very intricate and nontrivial matter. Having said that, taking $f_2 \simeq 1$ for stellar systems is well justified.

At the surface of the star there is no difference between NMC gravity and GR. However, if the emission happens at $r < R_s$, the measured value of redshift should be different from the general relativistic expectation. In fact,

$$\frac{1 + z_{NMC}(r)}{1 + z_{GR}(r)} = f_2(r). \tag{3.13}$$

If one can probe radiation from the interior of stars (possibly even the Sun) with sufficient precision, this formula may be very useful in constraining the model.

The weak coupling limit: a generalized TOV equation

Although the presence of a "strong" coupling may generate new and intriguing phenomena, it seems likely that for most applications one can assume only a "weak" coupling, and work with the corresponding equations. That is the subject of this section.

One starts by defining what is meant by "weak" coupling: looking at eq. (3.10), one concludes that the contribution of the NMC should be small if the derivative of the coupling function at a certain point r is negligible compared to the value of the function at the same point. Mathematically, this means that $rf_2'(r) \ll f_2(r)$, where the radial coordinate was added to make the relation dimensionless. Alternatively, one may write $RF_2(R) \ll f_2(R)$. Notice, however, that this does not necessarily imply $f_2 \simeq 1$. Indeed, any multiplicative constant is irrelevant for the inequalities above.

An important consequence of this assumption is that one may now neglect the troublesome terms containing F_2 in eqs. (2.12;2.13), which can then be simplified to (the choice $f_1(R) = 2\kappa R$ will be used when needed for the rest of this work)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \simeq 8\pi G f_2 T_{\mu\nu}, \tag{3.14}$$

$$R \simeq -8\pi G f_2 T, \tag{3.15}$$

where the gravitational constant G was made explicit. Defining $G_{eff} = Gf_2(r)$, eq. (3.14) is identical to the EFE with a "running" gravitational constant. Since $\frac{dG_{eff}}{dr} = G\frac{df_2}{dr}$ (or $\frac{dG_{eff}}{dR} = G\frac{df_2}{dR}$), the NMC may be related to a violation of the Strong Equivalence Principle, gauging it in a rather straightforward manner. Under this picture, the weak coupling condition essentially means that the violation is small. Moreover, notice that $\frac{f_2'(r)}{f_2(r)} = \frac{G_{eff}'(r)}{G_{eff}(r)}$, and so the added term in the equation of hydrostatic equilibrium may be related to the spatial variability of the gravitational constant.

Making use of eqs. (3.1)-(3.8), one finds that

$$\frac{\left(R_{tt} - \frac{1}{2} g_{tt} R\right)}{B} = \frac{1}{r^2} - \frac{1}{r^2 A} + \frac{A'}{rA^2}, \tag{3.16}$$

$$\frac{T_{tt}}{B} = \rho. \tag{3.17}$$

Eqs. (3.16)-(3.17), together with the tt component of eq. (3.14), lead to

$$\left(\frac{r}{A}\right)' = 1 - 8\pi G f_2 \rho r^2. \tag{3.18}$$

Integrating this equation from 0 to r, and assuming that $A(0)$ is finite and nonzero, yields

$$A(r) = \left[1 - \frac{2G_{eff} M^*(r)}{r}\right]^{-1}, \tag{3.19}$$

where

$$M^*(r) = M(r) - \frac{1}{f_2(r)} \int_0^r f_2'(s) M(s) ds, \tag{3.20}$$

$$M(r) = 4\pi \int_0^r \rho(s) s^2 ds. \tag{3.21}$$

Combining the $\theta\theta$ component of (3.14) with eqs. (3.10;3.15;3.19), one has

$$1-\left[1-\frac{2G_{eff}M^*(r)}{r}\right]\left[1-\frac{rp'(r)}{\rho+p}-\frac{rf_2'(r)}{f_2}\right]-\frac{G_{eff}M^*(r)}{r}=-4\pi G_{eff}r^2p, \tag{3.22}$$

which can be rearranged into

$$p'(r)=-\frac{G_{eff}}{r^2}\left[\rho(r)+p(r)\right]\left[4\pi r^3p(r)+M^*(r)\right]\left[1-\frac{2G_{eff}M^*(r)}{r}\right]^{-1}-\left[\rho(r)+p(r)\right]\frac{G'_{eff}(r)}{G_{eff}}. \tag{3.23}$$

This is the central result of the weak coupling limit. It states that under the condition $rf_2'(r)\ll f_2(r)$, i.e., if the effects of the coupling are small, the general relativistic Tolman-Oppenheimer-Volkoff equation approximately holds, but with an effective gravitational constant, G_{eff}, an effective mass within a sphere of radius r, $M^*(r)$, and an added term related to the spatial variation of G_{eff}.

Although the above equation is only suitable for a perturbative regime, there are a couple of facts worth being mentioned.

Firstly, the derivative of the effective mass term, $M^*(r)$, reads

$$\left[M^*(r)\right]'=4\pi r^2\rho(r)-\frac{f_2'(r)}{f_2(r)}M^*(r). \tag{3.24}$$

This suggests the definition of an effective density, ρ_{eff}, given by

$$\rho_{eff}=\rho-\frac{f_2'}{f_2}\frac{M^*(r)}{4\pi r^2}\simeq\rho-\frac{f_2'}{f_2}\frac{M(r)}{4\pi r^2}, \tag{3.25}$$

where the terms quadratic in $\frac{f_2'}{f_2}$ were neglected in the last step. Eq. (3.25) suggests that the gravitational effects one perceives are related not only to visible matter, enclosed in ρ, but also to a geometric correction brought about by the existence of a NMC, which may explain the hypothetical presence of dark matter in stars (the subject of dark matter mimicking in this theory will be treated more thoroughly in the last part of this chapter). Moreover, eqs. (3.20) and (3.25) provide a possible way to test the model, because although gravitationally one should feel the effects of $M^*(r)$ and ρ_{eff}, the contribution of visible matter is contained in $M(r)$ and ρ, and, as such, it may be possible to separate the effects of a NMC (if any).

Secondly, the last term on the RHS of eq. (3.23) may play a significant role in the weak coupling limit of the model. This is because the corrections to the remaining terms should average out. To illustrate this point, consider the case of a Newtonian star, $p\ll\rho$, $4\pi r^3p\ll M(r)$ and $\frac{2GM(r)}{r}\ll 1$, with a coupling function of the form $f_2(r)=e^{kr}$, with $kr\ll 1$. Additionally, $G_{eff}\simeq G$ and $M^*(r)\simeq M(r)$. These assumptions lead to an approximate generalized TOV equation (3.23) given by

$$p'(r)\simeq-\rho(r)\left[\frac{GM(r)}{r^2}+k\right], \tag{3.26}$$

which can be written in the form

$$\frac{d}{dr}\left[r^2\left(\frac{p'(r)}{\rho(r)}+k\right)\right]=-4\pi Gr^2\rho(r). \tag{3.27}$$

To solve this equation, one needs an equation of state, $p=F(\rho)$, and a set of initial conditions. Regarding the latter, a finite central density $\rho(0)$ will be assumed, as well as a vanishing central density gradient[2]. To fix $F(\rho)$, a *polytrope* stellar structure will be used, for which $F(\rho)=w\rho^\gamma$. By making the substitutions

$$r = \left(\frac{w\gamma}{4\pi G\left(\gamma-1\right)}\right)^{\frac{1}{2}}\rho(0)^{(\gamma-2)/2}\xi, \tag{3.28}$$

$$\rho = \rho(0)\theta^{1/(\gamma-1)}, \tag{3.29}$$

$$k = \frac{w\gamma}{\gamma-1}\rho(0)^{\gamma-1}K, \tag{3.30}$$

[2]There is a slight incorrection in this last assumption. In fact, one should have $\rho'(0)=-k\frac{\rho(0)}{F'(\rho(0))}$, but, for very small values of k, the difference is negligible, provided $F'(\rho(0))\neq 0$.

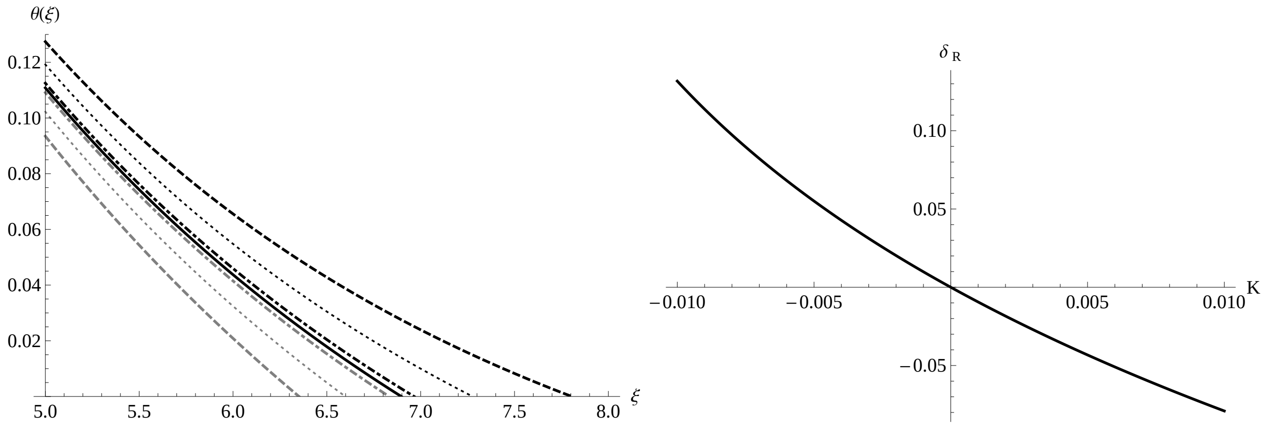

Figure 3.1: (Left panel) Comparison of $\theta(\xi)$ near $\xi_1(K)$ for $\gamma = \frac{4}{3}$ and different values of K. $|K| = 0.01$ - Dashed line. $|K| = 0.005$ - Dotted line. $|K| = 0.001$ - Dotdashed line. $K = 0$ - Full line. Graphs above (below) the full line ($K = 0$) and in black (gray) correspond to negative (positive) values of K. (Right panel) Plot of δ_K for K in the range -0.01 to 0.01 and $\gamma = \frac{4}{3}$.

eq. (3.27) takes the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\left(\frac{d\theta}{d\xi} + K\right)\right] + \theta^{1/(\gamma-1)} = 0, \tag{3.31}$$

with initial conditions

$$\theta(0) = 1, \tag{3.32}$$

$$\theta'(0) = 0. \tag{3.33}$$

This generalizes the usual *Lane-Emden equation* of GR, recovered when $K = 0$. For $\gamma > \frac{6}{5}$, the solution vanishes[3] for some finite $\xi_1(K)$, that is, $\theta(\xi_1(K)) = 0$, in which case the radius of the star is given by

$$R_s(K) = \left(\frac{w\gamma}{4\pi G(\gamma-1)}\right)^{\frac{1}{2}} \rho(0)^{(\gamma-2)/2}\xi_1(K). \tag{3.34}$$

The left panel of fig. (3.1) shows a graphic representation of the solution $\theta(\xi)$ near ξ_1 for $\gamma = \frac{4}{3}$ (other choices yield similar results) and different values of K. It is clear that positive values of K lead to smaller radii, while negative ones have the opposite effect. This is consistent with the physical interpretation of the NMC as a "geometric" fluid given above: if $K > 0$, then $f_2'(r) > 0$, and, according to eq. (3.10), the acceleration is directed inwards, reducing the radius. The same line of reasoning explains why a negative K should lead to larger radii.

The influence of the NMC may be measured using the parameter

$$\delta_R = \frac{R_s(K)}{R_s(0)} - 1 = \frac{\xi_1(K)}{\xi_1(0)} - 1. \tag{3.35}$$

A plot of δ_R as a function of K for $\gamma = \frac{4}{3}$ is shown in the right panel of fig. (3.1). Notice that even small values of K may originate significant changes in the radius. For example, $K = 10^{-2}$ leads to a radius prediction almost 8% smaller than the general relativistic case, $K = 0$, provided the weak coupling regime still holds (note that the condition $kr \ll 1$ now reads $\frac{1}{\sqrt{4\pi G}}\left(\frac{w\gamma}{\gamma-1}\right)^{3/2}\rho(0)^{(3\gamma-4)/2}\xi K \ll 1$). In fact, for small K, δ_R is well approximated by the linear function $-10K$. These results underscore the importance that a NMC may have on the dynamics of stars, even for a weak coupling.

[3] Rigorously, that is true only if $K = 0$. However, for small values of K, the statement is approximately correct

The general equations of stellar dynamics in nonminimally coupled gravity

Having derived the weak coupling equations, one now addresses the problem of establishing a mathematical framework for the general case.

Defining the auxiliary function

$$h(r) = -\frac{F_2\rho(r)}{\kappa} = -\frac{f_2'(r)\rho(r)}{\kappa R'(r)} \qquad R'(r) \neq 0, \tag{3.36}$$

the modified field equations (2.12) can be written as

$$(1+h)\,R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \triangle_{\mu\nu}h = \frac{1}{2\kappa}f_2 T_{\mu\nu}. \tag{3.37}$$

In general, for an arbitrary coupling function, one must solve these equations without neglecting any term (although in some cases it may be possible to consider simplifications, even when the coupling is not in a weak field regime. One such scenario will be discussed next). This is obviously an outstanding mathematical challenge, and an analytic solution seems completely out of reach. Indeed, even a numerical treatment of the equations is far from being a straightforward matter. Nevertheless, this section will serve to establish a series of formulas and results that may help simplify the calculations, and, ultimately, it will be shown that once one specifies an equation of state, a set of initial conditions (recall that eq. (3.37) is a fourth order system, and, as such, one requires four boundary values), and a coupling function, the field equations can be solved to yield $\rho(r)$, at least in principle.

The first equation to consider is the hydrostatic equilibrium condition, given by eq. (3.10), which establishes a relation between $B(r)$ and the three functions p,ρ and f_2. To do the same for $A(r)$, one must work with eq. (3.37). The nonzero components of $\triangle_{\mu\nu}h$ read

$$\triangle_{tt}h = \frac{B}{A}\left[h'' - \frac{B'}{2A}h' + \frac{2}{r}h'\right], \tag{3.38}$$

$$\triangle_{rr}h = -\left[\frac{B'}{2B} + \frac{2}{r}\right]h', \tag{3.39}$$

$$\triangle_{\theta\theta}h = \frac{r^2}{A}\left[-\frac{B'}{2B}h' - h'' + \frac{A'}{2A}h' - \frac{1}{r}h'\right], \tag{3.40}$$

$$\triangle_{\phi\phi}h = [\triangle_{\theta\theta}h]\sin^2(\theta). \tag{3.41}$$

It is possible to eliminate the second derivative of the function h by using the trace, eq. (2.13). Explicit calculation yields

$$h'' = \frac{1}{3}\left[\frac{f_2 T}{2\kappa} + R\,(1-h)\right]A - \frac{B'}{2B}h' + \frac{A'}{2A}h' - \frac{2}{r}h', \tag{3.42}$$

where $T = 3p - \rho$ is the trace of the stress-energy tensor. Defining the LHS of eq. (3.37) by $J_{\mu\nu}$, that is,

$$J_{\mu\nu} = (1+h)\,R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \triangle_{\mu\nu}h, \tag{3.43}$$

it is clear that

$$\frac{J_{rr}}{A} - \frac{J_{\theta\theta}}{r^2} = 0. \tag{3.44}$$

Multiplying by $(2rAB)^2$ and rearranging the terms, this equation reads

$$\alpha_1 A' + \alpha_2 A + \alpha_3 A^2 = 0, \tag{3.45}$$

$$\alpha_1 = B\left[2rB(1+h) + r^2B'(1+h) + 2r^2Bh'\right], \tag{3.46}$$

$$\alpha_2 = 4B^2(1+h) + 2rBB'(1+h) + r^2B'^2(1+h) + 4rB^2h' - 2r^2BB''(1+h) - 4r^2B^2h'', \tag{3.47}$$

$$\alpha_3 = -4B^2(1+h), \tag{3.48}$$

where the arguments of the functions were omitted. Eq. (3.45) is a Bernoulli differential equation, which can be solved exactly. Defining $\psi = \frac{\alpha_2}{\alpha_1}$ and $\chi = -\frac{\alpha_3}{\alpha_1}$, using the boundary conditions $B(R_s) = [A(R_s)]^{-1} = \left(1 - \frac{2GM}{R_s}\right)$, the metric coefficients may be expressed as

$$B(r) = \frac{1 - \frac{2GM}{R_s}}{[f_2(r)]^2} \exp\left[2 \int_r^{R_s} \frac{p'(s)}{\rho(s) + p(s)} ds\right], \tag{3.49}$$

$$A(r) = \frac{\exp\left[\int_r^{R_s} \psi(s) ds\right]}{\left(1 - \frac{2GM}{R_s}\right) + \int_r^{R_s} \chi(s) \exp\left[\int_s^{R_s} \psi(q) dq\right] ds}. \tag{3.50}$$

Notice that while eq. (3.49) explicitly gives $B(r)$ as a function of p, ρ and f_2, the same is not true for $A(r)$, because the auxiliary fields ψ and χ depend on h, which, in turn, depends on the Ricci scalar. Therefore, eq. (3.50) must be solved self-consistently. To conclude, one needs one more independent equation, which can be taken to be, for example, the tt component of the field equations (3.37).

Putting things into perspective, given a coupling function f_2 and an equation of state relating p and ρ, one must deduce a set of initial conditions (this task is far from being trivial for the general case), and solve one of the nonzero components of eq. (3.37), with h defined by (3.36), and the nontrivial metric coefficients given by eqs. (3.49) and (3.50), while checking the consistency of the latter. This justifies the claims made earlier about the mathematical difficulties involved.

Suppressive coupling and spacetime oscillations

The previous section showed that unless some simplification is assumed, the field equations of NMC gravity present a very complicated mathematical challenge, even when symmetries reduce the complexity of the problem. A special type of coupling that greatly simplifies calculations (but cannot be considered "weak") will now be presented, and its implications briefly discussed.

Suppose that at some point in spacetime $f_2 \simeq F_2 \simeq 0$ (there is no compelling reason to believe that one such situation should exist. However, it is no difficult to envisage coupling functions that tend to 0 when $R \to +\infty$. For example, $f_2(R) = e^{-R/R_0}$.). This assumption enables one to write the field equations (2.8) as

$$F_1 R_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} \simeq \triangle_{\mu\nu} F_1, \tag{3.51}$$

or, if $f_1(R) = 2\kappa R$, simply

$$R_{\mu\nu} \simeq 0, \tag{3.52}$$

where $R \simeq 0$ was used (something that follows from the trace equation). This is just the usual general relativistic vacuum case. The coupling function suppresses the effects of matter. However, as this happens, the curvature scalar tends to 0, and since it was argued before that $f_2 \simeq 1$ in this case, it seems that one has an inconsistency in the model. A possible way out of this problem is to let it go out of stationarity. If the coupling function and the other quantities are allowed to vary in time, it is easy to picture a solution: as some spacetime configuration pushes R to a value (possibly infinite) where $f_2 \simeq F_2 \simeq 0$, the suppressing coupling pushes R back to 0, and, consequently, bounces itself back to 1. The conclusion is simple: if the coupling suppresses the effects of matter, the geometry suppresses the effects of the coupling.

Suppose now that the configuration driving R to a value where $f_2 \simeq F_2 \simeq 0$ is a weak coupling one. As pointed out above, when f_2 gets sufficiently small, the curvature scalar must return to 0. But in doing so, the weak coupling limit is restored, and, eventually, the system will be driven towards a suppressing coupling regime once again. Provided the intermediate stage, when f_2 is crossing from 1 to 0 or the other way around, does not influence the situation significantly (something that should be true if the transition is sufficiently fast), an oscillatory motion will be established. If they exist, these oscillations in the "fabric" of spacetime may be detectable. Naturally, this detection depends heavily on the frequency and amplitude of the waves. Note that this oscillatory mechanism is different from the usual gravitational waves. For example, there is no *a priori* reason to think that it does not apply to spherically symmetric situations, which, according to GR, do not radiate gravitational waves.

A quantitative analysis, necessary to back the qualitative description presented above, is currently underway. Nevertheless, the crude example serves to make the point that NMC gravity encloses new and potentially interesting phenomena that may be accessible to experimental testing.

3.2 Nonstatic matter distributions

The previous section was dedicated to the study of static distributions of matter. While these have fundamental importance on the study of several astrophysical objects (most notably, stars), there are other situations that require a departure from this assumption. Galaxies and clusters of galaxies are good examples. This part of the thesis deals with the effects that a NMC may have on these cases.

The Newtonian limit: a generalized gravitational potential

It was previously noted that the extra acceleration in eq. (3.9) may be written simply as

$$f^{\nu}_{NMC} = -\frac{h^{\mu\nu}\nabla_{\mu} f_2}{f_2}. \tag{3.53}$$

Once more, the resemblance with the general relativistic pressure gradient acceleration is notorious. To explore the Newtonian limit of this model, one will assume that $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$, where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and $|\epsilon_{\mu\nu}| \ll 1$, $\frac{dt}{d\tau} \simeq 1$ and $|\frac{dx^i}{dt}| \ll 1$. Under these simplifications, the spatial components of the LHS of eq. (3.9) read approximately

$$u^{\mu}\nabla_{\mu} u^i \simeq \frac{d^2x^i}{dt^2} - \frac{1}{2}\nabla_i \epsilon_{tt}, \tag{3.54}$$

where the index i runs from 1 to 3 (the spatial coordinates). Additionally, the projection tensor may be simplified to $h^{\mu i} \simeq \eta^{\mu i} = \delta^{\mu i}$. Combining these results, the equation of motion for a fluid element in the Newtonian limit is given by

$$\frac{d^2x^i}{dt^2} \simeq \nabla_i \left[\frac{g_{tt}+1}{2} - \log\left(|f_2|\right)\right] - \frac{\nabla_i p}{\rho + p}, \tag{3.55}$$

where the identity $\epsilon_{tt} = g_{tt} + 1$ was used. The last term on the RHS of eq. (3.55) is just the usual pressure gradient force per unit mass. Moreover, the Newtonian gravitational potential is generally defined in GR by

$$\Phi_N = -\frac{g_{tt}+1}{2}. \tag{3.56}$$

However, there is now an unaccounted term in the equation of motion, related to the NMC. Since this term is geometric in nature (it depends solely on R), it seems quite natural to include it in a generalized gravitational potential:

$$\Phi_{NMC} = \Phi_N + \log\left(|f_2|\right). \tag{3.57}$$

Before venturing into the applications of this formula, it is interesting to see why it may be relevant by analyzing the dimensions involved. Consider the usual Newtonian gravitational pull of an object of mass M given by

$$g_N = -\nabla_r \Phi_N = -\frac{GM}{r^2}. \tag{3.58}$$

Dividing $g_{NMC} = -\frac{\nabla_r f_2}{f_2}$ by g_N, one may define the strength parameter δ_g:

$$\delta_g = \left|\frac{g_{NMC}}{g_N}\right| = \frac{c^2 r^2}{GM}\left|\frac{\nabla_r f_2}{f_2}\right|, \tag{3.59}$$

where the speed of light in vacuum c is now explicitly shown. The factor $\frac{c^2r^2}{GM}$ takes on different values for distinct astrophysical situations: it is approximately 10^{16} m at the surface of Earth, 10^{14} m at the surface of the Sun, and 10^{26} m for $r = 1$ kpc in a typical galaxy with $M \simeq 10^{11} M_{\odot}$ (not counting dark matter), where $M_\odot$ is the solar mass[4] . If the gradient term $\nabla_r f_2$ is sufficiently small, its effects should be noticeable only at very large scales, such as galaxies or clusters of galaxies. Indeed, if one takes $\nabla_r f_2 \simeq 10^{-26}$ m^{-1}, with $f_2 \simeq 1$, eq. (3.59) takes the values

$$\delta_g \simeq 10^{-10} \quad \text{at the surface of Earth,} \tag{3.60}$$
$$\delta_g \simeq 10^{-12} \quad \text{at the surface of the Sun,} \tag{3.61}$$
$$\delta_g \simeq 1 \quad \text{at r=1 kpc in a galaxy.} \tag{3.62}$$

These results show that the NMC may have an important influence in the dynamics of galaxies and other large-scale astrophysical objects, while, at the same time, having negligible effects at stellar systems level. Moreover, this can be achieved in a rather natural way, and using only a crude model that assumes $\nabla_r f_2 \simeq$ constant. More sophisticated choices should be able to produce even better results.

Mimicking dark matter profiles

Having derived a generalized gravitational potential in the form of eq. (3.57) and shown that its effects may be noticeable at galactic level without disrupting solar system dynamics, the possibility of simulating dark matter density profiles using this scheme will now be discussed. The idea of mimicking dark matter using NMC gravity has already been explored in Refs. [28, 29], though using a different approach that focus on power-law couplings of the form $f_2 = 1 + \left(\frac{R}{R_0}\right)^n$. The results presented in this work are, therefore, complementary to those.

The starting point of this derivation is Poisson's equation for gravity:

$$\nabla^2 \Phi_N = 4\pi G \rho_v, \tag{3.63}$$

where ρ_v is the visible matter density (it is assumed that no dark components exist). Suppose now one desires to generalize eq. (3.63) to include the effects a NMC present in the potential Φ_{NMC}. If it is assumed that the gravitational effects of dark matter are contained in the NMC, a rather straightforward way to do it is adding a mimicked dark matter density profile to the RHS of Poisson's equation, and connecting the additional term in (3.57) with it:

$$\nabla^2 \Phi_{NMC} = 4\pi G \left(\rho_v + \rho_{DM}\right), \tag{3.64}$$
$$\nabla^2 \log|f_2| = 4\pi G \rho_{DM}, \tag{3.65}$$

where ρ_{DM} is the replicated dark matter profile. The effect is similir - but not exactly equal - to what happens in the electrostatics of dielectrics: the diverge of the electric displacement vector makes reference only to free charges, $\nabla \cdot \boldsymbol{D} = \rho_f$, while the divergence of the electric field must account for both free and bound charges, $\epsilon_0 \nabla \cdot \boldsymbol{E} = \rho_f + \rho_b$. Here, the Laplacian of the usual Newtonian potential Φ_N relates only to the visible (true) matter, while the generalized Φ_{NMC} must also take into consideration the effects of the NMC. In electrostatics, the "background" dielectric provides the bound charges, while in this case the NMC provides the gravitational effects of a dark matter component.

Considering a spherically symmetric scenario, eq. (3.65) reads

$$\nabla_r \left(r^2 \frac{\nabla_r f_2}{f_2} \right) = 4\pi G r^2 \rho_{DM}, \tag{3.66}$$

or, provided $\lim_{r\to 0} r^2 \frac{\nabla_r f_2}{f_2} = 0$,

$$r^2 \frac{\nabla_r f_2}{f_2} = G \int_0^r 4\pi s^2 \rho_{DM}(s) ds = G M_{DM}(r), \tag{3.67}$$

where $M_{DM}(r)$ is the total mass of simulated dark matter inside a sphere of radius r, that is, if one had to specify the necessary mass value enclosed within a sphere of radius r to explain the gravitational effects not

[4] The Newtonian gravitational pull within a galaxy differs slightly from (3.58). In this case, the Hernquist density profile [40] was used to calculate g_N

$\rho_{DM}(r)$	$\log f_2(r)$	$\nabla_r f_2$ (m^{-1})
$4\rho_s \left(\frac{r}{r_s}\right)^{-1}\left(1+\frac{r}{r_s}\right)^{-2}$	$16\pi G r_s^3 \rho_s \frac{\log\left[\frac{r_s}{r+r_s}\right]}{r}$	7.2×10^{-28}
$\rho_s \left(1+\frac{r}{r_s}\right)^{-1}\left(1+\frac{r^2}{r_s^2}\right)^{-1}$	$\frac{\pi G r_s^2 \rho_s}{r}\left[2\left(r+r_s\right)\left(\arctan\left(\frac{r}{r_s}\right)-\log\left(r+r_s\right)\right)+\left(r-r_s\right)\log\left(r^2+r_s^2\right)\right]$	1.9×10^{-28}
$\rho_s e^{-d\left[\left(\frac{r}{r_s}\right)^{1/n}-1\right]}$	$\frac{4n\pi G\rho_s e^d r_s^3}{r d^{3n}}\left[\Gamma(3n,a)-\Gamma(3n)+\frac{r d^n}{r_s}\left(\Gamma(2n)-\Gamma(2n,a)\right)\right]$	1.9×10^{-27}

Table 3.1: (Left column) Common choices for dark matter profiles. From top to bottom: NFW profile [42], Burkert's profile [43] and Einasto's profile [44]. (Middle column) Calculated $\log f_2(r)$ for the different profiles. Multiplicative constants were disregarded. In the Einasto's case, $\Gamma(a,z)$ is the incomplete gamma function, defined by $\Gamma(a,z)=\int_z^{+\infty} t^{a-1}e^{-t}dt$, and $\Gamma(z)$ is the gamma function, defined by $\Gamma(z)=\Gamma(z,0)$. (Right column) Estimate of $\nabla_r f_2$ for typical values of the parameters taken from Ref. [41].

accounted for visible matter, it would be $M_{DM}(r)$. The total gravitational pull may then be written simply as

$$g = g_N + g_{NMC} = -\frac{G}{r^2}\left[M_v(r)+M_{DM}(r)\right], \tag{3.68}$$

where $M_v(r) = 4\pi\int_0^r s^2\rho_v(s)ds$.

Eq. (3.67) is a differential equation for f_2 that can be solved rather easily, yielding the solution

$$f_2(r) = f_2(r_0)\exp\left[\int_{r_0}^{r}\frac{GM_{DM}(s)}{s^2}ds\right]. \tag{3.69}$$

The multiplicative constant $f_2(r_0)$ is irrelevant in most cases, since physically measurable quantities generally depend on $\frac{\nabla_r f_2}{f_2}$.

Given a dark matter profile, eq. (3.69) can be used to calculate the explicit form of f_2. The results for common choices of ρ_{DM} [41] are summarized in the table (3.1), together with estimates for $\nabla_r f_2$ at $r = 1$ kpc for typical values of the various parameters involved. The results are within the expected limit. Note that no specific form for $f_2(R)$ is assumed. To do that, one would need to solve the field equations (possibly in a weak limit regime), express the curvature scalar R as a function of the radial coordinate r, invert the expression, and plug the result into eq. (3.69).

Chapter 4

Conclusions and Outlook

This thesis discusses the dynamics of perfect fluids in an alternative theory of gravity exhibiting a nonminimal coupling between curvature and matter.

The motivation to consider modifications of GR was briefly discussed in chapter 1, in which the alternative model presented was qualitatively introduced. This was followed by a quantitative introduction in chapter 2, where the base features (field equations, conservation law for $T_{\mu\nu}$ and equation of motion for perfect fluids) were summarized. The main results of this work were presented in chapter 3, and divided in two sections. In the first, static, spherically symmetric distributions of matter were considered, a situation that applies to the description of stars. It was shown that the nonminimal coupling generalizes the hydrostatic equilibrium equation of GR, a feature that may be used to predict differences in the gravitational redshift of radiation coming from within a star. Moreover, the specific form of the added term lead one to treat the coupling as a sort of geometric fluid. The weak coupling limit of the theory was considered next, and the main equations of stellar structure were derived in this regime. Possible implications for Newtonian stars were briefly discussed. The section concludes by establishing a mathematical framework to study the case of an arbitrary coupling, and by showing that such couplings may lead to new and exotic phenomena under specific circumstances. The second part of the chapter was devoted to the study of nonstatic fluids. The possibility of defining a generalized Newtonian potential to include the effects of the coupling was discussed, and it was argued that such modification naturally gives rise to a gravitational pull noticeable only at galactic distances. This was used to explain the dark matter halos of galaxies without actually resorting to particle models, something achieved through a mimicking mechanism.

Concluding, the model of gravity presented in this thesis seems to enclose some interesting and relevant features that may be useful in explaining the current challenges of experimental astrophysics and cosmology. Nevertheless, it is still largely unexplored, and work is under progress to develop the theory further. Regarding the subject of this thesis, there are at least a couple of points worth exploring: firstly, the effects that a strong coupling regime may have on the dynamics of stars. Secondly, the applicability of the dark matter mimicking mechanism introduced, that is, whether or not it can be a serious alternative to the particle models of dark matter.

Appendix A

Derivation of the conservation law for $T_{\mu\nu}$ and of the perfect fluid equation of motion

The conservation law for $T_{\mu\nu}$

For convenience, the field equations (2.8) are repeated here,

$$HR_{\mu\nu} - \frac{1}{2}f_1 g_{\mu\nu} = \triangle_{\mu\nu}H + f_2 T_{\mu\nu}, \tag{A.1}$$

where, as before, the identification $H \equiv F_1 + 2F_2\mathcal{L}_m$ is used.

In order to compute the covariant derivative of the equation above, several identities are necessary. Firstly, if the affine connection is taken to be torsion-free, the following formula holds for any scalar field F

$$\nabla_\mu \nabla_\nu F = \nabla_\nu \nabla_\mu F. \tag{A.2}$$

Moreover, if some function depends solely on the Ricci scalar, that is, $f \equiv f(R)$, then $\nabla_\mu f(R) = f'(R)\nabla_\mu R$. Another important result is

$$\nabla^\mu \triangle_{\mu\nu} F = (\Box\nabla_\nu - \nabla_\nu \Box)\, F = R_{\mu\nu}\nabla^\mu F, \tag{A.3}$$

which can be proved in the following way: the definition of the Riemann tensor reads $\nabla_\gamma\nabla_\delta X^\alpha - \nabla_\delta\nabla_\gamma X^\alpha = R^\alpha_{\beta\gamma\delta}X^\beta$. Using equation (A.2), one has $\Box\nabla_\nu F = \nabla_\mu\nabla^\mu\nabla_\nu F = \nabla_\mu\nabla_\nu\nabla^\mu F$, and taking $X^\alpha = \nabla^\alpha F$ in the equation above yields the desired relation. Finally, the identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\nabla^\mu R$, which follows from $\nabla^\mu G_{\mu\nu} = 0$, will also be of use.

If one now takes the covariant derivative of eq. (A.1) and plugs in the formulas presented above, the conservation law for $T_{\mu\nu}$ reads

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2}\left[g^{\mu\nu}\mathcal{L}_m - T^{\mu\nu}\right]\nabla_\mu R. \tag{A.4}$$

The equation of motion for perfect fluids

Having derived the conservation equation (A.4), one is now in a position to obtain the equation of motion for an element of perfect fluid. Starting with the usual form for the stress-energy tensor,

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}, \tag{A.5}$$

one can readily see that

$$\nabla_\mu T^{\mu\nu} = \left[u^\mu \nabla_\mu (\rho + p) + (\rho + p)\nabla_\mu u^\mu\right] u^\nu + (\rho + p)\, u^\mu \nabla_\mu u^\nu + g^{\mu\nu}\nabla_\mu p. \tag{A.6}$$

Contracting eq. (A.4) with u_ν, introducing eq. (A.6), and noting that $u_\nu u^\mu \nabla_\mu u^\nu = 0$, which follows directly from $u_\mu u^\mu = -1$, one has

$$u^\mu \nabla_\mu (\rho + p) + (\rho + p) \nabla_\mu u^\mu = u^\mu \nabla_\mu p - \frac{F_2}{f_2} (\mathcal{L}_m + \rho) u^\mu \nabla_\mu R. \tag{A.7}$$

Finally, plugging identity (A.7) into (A.6), rearranging the terms in eq. (A.4) and using $u^\mu \nabla_\mu u^\nu = \frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta$, where τ is an affine parameter, the equation of motion reads

$$\frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta = \frac{1}{\rho + p} \left[\frac{F_2}{f_2} (\mathcal{L}_m - p) \nabla_\mu R - \nabla_\mu p \right] h^{\mu\nu}. \tag{A.8}$$

Bibliography

[1] O. Bertolami, J. Páramos and S. Turyshev, *General Theory of Relativity: Will it survive the next decade?*, Lasers, Clocks, and Drag-Free: Technologies for Future Exploration in Space and Tests of Gravity, arXiv:0602016 [gr-qc] (2006).

[2] C. M. Will, *Was Einstein right? Testing Relativity at the centenary*, Annalen Phys. **15**, 19 (2005) arXiv:0504086 [gr-qc].

[3] F. Zwicky, *Die Rotverschiebung von extragalaktischen Nebeln*, Helv. Phys. Acta **6**, 110 (1933).

[4] F. Zwicky, *On the masses of nebulae and of clusters of nebulae*, The Astrophys. J. **86**, 3 (1937).

[5] V. Rubin and W. K. Ford, Jr, *Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions*, The Astrophys. J. **159**, 379 (1970).

[6] V. Rubin, N. Thonnard and W. K. Ford, Jr, *Rotational properties of 21 Sc Galaxies with a large range of luminosities and radii from NGC 4605 ($R = 4kpc$) to UGC 2885 ($R = 122kpc$)*, The Astrophys. J. **238**, 471 (1980).

[7] A. G. Riess *at al.*, *Observational Evidence from Supernovae for an accelerating Universe and a Cosmological Constant*, Astron. J. **116**, 1009 (1998).

[8] A. G. Riess *at al.*, *Type Ia supernova discoveries at $z > 1$ from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution*, Astrophys. J. **607**, 665 (2004).

[9] D. N. Spergel *at al.*, *Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology*, Astrophys. J. Suppl. **170**, 377 (2006).

[10] R. P. Feynman, F. B. Morinigo and W. G. Wagner, *Feynman Lectures on Gravitation*, Addison-Wesley (1995).

[11] E. Alvarez, *Quantum gravity: an introduction to some recent results*, Rev. Mod. Phys. **61**, 561 (1989).

[12] P. Kroupa *at al.*, *Local-Group tests of dark-matter Concordance Cosmology: Towards a new paradigm for structure formation?*, Astron. Astrophys. **523** 32 (2010).

[13] S. M. Carroll, *The Cosmological Constant*, Living Rev. Rel. **4**, 1 (2001).

[14] O. Bertolami, *The cosmological constant: a user's guide*, Int. J. Mod. Phys. **D 18**, 2303 (2009).

[15] H. Weyl, *Eine neue Erweiterung der Relativitatstheorie*, Ann. Phys. **59**, 101 (1919).

[16] T. P. Sotiriou and V. Faraoni, *$f(R)$ theories of gravity*, Rev. Mod. Phys. **82**, 451 (2010).

[17] S. Capozziello and V. Faraoni, *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics*, Springer (2011).

[18] S. Nojiri and S. D. Odintsov, *Introduction to Modified Gravity and Gravitational Alternative for Dark Energy*, Int. J. Geom. Meth. Phys. **4**, 115 (2007).

[19] A. D. Dolgov and M. Kawasaki, *Can modified gravity explain accelerated cosmic expansion?*, Phys. Lett. **B 573**, 1 (2003).

[20] V. Faraoni, *Matter instability in modified gravity*, Phys. Rev. **D 74**, 104017 (2006).

[21] V. Faraoni, *de Sitter space and the equivalence between $f(R)$ and scalar-tensor gravity*, Phys. Rev **D 75**, 067302 (2007).

[22] T. Sotiriou, V. Faraoni, *Modified gravity with R-matter coupling and (non-)geodesic motion*, Class. Quant. Grav. **25**, 205002 (2008).

[23] O. Bertolami, C. G. Boehmer, T. Harko and F. S. N. Lobo, *Extra force in $f(R)$ modified theories of gravity*, Phys. Rev. **D 75**, 104016 (2007).

[24] G. Allemandi, A. Borowiec, M. Francaviglia and S. D. Odintsov, *Dark energy dominance and cosmic acceleration in first-order formalism*, Phys. Rev **D 72**, 063505 (2005).

[25] O. Bertolami and J. Páramos, *On the non-minimal gravitational coupling to matter*, Class. Quantum Grav. **25**, 245017 (2008).

[26] O. Bertolami, F. S. N. Lobo and J. Páramos, *Nonminimal coupling of perfect fluids to curvature*, Phys. Rev. **D 78**, 064036 (2008).

[27] O. Bertolami and J. Páramos, *Do $f(R)$ theories matter?*, Phys. Rev. **D 77**, 084018 (2008).

[28] O. Bertolami and J. Páramos, *Mimicking dark matter through a non-minimal gravitational coupling with matter*, Journal of Cosmology and Astroparticle Physics **3**, 009 (2010).

[29] O. Bertolami, P. Frazão and J. Páramos, *Mimicking dark matter in clusters through a non-minimal gravitational coupling with matter: the case of the Abell cluster A586*, arXiv:1111.3167 [gr-qc] (2011).

[30] O. Bertolami, P. Frazão and J. Páramos, *Accelerated expansion from a non-minimal gravitational coupling to matter*, Phys. Rev. **D 81**, 104046 (2010).

[31] O. Bertolami and J. Páramos, *Mimicking the cosmological constant: Constant curvature spherical solutions in a non-minimally coupled model*, Phys. Rev **D 84**, 064022 (2011).

[32] O. Bertolami, P. Frazão and J. Páramos, *Reheating via a generalized non-minimal coupling of curvature to matter*, Phys. Rev. **D 83**, 044010 (2011).

[33] O. Bertolami, M. C. Sequeira, *Energy Conditions and Stability in $f(R)$ theories of gravity with non-minimal coupling to matter*, Phys. Rev. **D 79**, 104010 (2009).

[34] O. Bertolami, R. Z. Ferreira, *Traversable Wormholes and Time Machines in non-minimally coupled curvature-matter $f(R)$ theories*, arXiv:1203.0523 [gr-qc] (2012).

[35] O. Bertolami and A. Martins, *Dynamics of perfect fluids in nonminimally coupled gravity*, Phys. Rev. **D 85**, 024012 (2012).

[36] B. F. Schutz, *Perfect Fluids in General Relativity: Velocity Potentials and a Variational Principle*, Phys. Rev. **D 2**, 2762 (1970).

[37] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge University Press (1973).

[38] J. D. Brown, *Action functionals for relativistic perfect fluids*, Class. Quantum Grav. **10**, 1579 (1993).

[39] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, Inc. (1972).

[40] L. Hernquist, *An Analytical Model for Spherical Galaxies and Bulges*, The Astroph. J. **356**, 359 (1990).

[41] D. Merritt, A. W. Graham, B. Moore, J. Diemand and B. Terzić, *Empirical models for dark matter halos. I. Nonparametric construction of density profiles and comparison with parametric models*, The Astron. J. **132**, 2685 (2006).

[42] J. F. Navarro, C. S. Frenk and S. D. M. White, *The assembly of galaxies in a hierarchically clustering universe*, Mon. Not. of the Royal Astron. Soc. **275**, 56 (1995).

[43] A. Burkert, *The Structure of Dark Matter Halos in Dwarf Galaxies*, The Astroph. Jour. **446**, L25 (1995).

[44] J. Einasto, *Influence of the atmospheric and instrumental dispersion on the brightness distribution in a galaxy*, Trudy Inst. Astrofiz. Alma-Ata **5**, 87 (1965).