Convergence of Numerical Methods for Viscosity Solutions through the adjoint method

Tiago Salvador, Diogo A. Gomes (Advisor)

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Abstract

We present a new proof for the rate of convergence of numerical schemes for the one dimensional time dependent Hamilton-Jacobi equation, a result well-known in literature. This is done by generalizing to problems with boundary conditions the proof of Cagnetti, Gomes and Tran (2012, [5]) for the periodic setting using the adjoint method.

Keywords: Viscosity Solutions, Adjoint Method, Hamilton-Jacobi equation, Numerical Scheme.

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1 Introduction

In this paper we study semi-discrete schemes for viscosity solutions of Hamilton-Jacobi equations. More precisely, consider the problem

\[ \begin{cases} 
  u_t + H(u_x) = 0, & \text{in } \mathcal{O}, \\
  u = g, & \text{on } \partial_p \mathcal{O},
\end{cases} \]

where \( \mathcal{O} = (0, 1) \times (0, +\infty) \) is the parabolic cylinder, \( \partial_p \mathcal{O} = \mathcal{O} \setminus \mathcal{O} \) is the parabolic boundary, \( g \in C^2(\partial_p \mathcal{O}) \) and \( H : \mathbb{R} \to \mathbb{R} \) is a smooth convex and coercive Hamiltonian. Our idea is to approximate the term \( u_x \) by \( \delta_h u \) and \( \delta_{-h} u \) where

\[ \delta_h u(x, t) = \frac{u(x + h, t) - u(x, t)}{h} \]

for \( 0 < h < 1 \). This approach leads to the following semi-discrete approximation schemes

\[ \begin{cases} 
  u_t^h + F (-\delta_h u^h, \delta_{-h} u^h) = 0, & \text{in } \mathcal{O}, \\
  u^h = \tilde{g}, & \text{on } \partial_p \mathcal{O}^h,
\end{cases} \]

where \( \mathcal{O}^h = [-h, 1 + h] \times [0, +\infty) \), \( \partial_p \mathcal{O}^h = \mathcal{O}^h \setminus \mathcal{O} \), \( \tilde{g} \) is a \( C^2 \) extension of \( g \) and \( F \in C^2(\mathbb{R}^2) \) satisfying the following properties:
(F1) $F$ is convex;

(F2) $F(\cdot, q)$ is increasing for each $q \in \mathbb{R}$ and $F(p, \cdot)$ is increasing for each $p \in \mathbb{R}$;

(F3) $F(-p, p) = H(p)$ for every $p \in \mathbb{R}$.

In [5], Cagnetti, Gomes and Tran consider the problem (1) but in the periodic setting, that is, they take $O = \mathbb{T} \times (0, +\infty)$ and $\partial_p O = \mathbb{T} \times \{t = 0\}$ where $\mathbb{T}$ is the one dimensional torus. The main result of [5] is a new proof for the rate of convergence of the solutions of the approximation schemes (2) to the solution of (1) (in the periodic setting) using the adjoint method recently introduced by Evans in [8]. By considering the torus they avoid dealing with the boundary conditions. Here we extend their result to the interval $[0, 1]$ and therefore we will have to deal with the boundary conditions.

Our main result is then the following.

**Theorem 1.1.** For every $T \in (0, +\infty)$ and $0 < h < 1$ there exists a positive constant $C = C(T)$, independent of $h$, such that

$$
\sup_{t \in [0, T]} \|u(\cdot, t) - u_h(\cdot, t)\|_{L^\infty((0, 1))} \leq C\sqrt{h}.
$$

The key result here it’s not the convergence in itself but instead the rate of convergence. The convergence of viscosity solutions of approximation schemes of fully nonlinear second-order elliptic or parabolic PDE’s is a rather classic result. In [3], Barles and Souganidis prove that a monotone, stable and consistent scheme is convergent. Although their result is proved for fully discrete schemes, a similar result can be obtained for the semi-discrete schemes considered here.

## 2 Viscosity Solutions for discrete schemes

Our goal in this Section is to present the theory of viscosity solutions and some results for the semi-discrete numerical schemes that arise in the approximation of (1).

First we introduce some notation. Our working domain will be $O := \Omega \times (0, +\infty)$ where $\Omega \subset \mathbb{R}$ is a bounded open interval (later we will take $\Omega = (0, 1)$). Plus, since we will discretize in space, it is also useful to define $\Omega^h := \{x \pm h : x \in \Omega\}$ and $O^h := \Omega^h \times [0, +\infty)$ for every $h > 0$. As for the boundary we will use the notation $\partial_p O^h$ to denote the set $O^h \setminus O$.

We will focus on the following Dirichlet problem:

$$
\begin{aligned}
\begin{cases}
  u_t + N(-\delta_h u, \delta_{-h} u) &= 0, &\text{in } O, \\
  u &= g, &\text{on } \partial_p O^h,
\end{cases}
\end{aligned}
$$

where $0 < h < h_0$ with $h_0$ fixed a priori and less or equal than the length of $\Omega$, $N : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ are continuous functions. Moreover, we assume that $N$ is *degenerate elliptic* and by that mean that $N$ is increasing in each variable. This is a necessary property to guarantee the consistency of the definition of viscosity solution that we introduce next. Additionally, we also need to consider maximum/minimum points global in $x$ and local in $t$. Therefore to make this exposition easier to understand we introduce the following definition.
Definition 2.1. Let \( f : \Omega^h \rightarrow \mathbb{R} \) be a function.

(i) We say that \((x_0, t_0) \in \Omega^h\) is a viscosity maximum point of \(f\) if there is \(\Delta t > 0\) such that for any \((x, t) \in N_{\Delta t}^t := \{(x, t) \in \Omega^h : |t - t_0| \leq \Delta t\}, f(x, t) \leq f(x_0, t_0)\).

(ii) We say that \((x_0, t_0) \in \Omega^h\) is a viscosity minimum point of \(f\) if \((x_0, t_0)\) is a viscosity maximum point of \(-f\).

As for the test functions there is no need to consider \(C^2\) functions as test functions since we merely have a time derivative in our equation. Hence we will consider test functions belonging to the set \(T(\Omega^h)\) defined as
\[
T(\Omega^h) := \{\varphi \in C(\Omega^h) : \varphi(x, \cdot) \in C^1((0, +\infty)) \text{ for } x \in \Omega\}.
\]

We can now give the definition of viscosity solution of (3).

Definition 2.2. Let \( u : \Omega^h \rightarrow \mathbb{R} \) be a function. Then:

(i) \(u\) is a viscosity subsolution of the problem (3) if and only if \(u^* \leq g\) in \(\partial_p \Omega^h\) and for all \(\varphi \in T(\Omega^h)\) if \((x_0, t_0) \in \Omega\) is a viscosity maximum point of \(u^* - \varphi\) then
\[
\varphi_t(x_0, t_0) + N(-\delta_h \varphi(x_0, t_0), \delta_{-h} \varphi(x_0, t_0)) \leq 0.
\]

(ii) \(u\) is a viscosity supersolution of the problem (3) if and only if \(u_\ast \geq g\) in \(\partial_p \Omega^h\) and for all \(\varphi \in T(\Omega^h)\) if \((x_0, t_0) \in \Omega\) is a viscosity minimum point of \(u_\ast - \varphi\) then
\[
\varphi_t(x_0, t_0) + N(-\delta_h \varphi(x_0, t_0), \delta_{-h} \varphi(x_0, t_0)) \geq 0.
\]

The function \(u\) is said to be a viscosity solution of (3) if it is both a viscosity subsolution and supersolution of (3).

As mentioned before, the definition of viscosity solution just given is consistent with the one of classical solution, a fact that we illustrate in the next Proposition.

Proposition 2.3. Suppose that \(u \in T(\Omega^h)\). Then \(u\) is a classical solution of (3) if and only if \(u\) is a viscosity solution of (3).

Proof. See [9].

We present next the comparison principle, for which, interestingly enough, we do not need additional hypothesis on \(N\).

Theorem 2.4. Let \(u \in USC(\Omega^h)\) and \(v \in LSC(\Omega^h)\) be, respectively, a subsolution and a supersolution of the problem (3). Then \(u \leq v\) in \(\Omega^h\).

Proof. See [9].
Theorem 2.5. Assume that there is a subsolution \( \underline{u} \) and a supersolution \( \overline{u} \) of (3) that satisfy the boundary condition \( \underline{u}(x,t) = \overline{u}(x,t) = g(x,t) \) for \( (x,t) \in \partial_p \Omega^h \) and such that \( \overline{u}(x,t) < +\infty \) for any \( (x,y) \in \Omega^h \). Then

\[
\underline{u}(x,t) = \sup \{ \underline{u}(x,t) : \underline{u} \leq \overline{u} \text{ and } \overline{u} \text{ is a subsolution of (3)} \}
\]
is the unique continuous solution of (3).

Proof. See [9]. \( \square \)

3 Convergence through the adjoint method

In this Section we outline the proof of our main result, Theorem 1.1, where we will use the adjoint method adapting the ideas from [5]. We refer the reader to [9] for all the proofs of the results presented below.

Ideally we would like to apply the adjoint method directly to the functions \( u^h \) solution of (2). However, for that to be the case \( u^h \) has to be smooth enough and the Perron method does not give us the needed regularity. Hence we fix \( T > 0 \) and consider instead the solution \( u^{h,\varepsilon} \) of

\[
\begin{aligned}
& \begin{cases} 
  u_t - \varepsilon \Delta u = -F(-\delta_h u, \delta_{-h} u), & \text{in } \partial_p \Omega^h, \\
  u = \tilde{g}, & \text{on } \partial_p \Omega^h, \\
\end{cases} \\
& (\Omega = (0,1), \partial_p \Omega^h = \Omega_h \times [0,T]) \text{ and } \partial_p \Omega^h = \Omega^h \setminus \partial_p \Omega.
\end{aligned}
\]

where \( \Omega = (0,1), \partial_p \Omega = \Omega \times (0,T), \partial_p \Omega^h = \Omega_h \times [0,T] \) and \( \partial_p \Omega = \partial_p \Omega^h \setminus \partial_p \Omega \).

Let us consider the formal linearized operator \( L^{h,\varepsilon} \) given by

\[
v \mapsto L^{h,\varepsilon} v = u_t - \varepsilon \Delta u - D_p F(\delta_h v) + D_q F(\delta_{-h} v),
\]

where \( D_p F \) and \( D_q F \) are evaluated at \( (-\delta_h u^{h,\varepsilon}, \delta_{-h} u^{h,\varepsilon}) \) which we omit to simplify the notation.

In order to apply the adjoint method to \( u^{h,\varepsilon} \) we need also to consider, for each \( 0 < h < 1, \varepsilon > 0, t_1 \in [0,T] \) and \( \varphi \in C^\infty(\Omega^h) \) with \( \text{supp} \varphi \subseteq \Omega, \varphi \geq 0 \) and \( \int_\Omega \varphi(x) \, dx = 1 \), the adjoint variable \( \sigma^{h,\varepsilon,t_1,\varphi} \) solution of

\[
\begin{aligned}
& \begin{cases} 
  -\sigma_t - \varepsilon \Delta \sigma = -\delta_{-h}(\sigma D_p F) + \delta_h(\sigma D_q F), & \text{in } \Omega_0, \\
  \sigma(x,t_1) = \varphi(x)1_{\{t=t_1\}}, & \text{on } \partial_p \Omega^h_0, \\
\end{cases} \\
& (\Omega_0 = \Omega \times [0,t_1], \Omega_0^h = \Omega_h \times [0,t_1]) \text{ and } \partial_p \Omega^h_0 = \Omega^h_0 \setminus \partial_p \Omega_0.
\end{aligned}
\]

Proposition 3.1. Let \( 0 < h < 1, \varepsilon > 0 \). Then there exists a unique solution \( u^{h,\varepsilon} \) of (4) with

\[
u^{h,\varepsilon} \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \text{ and } (u^{h,\varepsilon})' \in L^2(0,T; L^2(\Omega)).
\]

Proposition 3.2. Let \( 0 < h < 1, \varepsilon > 0, t_1 \in [0,T] \) and \( \varphi \in C^\infty(\Omega) \) with \( \text{supp} \varphi \subseteq \Omega, \varphi \geq 0 \) and \( \int_\Omega \varphi(x) \, dx = 1 \). Then there exists a unique solution \( \sigma^{h,\varepsilon,t_1,\varphi} \) of (5) with

\[
\sigma^{h,\varepsilon,t_1,\varphi} \in L^2(0,t_1; H^2(\Omega)) \cap L^\infty(0,t_1; H^1(\Omega)) \text{ and } (\sigma^{h,\varepsilon,t_1,\varphi})' \in L^2(0,t_1; L^2(\Omega)).
\]
It turns out that $\sigma^{h,\varepsilon,t_1,\psi}$ has some nice properties.

**Proposition 3.3.** Let $0 < h < 1$, $\varepsilon > 0$, $t_1 \in [0,T]$ and $\psi \in C^\infty(\Omega)$ with $\text{supp}\,\psi \subseteq \Omega$, $\psi \geq 0$ and $\int_\Omega \psi(x) \, dx = 1$. Then for every $t \in [0,t_1]$, $\sigma^{h,\varepsilon,t_1,\psi}(s,t) \geq 0$ and

$$
\int_0^{t_1} \sigma^{h,\varepsilon,t_1,\psi}(x,t) \, dx = \varepsilon \int_t^{t_1} \left( \sigma^{h,\varepsilon,t_1,\psi}(0,s) - \sigma^{h,\varepsilon,t_1,\psi}(1,s) \right) \, ds + \frac{1}{h} \int_1^{h} \int_0^{t_1} \sigma^{h,\varepsilon,t_1,\psi} D_F \, dx \, ds + \frac{1}{h} \int_0^{t_1} \sigma^{h,\varepsilon,t_1,\psi} D \, dx \, ds = 1.
$$

From the regularity of $u^{h,\varepsilon}$ we can prove the following Propositions.

**Proposition 3.4.** The following equalities are satisfied in the weak sense in $\Omega \times [0,T]$:

$$
\begin{align*}
L^{h,\varepsilon} u^{h,\varepsilon}_x &= 0, \\
L^{h,\varepsilon} u^{h,\varepsilon}_{xx} + D_{pq} \Delta^{h,\varepsilon} + D_{pq} \nabla \nabla^{h,\varepsilon} &= 0, \\
L^{h,\varepsilon} u^{h,\varepsilon} + \frac{1}{h} D^{h,\varepsilon} &= 0, \\
\left( u^{h,\varepsilon}_{xx} \right)_{t_1} &= 0,
\end{align*}
$$

where $u^{h,\varepsilon} = (u^{h,\varepsilon})^2/2$ and $u^{h,\varepsilon}_x = \partial u^{h,\varepsilon}/\partial x$.

**Proposition 3.5.** Let $0 < h < 1$ and $\varepsilon > 0$. Then for every $t \in [0,T]$

$$
\begin{align*}
\|u^{h,\varepsilon}_x(t)\|_{L^\infty(\Omega)} &\leq \|\tilde{g}\|_{L^\infty([-1,2] \times [0,T])}, \\
\|u^{h,\varepsilon}_{xx}(t)\|_{L^\infty([-1,2] \times [0,T])} &\leq \|\tilde{g}\|_{L^\infty([-1,2] \times [0,T])}, \\
\|\delta \tilde{h} u^{h,\varepsilon}(t)\|_{L^\infty(\Omega)} &\leq \|\tilde{g}\|_{L^\infty([-1,2] \times [0,T])}.
\end{align*}
$$

In particular,

$$
\begin{align*}
(u^{h,\varepsilon}_x)_{t_1} &\geq h \|\tilde{g}\|_{L^\infty([-1,2] \times [0,T])}, \\
(u^{h,\varepsilon}_{xx})_{t_1} &\geq h \|\tilde{g}\|_{L^\infty([-1,2] \times [0,T])}.
\end{align*}
$$

**Proposition 3.6.** There exists a positive constant $C$, independent of $h$ and $\varepsilon$, such that

$$
\|u^{h,\varepsilon}_x(t)\|_{L^\infty(\Omega)} \leq \frac{1}{\sqrt{h}} C(1 + t),
$$

for every $0 < h < 1$ and $t \in (0, +\infty)$.

Finally, we can prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $0 < h_1 < h_2 < 1$ and $\varepsilon > 0$. We have

$$
u^{h_2,\varepsilon}(x,t) - u^{h_1,\varepsilon}(x,t) = \int_{h_1}^{h_2} u^{h,\varepsilon}_x(x,t) \, dh.
$$

Hence

$$
\|u^{h_2,\varepsilon}(t) - u^{h_1,\varepsilon}(t)\|_{L^\infty(\Omega)} \leq C(1 + t) \int_{h_1}^{h_2} \frac{1}{\sqrt{h}} dh \leq C(1 + t) \left| \sqrt{h_2} - \sqrt{h_1} \right|.
$$
Due to the stability of weak solutions we have that $u^{h,\varepsilon} \rightarrow u^h$ as $\varepsilon \rightarrow 0$. Hence by taking the limit as $\varepsilon \rightarrow 0$ in the previous inequality, we get
\[
\| u^{h_2}(\cdot,t) - u^{h_1}(\cdot,t) \|_{L^\infty(\Omega)} \leq C(1 + t) \left| \sqrt{h_2} - \sqrt{h_1} \right|.
\]
Hence the functions $u^h$ form a Cauchy sequence in a Banach space and therefore are convergent as $h \rightarrow 0$ which allows us to obtain
\[
\sup_{t \in [0,T]} \| u(\cdot,t) - u^h(\cdot,t) \|_{L^\infty(\Omega)} \leq C\sqrt{h},
\]
for some positive constant $C = C(T)$ as desired. \qed

References


